2-dimensional stable pairs on 4-folds

Amin Gholampour

Joint work with Yunfeng Jiang and Jason Lo

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- **Examples**: A general cubic surface in $\mathbb{P}^3$ contains 27 lines (Cayley 1849); there are 80,160 twisted cubics meeting 12 general lines, and 5,819,539,783,680 twisted cubics tangent to 12 general quadric surfaces in $\mathbb{P}^3$ (Schubert 1879); a general quintic threefold in $\mathbb{P}^4$ contains 2875 lines and 609,250 conics (Katz 1986) etc.

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- In Donaldson-Thomas theory (for X a 3-fold) the compactification is achieved by considering the *Hilbert scheme* of 1-dimensional subschemes of X in class β. So the curves are free to get any types of singularities, to become reducible and non-reduced, and even to have 0-dimensional components (roaming points).

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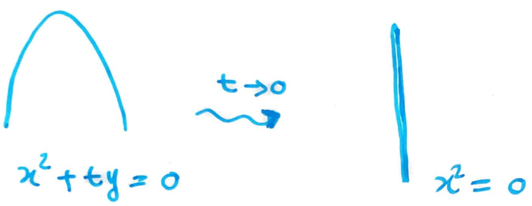
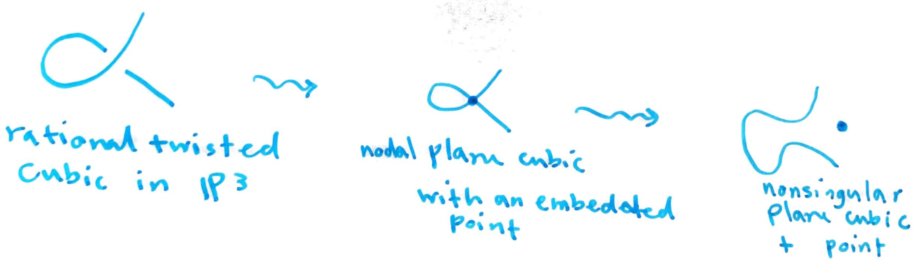
- I will talk about another curve counting theory in the next Section. Because of such complications in the boundaries, these deformations invariants numbers (DT invariants, GW invariants,...) may differ from the actual counts of the curves (they are called the virtual counts of curves in class β).

Examples of degenerations





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  C can be non-reduced or reducible but is not allowed to have embedded points.
- **Example**: Explain the limit of stable pairs in the figure above.

- Let ω be a fixed very ample line bundle on X (i.e. a choice of embedding $X \subset \mathbb{P}^N$), and $q(k) \in \mathbb{Q}[k]$ be a polynomial with positive leading coefficient.
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  Suppose (F, s) is a pair of a 1-dimensional sheaf F on X and $s \in \Gamma(X, F)$.
- The Hilbert polynomial of F with respect to ω is given by
  $\chi(F(k)) = (\mathrm{ch}_2(F) \cdot \omega)k + \chi(F)$.
  If F is pure $\mathrm{ch}_2(F) = [\mathrm{Supp}(F)]$. Denote $r(F) := \mathrm{ch}_2(F) \cdot \omega$, i.e. the leading coefficient of the Hilbert polynomial.

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- The pair (F, s) is said to be (ω, q)-semistable if F is pure and for any nonzero proper subsheaf $G \subset F$
  $\dfrac{\chi(G(k))}{r(G)} \leqslant \dfrac{\chi(F(k)) + q(k)}{r(F)} \quad k \gg 0$, and in case s factors through G
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- Pandharipande-Thomas: if $q(k) \gg 0$ then
  (ω, q)-semistability $\Leftrightarrow$ (ω, q)-stability $\Leftrightarrow$ PT-stability.

- Fix $n \in \mathbb{Z}$, $\beta \in H_2(X, \mathbb{Z})$. Le Potier constructed the moduli space $P_n^{(\omega,q)}(X, \beta)$ of semistable pairs (F, s), such that the Hilbert polynomial of F is $(\beta \cdot \omega)k + n$.

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- To define PT invariants one need to be able to "integrate" over this moduli space.

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- Obstruction theory of such a pair is governed by the Ext groups $\operatorname{Ext}^i(I[-1], F)$ for $i \geqslant 0$:
  There is an obstruction class in $\operatorname{Ext}^1(I[-1], F)$ whose vanishing is equivalent to the existence of an infinitesimal extension of (F, s), and if the obstruction class is 0 then the infinitesimal deformations form a torsor (principal homogeneous space) for $\operatorname{Ext}^0(I[-1], F)$.

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- The latter obstruction theory is perfect.
  Behrend-Fantechi: There is a virtual fundamental class $[P_n(X, \beta)]^{\mathrm{vir}} \in A_{\mathrm{vd}}(P_n(X, \beta))$,
  where $\mathrm{vd} := \mathrm{ext}^1(I, I)_0 - \mathrm{ext}^2(I, I)_0 = -K_X \cdot \beta$ is called the *virtual dimension* of $P_n(X, \beta)$.

- PT invariants are defined by integrating against $[P_n(X,\beta)]^{\mathrm{vir}}$.
  If X is Calabi-Yau then $K_X \cong \mathcal{O}_X$ and so the virtual dimension is 0, and
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  Similarly, define $Z_\beta^{DT}(X,q) = \sum_n I_{n,\beta} q^n \in \mathbb{Z}((q))$, $Z_\beta^{GW}(X,u) = \sum_g N^\bullet_{g,\beta} u^{2g-2} \in \mathbb{Q}((u))$.

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- Conjecture: $Z^{PT}(q)$ and $Z^{DT}(q)/M(-q)^{\chi(X)}$ are the Laurent expansion of a rational function in q invariant under $q \leftrightarrow q^{-1}$, where $M(q) = \prod_{n\geqslant 1}(1-q^n)^{-n}$ is the MacMahon function.

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- Conjecture: After the variable change $-q = e^{iu}$
  $Z^{DT}(q)/M(-q)^{\chi(X)} = Z^{PT}(q) = Z^{GW}(u)$.

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- There are highly nontrivial relations among GW, DT and PT invariants.
  GW $\leftrightarrow$ PT $\leftrightarrow$ DT.
- For a fixed $\beta \neq 0$ define the generating series of PT invariants (X Calabi-Yau)
  $Z_\beta^{PT}(X,q) = \sum_n P_{n,\beta} q^n \in \mathbb{Z}((q))$.
  Similarly, define $Z_\beta^{DT}(X,q) = \sum_n I_{n,\beta} q^n \in \mathbb{Z}((q))$, $Z_\beta^{GW}(X,u) = \sum_g N^\bullet_{g,\beta} u^{2g-2} \in \mathbb{Q}((u))$.
- Conjecture: $Z^{PT}(q)$ and $Z^{DT}(q)/M(-q)^{\chi(X)}$ are the Laurent expansion of a rational function in q invariant under $q \leftrightarrow q^{-1}$, where $M(q) = \prod_{n \geqslant 1}(1-q^n)^{-n}$ is the MacMahon function.
- Conjecture: After the variable change $-q = e^{iu}$
  $Z^{DT}(q)/M(-q)^{\chi(X)} = Z^{PT}(q) = Z^{GW}(u)$.
- These conjectures have been proven in many important special cases.
  Formulated by Maulik-Nekrasov-Okounkov-Pandharipande and Pandharipande-Thomas.
  Maulik-Oblomkov-Okounkov-Pandharipande proved DT/GW correspondence in Toric case.
  Bridgeland gave a proof of DT/PT correspondence in Calabi-Yau case using the language of *motivic Hall algebras*.
  Pandharipande-Pixton gave a proof of GW/PT correspondence for Calabi-Yau complete intersections.
  Toda formulated and proved higher rank version of DT/PT correspondence.
  Many other people have made significant contributions.

Le Potier's stability condition

- X nonsingular projective 4-fold over $\mathbb{C}$ with a choice of a very ample line bundle ω.

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- A *2-dimensional stable pair* (F, s) on X in class ch consists of

$$\begin{cases} F & \text{pure 2-dimensional sheaf and } \mathrm{ch}(F) = \mathrm{ch}, \\ s \in \Gamma(X, F) & \text{is a global section of } F, \\ \mathrm{Coker}(\mathcal{O}_X \xrightarrow{s} F) & \text{is at most 1-dimensional.} \end{cases}$$

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  If S is Cohen-Macaulay $\mathrm{Coker}(\mathcal{O}_X \xrightarrow{s} F)$ is pure 1-dimensional.
- As in the case of PT pairs the stability of 2d stable pairs may be realized as a limit of Le Potier's stability (i.e when $q(k) \gg 0$).
  As a result, there is a fine moduli space $P(X, \mathrm{ch})$ (independent of ω, q) for the 2d stable pairs in class ch, which is a projective scheme over $\mathbb{C}$.

- <u>Goal</u>: Identify $P(X, \mathrm{ch})$ with a moduli space of objects in $D^b(X)$.
  Think of a 2d stable pair (F, s) as a 2-term complex $J := [\mathcal{O}_X \xrightarrow{s} F]$ with $\mathcal{O}_X$ is in degree -1 and F is in degree 0.
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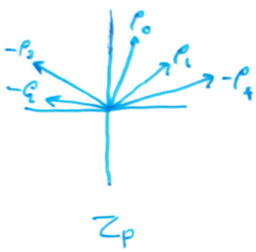

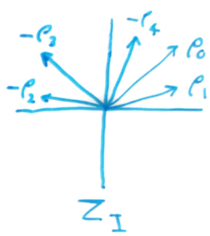

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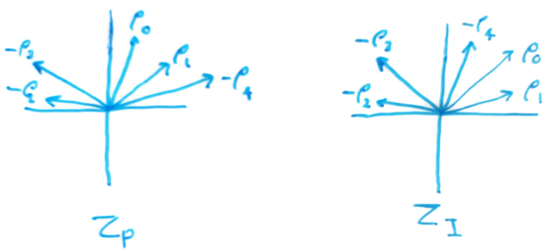

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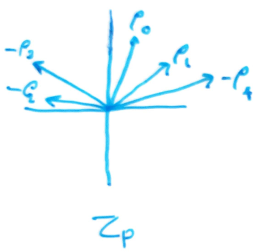

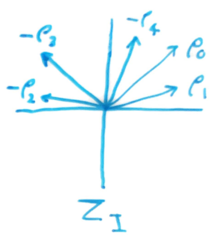

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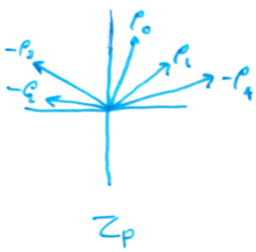

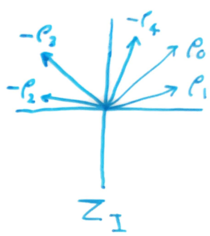

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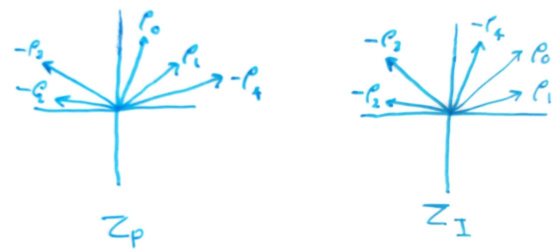

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- $Z_I \mid Z_p$-wall-crossing: Is interpreted as $-\rho_4$ rotating clockwise from its location in the right hand picture arrangement past ρ_0 and then past ρ_1.
- We have also proven higher rank versions of these results.

Categorical relations

- For full subcategories $\mathcal{C}_1, \mathcal{C}_2 \subset D^b(X)$ let $[\mathcal{C}_1, \mathcal{C}_2] \subset D^b(X)$ (resp. $[\![\mathcal{C}_1, \mathcal{C}_2]\!] \subset D^b(X)$) be the full subcategory consisting of $E \in D^b(X)$ fitting in an exact triangle
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- **Theorem**: Suppose $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ are two torsion pairs in $\mathrm{Coh}^{\leqslant 1}(X)$ such that $\mathcal{T}_1 \subset \mathcal{T}_2$ then $[\![\mathcal{T}_2 \cap \mathcal{F}_1, \mathrm{Coh}_{\mu}^{\mathcal{T}_2}(X)]\!] = [\![\mathrm{Coh}_{\mu}^{\mathcal{T}_1}(X), \mathcal{T}_2 \cap \mathcal{F}_1]\!]$.

Categorical relations

- For full subcategories $\mathcal{C}_1, \mathcal{C}_2 \subset D^b(X)$ let $[\mathcal{C}_1, \mathcal{C}_2] \subset D^b(X)$ (resp. $[\![\mathcal{C}_1, \mathcal{C}_2]\!] \subset D^b(X)$) be the full subcategory consisting of $E \in D^b(X)$ fitting in an exact triangle

$$E_1 \longrightarrow E \longrightarrow E_2 \xrightarrow{[1]} E_1$$

where $E_1 \in \mathcal{C}_1$ and $E_2 \in \mathcal{C}_2$ (resp. and $\mathrm{Hom}(\mathcal{C}_1, \mathcal{C}_2) = 0$).
- For any $\mu \in \mathbb{R}$ and a *torsion pair* $(\mathcal{T}, \mathcal{F})$ in $\mathrm{Coh}^{\leqslant 1}(X)$ (e.g. $(\mathrm{Coh}^{=0}(X), \mathrm{Coh}^{=1}(X))$) let $\mathrm{Coh}_\mu^{\mathcal{T}}(X) := \{E \in D^{[-1,0]}(X) \mid h^{-1}(E)\ \mu_\omega\text{-ss t.f. of slope } \mu, \quad h^0(E) \in \mathcal{T}, \quad \mathrm{Hom}(\mathcal{T}, E) = 0\}$.
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- **Corollary 1**: $[\![\mathcal{T}, \mathrm{Coh}_\mu^{\mathcal{T}}(X)]\!] = [\![\mathrm{Coh}_\mu(X)[1], \mathcal{T}]\!]$.
(Take $\mathcal{T}_1 = 0, \mathcal{F}_1 = \mathrm{Coh}^{\leqslant 1}(X), \mathcal{T}_2 = \mathcal{T}$.)

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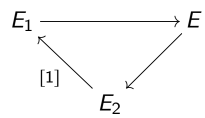

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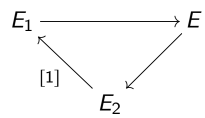
- **Corollary 1**: $[\![\mathcal{T}, \mathrm{Coh}^{\mathcal{T}}_{\mu}(X)]\!] = [\![\mathrm{Coh}_{\mu}(X)[1], \mathcal{T}]\!]$.
(Take $\mathcal{T}_1 = 0, \mathcal{F}_1 = \mathrm{Coh}^{\leqslant 1}(X), \mathcal{T}_2 = \mathcal{T}$.)
- **Corollary 2**: For any $b \in \mathbb{R}$ $\quad [\![\mathcal{C}_{(-\infty,b)}(X), \mathrm{Coh}^{\mathrm{Coh}^{\leqslant 1}(X)}_{\mu}(X)]\!] = [\![\mathrm{Coh}^{\mathcal{C}_{[b,\infty]}(X)}_{\mu}(X), \mathcal{C}_{(-\infty,b)}(X)]\!]$,
where for any interval $I \quad \mathcal{C}_I(X) := \left\langle G \in \mathrm{Coh}^{\leqslant 1}(X) \mid G \text{ is } \overline{\mu}_\omega\text{-ss}, \quad \overline{\mu}_\omega(G) \in I \right\rangle$,
and $\overline{\mu}_\omega(-) := \dfrac{\mathrm{ch}_4(-)}{\omega \cdot \mathrm{ch}_3(-)}$.
(Take $\mathcal{T}_1 = \mathcal{C}_{[b,\infty]}(X), \mathcal{F}_1 = \mathcal{C}_{(-\infty,b)}(X), \mathcal{T}_2 = \mathrm{Coh}^{\leqslant 1}(X), \mathcal{F}_2 = 0$.)

Categorical relations

- For full subcategories $\mathcal{C}_1, \mathcal{C}_2 \subset D^b(X)$ let $[\mathcal{C}_1, \mathcal{C}_2] \subset D^b(X)$ (resp. $[\![\mathcal{C}_1, \mathcal{C}_2]\!] \subset D^b(X)$) be the full subcategory consisting of $E \in D^b(X)$ fitting in an exact triangle



  where $E_1 \in \mathcal{C}_1$ and $E_2 \in \mathcal{C}_2$ (resp. and $\operatorname{Hom}(\mathcal{C}_1, \mathcal{C}_2) = 0$).
- For any $\mu \in \mathbb{R}$ and a *torsion pair* $(\mathcal{T}, \mathcal{F})$ in $\operatorname{Coh}^{\leqslant 1}(X)$ (e.g. $(\operatorname{Coh}^{=0}(X), \operatorname{Coh}^{=1}(X))$) let $\operatorname{Coh}_{\mu}^{\mathcal{T}}(X) := \{E \in D^{[-1,0]}(X) \mid h^{-1}(E)\ \mu_{\omega}\text{-ss t.f. of slope } \mu, \quad h^0(E) \in \mathcal{T}, \quad \operatorname{Hom}(\mathcal{T}, E) = 0\}$.
- **Theorem**: Suppose $(\mathcal{T}_1, \mathcal{F}_1)$ and $(\mathcal{T}_2, \mathcal{F}_2)$ are two torsion pairs in $\operatorname{Coh}^{\leqslant 1}(X)$ such that $\mathcal{T}_1 \subset \mathcal{T}_2$ then $[\![\mathcal{T}_2 \cap \mathcal{F}_1, \operatorname{Coh}_{\mu}^{\mathcal{T}_2}(X)]\!] = [\![\operatorname{Coh}_{\mu}^{\mathcal{T}_1}(X), \mathcal{T}_2 \cap \mathcal{F}_1]\!]$.
- **Corollary 1**: $[\![\mathcal{T}, \operatorname{Coh}_{\mu}^{\mathcal{T}}(X)]\!] = [\![\operatorname{Coh}_{\mu}(X)[1], \mathcal{T}]\!]$.
  (Take $\mathcal{T}_1 = 0, \mathcal{F}_1 = \operatorname{Coh}^{\leqslant 1}(X), \mathcal{T}_2 = \mathcal{T}$.)
- **Corollary 2**: For any $b \in \mathbb{R}$ $\quad [\![\mathcal{C}_{(-\infty,b)}(X), \operatorname{Coh}_{\mu}^{\operatorname{Coh}^{\leqslant 1}(X)}(X)]\!] = [\![\operatorname{Coh}_{\mu}^{\mathcal{C}_{[b,\infty]}(X)}(X), \mathcal{C}_{(-\infty,b)}(X)]\!]$,
  where for any interval I $\quad \mathcal{C}_I(X) := \big\langle G \in \operatorname{Coh}^{\leqslant 1}(X) \mid G \text{ is } \overline{\mu}_{\omega}\text{-ss}, \quad \overline{\mu}_{\omega}(G) \in I \big\rangle$,
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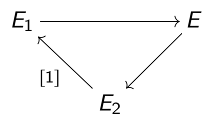
  (Take $\mathcal{T}_1 = \mathcal{C}_{[b,\infty]}(X), \mathcal{F}_1 = \mathcal{C}_{(-\infty,b)}(X), \mathcal{T}_2 = \operatorname{Coh}^{\leqslant 1}(X), \mathcal{F}_2 = 0$.)
- We show that for any fixed and effective class $\gamma \in H^4(X, \mathbb{Z})$
  $\operatorname{Coh}_0^{\operatorname{Coh}^{\leqslant 1}(X)}(X; -1, \mathcal{O}_X, \gamma)$ consists precisely of 2d stable pairs on X.
  Also, $\operatorname{Coh}_0(X; 1, \mathcal{O}_X, -\gamma)$ consists precisely of the ideal sheaves of 2d subschemes of X.

- For full subcategories $\mathcal{C}_1, \mathcal{C}_2 \subset D^b(X)$ let $[\mathcal{C}_1, \mathcal{C}_2] \subset D^b(X)$ (resp. $[\![\mathcal{C}_1, \mathcal{C}_2]\!] \subset D^b(X)$) be the full subcategory consisting of $E \in D^b(X)$ fitting in an exact triangle



where $E_1 \in \mathcal{C}_1$ and $E_2 \in \mathcal{C}_2$ (resp. and $\mathrm{Hom}(\mathcal{C}_1, \mathcal{C}_2) = 0$).
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- **Corollary 1**: $[\![\mathcal{T}, \mathrm{Coh}_{\mu}^{\mathcal{T}}(X)]\!] = [\![\,\mathrm{Coh}_{\mu}(X)[1], \mathcal{T}]\!]$.
(Take $\mathcal{T}_1 = 0, \mathcal{F}_1 = \mathrm{Coh}^{\leqslant 1}(X), \mathcal{T}_2 = \mathcal{T}$.)
- **Corollary 2**: For any $b \in \mathbb{R}$ $\quad [\![\mathcal{C}_{(-\infty,b)}(X), \mathrm{Coh}_{\mu}^{\mathrm{Coh}^{\leqslant 1}(X)}(X)]\!] = [\![\,\mathrm{Coh}_{\mu}^{\mathcal{C}_{[b,\infty]}(X)}(X), \mathcal{C}_{(-\infty,b)}(X)]\!]$,
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- We show that for any fixed and effective class $\gamma \in H^4(X, \mathbb{Z})$
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Also, $\mathrm{Coh}_0(X; 1, \mathcal{O}_X, -\gamma)$ consists precisely of the ideal sheaves of 2d subschemes of X.
- In Corollary 1, take $\mathcal{T} = \mathrm{Coh}_{[0,\infty]}(X)$, $\mu = 0$, and in Corollary 2 set $b = 0 = \mu$, and then restrict:
$[\![\mathcal{C}_{[0,\infty]}(X), \mathrm{Coh}_0^{\mathcal{C}_{[0,\infty]}(X)}(X; -1, \mathcal{O}_X, \gamma)]\!] = [\![\,\mathrm{Coh}_0(X; 1, \mathcal{O}_X, -\gamma)[1], \mathcal{C}_{[0,\infty]}(X)]\!]$.
$[\![\mathcal{C}_{(-\infty,0)}(X), \mathrm{Coh}_0^{\mathrm{Coh}^{\leqslant 1}(X)}(X; -1, \mathcal{O}_X, \gamma)]\!] = [\![\,\mathrm{Coh}_0^{\mathcal{C}_{[0,\infty]}(X)}(X; -1, \mathcal{O}_X, \gamma), \mathcal{C}_{(-\infty,0)}(X)]\!]$.

Boundedness results

- Suppose γ is a reduced effective class such that any pure 2-dimensional subscheme of X in class γ is Cohen-Macaulay (i.e. satisfies S_2).

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- The following subsets of $\mathbb{Q}$ are bounded below:

$$B_I := \{\omega \cdot \mathrm{ch}_3(E) \colon E \in \mathrm{Coh}_0(X; 1, \mathcal{O}_X, -\gamma)[1]\},$$
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- For a fixed $\beta \in H^6(X, \mathbb{Q})$, the following subsets of $\mathbb{Q}$ are

$$A_I(\beta) := \{\mathrm{ch}_4(E) \colon E \in \mathrm{Coh}_0(X; 1, \mathcal{O}_X, -\gamma)[1] \ \& \ \mathrm{ch}_3(E) = \beta\} \quad \text{bounded below},$$
$$A_P(\beta) := \{\mathrm{ch}_4(E) \colon E \in \mathrm{Coh}_0^{\mathrm{Coh}^{\leqslant 1}(X)}(X; -1, \mathcal{O}_X, \gamma) \ \& \ \mathrm{ch}_3(E) = \beta\} \quad \text{bounded above},$$
$$A_T(\beta) := \{\mathrm{ch}_4(E) \colon E \in \mathrm{Coh}_0^{\mathcal{C}_{[0,\infty]}(X)}(X; -1, \mathcal{O}_X, \gamma) \ \& \ \mathrm{ch}_3(E) = \beta\} \quad \text{bounded}.$$

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- By the above results

$$m := \min(0, \inf B_I, \inf B_P) \in \mathbb{Q}.$$
$$l(\beta) := \min(0, \inf A_I(\beta), \inf A_T(\beta)) \in \mathbb{Q},$$
$$u(\beta) := \max(0, \sup A_P(\beta), \sup A_T(\beta)) \in \mathbb{Q}.$$

- Let $\mathcal{A}_0(X) := \langle \mathrm{Coh}_{\leqslant 0}(X)[1], \mathrm{Coh}_{>0}(X) \rangle$, where $\mathrm{Coh}_{\leqslant 0}(X) := \big\langle G \in \mathrm{Coh}\, X) \mid G$ is μ_ω-ss, $\quad \mu_\omega(G) \leqslant 0 \big\rangle$, and similarly for $\mathrm{Coh}_{>0}(X)$.

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- $\mathcal{A}_0(X)$ is the heart of a bounded t-structure, and contains all the objects under discussion.
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- Denote by $H(\mathcal{A}_0) := K(\mathrm{St}/\mathcal{O}bj(\mathcal{A}_0))$ the *motivic Hall algebra* of $\mathcal{A}_0$. It is the $\mathbb{Q}$-vector space span of the isomorphism classes $[\mathcal{M} \to \mathcal{O}bj(\mathcal{A}_0)]$, where $\mathcal{M}$ is a finite type stack with affine geometric stabilizers, modulo certain relations such as: $[\mathcal{M}_1 \coprod \mathcal{M}_2 \to \mathcal{O}bj(\mathcal{A}_0)] = [\mathcal{M}_1 \to \mathcal{O}bj(\mathcal{A}_0)] + [\mathcal{M}_2 \to \mathcal{O}bj(\mathcal{A}_0)]$,...(geometric bijections and Zariski fibrations). Let $H_{\mathrm{ch}}(\mathcal{A}_0)$ be the span of $[\mathcal{M} \to \mathcal{O}bj_{\mathrm{ch}}(\mathcal{A}_0) \subset \mathcal{O}bj(\mathcal{A}_0)]$.

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- $H(\mathcal{A}_0)$ has $K(\mathrm{St}/\mathbb{C})$-module structure. It is also equipped with an associative product $\star$ defined by means of the stack of short exact sequences in $\mathcal{A}_0(X)$, denoted by $\mathcal{E}x(\mathcal{A}_0)$: $[\mathcal{M}_1 \to \mathcal{O}bj(\mathcal{A}_0)] \star [\mathcal{M}_2 \to \mathcal{O}bj(\mathcal{A}_0)] = [\mathcal{M}_3 \to \mathcal{O}bj(\mathcal{A}_0)]$, where $\mathcal{M}_3$ is the fiber product as in

$$\begin{array}{ccccc} \mathcal{M}_3 & \longrightarrow & \mathcal{E}x(\mathcal{A}_0) & \xrightarrow{p_M} & \mathcal{O}bj(\mathcal{A}_0) \\ \downarrow & & \downarrow{\scriptstyle (p_L, p_R)} & & \\ \mathcal{M}_1 \times \mathcal{M}_2 & \longrightarrow & \mathcal{O}bj(\mathcal{A}_0) \times \mathcal{O}bj(\mathcal{A}_0) & & \end{array}$$

Hall algebra relations

- Let $\mathcal{A}_0(X) := \langle \mathrm{Coh}_{\leqslant 0}(X)[1], \mathrm{Coh}_{>0}(X) \rangle$, where $\mathrm{Coh}_{\leqslant 0}(X) := \big\langle G \in \mathrm{Coh}\, X) \mid G \text{ is } \mu_\omega\text{-ss}, \quad \mu_\omega(G) \leqslant 0 \big\rangle$, and similarly for $\mathrm{Coh}_{>0}(X)$.
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- For $\mathrm{ch} = (-1, 0, \gamma, \beta, \xi)$ or $\mathrm{ch} = (0,0,0,\beta,\xi)$ define the completions

$$\widehat{H}(\mathcal{A}_0) := \prod_{\substack{\omega \cdot \beta \geqslant m \\ \xi \geqslant l(\beta)}} H_{\mathrm{ch}}(\mathcal{A}_0), \qquad \widetilde{H}(\mathcal{A}_0) := \prod_{\substack{\omega \cdot \beta \geqslant m \\ \xi \leqslant u(\beta)}} H_{\mathrm{ch}}(\mathcal{A}_0),$$

where $m, l(\beta), u(\beta) \in \mathbb{Q}$ were defined in the previous page. By the boundedness results the following memberships hold.

Hall algebra relations

- Let $\mathcal{A}_0(X) := \langle \mathrm{Coh}_{\leqslant 0}(X)[1], \mathrm{Coh}_{>0}(X)\rangle$, where
  $\mathrm{Coh}_{\leqslant 0}(X) := \big\langle G \in \mathrm{Coh}\, X) \mid G \text{ is } \mu_\omega\text{-ss}, \quad \mu_\omega(G) \leqslant 0\big\rangle$, and similarly for $\mathrm{Coh}_{>0}(X)$.
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- Denote by $H(\mathcal{A}_0) := K(\mathrm{St}/\mathcal{O}bj(\mathcal{A}_0))$ the *motivic Hall algebra* of $\mathcal{A}_0$.
  It is the $\mathbb{Q}$-vector space span of the isomorphism classes $[\mathcal{M} \to \mathcal{O}bj(\mathcal{A}_0)]$, where $\mathcal{M}$ is a finite type stack with affine geometric stabilizers, modulo certain relations such as:
  $[\mathcal{M}_1 \coprod \mathcal{M}_2 \to \mathcal{O}bj(\mathcal{A}_0)] = [\mathcal{M}_1 \to \mathcal{O}bj(\mathcal{A}_0)] + [\mathcal{M}_2 \to \mathcal{O}bj(\mathcal{A}_0)]$,...(geometric bijections and Zariski fibrations).
  Let $H_{\mathrm{ch}}(\mathcal{A}_0)$ be the span of $[\mathcal{M} \to \mathcal{O}bj_{\mathrm{ch}}(\mathcal{A}_0) \subset \mathcal{O}bj(\mathcal{A}_0)]$.
- $H(\mathcal{A}_0)$ has $K(\mathrm{St}/\mathbb{C})$-module structure. It is also equipped with an associative product $\star$ defined by means of the stack of short exact sequences in $\mathcal{A}_0(X)$, denoted by $\mathcal{E}x(\mathcal{A}_0)$:
  $[\mathcal{M}_1 \to \mathcal{O}bj(\mathcal{A}_0)] \star [\mathcal{M}_2 \to \mathcal{O}bj(\mathcal{A}_0)] = [\mathcal{M}_3 \to \mathcal{O}bj(\mathcal{A}_0)]$, where $\mathcal{M}_3$ is the fiber product as in

$$\begin{array}{ccccc} \mathcal{M}_3 & \longrightarrow & \mathcal{E}x(\mathcal{A}_0) & \xrightarrow{p_M} & \mathcal{O}bj(\mathcal{A}_0) \\ \downarrow & & \downarrow{\scriptstyle (p_L, p_R)} & & \\ \mathcal{M}_1 \times \mathcal{M}_2 & \longrightarrow & \mathcal{O}bj(\mathcal{A}_0) \times \mathcal{O}bj(\mathcal{A}_0) & & \end{array}$$

- For $\mathrm{ch} = (-1, 0, \gamma, \beta, \xi)$ or $\mathrm{ch} = (0, 0, 0, \beta, \xi)$ define the completions

$$\widehat{H}(\mathcal{A}_0) := \prod_{\substack{\omega \cdot \beta \geqslant m \\ \xi \geqslant l(\beta)}} H_{\mathrm{ch}}(\mathcal{A}_0), \qquad \widetilde{H}(\mathcal{A}_0) := \prod_{\substack{\omega \cdot \beta \geqslant m \\ \xi \leqslant u(\beta)}} H_{\mathrm{ch}}(\mathcal{A}_0),$$

  where $m, l(\beta), u(\beta) \in \mathbb{Q}$ were defined in the previous page. By the boundedness results the following memberships hold.
- Let

$$\delta_{\mathcal{C}}^{\geqslant 0} \in \widehat{H}(\mathcal{A}_0), \quad \delta_{\mathcal{C}}^{<0} \in \widetilde{H}(\mathcal{A}_0), \quad \delta_I(\gamma) \in \widehat{H}(\mathcal{A}_0)$$
$$\delta_P(\gamma) \in \widetilde{H}(\mathcal{A}_0), \quad \widetilde{H}(\mathcal{A}_0) \ni \delta_T(\gamma) \in \widehat{H}(\mathcal{A}_0),$$

  which respectively correspond to the moduli stacks of objects in the categories

$$\mathcal{C}_{[0,\infty]}(X), \quad \mathcal{C}_{(-\infty,0)}(X), \quad \mathrm{Coh}_0(X; -1, \mathcal{O}_X, \gamma)[1],$$
$$\mathrm{Coh}_0^{\mathrm{Coh}^{\leqslant 1}(X)}(X; -1, \mathcal{O}_X, \gamma), \quad \mathrm{Coh}_0^{\mathcal{C}_{[0,\infty]}(X)}(X; -1, \mathcal{O}_X, \gamma).$$

Hall algebra relations

- Let $\mathcal{A}_0(X) := \langle \mathrm{Coh}_{\leqslant 0}(X)[1], \mathrm{Coh}_{>0}(X)\rangle$, where
  $\mathrm{Coh}_{\leqslant 0}(X) := \big\langle G \in \mathrm{Coh}\, X) \mid G \text{ is } \mu_\omega\text{-ss}, \quad \mu_\omega(G) \leqslant 0\big\rangle$, and similarly for $\mathrm{Coh}_{>0}(X)$.
- $\mathcal{A}_0(X)$ is the heart of a bounded t-structure, and contains all the objects under discussion.
  Let $\mathcal{O}bj(\mathcal{A}_0)$ be the stack of the objects in $\mathcal{A}_0(X)$, and $\mathcal{O}bj_{\mathrm{ch}}(\mathcal{A}_0) \subset \mathcal{O}bj(\mathcal{A}_0)$ be the substack of the objects with Chern character ch.
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- The categorical relations of the last page imply
  Theorem:

$$\begin{aligned}
\delta_I(\gamma) \star \delta_{\mathcal{C}}^{\geqslant 0} &= \delta_{\mathcal{C}}^{\geqslant 0} \star \delta_T(\gamma) \quad \text{in } \widehat{H}(\mathcal{A}_0), \\
\delta_{\mathcal{C}}^{<0} \star \delta_P(\gamma) &= \delta_T(\gamma) \star \delta_{\mathcal{C}}^{<0} \quad \text{in } \widetilde{H}(\mathcal{A}_0).
\end{aligned}$$

- As for PT theory the natural obstruction theory of a 2d pair (F, s) that is governed by $\mathrm{Ext}^i(J[-1], F)$, where $J = [\mathcal{O}_X \to F]$, is not perfect.
  By the identification $P(X, \mathrm{ch}) \cong \mathcal{M}_{\mathcal{O}_X}(X, \mathrm{ch}')$ we can instead use the fixed-determinant obstruction theory of the object $J \in D^b(X)$ that is governed by $\mathrm{Ext}^i(J, J)_0$.

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- Now suppose X is a Calabi-Yau 4-fold. It can be shown that $\operatorname{Ext}^i(J, J)_0 \neq 0$ only for $i = 1, 2, 3$. By Serre duality $\operatorname{Ext}^1(J, J)_0 \cong \operatorname{Ext}^3(J, J)_0^*, \qquad \operatorname{Ext}^2(J, J)_0 \cong \operatorname{Ext}^2(J, J)_0^*.$

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- Let $\pi\colon P(X, \mathrm{ch}) \times X \to P(X, \mathrm{ch})$ be the projection and $\mathbb{J} \in D^b(P(X, \mathrm{ch}) \times X)$ be the universal stable pair.
  There is an obstruction theory $\mathbb{E} := R\pi_* R\mathcal{H}om(\mathbb{J}, \mathbb{J})_0[3] \to \mathbb{L}_P$, which is *symmetric* i.e. it comes with a natural isomorphism $\mathbb{E}^\vee \xrightarrow{\cong} \mathbb{E}[-2]$.

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- Oh-Thomas recent theory gives a virtual fundamental class $[P(X, \mathrm{ch})]^{\mathrm{vir}} \in A_{\mathrm{vd}_P/2}(P(X, \mathrm{ch}), \mathbb{Z}[\frac{1}{2}])$, where $\mathrm{vd}_P := 2\,\mathrm{ext}^1(J, J)_0 - \mathrm{ext}^2(J, J)_0 = 2(\xi + \gamma \cdot \mathrm{td}_2(X)) - \gamma^2$.
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- Oh-Thomas construct a localization of $\sqrt{e}(-)$ to the zero set of an isotropic section of an $\text{SO}(2n, \mathbb{C})$-bundle E as well as the *square root Gysin operators* $\sqrt{0_E^!}\colon A_*(C, \mathbb{Z}[\frac{1}{2}]) \to A_{*-n}(Z, \mathbb{Z}[\frac{1}{2}])$, where $C \subset E$ is an isotropic subcone with the zero section Z.

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- Oh-Thomas theory also gives a virtual fundamental class of the Hilbert scheme of 2-dimensional subschemes of X in class ch with the same virtual dimension as vd_P, and hence one can define the invariants by integrating against it. In a work in progress Bae-Kool-Park have found 2d pair/ideal correspondences among these numerical invariants.

- Let S be a nonsingular projective surface, V be a rank 2 vector bundle on S such that $\wedge^2 V \cong K_S$.
  Then $X := \mathrm{tot}(V) \xrightarrow{p} S$ is a quasi-projective Calabi-Yau 4-fold. Let $\mathrm{ch} = (0, 0, [S], \beta, \xi)$ be a compactly supported class.

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- Case 1: If $H^0(V) = 0 = H^{i>0}(\mathcal{O}_S)$ then $P(X, \mathrm{ch})$ is projective and identified with the nested Hilbert scheme $S_\beta^{[n]}$ (n is determined by ξ, β).
  Assume any $L \in \mathrm{Pic}_\beta(S)$ is $(n-1)$-very ample (i.e. the natural map $H^0(L) \to H^0(L|_Z)$ is surjective for any $Z \in S^{[n]}$) then $P(X, \mathrm{ch})$ is smooth and Ob_P is a vector bundle of rank $4n + e(V) - 2$ with fiber over $(Z, D) \in S_\beta^{[n]}$ identified with $\mathrm{Ext}^1(I_Z, V \otimes I_Z)$.

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- Case 2: If $V = L_1 t \oplus L_2 t^{-1}$ is a split rank 2 vector bundle equipped with the $\mathbb{C}^*$-action then $P(X, \mathrm{ch})^{\mathbb{C}^*} \cong S_\beta^{[n]} \hookrightarrow S^{[n]} \times S_\beta$ and the pushforward of $[P(X, \mathrm{ch})^{\mathbb{C}^*}]^{\mathrm{vir}}$ is $c_n(\mathrm{CO}_\beta^{[n]}) \cap [S^{[n]}] \times [S_\beta]^{\mathrm{vir}}$, where $\mathrm{CO}_\beta^{[n]}$ is a rank n tautological bundle over $S^{[n]} \times S_\beta$. As a result, using Oh-Thomas virtual localization formula the 2d stable pair invariants can be expressed in terms of known integrals over $S^{[n]}$ and Seiberg-Witten invariants of S.

- Let Y be a nonsingular Fano projective threefold, $\mathrm{ch}_Y(0, \gamma, \beta, \xi)$ be a Chern character vector, $P(Y, \mathrm{ch}_Y)$ Le Potier's moduli space of 2d stable pairs on Y.

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  $P(X,\mathrm{ch})$ is proper and contains $P(Y,\mathrm{ch}_Y)$ as a closed subscheme.
- If γ is a reduced class then $P(X,\mathrm{ch}) \cong P(Y,\mathrm{ch}_Y)$ so $P(Y,\mathrm{ch}_Y)$ inherits a symmetric obstruction theory (and hence a virtual fundamental class) $\mathbb{E} \to \mathbb{L}_P$ from $P(X,\mathrm{ch})$.
  If $\mathbb{J}_Y = [\mathcal{O}_{P\times Y} \to \mathbb{G}]$ is the universal stable pair on $P(Y,\mathrm{ch}_Y) \times Y$ then
  $\mathbb{E} \cong R\pi_* R\mathcal{H}om(\mathbb{J}_Y[-1],\mathbb{G})[2] \oplus R\pi_* R\mathcal{H}om(\mathbb{J}_Y[-1],\mathbb{G})^\vee$.

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e.g. $X = \mathrm{Ab} \times \mathrm{Ab}$ and $X = \mathrm{Ab} \times \mathrm{K3}$.
- Let Ab be a fiber of p and $\mathrm{ch} = \mathrm{ch}(\mathcal{O}_{\mathrm{Ab}}^{\oplus n})$ for some n. Then
$B^{[n]} \xrightarrow{\cong} P(X, \mathrm{ch}) \qquad Z \mapsto [\mathcal{O}_X \to \mathcal{O}_{p^{-1}(Z)}]$.
Moreover, $\mathrm{vd}_{\mathrm{P}} = 0$ and $\mathrm{Ob}_{\mathrm{P}} \cong T_{B^{[n]}} \oplus \Omega_{B^{[n]}}$ and the corresponding stable pair invariant is
$\deg[P(X, \mathrm{ch})]^{\mathrm{vir}} = e(B^{[n]})$.