CENTRAL LIMIT THEOREM FOR RECURRENT RANDOM WALKS ON A STRIP WITH BOUNDED POTENTIAL

D. DOLGOPYAT AND I. GOLDSHEID

Abstract. We prove that a recurrent random walk (RW) in i.i.d. random environment (RE) on a strip which does not obey the Sinai law exhibits the Central Limit asymptotic behaviour.

We also show that there exists a collection of proper subvarieties in the space of transition probabilities such that

- If RE is stationary and ergodic and the transition probabilities are concentrated on one of subvarieties from our collection then the CLT holds;
- If the environment is i.i.d then the above condition is also necessary for the CLT.

As an application of our techniques we prove the CLT for the quasiperiodic environments with Diophantine frequencies in the recurrent case and complement this result by proving that in the transient case the CLT holds for all strictly ergodic environments.

All these results are valid for one-dimensional RWRE with bounded jumps as a particular case of the strip model.

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1. Introduction: brief history of the problem, motivations, informal description of results.

It is well known that one dimensional RWRE exhibit features which are very different from those of classical random walks. This fact was first discovered in 1975 by Solomon ([27]) and by Kesten, Kozlov, and Spitzer ([17]) for transient random walks on $\mathbb{Z}$ with jumps to nearest neighbours. In 1982, Sinai ([25]) found one of the most striking manifestations of that: he proved that for recurrent nearest neighbour RWRE the correct scaling is $\ln^2 n$.

Transient RWRE were then studied in a number of papers. The overview of this development can be found in [12, 13] and more recent papers [8, 9]. However, here we shall mainly discuss the recurrent case as the main subject of this paper.

Methods used in [25] (as well as in [27, 17]) rely heavily on the fact that the random walk is on $\mathbb{Z}$ and is allowed to jump *only* to the nearest sites. Hence the natural question asked by Sinai in his paper: would it be possible to extend his (and other) results to more general models such as RW on $\mathbb{Z}$ with bounded jumps.

In 1984, Key [20] found a recurrence criterion for RWRE on $\mathbb{Z}$ for the so called $[-l, r]$ model, where r and l are the maximal lengthes of possible jumps of the walk to the right and to the left respectively. Key's criterion was stated in terms of properties of the "middle" Lyapunov exponents of products of random matrices constructed from the parameters of the environment. This approach was developed by Letchikov [21] who in 1998 obtained a partial answer to Sinai's question. He proved that recurrent RWs on $\mathbb{Z}$ with bounded jumps in i.i.d. environment exhibit the Sinai behaviour if the probabilities of jumps of length 1 dominate the probabilities of other jumps.

A comprehensive review of further results obtained by means of these techniques can be found in papers [5], [6], and [7] by Brémont. Comments on the relation between the relevant Brémont's results and the results of this work will be provided later. Here, we mention only that these papers analyze the $[-l, r]$ model ($[-l, 1]$ model in [5]) and that this model reduces to the study of the walks on the strip.

In 2000, Bolthausen and Goldsheid ([2]) introduced RWRE on a strip. They also reduced the study of the RWRE with bounded jumps on $\mathbb{Z}$ to that of RW on a strip and proved the recurrence and transience criterion for the strip model. The technique used in [2] is completely different from that of [20] and [21].

The approach of [2] was developed in [13] where conditions for the Law of Large Numbers and the CLT for transient RWs were provided.

A complete answer to Sinai's question was obtained in [3] where further development of methods from [2] and [13] allowed authors to prove that, unless the parameters of the environment belong to a certain algebraic subvariety, recurrent random walks in i.i.d. environments obey the Sinai behaviour. The description of this subvariety is quite explicit. In particular, this description was used in [3] to show that recurrent finite range RWs in i.i.d. environments on $\mathbb{Z}$ exhibit either the Sinai behaviour or the CLT behaviour. Moreover, the CLT alternative takes place if and only if the walk is a martingale (and hence non-martingale recurrent walks obey the Sinai behaviour).

Until now, it was unclear whether a similar alternative holds for the walks on a strip. One of the goals of this work is to complete the picture by proving that

• *in recurrent i.i.d. environments on a strip there is an alternative: either the walk exhibits the Sinai behaviour or it satisfies the classical Central Limit Theorem.*

This statement completes the classification of possible limiting distributions of the RWRE on the strip (the limiting distributions in the transient case were obtained in [9]).

The above result is a corollary of a theorem which provides

• *a criterion for the CLT for recurrent RWs in ergodic random environments.*

In fact

• *the method we use allows us to establish the CLT for a wide class of recurrent RWs in stationary ergodic environments on a strip.*

In particular we show that

• *recurrent RWs in Diophantine quasi-periodic random environments generated by sufficiently smooth functions satisfy the CLT.*

Finally, we complement this statement by extending to the strip model the result which was proved in [12] for walks on $\mathbb{Z}$ with nearest neighbour jumps by proving that

• *transient RWs on a strip in environments generated by continuous uniquely ergodic transformations of a compact metric space always satisfy the CLT with positive drift.*

Note that the last two statements provide complete classification of the walks in Diophantine quasi-periodic environments. We would like to emphasize that in the transient case no smoothness of the uniquely ergodic transformation is required (in contrast to the recurrent case).

2. Definition of the model and some preparatory facts.

2.1. **The Model.** We recall the definition of the RWRE on a strip from [2]. Consider a strip $\mathbb{S} = \mathbb{Z} \times \{1, \dots, m\}$ and a random walk on $\mathbb{S}$. Let $L_n = \{(n, i) : 1 \le i \le m\}$ be layer n of the strip. In our model, the walk is allowed to jump from any point $(n, i) \in L_n$ only to points in L_{n-1}, or L_n, or L_{n+1}. To define the corresponding transition kernel consider a sequence of triples (P_n, Q_n, R_n), $-\infty < n < \infty$, of $m \times m$ non-negative matrices such that for all $n \in \mathbb{Z}$ the sum $P_n + Q_n + R_n$ is a stochastic matrix. That is,

$$(P_n + Q_n + R_n)\mathbf{1} = \mathbf{1}, \tag{2.1}$$

where $\mathbf{1}$ is a column vector whose components are all equal to 1. The matrix elements of P_n are denoted $P_n(i, j)$, $1 \le i, j \le m$, and similar notations are used for Q_n and R_n. We now set

$$\mathcal{Q}(z, z_1) \stackrel{\text{def}}{=} \begin{cases} P_n(i,j) & \text{if} \quad z = (n,i),\ z_1 = (n+1, j), \\ R_n(i,j) & \text{if} \quad z = (n,i),\ z_1 = (n, j), \\ Q_n(i,j) & \text{if} \quad z = (n,i),\ z_1 = (n-1, j), \\ 0 & \text{otherwise}, \end{cases} \tag{2.2}$$

From now on we suppose that each such sequence is a realization of a strictly stationary ergodic process and let $(\Omega, \mathcal{F}, \mathbb{P}, T)$ be the corresponding dynamical system with Ω denoting the space of all sequences $\omega = (\omega_n)_{n=-\infty}^{\infty} = ((P_n, Q_n, R_n))_{n=-\infty}^{\infty}$ of triples described above, $\mathcal{F}$ being the corresponding natural σ-algebra, $\mathbb{P}$ denoting the probability measure on $(\Omega, \mathcal{F})$, and T being a shift operator on Ω defined by $(T\omega)_n = \omega_{n+1}$.

For a fixed ω we define a random walk $\xi(t) = (X(t), Y(t))$, $t \ge 0$, on $\mathbb{S}$ in the usual way: for any starting point $z = (n, i) \in \mathbb{S}$ and fixed ω the law $P_{\omega,z}$ for the

Markov chain $\xi(\cdot)$ is given by

$$P_{\omega,z}\left(\xi(1)=z_1,\ldots,\xi(t)=z_t\right) \stackrel{\text{def}}{=} \mathcal{Q}_\omega(z,z_1)\mathcal{Q}_\omega(z_1,z_2)\cdots\mathcal{Q}_\omega(z_{t-1},z_t). \tag{2.3}$$

We call ω the *environment* or the *random environment* on the strip $\mathbb{S}$. Denote by Ξ_z the set of trajectories $\xi(\cdot)$ starting at z. $P_{\omega,z}$ is the so called quenched probability measure on Ξ_z. The semi-direct product $\mathbb{P}(d\omega)P_{\omega,z}(d\xi)$ of $\mathbb{P}$ and $P_{\omega,z}$ is defined on the direct product $\Omega\times\Xi_z$ and is called the annealed measure. The corresponding mathematical expectations are denoted by $\mathbb{E}$ and $E_{\omega,z}$.

Remark 2.1. The study of one-dimensional RW with bounded jumps in RE on $\mathbb{Z}$ can be reduced to the study of the above model. The explanation of this fact was given in [2] and later in [13] and [3] and shall not be repeated here.

Denote by $\mathcal{J}$ the following set of triples of $m\times m$ matrices:

$$\mathcal{J} \stackrel{\text{def}}{=} \left\{(P,Q,R)\,:\, P\geq 0,\, Q\geq 0,\, R\geq 0 \ \text{ and } \ (P+Q+R)\mathbf{1}=\mathbf{1}\right\}.$$

Let $\mathcal{J}_0=\mathcal{J}_0(\mathbb{P})\subset\mathcal{J}$ be the support of the probability distribution of the random triple (P_n,Q_n,R_n) defined above (obviously, this support does not depend on n).

Since $\Omega=\mathcal{J}^{\mathbb{Z}}$, it can be endowed by a metric (in many ways). We shall make use of the following metric. For $\omega'=\{(P'_n,Q'_n,R'_n)\}$, $\omega''=\{(P''_n,Q''_n,R''_n)\}$ set

$$\mathbf{d}(\omega',\omega'')=\sum_{n\in\mathbb{Z}}\frac{\|P'_n-P''_n\|+\|Q'_n-Q''_n\|+\|R'_n-R''_n\|}{2^{|n|}}. \tag{2.4}$$

Below, whenever we say that a function defined on Ω is continuous we mean that it is continuous with respect to the topology induced by the metric $\mathbf{d}(\cdot,\cdot)$.

Here and below for a vector $x=(x_i)$ and a matrix $A=(a(i,j))$ we set $\|x\|\stackrel{\text{def}}{=}\max_i|x_i|$ which implies $\|A\|=\sup_{\|x\|=1}\|Ax\|=\max_i\sum_j|a(i,j)|$. We say that A is strictly positive (and write $A>0$), if all its matrix elements satisfy $a(i,j)>0$. A is called non-negative (and we write $A\geq 0$), if all $a(i,j)$ are non negative. A similar convention applies to vectors. Note that if A is a non-negative matrix then $\|A\|=\|A\mathbf{1}\|$.

The following two assumptions **C1** and **C2** listed below will be referred to as Condition **C** which is supposed to be satisfied throughout the paper.

Condition C:

C1: (P_n,Q_n,R_n), $-\infty<n<\infty$, is an ergodic sequence (equivalently, T is an ergodic transformation of Ω).

C2: There is an $\varepsilon>0$ and a positive integer number $k_0<\infty$ such that for any $(P,Q,R)\in\mathcal{J}_0$ and all $i,\,j\in[1,m]$

$$||R^{k_0}||\leq 1-\varepsilon,\quad ((I-R)^{-1}P)(i,j)\geq\varepsilon,\quad ((I-R)^{-1}Q)(i,j)\geq\varepsilon. \tag{2.5}$$

Observe that $((I-R_n)^{-1}P_n)(i,j)$ is the probability that the walker starting from (n,i) arrives to $(n+1,j)$ at her fist exit from the layer L_n. The meaning of $((I-R_n)^{-1}Q_n)(i,j)$ is similar.

We note that condition (2.5) is trivially satisfied if for all (i,j) we have

$$P(i,j) \geq \varepsilon, \quad Q(i,j) \geq \varepsilon, \quad R(i,j) \geq \varepsilon. \tag{2.6}$$

However (2.6) never holds for the environments coming from one dimensional walks with bounded jumps while (2.5) holds in that case under mild non-degeneracy conditions. We refer to [3] for more discussion.

2.2. **Matrices ζ_n, A_n, α_n and some related quantities.** We are now in a position to recall the definitions of several objects most of which were first introduced and studied in [2], [3] and which will play a crucial role in this work.

For a given $\omega \in \Omega$, define a sequence of $m \times m$ stochastic matrices ζ_n as follows. Fix an integer a and a stochastic matrix ψ. For $n \geq a$ define matrices ψ_n as follows. Put $\psi_a = \psi$ and for $n > a$ define recursively

$$\psi_n = \psi_n(a,\psi) = (I - R_n - Q_n\psi_{n-1})^{-1}P_n, \quad n = a+1,\, a+2,\, \ldots. \tag{2.7}$$

It is easy to show (see [2]) that matrices ψ_n are stochastic. Next, for a fixed n define

$$\zeta_n = \lim_{a\to-\infty} \psi_n. \tag{2.8}$$

As shown in [2, Theorem 1] the limit (2.8) exists and is independent of the choice of the initial matrix ψ.

Next, we define probability row-vectors $\sigma_n = \sigma_n(\omega) = (\sigma_n(\omega,1),\ldots,\sigma_n(\omega,m))$ which are associated with the matrices ζ_n. Let $\tilde\sigma$ be an arbitrary probability row-vector (by which we mean that $\tilde\sigma \geq 0$ and $\sum_{i=1}^m \tilde\sigma(i) = 1$). Set

$$\sigma_n \stackrel{\text{def}}{=} \lim_{a\to-\infty} \tilde\sigma\zeta_a\ldots\zeta_{n-1}. \tag{2.9}$$

By the standard contraction property of the product of stochastic matrices, this limit exists and does not depend on the choice of the sequence $\tilde\sigma_a$ (see [13, Lemma 1]). Vectors σ_n could be equivalently defined as the unique sequence of probability vectors satisfying the infinite system of equations

$$\sigma_n = \sigma_{n-1}\zeta_{n-1}, \quad n \in \mathbb{Z}. \tag{2.10}$$

Combining (2.9) with standard contracting properties of stochastic matrices ζ we obtain for $k > n$ that

$$\zeta_n\ldots\zeta_{k-1} = (\sigma_k(1)\mathbf{1},\ldots,\sigma_k(m)\mathbf{1}) + \mathcal{O}\left(\theta^{k-n}\right) \text{ with } \left|\mathcal{O}\left(\theta^{k-n}\right)\right| \leq C\theta^{k-n}, \tag{2.11}$$

where $0 \leq \theta < 1$ and C depend only on the ε from (2.5).

Define

$$\alpha_n = Q_{n+1}(I - R_n - Q_n\zeta_{n-1})^{-1}, \quad A_n = (I - R_n - Q_n\zeta_{n-1})^{-1}Q_n. \tag{2.12}$$

Note that $\alpha_nP_n = Q_{n+1}\zeta_n$ and hence

$$\alpha_n = Q_{n+1}(I - R_n - \alpha_nP_n)^{-1}. \tag{2.13}$$

Matrices $\alpha_n,\ A_n$ are positive and therefore we can set

$$v_n = \lim_{a\to-\infty} \frac{A_nA_{n-1}\dots A_{a+1}\tilde{v}_a}{\|A_nA_{n-1}\dots A_{a+1}\tilde{v}_a\|}. \tag{2.14}$$

As explained in [3, Theorem 4] this limit exists and does not depend on the choice of the sequence of vectors $\tilde{v}_a \geq 0, ||\tilde{v}_a|| = 1$.

Similarly, for any sequence of row-vectors $\tilde{l}_a \geq 0$, $\|\tilde{l}_a\| = 1$, define

$$l_n = \lim_{a\to\infty} \frac{\tilde{l}_a\alpha_{a-1}\dots\alpha_n}{\left\|\tilde{l}_a\alpha_{a-1}\dots\alpha_n\right\|}. \tag{2.15}$$

Set

$$\lambda_k = \|A_kv_{k-1}\| \text{ and } \tilde{\lambda}_k = \|l_{k+1}\alpha_k\| \tag{2.16}$$

then obviously

$$l_{k+1}\alpha_k = \tilde{\lambda}_k l_k, \quad A_kv_{k-1} = \lambda_k v_k \tag{2.17}$$

and for any $n \geq k$ we have

$$\|A_nA_{n-1}\dots A_kv_{k-1}\| = \lambda_n\dots\lambda_k, \quad \|l_{n+1}\alpha_n\alpha_{n-1}\dots\alpha_k\| = \tilde{\lambda}_n\dots\tilde{\lambda}_k. \tag{2.18}$$

Remark 2.2. It should be emphasized that the proof provided in [2], [3] of the existence of the limits (2.8) and (2.14) in fact works for *all* (and not just almost all) sequences ω satisfying (2.5). If we define

$$\begin{aligned} &\zeta(\omega) = \zeta_0(\omega), \quad A(\omega) = A_0(\omega), \quad \alpha(\omega) = \alpha_0(\omega), \quad \sigma(\omega) = \sigma_0(\omega) \\ &v(\omega) = v_0(\omega), \quad l(\omega) = l_0(\omega) \quad \lambda(\omega) = \lambda_0(\omega), \quad \tilde{\lambda}(\omega) = \tilde{\lambda}_0(\omega) \end{aligned} \tag{2.19}$$

then

$$\begin{aligned} &\zeta_n = \zeta(T^n\omega), \quad A_n = A(T^n\omega), \quad \alpha_n = \alpha(T^n\omega), \quad \sigma_n(\omega) = \sigma(T^n\omega), \\ &v_n = v(T^n\omega), \quad l_n = l(T^n\omega), \quad \lambda_n = \lambda(T^n\omega), \quad \tilde{\lambda}_n = \tilde{\lambda}(T^n\omega). \end{aligned} \tag{2.20}$$

Moreover, the functions $\zeta(\cdot)$, $v(\cdot)$, $l(\cdot)$ are continuous in ω. The continuity of all other functions is implied by the continuity of ζ, v, and l. In fact, we have a stronger result, namely the above functions are Hölder with respect to the metric $\mathbf{d}$ defined by (2.4), see Lemma A.2. This regularity plays important role in our analysis.

Remark 2.3. Note that $m = 1$ corresponds to the random walks on $\mathbb{Z}$ with jumps to the nearest neighbours. In this case $p_n = P_\omega(\xi(t+1) = n+1|\xi(t) = n)$ and $q_n = 1 - p_n$. The above formulae now become very simple, namely $\psi_n = \zeta_n = 1$, $v_n = l_n = 1$, $A_n = \lambda_n = \frac{q_n}{p_n}$, $\alpha_n = \tilde{\lambda}_n = \frac{q_{n+1}}{p_n}$.

2.3. **Recurrence and transience criteria.** The following recurrence and transience criteria were proved in [2].

Theorem 2.4 ([2], Theorem 2.)**.** *Suppose that Condition* **C** *is satisfied. Then for* $\mathbb{P}$*-almost all* ω *the following holds:*
RW is recurrent, that is $P_{\omega,z}(\liminf_{t\to\infty} X_t = -\infty$ *and* $\limsup_{t\to\infty} X_t = \infty) = 1$, *iff* $\mathbb{E}(\ln\lambda) = 0$
RW is transient to the right, that is $P_{\omega,z}(X_t \to +\infty$ *as* $t \to \infty) = 1$, *iff* $\mathbb{E}(\ln\lambda) < 0$,
RW is transient to the left, that is $P_{\omega,z}(X_t \to -\infty$ *as* $t \to \infty) = 1$, *iff* $\mathbb{E}(\ln\lambda) > 0$.

3. Statement of results

3.1. **The Central Limit Theorem.** We shall now state sufficient conditions under which the asymptotic behaviour of a recurrent RW on a strip is described by the CLT. As far as we are aware of, the only result of this kind was previously established by Brémont in [5] for the $[-l, 1]$ model which is a very particular case of our model (as has already been mentioned in the Introduction).

In subsection 3.2 we show how to apply this result to independent and to quasiperiodic environment.

Theorem 3.1. *Consider an ergodic environment satisfying* (2.5)*. Assume that there exist functions* $\beta, \tilde\beta$ *such that*

$$\lambda = \frac{\beta(T\omega)}{\beta(\omega)} \text{ and } \mathbb{E}(\beta^3 + \beta^{-3}) < \infty \tag{3.1}$$

and

$$\tilde\lambda = \frac{\tilde\beta(T\omega)}{\tilde\beta(\omega)} \text{ and } \mathbb{E}(\tilde\beta^3 + \tilde\beta^{-3}) < \infty. \tag{3.2}$$

Then there is a constant $D > 0$ *such that for* $\mathbb{P}$*-almost all environments*

$$\frac{X_n}{\sqrt{n}} \Rightarrow \mathcal{N}(0, D).$$

Remark 3.2. If conditions (3.1), (3.2) are satisfied then it follows from (2.18) that for any $n \ge k$

$$\begin{aligned} \|A_n A_{n-1} \dots A_k v_{k-1}\| &= \lambda_n \dots \lambda_k = \frac{\beta(T^{n+1}\omega)}{\beta(T^k\omega)}, \\ \|l_{n+1}\alpha_n\alpha_{n-1}\dots\alpha_k\| &= \tilde\lambda_n \dots \tilde\lambda_k = \frac{\tilde\beta(T^{n+1}\omega)}{\tilde\beta(T^k\omega)}. \end{aligned} \tag{3.3}$$

The following definition of random potential was used in [3] and is analogous to the one introduced in [26].

Definition. A *potential* is a random function of n defined by

$$\mathcal{P}_n(\omega) \equiv \mathcal{P}_n \stackrel{\text{def}}{=} \begin{cases} \ln ||A_n...A_1|| & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \\ -\ln ||A_0...A_{n+1}|| & \text{if } n \leq -1 \end{cases} \tag{3.4}$$

Condition (2.5) implies that all matrix elements of matrices A_n are uniformly separated from 0. This implies that $\mathcal{P}_n$ is bounded if and only if $\ln \|A_n \dots A_k v_{k-1}\|$ is bounded which, in turn, is equivalent to (3.1) with bounded β. In one direction, this statement is immediate due to (3.3). The other direction is implied by a well known result stated in Lemma C.1 in Appendix C.

Conditions (3.1) and (3.2) may still appear artificial. In fact, as shown in [10], they are necessary and sufficient for the existence of the invariant measure on the space of environments which in turn is one of the basic ingredients of the proof of Theorem 3.1. Moreover, as will be seen in the next subsection, these conditions can be checked for some interesting classes of environments.

3.2. **Applications.** The following lemma describes one of the most important classes of environments for which conditions (3.1) and (3.2) are satisfied.

Lemma 3.3. *(a) For ergodic environments satisfying* (2.5) *conditions* (3.1) *and* (3.2) *are equivalent. Moreover, there is a constant* $c > 0$ *such*

$$c^{-1}\tilde{\beta}(\omega) \leq \beta(\omega) \leq c\tilde{\beta}(\omega). \tag{3.5}$$

(b) For i.i.d. environments satisfying (2.5) *conditions* (3.1) *and* (3.2) *hold iff the RW is recurrent but does not exhibit the Sinai behavior. In this case the functions* $\beta, \tilde{\beta}$ *can be chosen to be continuous.*

Corollary 3.4. *A recurrent random walk on a strip in an i.i.d. environment either exhibits the Sinai behavior, or satisfies the CLT.*

To give more examples of environments satisfying conditions of Theorem 3.1 we need the following definition. Call a set $\Lambda \subset \mathcal{J}$ *admissible* if there exists an i.i.d. environment $\mathbb{P}$ such that $\mathcal{J}_0(\mathbb{P}) = \Lambda$ and the corresponding random walk is recurrent and satisfies the CLT. Note that due to the continuity of functions $\beta, \tilde{\beta}$ equations (3.1) and (3.2) hold for all (not merely almost all) environments in $\Lambda^{\mathbb{Z}}$. Thus Theorem 3.1 implies the following corollary.

Corollary 3.5. *If* Λ *is admissible and* $\tilde{\mathbb{P}}$ *is a stationary ergodic measure on* $\Lambda^{\mathbb{Z}}$ *then* X_n *is recurrent and satisfies the CLT for* $\tilde{\mathbb{P}}$ *almost every* ω.

Lemma 3.6. *Suppose that there is a vector* $f = \{f_k\}_{k=1}^m$ *such that* $M_n = X_n + f_{Y_n}$ *is a martingale. Then* (3.1) *and* (3.2) *hold.*

Corollary 3.7. *The CLT holds for ergodic one dimensional environments where the position of the walker is a martingale.*

We have already mentioned above that the results of [3] show that the CLT behavior of recurrent walks is exceptional for the i.i.d environments. The same need not be the case in other settings. For example, consider quasiperiodic random walks. Namely, we assume that $\omega \in \mathbb{T}^d$ and

$$(P_n, Q_n, R_n)(\omega) = (\bar{P}, \bar{Q}, \bar{R})(\omega + n\gamma),$$

where γ is a vector in $\mathbb{R}^d$, and $(\bar{P}, \bar{Q}, \bar{R})$ are C^r matrix valued functions on $\mathbb{T}^d$. We assume that γ is Diophantine, that is there are constants C, σ such that for each $k \in \mathbb{Z}^d, \tilde{k} \in \mathbb{Z}$

$$|\langle \gamma, k\rangle - \tilde{k}| \geq C|k|^{-\sigma}. \tag{3.6}$$

Theorem 3.8. *Assume that the walk is recurrent and*

$$r > d + \sigma. \tag{3.7}$$

Then (3.1) *holds (and hence the random walk satisfies the CLT).*

Extending a result from [12] to the strip we, in particular, obtain a complete description of RW in Diophantine quasiperiodic environments.

To formulate this extension we consider the following setting. Suppose that

$$(P_n, Q_n, R_n)(\omega) = (\bar{P}, \bar{Q}, \bar{R})(f^n\omega) \tag{3.8}$$

where f is a homeomorphism of a space Ω and $(\bar{P}, \bar{Q}, \bar{R})$ are continuous matrix valued functions on Ω. Recall that a map $f : \Omega \to \Omega$ is called *uniquely ergodic* if for any continuous real valued function Φ the following limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \Phi(f^n\omega) \tag{3.9}$$

exists for all $\omega \in \Omega$ and does not depend on ω. We recall that if Ω is a compact space then the unique ergodicity of f is equivalent to uniform in $\omega \in \Omega$ convergence of the averages (3.9). If f in (3.8) is uniquely ergodic we call (P_n, Q_n, R_n) a uniquely ergodic environment.

The next result was proven in [12] for the one-dimensional nearest neighbour walk (the case $m = 1$). In the Appendix, we prove it for arbitrary strip.

Theorem 3.9. *A transient RW on a strip in a uniquely ergodic environment generated by a continuous* $(\bar{P}, \bar{Q}, \bar{R})$ *satisfies the CLT.*

A more precise statement of this result including the normalization is given in Theorem B.1.

Corollary 3.10. *RW on a strip in a Diophantine quasiperiodic environment always satisfies the CLT.*

Proof. If the RW is recurrent the result follows from Theorems 3.1 and 3.8 and if it is transient then it follows from Theorem 3.9. □

Remark 3.11. Alili in [1] proved the CLT for RW in smooth Diophantine quasiperiodic environments with jumps to nearest neighbours on $\mathbb{Z}$. Brémont in [7] extended this result to RW with bounded jumps (the $[-l, r]$ model) in a quasiperiodic environments generated by a smooth enough function on the torus. In the recurrent regime, Brémont's result is a particular case of Theorems 3.1 and 3.8. In the transient regime, Theorem 3.9 gives a much more general result as it works for all uniquely ergodic environments and requires only continuity of the generating probabilities.

Lemma 3.3 and Corollaries 3.4 and 3.5 lead naturally to the question of characterizing the admissible sets. By Corollary 3.5 a subset of an admissible set is admissible. Recall that the Zariski closure $\bar{\mathcal{A}}$ of a set $\mathcal{A}$ is the smallest algebraic variety containing $\mathcal{A}$. The next result shows that maximal admissible sets are algebraic subvarieties.

Lemma 3.12. *The Zariski closure $\bar{\Lambda}$ of an admissible set Λ is admissible.*

3.3. **Organization of the paper.** Our main result, Theorem 3.1, is proven in Sections 4–7. Namely, Section 4 describes the main ingredients of the proof, Section 5 presents, in the case of the nearest neighbour RWs on $\mathbb{Z}$, the simplest version of the formulae for the density of the invariant measure and the martingale which play a major role in the proof of the main result. Section 6 constructs the invariant measure for the environment viewed from the particle, and Section 7 proves the existence of a martingale which is asymptotically linear with respect to the $\mathbb{Z}$-coordinate of the walk (the latter is often called the harmonic coordinate for the system). The uniqueness of the martingale is established in Section 8. Section 9 contains the proof of Lemma 3.3. Lemma 3.12 is proven in Section 10. Section 11 contains the proof of Lemma 3.6. Two sections deal with quasiperiodic environments. Namely, Theorem 3.8 is proven in Section 12 and Theorem 3.9 is established in Appendix B.

4. Main ingredients in the proof of the CLT.

The proof of Theorem 3.1 consists of the following ingredients.

The environment seen by the particle is the random sequence (ω_n, Y_n), $n \geq 0$, where $\omega_n = T^{X_n}\omega$ and $\xi_n = (X_n, Y_n)$ is the position of the walk at time n.

Lemma 4.1. *If* (3.2) *holds then the environment seen by the particle has an invariant measure μ with bounded density ρ with respect to the original environment measure.*

Lemma 4.2. *The process (ω_n, Y_n) is ergodic with respect to μ.*

Lemma 4.2 is a well known result. Its proof can be found in [4, Theorem 1.2].

Lemma 4.3. *If* (3.1) *holds then there is a function $M(x, y) = M_\omega(x, y)$ such that*

(1) *For almost all ω $M_n = M_\omega(X_n, Y_n)$ is a martingale;*

(2) *The increments of* M_n *are square integrable with respect the measure* $\mu(d\omega)P_\omega$*;*
(3) *For a.e.* ω, *the ratio* $\frac{M_\omega(x,y)}{x} \to c,\ c \neq 0$, *for all* $y \in \{1 \dots m\}$ *as* $|x| \to \infty$.

Lemmas 4.1 and 4.3 imply Theorem 3.1 in a standard way which we now recall for completeness.

Proof of Theorem 3.1. Observe that Lemma 4.3 implies that

$$\frac{X_n}{\sqrt{n}} = \frac{M_n}{c\sqrt{n}}(1+o(1)) + o(1) \text{ as } n \to \infty. \tag{4.1}$$

Indeed, if $|X_n| \geq n^{1/4}$ then (4.1) holds due to Lemma 4.3(3) while if $|X_n| \leq n^{1/4}$ then (4.1) holds since both the RHS and the LHS are $o(1)$. Due to (4.1) it suffices to prove the CLT for M_n. By [15] it suffices to show that $\frac{D_n}{n}$ converges for $\mathbb{P}$-almost all ω to a non-random limit, where

$$\begin{aligned} D_n &= \sum_{k=0}^{n-1} \mathbb{E}_\omega \left([M(X_{k+1}, Y_{k+1}) - M(X_k, Y_k)]^2 \,|(X_0, Y_0) \dots (X_k, Y_k) \right) \\ &= \sum_{k=0}^{n-1} \mathbb{E}_\omega \left([M(X_{k+1}, Y_{k+1}) - M(X_k, Y_k)]^2 \,|(X_k, Y_k) \right) \end{aligned}$$

but this convergence follows immediately from the ergodicity of the (ω_n, Y_n) process. □

5. Nearest neighbour walks on $\mathbb{Z}$.

Below we present proofs of Lemmas 4.1 and 4.3 in the case of the nearest neighbour walks on $\mathbb{Z}$ where the formulae for ρ_n and M_n are simple. They may seem to be a result of a guess rather than a derivation. In fact, we borrow the form of ρ_n from [26] and the formula for M_n results from the analysis of a solution to (5.1) considered, for example, in [11]. (Of course, they could also be obtained as simplified versions of formulae for ρ_n and M_n we derive in Sections 6 and 7.)

Note that in the case of walks on $\mathbb{Z}$ (see Remark 2.3), condition (3.1) takes the form

$$A_n = \frac{q_n}{p_n} = \lambda_n = \frac{\beta_{n+1}}{\beta_n}, \text{ where } \beta_n = \beta(T^n\omega).$$

Proof of Lemma 4.1 for $\mathbb{Z}$. Let ρ be the density of the invariant measure and $\rho_n(\omega) = \rho(T^n\omega)$. Then ρ satisfes

$$\rho_n = p_{n-1}\rho_{n-1} + q_{n+1}\rho_{n+1}.$$

We claim that this equation has a solution of the form $\rho_n = \frac{1}{\beta_n q_n}$. Indeed

$$p_{n-1}\rho_{n-1} + q_{n+1}\rho_{n+1} = \frac{p_{n-1}}{q_{n-1}\beta_{n-1}} + \frac{1}{\beta_{n+1}} = \frac{1}{\beta_n} + \frac{p_n}{q_n\beta_n} = \frac{1}{\beta_n}\left(1 + \frac{p_n}{q_n}\right) = \frac{1}{q_n\beta_n} = \rho_n.$$

□

Proof of Lemma 4.3 for $\mathbb{Z}$. If $X_t,\ t \geq 0$, is the nearest neighbor walk on $\mathbb{Z}$ in random environment ω then $M_\omega(X_t)$ is a martingale if the sequence $\{M_n = M_\omega(n),\ n \in \mathbb{Z}\}$ satisfies the equation

$$M_n = p_n M_{n+1} + q_n M_{n-1}. \tag{5.1}$$

The space of solutions to (5.1) is two-dimensional and we claim that a solution linearly independent of $M_n \equiv 1$ has the form

$$M_n = \begin{cases} \sum_{j=1}^{n} \beta_j & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -\sum_{j=n+1}^{0} \beta_j & \text{if } n \leq -1. \end{cases}$$

Let us check this claim, say for $n \geq 1$. In this case

$$p_n M_{n+1} + q_n M_{n-1} = p_n(M_n + \beta_{n+1}) + q_n(M_n - \beta_n) = M_n + p_n\beta_{n+1} - q_n\beta_n = M_n.$$

□

6. Environment seen by the particle.

Proof of Lemma 4.1. We will construct the density $\rho : \Omega \times [1, \dots, m] \mapsto \mathbb{R}$ as a solution of (6.1) below. Denote by $\rho = \rho(\omega)$ the row-vector with components $\rho(\omega, i)$ and let $\rho_n = \rho(T^n\omega)$ be a vector with components $\rho_n(i) = \rho(T^n\omega, i)$. For ρ to be a density of the invariant measure of the Markov chain $(T^{X_t}\omega, Y_t),\ t \geq 0$, the corresponding vectors ρ_n should satisfy the system of equations

$$\rho_n = \rho_{n+1}Q_{n+1} + \rho_n R_n + \rho_{n-1}P_{n-1}, \quad -\infty < n < \infty. \tag{6.1}$$

The restriction of this equation to a finite strip $a \leq n \leq b$ was analyzed in [2, section 3]. The solution found there satisfies certain (reflecting) boundary conditions and has a meaning different from the one we are interested in here.

However, we borrow from [2] the following fact. For any m-dimensional vector h set $\rho_b^h = h$ and define ρ_n^h for $n \leq b-1$ by the recurrence relation $\rho_n^h = \rho_{n+1}^h \alpha_n$, where the matrices α_n are defined in (2.12). Then the vectors ρ_n^h solve (6.1) for all $n \leq b-1$. For the sake of completeness, we shall check this statement. Obviously, if $n \leq b-1$ then

$$\rho_n^h = h\alpha_{b-1} \dots \alpha_n \tag{6.2}$$

and hence

$$\begin{aligned} \rho_{n+1}^h Q_{n+1} + \rho_n^h R_n + \rho_{n-1}^h P_{n-1} &= h\alpha_{b-1} \dots \alpha_{n+1}(Q_{n+1} + \alpha_n R_n + \alpha_n\alpha_{n-1}P_{n-1}) \\ &\overset{(*)}{=} h\alpha_{b-1} \dots \alpha_{n+1}\alpha_n = \rho_n^h, \end{aligned}$$

where $(*)$ follows from the relation $\alpha_n = Q_{n+1} + \alpha_n R_n + \alpha_n\alpha_{n-1}P_{n-1}$ which in turn is equivalent to (2.13).

Next, note that for vectors l_n defined in (2.15) it follows from (2.17) and condition (3.2) that

$$l_{n+1}\alpha_n = \tilde{\lambda}_n l_n = \frac{\tilde{\beta}(T^{n+1}\omega)}{\tilde{\beta}(T^n\omega)} l_n \text{ and so } \frac{1}{\tilde{\beta}(T^{n+1}\omega)} l_{n+1}\alpha_n = \frac{1}{\tilde{\beta}(T^n\omega)} l_n. \tag{6.3}$$

Remember that $l_n = l(T^n\omega)$. Set

$$\rho(\omega) = \frac{1}{Z\tilde{\beta}(\omega)} l(\omega), \text{ where } Z = \mathbb{E}[\frac{1}{\tilde{\beta}(\omega)} \sum_{i=1}^{m} l(\omega, i)]. \tag{6.4}$$

Then the second equation in (6.3) has the form $\rho_n = \rho_{n+1}\alpha_n$, where $\rho_n = \rho(T^n\omega)$ for all $n \in \mathbb{Z}$. Hence, the ρ_n, $n \in \mathbb{Z}$, solve (6.1) which means that ρ defined by (6.4) is the density of the invariant measure of our Markov chain. □

7. Construction of the martingale.

Proof of Lemma 4.3. Let $\mathfrak{m}_n$ denote a vector with components $\mathfrak{m}_n(i) = M(n,i)$. For the process $M(X_t, Y_t)$, $t \geq 0$, to be a martingale with respect to the measure $P_{\omega,z}$, the vectors $\mathfrak{m}_n$ should satisfy the equation

$$\mathfrak{m}_n = P_n\mathfrak{m}_{n+1} + R_n\mathfrak{m}_n + Q_n\mathfrak{m}_{n-1}. \tag{7.1}$$

Solution to this equation on a finite part of the strip, $a \leq n \leq b$, was analyzed in [2]. In order to make our proof more self-contained, we reproduce some calculations from [2]. Namely, define a sequence of $m \times m$ matrices φ_n, $n \geq a+1$ by setting $\varphi_a = 0$ and computing φ_n recursively

$$\varphi_n = (I - R_n - Q_n\varphi_{n-1})^{-1}P_n, \quad \text{if } n \geq a. \tag{7.2}$$

The solutions to (7.1) with boundary conditions $\mathfrak{m}_a = 0$, $\mathfrak{m}_b = \mathfrak{f}$ can be presented in the following form:

$$\mathfrak{m}_n = \varphi_n\varphi_{n+1}\ldots\varphi_{b-1}\mathfrak{f}, \quad a \leq n \leq b. \tag{7.3}$$

For $n = a$ or $n = b$ this statement is obvious and for $a < n < b$ it can be verified by substituting the right hand side of (7.3) into (7.1).

Next set $\Delta_n = \zeta_n - \varphi_n$, where ζ_n are matrices defined in (2.7), (2.8). Following [2], we present this difference as

$$\begin{aligned}\Delta_n &= (I - R_n - Q_n\zeta_{n-1})^{-1}P_n - (I - R_n - Q_n\varphi_{n-1})^{-1}P_n \\ &= (I - R_n - Q_n\zeta_{n-1})^{-1}Q_n\Delta_{n-1}(I - R_n - Q_n\varphi_{n-1})^{-1}P_n = A_n\Delta_{n-1}\varphi_n.\end{aligned} \tag{7.4}$$

Iterating the last relation gives, (cf. [2, equation (2.13)]) that if $|n| < b$ then

$$\Delta_n = A_n \ldots A_{-b+1}\Delta_{-b}\varphi_{-b+1}\ldots\varphi_n. \tag{7.5}$$

In order to construct a linearly growing solution of (7.1) we consider the solution $\mathfrak{m}_n$ corresponding to $\mathfrak{f} = \mathbf{1}$ and study some related limits of this solution as $a \to -\infty, b \to \infty$ so that $|a| \gg b$.

It follows from (3.1) that $\mathbb{E}(\ln \lambda) = 0$ and hence by Theorem 2.4 the walk is recurrent. Recall (see [2, formula (2.3)]) that $\varphi_n(i,j)$ is the $P_{\omega,(n,i)}$-probability that a RW starting from (n,i) reaches layer $n+1$ before layer a and that it hits layer $n+1$ at $(n+1,j)$. So, due to recurrence, we have that for any i

$$P_{\omega,(n,i)}\{\text{reach layer } n+1 \text{ before } a\} = \sum_{j=1}^{m} \varphi_n(i,j) \to 1 \text{ as } a \to -\infty.$$

Since $\Delta_a = \zeta_a$, (7.4) implies that $\Delta_n \geq 0$. Therefore

$$\|\Delta_n\| = \|\Delta_n \mathbf{1}\| = \|(\zeta_n - \varphi_n)\mathbf{1}\| = 1 - \min_i \sum_{j=1}^{m} \varphi_n(i,j) \to 0 \text{ as } a \to -\infty.$$

Hence given ε, b there exists L such that if $|a| \geq L$ and $|n| \leq b$ then $||\Delta_n|| \leq \varepsilon$. Let $\varepsilon_b = ||\Delta_{-b}||$.

Condition (3.1) implies that $\|A_n \dots A_{-b}\|$ is uniformly bounded and hence $||\Delta_n|| \leq C\varepsilon_b$. Next, (7.4) also gives

$$\Delta_n = A_n \Delta_{n-1}(\zeta_n - \Delta_n) = A_n \Delta_{n-1}\zeta_n - A_n \Delta_{n-1}\Delta_n. \tag{7.6}$$

Substituting $\Delta_{n-1} = A_{n-1}\Delta_{n-2}\zeta_{n-1} - A_{n-1}\Delta_{n-2}\Delta_{n-1}$ only in the term $A_n\Delta_{n-1}\zeta_n$ we obtain

$$\Delta_n = A_n A_{n-1}\Delta_{n-2}\zeta_{n-1}\zeta_n - A_n A_{n-1}\Delta_{n-2}\Delta_{n-1}\zeta_n - A_n \Delta_{n-1}\Delta_n.$$

Continuing this process we obtain

$$\Delta_n = A_n \dots A_{-b+1}\Delta_{-b}\zeta_{-b+1}\dots\zeta_n - \sum_{k=-b}^{n-1} A_n \dots A_{k+1}\Delta_k \Delta_{k+1}\zeta_{k+2}\dots\zeta_n, \tag{7.7}$$

where by convention $A_n \dots A_{k+1} = I$ if $k+1 < n$ and $\zeta_{k+2}\dots\zeta_n = I$ if $k+2 > n$. Equality (7.7) implies

$$\Delta_n = A_n \dots A_{-b+1}\Delta_{-b}\zeta_{-b+1}\dots\zeta_n + \mathcal{O}(\varepsilon_b^2 b). \tag{7.8}$$

Applying similar reasoning to (7.3) with $\mathfrak{f} = \mathbf{1}$ and $\varphi_j = \zeta_j - \Delta_j$ gives

$$\begin{aligned}\mathfrak{m}_n &= \mathbf{1} - \sum_{n \leq k \leq b-1} \zeta_n \dots \zeta_{k-1}\Delta_k \zeta_{k+1}\dots\zeta_{b-1}\mathbf{1} + \mathcal{O}(\varepsilon_b^2 b^2) \\ &= \mathbf{1} - \sum_{n \leq k \leq b-1} \zeta_n \dots \zeta_{k-1}\Delta_k \mathbf{1} + \mathcal{O}(\varepsilon_b^2 b^2).\end{aligned}$$

Substituting (7.8) into the last equation gives

$$\mathfrak{m}_n = \mathbf{1} - \sum_{n \leq k \leq b-1} \zeta_n \dots \zeta_{k-1} A_k \dots A_{-b+1} w_{-b} + \mathcal{O}(\varepsilon_b^2 b^2)$$

where $w_{-b} = \Delta_{-b}\mathbf{1}$. Now $\bar{\mathfrak{m}}_n^{a,b} = \frac{\mathbf{1}-\mathfrak{m}_n}{||w_{-b}||}$ satisfies (7.1) since it is a linear combination of two solutions. Note that

$$\bar{\mathfrak{m}}_n^{a,b} = \sum_{n\le k\le b-1} \zeta_n \dots \zeta_{k-1} A_k \dots A_{-b+1} u_{-b} + \mathcal{O}(\varepsilon b^2), \tag{7.9}$$

where $u_{-b} = w_{-b}/||w_{-b}||$ and we have used that $\varepsilon_b \le \varepsilon$.

We shall now compute the limit of $\bar{\mathfrak{m}}_n^{a,b}$ as $a \to -\infty$. To this end note that

$$\lim_{a\to-\infty} u_{-b} = \lim_{a\to-\infty} \frac{A_{-b}\dots A_{a+1}\zeta_a\varphi_{a+1}\dots\varphi_{-b}\mathbf{1}}{\|A_{-b}\dots A_{a+1}\zeta_a\varphi_{a+1}\dots\varphi_{-b}\mathbf{1}\|} = v_{-b}, \tag{7.10}$$

where we first use (7.5) and then proceed as in (2.14) with $\tilde{v}_a = \frac{\zeta_a\varphi_{a+1}\cdots\varphi_{-b}\mathbf{1}}{\|\zeta_a\varphi_{a+1}\cdots\varphi_{-b}\mathbf{1}\|}$. Passing to the limit $a \to -\infty$ in (7.9) we obtain the following solution on $(-b, b)$

$$\bar{\mathfrak{m}}_n^{b} = \sum_{n\le k\le b-1} \zeta_n \dots \zeta_{k-1} A_k \dots A_{-b+1} v_{-b}.$$

By (2.18) and (3.3) we have

$$A_k \dots A_{-b+1} v_{-b} = \lambda_k \dots \lambda_{-b+1} v_k = \frac{\beta(T^{k+1}\omega)}{\beta(T^{-b+1}\omega)} v_k \tag{7.11}$$

and by (2.11)

$$\zeta_n \dots \zeta_{k-1} v_k = (\sigma_k(1)\mathbf{1}, \dots, \sigma_k(m)\mathbf{1}) v_k + \mathcal{O}\left(\theta^{k-n}\right) = (\sigma_k v_k)\mathbf{1} + \mathcal{O}\left(\theta^{k-n}\right), \tag{7.12}$$

where here and below we denote $(\sigma_k v_k) \overset{\text{def}}{=} \sum_{i=1}^m \sigma_k(i) v_k(i)$. We thus see that

$$\beta(T^{-b+1}\omega)\bar{\mathfrak{m}}_n^b = \sum_{k=n}^{b-1} \beta(T^{k+1}\omega)(\sigma_k v_k)\mathbf{1} + \sum_{k=n}^{b-1} \beta(T^{k+1}\omega)\mathcal{O}\left(\theta^{k-n}\right)$$

is also a solution to (7.1) on $(-b, b)$ and so is

$$\begin{aligned}\hat{\mathfrak{m}}_n^b &\overset{\text{def}}{=} \beta(T^{-b+1}\omega)\bar{\mathfrak{m}}_n^b - \sum_{k=0}^{b-1} \beta(T^{k+1}\omega)(\sigma_k v_k)\mathbf{1} \\ &= -\sum_{k=0}^{n-1} \beta(T^{k+1}\omega)(\sigma_k v_k)\mathbf{1} + \sum_{k=n}^{b-1} \beta(T^{k+1}\omega)\mathcal{O}\left(\theta^{k-n}\right)\end{aligned}$$

The series $\sum_{k=n}^{\infty} \beta(T^{k+1}\omega)\mathcal{O}\left(\theta^{k-n}\right)$ converges absolutely because of (2.11) (note that the terms of the last sum do not depend on b). Hence setting $M(x,\cdot) = \lim_{b\to\infty} \hat{\mathfrak{m}}_x^b$ we obtain a solution

$$M(x,\cdot) = \sum_{k=0}^{x-1} \beta(T^{k+1}\omega)(\sigma_k v_k)\mathbf{1} + \mathcal{B}(T^x\omega), \tag{7.13}$$

where

$$\mathcal{B}(\omega) = \sum_{k=0}^{\infty} \beta(T^{k+1}\omega)(\zeta_0 \dots \zeta_{k-1} v_k - (\sigma_k v_k)\mathbf{1}).$$

It remains to check statements (2) and (3) of Lemma 4.3.

Denote by $\mathbb{E}^\mu$ the expectation with respect to the measure μ. To check that (2) holds, we have to show that

$$\mathcal{D} \stackrel{\text{def}}{=} \mathbb{E}^\mu \left(E_\omega \left(M_\omega(X_{t+1}, Y_{t+1}) - M_\omega(X_t, Y_t) \right)^2 \right) \tag{7.14}$$

$$= \mathbb{E}^\mu \left(E_\omega \left(M_\omega(X_1, Y_1) - M_\omega(X_0, Y_0) \right)^2 \right) < \infty. \tag{7.15}$$

Note that equality (7.15) holds since μ is an invariant measure of the Markov chain $(T^{X_t}\omega, Y_t),\ t \geq 0$. Now $\mathcal{D}$ can be presented as

$$\mathcal{D} = \mathbb{E} \left(\sum_{i=1}^{m} \rho(\omega, i) \sum_{s=0,\pm1; 1 \leq j \leq m} \mathcal{Q}((0,i),(s,j))(M(0,i) - M(s,j))^2 \right),$$

where $\mathcal{Q}((0,i),(s,j))$ is defined by (2.2). Equation (7.13) implies that

$$|M(0,i) - M(s,j)| \leq C \sum_{k=0}^{\infty} \theta^k \beta(T^k\omega),$$

where, as before, C and θ depend only on the ε from (3.1). This inequality, together with (6.4) and (3.5), implies

$$\mathcal{D} \leq C\mathbb{E} \left(\sum_{k \geq 0, j \geq 0} \theta^{k+j} \beta^{-1}(\omega)\beta(T^k\omega)\beta(T^j\omega) \right).$$

But $\mathbb{E}(\beta^{-1}(\omega)\beta(T^k\omega)\beta(T^j\omega)) \leq \frac{1}{3}\mathbb{E}(\beta^{-3}(\omega) + \beta^3(T^k\omega) + \beta^3(T^j\omega)) = \frac{1}{3}\mathbb{E}(\beta^{-3}(\omega) + 2\beta^3(\omega))$ and this finishes the proof of Property (2).

Remark 7.1. Note that in the case of a RWRE on $\mathbb{Z}$ with nearest neighbour jumps condition (3.1) can be replaced by $\mathbb{E}(\beta(\omega) + \beta^{-1}(\omega)) < \infty$. On a strip, we need the stronger requirement (3.1) because of the term $\mathcal{B}$ in (7.13).

Finally, Property (3) follows from the ergodic theorem. □

8. The Liouville Theorem.

The construction of the martingale in the previous section was based on a choice of two particular solutions of the martingale equation on finite intervals. The following lemma shows that the final result is essentially unique. And even though this lemma is not used in the rest of the paper, it provides an important contribution to the understanding of the whole picture.

Let $\mathfrak{M}$ denote the space of martingales satisfying conditions (1)–(2) of Lemma 4.3 and such that if $M(\cdot,\cdot) \in \mathfrak{M}$ then

$$\lim_{x\to\pm\infty} \frac{M(x,y)}{x} = 1.$$

(Clearly, we can scale the martingale from Lemma 4.3 to achieve this condition.)

Lemma 8.1. *If $M_1, M_2 \in \mathfrak{M}$ then $M_1 - M_2 = \mathrm{Const}$.*

Proof. Let $\bar{M}(x,y) = M_1(x,y) - M_2(x,y)$. Then $\bar{M}(x,y)$ grows sublinearly. Hence by Theorem 3.1, for almost all ω, $\frac{\bar{M}(X_n,Y_n)}{\sqrt{n}} \to 0$ in probability with respect to the $P_{\omega,z}$ measure on the space of trajectories. On the other hand the proof of Theorem 3.1 shows that $\frac{\bar{M}(X_n,Y_n)}{\sqrt{n}} \to 0$ iff

$$\bar{D}_n = \sum_{k=0}^{n-1} \mathbb{E}\left(E_\omega\left(\left[\bar{M}(X_{k+1},Y_{k+1}) - \bar{M}(X_k,Y_k)\right]^2 |X_k\right)\right) \equiv 0,$$

which in turn implies that $\bar{M}$ is a constant. □

9. Equivalent conditions for boundedness of the potential.

Proof of Lemma 3.3. (a) Define $\mathfrak{a}_n = I - R_n - Q_n\zeta_{n-1}$. In these notations, we have $A_n = \mathfrak{a}_n^{-1}Q_n$ and $\alpha_{n-1} = Q_n\mathfrak{a}_{n-1}^{-1}$ and hence $\mathfrak{a}_nA_n = \alpha_{n-1}\mathfrak{a}_{n-1}$. Multiplying both parts of the last equality by vectors l_n and v_{n-1} we obtain

$$l_n\mathfrak{a}_nA_nv_{n-1} = l_n\alpha_{n-1}\mathfrak{a}_{n-1}v_{n-1}$$

and this, by (2.16) and (2.17), gives

$$(l_n\mathfrak{a}_nv_n)\,\lambda_n = (l_{n-1}\mathfrak{a}_{n-1}v_{n-1})\tilde{\lambda}_n.$$

This implies the following relation between $\beta(\omega)$ in (3.1) and $\tilde{\beta}(\omega)$ in (3.2):

$$\tilde{\beta}(\omega) = \beta(\omega)(l(T^{-1}\omega)\mathfrak{a}(T^{-1}\omega)v(T^{-1}\omega)). \tag{9.1}$$

(We use hear the fact that, due to ergodicity, the existence of β (or $\tilde{\beta}$) implies that it is unique up to a multiplication by a constant.)

(b) Suppose that the walk is recurrent but does not exhibit the Sinai behavior. Note that

$$\lambda_{i+1}\ldots\lambda_k = \exp\left(\sum_{j=i+1}^{k} \ln\lambda_j\right) = \exp\left(\sum_{j=i}^{k-1} \ln\|A_{j+1}v_j\|\right)$$

Applying [3, Theorem 6] to the additive functional

$$\sum_{j=i}^{k-1} \ln\|A_{j+1}v_j\|$$

we obtain (3.1) with continuous β. The argument for (3.2) follows from (9.1).

Conversely if (3.1) and (3.2) hold then the RW does not exhibit the Sinai behavior by Theorem 3.1. □

10. Periodic boundary conditions.

Here we describe a criterion for recurrence and the CLT in terms of periodic approximations to our random environment. The results are analogous to the Livsic theory for hyperbolic dynamical systems (cf. [23, 24]). Given N let $\pi^N(n,y)$ denote the invariant measure for the random walk on $[0, N-1]\times[1\dots m]$ with periodic boundary conditions. Let π_n^N denote the vector with components $\pi_n^N(y) = \pi^N(n,y)$.

Proposition 10.1. *Condition* (3.1) *holds if and only if for each N and for each $((P_n, Q_n, R_n))_{n=0}^{N-1} \in \mathcal{J}_0^N$ the following identity holds*

$$\pi_0^N Q_0 \mathbf{1} = \pi_{N-1}^N P_N \mathbf{1}. \tag{10.1}$$

The proof consists of two steps.

Lemma 10.2. (3.1) *holds if and only if for each N and for each environment ω such that $T^N\omega = \omega$ we have*

$$\lambda_0\lambda_1\dots\lambda_{N-1} = 1. \tag{10.2}$$

Proof. By Lemma C.1 we need to show that (10.2) is equivalent to

$$\sum_{j=0}^{n-1} \ln\lambda(T^j\omega) \tag{10.3}$$

being uniformly bounded in $\omega\in\Omega$ and $n\in\mathbb{N}$

(a) If (10.3) is bounded for each ω it is in particular bounded for periodic ω and hence

$$\sum_{j=0}^{kN-1} \ln\lambda(T^j\omega) = k\left[\sum_{j=0}^{N-1}\ln\lambda_j\right]$$

is uniformly bounded in k which is only possible if (10.2) holds.

(b) Suppose that (10.2) holds. Given ω, N let $\tilde\omega$ be the environment such that $\tilde\omega_j = \omega_0$ for $j\in\{0,\dots N-1\}$ and such that $\tilde\omega$ is periodic with period N. Then due to Lemma A.2

$$\left|\sum_{j=0}^{N-1}\ln\lambda(T^j\omega)\right| = \left|\sum_{j=0}^{N-1}\ln\lambda(T^j\omega) - \sum_{j=0}^{N-1}\ln\lambda(T^j\tilde\omega)\right| = \left|\sum_{j=0}^{N-1}\left[\ln\lambda(T^j\omega) - \ln\lambda(T^j\tilde\omega)\right]\right| \le$$

$$\mathrm{Const}\sum_{j=0}^{N-1}\mathbf{d}^\alpha(T^j\omega, T^j\tilde\omega) \le \mathrm{Const}\sum_{j=0}^{N-1}2^{-\alpha(\min(j,N-j))} \le \mathrm{Const}$$

giving that (10.3) is bounded. □

Lemma 10.3. *For each periodic environment* (10.1) *and* (10.2) *are equivalent.*

Proof. Applying [2, Theorem 2] to the periodic environment we see that the recurrence is equivalent to (10.2). On the other hand considering our walker only at times when her position is divisible by N we obtain a random walk in a constant environment. In this case the recurrence is equivalent to the vanishing of drift which is what condition (10.1) says. □

Proof of Lemma 3.12. For given matrices

$$(P_0, Q_0, R_0), (P_1, Q_1, R_1), \dots, (P_{N-1}, Q_{N-1}, R_{N-1})$$

the entries $\pi^N(n, y)$ are rational functions of the coefficients. Accoridngly equation (10.1) can be written as

$$\mathbb{F}_N((P_0, Q_0, R_0), (P_1, Q_1, R_1) \dots (P_{N-1}, Q_{N-1}, R_{N-1})) = 0$$

where $\mathbb{F}_N$ is a certain polinomial. In other words (3.1) holds if and only iff $\mathbb{F}_N$ vanishes on Λ^N. But then it also vanishes on $\bar{\Lambda}^N$ and hence $\bar{\Lambda}$ is also admissible. □

11. Stationary case.

Proof of Lemma 3.6. The condition that $M_n = X_n + f_{Y_n}$ is a martingale is equivalent to

$$f = (P + R + Q)f + (P - Q)\mathbf{1} \tag{11.1}$$

Let $\mathcal{J}_{\varepsilon,f}$ be the set of all triples $(P, Q, R) \in \mathcal{J}$ satisfying (2.5) and (11.1). Consider the random environment where (P_n, R_n, Q_n) are iid and are uniformly distributed on $\mathcal{J}_{\varepsilon,f}$. Then by [15, Theorem 4.1] given $\bar{\varepsilon}$ there exists δ such that

$$\mathbb{P}(|X_n| > \delta\sqrt{n}) > 1 - \bar{\varepsilon}$$

for large n. Accordingly X_n does not exhibit the Sinai behavior. Therefore by Lemma 3.3, (3.1) and (3.2) are satisfied for all environments in $(\mathcal{J}_{\varepsilon,f})^{\mathbb{Z}}$. □

12. Quasiperiodic case: proof of Theorem 3.8

Note that by stationarity there exist functions $\bar{\zeta}, \bar{A}, \bar{\alpha}, \bar{v}, \bar{l}, \bar{\lambda}, \bar{\bar{\lambda}}$ on $\mathbb{T}^d$ such that

$$\zeta_n(\omega) = \bar{\zeta}(\omega + n\gamma), \quad A_n(\omega) = \bar{A}(\omega + n\gamma), \quad \alpha_n(\omega) = \bar{\alpha}(\omega + n\gamma),$$

$$v_n(\omega) = \bar{v}(\omega+n\gamma), \quad l_n(\omega) = \bar{l}(\omega+n\gamma), \quad \lambda_n(\omega) = \bar{\lambda}(\omega+n\gamma), \quad \tilde{\lambda}_n(\omega) = \bar{\bar{\lambda}}(\omega+n\gamma).$$

Lemma 12.1. *The functions $\bar{\zeta}, \bar{A}, \bar{\alpha}, \bar{v}, \bar{l}, \bar{\lambda}$ and $\bar{\bar{\lambda}}$ are C^r smooth.*

This lemma is proven in Appendix A.

Next, by Theorem 2.4 ([2, Theorem 2]) recurrence is equivalent to

$$\int_{\mathbb{T}^d} \ln \bar{\lambda}(\omega) d\omega = 0 \tag{12.1}$$

Now [18] tells us that if $\Phi \in C^r(\mathbb{T}^d)$ has zero mean and (3.6) and (3.7) are satisfied then there is $\tilde{\Phi} \in C^0(\mathbb{T}^d)$ such that

$$\Phi(\omega) = \tilde{\Phi}(\omega+\gamma) - \tilde{\Phi}(\omega) \text{ and hence} \sum_{k=0}^{n-1} \Phi(\omega + k\gamma) = \tilde{\Phi}(\omega + n\gamma) - \tilde{\Phi}(\omega). \tag{12.2}$$

Applying (12.2) with $\Phi = \ln \bar{\lambda}$ we obtain (3.1).

Appendix A. The Invariant Section Theorem

The following result is useful for ascertaining the regularity of auxiliary sequences of matrices considered in this paper.

Let X and Y be metric spaces.

Consider a skew product transformation $F : X \times Y \mapsto X \times Y$ given by $F(x, y) = (f(x), g(x, y))$ and such that

(1) F is a continuous transformation;
(2) $f(x)$ is a homeomorphism;
(3) $g(x, \cdot) : Y \to Y$ is a fiber contraction, that is, there exists $\theta < 1$ such that
$$d(g(x, y'), g(x, y'')) \le \theta d(y', y'').$$

Proposition A.1. *(*[16, Theorem 3.5]*) (a) F admits an invariant section. That is, there exists a map $\Gamma : X \to Y$ such that $g(x, \Gamma(x)) = \Gamma(f(x))$.*

(b) If F is C^α, f and f^{-1} are Lipshitz, and $\theta[Lip(f^{-1})]^\alpha < 1$ then Γ belongs to a Hölder space C^α.

(c) If X is a manifold and Y is a manifold with boundary and $g(x, \cdot) : Y \to Int(Y)$ for each $x \in X$ and if F is a C^r diffeomorphism such that

$$\theta[\mathrm{Lip}(f^{-1})]^r < 1 \tag{A.1}$$

then Γ is C^r smooth.

Lemma A.2. *The maps*

$$\omega \to \zeta(\omega),\ \omega \to A(\omega),\ \omega \to \alpha(\omega),\ \omega \to v(\omega), \omega \to l(\omega),\ \omega \to \lambda(\omega),\ \omega \to \tilde{\lambda}(\omega)$$

defined by (2.19) *are Hölder continuous with respect to the metric* $\mathbf{d}$ *defined by* (2.4).

Proof. We start with the smoothness of ζ. To this end we apply Proposition A.1 to the map F_1 defined on the product of $\Omega \times Z$, where Z is the space of stochastic matrices by the formula

$$F_1(\omega, \zeta) = (T\omega, (I - Q(\omega)\zeta - R(\omega))^{-1}P(\omega)).$$

Thus f is a shift T and so $\mathrm{Lip}(T^{-1}) = 2$. On the other hand due to [9, Proposition D.1], there are constants $\bar{K} > 0, \bar{\theta} < 1$ which depend only on the width of the strip m and on ε in (2.5) such that

$$d(F_1^n(\omega, \zeta'), F_1^n(\omega, \zeta'')) \le \bar{K}\bar{\theta}^n d(\zeta', \zeta'').$$

Applying Proposition A.1 to $F_1^{n_0}$ where n_0 is such that $\bar{K}\bar{\theta}^{n_0} < 1$ we get that ζ is C^α where α is such that

$$2^{\alpha n_0} \bar{K} \bar{\theta}^{n_0} < 1.$$

(Since n_0 can be arbitrarily large we can optimize with respect to n_0 and conclude that $\omega \to \zeta(\omega)$ is C^α provided that $2^\alpha \bar{\theta} < 1$.)

Since $\omega \to \zeta(\omega)$ is C^α, (2.12) shows that $\omega \to A(\omega)$ and $\omega \to \alpha(\omega)$ is C^α as well.

Next, $A(\omega)$ are positive matrices and therefore preserve the positive cone in $\mathbb{R}^m$. Moreover they act as contractions in the so called Hilbert metric (see e.g [22]). Consider now the map F_2 acting on $\Omega \times \mathbb{S}_+^{m-1}$ by the formula

$$F_2(\omega, v) = \left(T\omega, \frac{A(\omega)v}{||A(\omega)v||} \right),$$

where $\mathbb{S}_+^{m-1}$ is the set of unit vectors with positive coordinates. This map is a fiber contraction in the metric induced on $\mathbb{S}_+^{m-1}$ in a natural way by the Hilbert metric. Thus Proposition A.1 implies that $\omega \to v(\omega)$ is C^α. The Hölder property of $\omega \to l(\omega)$ is established similarly by looking at the projective action of α.

Finally the Hölder property of $\lambda(\omega)$ follows from the Hölder property of A and v, and the Hölder property of $\tilde{\lambda}(\omega)$ follows from the Hölder property of α and l. □

Proof of Lemma 12.1. The proof of Lemma 12.1 is similar to the proof of Lemma A.2 except that now we apply Proposition A.1 to skew products with the base map being toral translation $f(\omega) = \omega + \gamma$ rather than the shift of Ω. Thus $f^{-1}(\omega) = \omega - \gamma$ is an isometry and thus $\text{Lip}(f^{-1}) = 1$. Accordingly (A.1) holds for all r implying that $\bar{\zeta}, \bar{A}, \bar{\alpha}, \bar{v}, \bar{l}, \bar{\lambda}$ and $\bar{\tilde{\lambda}}$ are C^r smooth. □

Appendix B. CLT for transient uniquely ergodic environments.

In this section we consider uniquely ergodic environments defined by (3.8).

Then by stationarity there exist functions $\bar{\zeta}, \bar{A}, \bar{v}, \bar{\lambda}(\omega) = \ln ||\bar{A}(\omega)\bar{v}(f^{-1}(\omega)||$ on Ω such that

$$\zeta_n(\omega) = \bar{\zeta}(f^n\omega), \quad A_n(\omega) = \bar{A}(f^n\omega), \quad v_n(\omega) = \bar{v}(f^n\omega), \quad \lambda_n(\omega) = \bar{\lambda}(f^n\omega).$$

Applying C^0 Invariant Section Theorem (Proposition A.1(a) and Lemma A.2) we conclude similarly to Section 12 that the above functions $\bar{\zeta}$, $\bar{A}$, $\bar{v}$ and hence also $\bar{\lambda}$ are continuous.

Without loss of generality we assume that $X_t \to +\infty$ as $t \to \infty$ and hence $\boldsymbol{\lambda} = \mathbb{E}(\ln \bar{\lambda}) < 0$. We recall the general results proven in [13] for ergodic environments such that

$$\text{(B.1)} \qquad \mathbb{E}\left(||A_n(\omega)\dots A_2(\omega)A_1(\omega)v_0(\omega)||^2\right) = \mathbb{E}\left(\left[\bar{\lambda}(f^{n-1}\omega)\dots\bar{\lambda}(f\omega)\bar{\lambda}(\omega)\right]^2\right)$$

decays exponentially as $n \to \infty$. In our case this assumption is satisfied. Indeed, due to the unique ergodicity

$$\frac{\sum_{i=0}^{n-1} \ln \bar{\lambda}(f^j\omega)}{n} \to \boldsymbol{\lambda} \text{ as } n \to \infty \text{ uniformly in } \omega. \tag{B.2}$$

Hence for any $\varepsilon > 0$ there is N_ε such that for all $n > N_\varepsilon$ and all $\omega \in \Omega$ there is $\epsilon(n, \omega)$ satisfying $|\epsilon(n, \omega)| \leq \varepsilon$ and such that

$$\begin{aligned} &||A_n(\omega) \dots A_2(\omega)A_1(\omega)v_0(\omega)|| = \bar{\lambda}(\omega)\bar{\lambda}(f\omega) \dots \bar{\lambda}(f^{n-1}\omega) = \\ &\exp\left(\sum_{i=0}^{n-1} \ln \bar{\lambda}(f^j\omega)\right) = \exp(n(\boldsymbol{\lambda} + \epsilon(n, \omega))) \end{aligned} \tag{B.3}$$

which implies the exponential decay of (B.1).

It is clear that the asymptotic properties of the walk do not depend on its starting point. However, some formulae simplify significantly if the initial distribution is chosen as follows. We define the initial distribution of ξ by setting

$$P_{\omega,(0,\cdot)}\{\xi_\omega(0) = (0, i)\} = \sigma_0(i), \quad 1 \leq i \leq m,$$

where σ_0 is defined by (2.9). Let us list some properties of the vectors σ_n which will be used below. It follows directly from the above definition of σ_n that $\sigma_n = \sigma_k\zeta_k \dots \zeta_{n-1}$ for any $k < n$ (here we also use the relation $\zeta_n(\omega) = \zeta(f^n\omega)$). Next, $\sigma_0(\omega)$ is a continuous function of ω. This fact follows form Proposition A.1 applied to $\Omega \times U$, where U is the set of probability vectors of dimension m. The related skew product transformation is given by $(\omega, y) \mapsto (f\omega, y\zeta(\omega))$. The related fiber contraction property is the standard property of stochastic matrices ζ with $\zeta(i, j) \geq \epsilon$, where $\epsilon > 0$ and depends only on the ε from (2.5) (see [13, equations (1.9) and (1.17)]).

So, from now on $\xi(t) = \xi_\omega(t) = (X(t), Y(t))$ is the walk in RE ω starting from a random point in layer 0. More precise notations, such as e. g. $\xi_{\omega,(0,\cdot)}(t)$ will also be used where appropriate. The same convention applies to P_ω and $\mathbb{E}_\omega$.

Let $\mathfrak{t}_n$ be the hitting time, by the RW, of layer n, $\mathfrak{t}_n = \min\{t : X_t = n\}$. Recall that if a RW is recurrent or transient to the right then the entries of the matrix ζ_n have the following probabilistic meaning:

$$\zeta_n(i, j) = P_{\omega,(n,i)} \left(\text{RW starting from } (n, i) \text{ hits } L_{n+1} \text{ at } (n + 1, j)\right).$$

Since ξ is a Markov chain, it follows for $n \geq 1$ that

$$P_{\omega,(0,\cdot)}\{\xi(\mathfrak{t}_n) = (n, i)\} = \sigma_n(i), \quad 1 \leq i \leq m.$$

It is proven in [13] that if (B.1) holds then there are positive constants $\mathbf{v}$ and σ such that with probability 1

$$\frac{\mathbb{E}_\omega(\mathfrak{t}_n)}{n} \to \frac{1}{\mathbf{v}} \tag{B.4}$$

and

$$\frac{\mathfrak{t}_n - \mathbb{E}_\omega(\mathfrak{t}_n)}{\sigma\sqrt{n}} \text{ converges to a standard normal distribution.} \tag{B.5}$$

Define $\mathfrak{b}_n$ by the condition

$$\mathbb{E}_\omega(\mathfrak{t}_{\mathfrak{b}_n-1}) \le n \le \mathbb{E}_\omega(\mathfrak{t}_{\mathfrak{b}_n}).$$

We are now in a position to prove the precise version of Theorem 3.9.

Theorem B.1. *$\sigma\frac{X_n-\mathfrak{b}_n}{\sqrt{n}}$ converges to a standard normal distribution almost surely.*

Proof. We need two strengthenings of (B.4) for uniquely ergodic environments. First, as $k \to \infty$

$$\frac{\mathbb{E}_\omega(\mathfrak{t}_k)}{k} \to \frac{1}{\mathbf{v}} \text{ uniformly in } \omega \tag{B.6}$$

and hence

$$\frac{\mathbb{E}_\omega(\mathfrak{t}_{\mathfrak{i}+k} - \mathfrak{t}_\mathfrak{i})}{k} = \frac{\mathbb{E}_{f^\mathfrak{i}\omega}(\mathfrak{t}_k)}{k} \to \frac{1}{\mathbf{v}} \text{ uniformly in } \mathfrak{i} \text{ and } \omega. \tag{B.7}$$

Second

$$\mathbb{E}_\omega(\mathfrak{t}_1) \text{ is bounded} \tag{B.8}$$

and hence

$$\mathbb{E}_\omega(\mathfrak{t}_{\mathfrak{b}_n}) = n + O(1). \tag{B.9}$$

The proofs of (B.6) and (B.8) will be given later. Let us first see how those facts imply the theorem. Fix a large K. Given x we have

$$\mathbb{P}_\omega(\mathfrak{t}_{\mathfrak{b}_n+\sqrt{n}x+K\ln n} \le n) - \mathbb{P}_\omega(\mathcal{A}_{n,K}) \le \mathbb{P}_\omega\left(X_n - \mathfrak{b}_n \ge \sqrt{n}x\right) \le \mathbb{P}_\omega(\mathfrak{t}_{\mathfrak{b}_n+\sqrt{n}x} \le n)$$

where $\mathcal{A}_{n,K}$ is the event that X returns to level n after visiting level $n+K\ln n$. By [9, Lemma 3.2] if K is large enough then there is $\varepsilon > 0$ such that with probability 1

$$\mathbb{P}_\omega(\mathcal{A}_{n,K}) \le C(\omega)e^{-\varepsilon K \ln n} = \frac{C(\omega)}{n^{\varepsilon K}}.$$

Therefore to complete the proof of the CLT for X it suffices to obtain the asymptotic behaviour of $\mathbb{P}_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n} \le n)$ under the assumption that $k_n/\sqrt{n} \to x$. Next,

$$\mathbb{P}_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n} \le n) = \mathbb{P}_\omega\left(\frac{\mathfrak{t}_{\mathfrak{b}_n+k_n} - \mathbb{E}_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n})}{\sqrt{n}} \le \frac{n - \mathbb{E}_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n})}{\sqrt{n}}\right)$$

Equations (B.7) and (B.9) show that

$$\lim_{n\to\infty} \frac{n - \mathbb{E}_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n})}{\sqrt{n}} = -x$$

so (B.5) gives

$$\lim_{n\to\infty} \mathbb{P}_\omega(\mathfrak{t}_{\mathfrak{b}_n+k_n} \leq n) = \int_{-\infty}^{-x/\sigma} \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds$$

proving the CLT for X.

It remains to establish (B.6) and (B.8).

Denote by $\mathfrak{e}_n$ the column vector whose i^{th} coordinate $\mathfrak{e}_n(i)$ is $\mathbb{E}_{\omega,(0,i)}(\mathfrak{t}_n)$ (the expectation of $\mathfrak{t}_n$ conditioned on $\xi(0) = (0, i)$). By [13, equation (4.27)] for $n \geq 1$

$$\mathfrak{e}_n = \sum_{j=0}^{n-1} \zeta_0 \dots \zeta_j \sum_{i=0}^{-\infty} A_j \dots A_{j-i+1}(I - Q_{j-i}\zeta_{j-i-1} - R_{j-i})^{-1}\mathbf{1},$$

where we use the following conventions: for any k $\zeta_k \dots \zeta_k = I$, $A_j \dots A_{j+i-1} = I$ if $i = 0$, and $A_j \dots A_{j-i+1} = A_j$ if $i = 1$. Since $\mathbb{E}_\omega(\mathfrak{t}_n) = \sum_{i=1}^m \sigma_0(i)\mathfrak{e}_n(i)$, we have

$$\mathbb{E}_\omega(\mathfrak{t}_n) = \sigma_0\mathfrak{e}_n = \sum_{j=0}^{n-1} \sigma_j \sum_{i=0}^{\infty} A_j \dots A_{j-i+1}(I - Q_{j-i}\zeta_{j-i-1} - R_{j-i})^{-1}\mathbf{1}$$

and in particular

$$\mathbb{E}_\omega(\mathfrak{t}_1) = \sigma_0 \sum_{i=0}^{\infty} A_0 \dots A_{-i+1}(I - Q_{-i}\zeta_{-i-1} - R_{-i})^{-1}\mathbf{1}.$$

Hence

$$\mathbb{E}_\omega(\mathfrak{t}_1) \leq \text{Const} \sum_{i=0}^{\infty} ||A_0 \dots A_{-i+1}||.$$

Due to (B.2) and (B.3)

$$||A_i \dots A_0|| \leq \text{Const} \prod_{k=0}^{i-1} \bar{\lambda}(f^k\omega) \tag{B.10}$$

proving (B.8). Next, denote $\mathfrak{u}(\omega) = \mathbb{E}_\omega(\mathfrak{t}_1)$. Obviously $\mathbb{E}_\omega(\mathfrak{t}_n) = \sum_{j=0}^{n-1} \mathfrak{u}(f^j\omega)$ and since $\mathfrak{u}$ is continuous the unique ergodicity implies that $\frac{1}{n}\sum_{j=0}^{n-1} \mathfrak{u}(f^j\omega)$ converges uniformly uniformly in ω which proves (B.6). □

Note that Theorem B.1 requires a random centering by $\mathfrak{b}_n(\omega)$. On the other hand if f is a translation on $\mathbb{T}^d$, $(\bar{P}, \bar{Q}, \bar{R})$ are C^r, and (3.6) and (3.7) hold then σ_0 and hence $\mathfrak{u}$ are C^r. We now set $\bar{\mathfrak{u}} = \int_{\mathbb{T}^d} \mathfrak{u}(\omega)d\omega$ and apply (12.2) to $\mathfrak{u} - \bar{\mathfrak{u}}$. This gives $\mathfrak{u}(\omega) = \bar{\mathfrak{u}} + \hat{\Phi}(\omega + \gamma) - \hat{\Phi}(\omega)$, where $\hat{\Phi}$ is continuous and hence

$$\mathbb{E}_\omega(\mathfrak{t}_n) = n\bar{\mathfrak{u}} + \hat{\Phi}(\omega + n\gamma) - \hat{\Phi}(\omega) = \frac{n}{\mathbf{v}} + O(1), \text{ where } \frac{1}{\mathbf{v}} = \bar{\mathfrak{u}}$$

Accordingly $\mathfrak{b}_n = \mathbf{v}n + O(1)$ and we obtain

Corollary B.2. *In the quasiperiodic environment satisfying* (3.6)*,* (3.7) *and* $\mathbf{v} \neq 0$

$$\frac{\sigma(X_n - n\mathbf{v})}{\sqrt{n}}$$

converges to a standard normal distribution where σ *is the constant from* (B.5)*.*

Appendix C. Bounded ergodic sums.

The following lemma is a variation of the Gottschalk-Hedlund Theorem [14, Theorem 14.11]. We include the proof of this lemma for the sake of completeness and because it is very short.

Lemma C.1. *Let* T *be an ergodic transformation and* Φ *be a measurable function. Then there exists a constant* K *such that for almost all* ω

$$\left|\sum_{j=0}^{n-1} \Phi(T^j\omega)\right| \leq K \tag{C.1}$$

iff there exists a bounded function $\tilde{\Phi}$ *such that*

$$\Phi(\omega) = \tilde{\Phi}(T\omega) - \tilde{\Phi}(\omega) \tag{C.2}$$

Proof. (C.2) implies (C.1) since in that case $\sum_{j=0}^{n-1} \Phi(T^j\omega) = \tilde{\Phi}(T^n\omega) - \tilde{\Phi}(\omega)$. Conversely, if (C.1) holds then one can set $\tilde{\Phi}(\omega) = -\lim\inf_{n\to\infty} \sum_{j=0}^{n-1} \Phi(T^j\omega)$. □

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Dmitry Dolgopyat: Department of Mathematics and Institute of Physical Science and Technology, University of Maryland, College Park, MD, 20742, USA

Ilya Goldsheid: School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, Great Britain