FERMI ACCELERATION.

DMITRY DOLGOPYAT

Abstract. Fermi acceleration is a mechanism, first suggested by Enrico Fermi in 1949, to explain heating of particles in cosmic rays. Fermi studied charged particles being reflected by the moving interstellar magnetic field and either gaining or losing energy, depending on whether the "magnetic mirror" is approaching or receding. In a typical environment, Fermi argued, the probability of a head-on collision is greater than a head-tail collision, so particles would, on average, be accelerated. Since then Fermi acceleration has been used to explain a number of natural phenomena and several simple mathematical models demonstrating Fermi acceleration have been proposed. We describe these models and explain why they do or do not exhibit Fermi acceleration. We also mention some models where the answer is not known.

Contents

1. Introduction.

Consider a particle moving in a stationary force field. We want to understand how the particle's velocity changes with time. Naively one can expect that, in a stationary environment, the particle gets into

2000 *Mathematics Subject Classification.* Primary 70H11, 70K65, 60K37. Secondary 60J60.

This paper is an extended version of the lecture presented at the Winter Meeting of Canadian Mathematical Society on December 11, 2006. I thank Giovanni Forni and Konstantin Khanin for inviting me to deliver this lecture.

equilibrium with the environment after some time and so its velocity (temperature) becomes a stationary process. While this is indeed possible, especially in a dissipative setting, for Hamiltonian systems there is another regime. If the size of the system the particle interacts with is infinite (or very large) it can take particle infinite (or very large) time to get into equilibrium and its behavior during the transition regime is of interest. One of the first papers on this subject was work of Enrico Fermi on the origin of cosmic radiation [22]. Fermi wanted to explain abundance of high energy particles in the space. He argued that the particle can gain energy by passing through the non-uniform magnetic field. Fermi observed that if the particle meets a region of high field intensity moving towards the particle, then the particle accelerates, while if the fast particle passes a region of high field intensity moving away from the particle, then the particle decelerates. He concluded

> *The net result will be the average gain [of energy], primary for the reason that head-on collisions are more frequent than overtaking collisions since the relative velocity is larger in the former case.*

In other words the particle is more likely to accelerate than decelerate for the same reason that a motorist driving on a highway sees more cars going towards him than away from him.

Let us remark that if we consider not only head-on and overtaking collisions but sideway collisions as well then this mechanism is well known in mathematics. Namely, consider a simple random walk on $\mathbb{Z}^d : X_{n+1} = X_n + \xi_n$ where ξ_n are independent and take values $\pm e_j$, $j = 1 \dots d$. Suppose we want to know how far is the walker from the origin. If the walker is far from the origin the spheres, which are level sets of the distance function, are well approximated by the tangent planes and so in this approximation the average distance of the walker from the origin remains the same:

$$d(X_{n+1}, 0) = d(X_n, 0) + \frac{< X_n, \xi_n >}{d(X_n, 0)} + O\left(\frac{1}{d(X_n, 0)}\right).$$

and

$$\mathbb{E}(d(X_{n+1}, 0)|X_0 \dots X_n) = d(X_n, 0) + O\left(\frac{1}{d(X_n, 0)}\right).$$

However taking into account the higher order terms shows that the mean distance from the origin grows with time, because spheres are convex. More precisely the Central Limit Theorem implies that

$$\mathbb{E}(d(X_n, 0)) \sim c_d \sqrt{n}.$$

It was quickly realized that the particular features of the problem studied by Fermi were not essential for his argument and the same reasoning can be used to explain a wide range of natural phenomena. The goal of this survey is to discuss mathematical problems related to Fermi acceleration and to illustrate how it is related to some of the greatest achievements of the theory of dynamical systems in the 20th century. More specifically, we describe several simple mechanical models which were proposed to illustrate this phenomenon and explain why they do or do not exhibit Fermi acceleration. But to do so we first need to single out essential features of Fermi acceleration mechanism. In my opinion those features are the following.

- The mechanism is stochastic, that is, different initial conditions can lead to very different energy growth patterns;
- The acceleration is second order, that is, energy is a non-linear function of time. Often one can expect anomalous exponents.

On the other hand there are some features of the Fermi's original model (for example relativistic effects and magnetic fields) which we will not touch in this survey since there are simpler models with similar behavior and so most of the mathematical literature does not deal with them. We just note that often these can be incorporated into the model at the price of greater technical sophistication and refer the reader to the surveys [52, 62, 63] for more details.

2. **Stochastic models.**

Before discussing deterministic systems we describe some stochastic models which tell us that results can be expected.

Consider the equation $\ddot{x} = F(x,t)$ with

$$F(x,t) = \sum_j \phi_j(x - x_j, t - t_j)$$

where (x_j, t_j) is a Poisson process on $\mathbb{R}^d \times \mathbb{R}$ and $\phi_j(x,t)$ are independent functions supported on a cube of side 1 with rotation invariant distribution. We assume that for any j, $||\phi_j||_{C^2} \leq K$.

Suppose that $v(t_0) \gg 1$. Would the particle accelerate or decelerate? Rescale time $s = t/\varepsilon$ where $\varepsilon = 1/v(t_0)$. Then

$$\frac{\partial^2 x}{\partial s^2} = \varepsilon^2 F(x, \varepsilon s).$$

Thus

$$v(T) - v(0) = \varepsilon^2 \left(\int_0^T F(x(s), \varepsilon s) ds \right).$$

If F was independent of x we could apply the Central Limit Theorem. Here we use a non-linear version of the Central Limit Theorem called the *diffusion approximation* established in [30, 10]. We restate their results in our original variables.

Theorem 1. *Suppose $x(0) = x_0$, $v(0) = v_0$. Denote $K(t) = \frac{v^2(t)}{2}$. Then as $c \to \infty$ the process*

$$\frac{K^{3/2}(c\tau)}{c}$$

converges weakly to a square Bessel process $J(\tau)$ of dimension $\frac{2d}{3}$ started from 0.

Recall that the square Bessel process of dimension $\mathbf{n}$ is a solution of the following stochastic differential equation

$$dJ = \mathbf{n}d\tau + 2\sqrt{J}dW(\tau). \tag{1}$$

The appearance of square Bessel processes in Theorem 1 is not accidental.

Theorem 2. *Let $J(\tau)$ be the process with the following properties.*

1. *$J(\tau)$ is Markov;*
2. *$J(\tau)$ has continuous paths;*
3. *$J(c\tau)$ has the same distribution as $cJ(\tau)$.*

Then $J(\tau)$ is a square Bessel process (up to time rescaling).

Proof. Make a change of variables $V = \ln J$, $d\sigma = \frac{d\tau}{J}$. Then $V(\sigma)$ is a stationary process with continuous paths and independent increments. By Paul Levi's Theorem $V = a\sigma + bW(\sigma)$. Passing to our original variables we obtain (1), up to rescaling. □

Remark. If we lift the restriction that J is continuous then more processes then the set of possible limits becomes larger but these are well understood [36].

We denote by $\mathcal{B}_{\alpha,\mathbf{n}}$ the α-power of the square Bessel process of dimension $\mathbf{n}$. Thus $\mathcal{B}_{\alpha,\mathbf{n}} = \mathcal{K}^\alpha$, where $\mathcal{K}$ satisfies (1).

Our notation is justified by the following formulas
$\mathcal{B}_{\alpha,\mathbf{n}}(t) \sim t^\alpha$,
$(\mathcal{B}_{\alpha,\mathbf{n}})^\beta = \mathcal{B}_{\alpha\beta,\mathbf{n}}$
If $w_1(t), w_2(t) \dots w_d(t)$ are independent Brownian Motions then

$$\sum_{j=1}^{d} w_j^2(t) \sim \mathcal{B}_{1,d}.$$

Because they have so much symmetry Bessel processes are studied extensively. Let us mention some of their properties ([53], Section XI).

Proposition 2.1. *(a) $J(\tau)$ is recurrent if $\mathbf{n} \leq 2$ and transient if $\mathbf{n} > 2$.*

(b)
$$\mathbb{P}_x(a \leq J(\tau) \leq b) = \int_a^b p(x, y, \tau) dy$$

where

$$p(x, y, \tau) = \begin{cases} \frac{1}{2}\left(\frac{y}{x}\right)^{n/2-1} e^{-(x+y)/2\tau} I_{n/2-1}\left(\frac{\sqrt{xy}}{\tau}\right) & \text{if } x \neq 0 \\ \left(\frac{1}{2\tau}\right)^{n/2} \frac{1}{\Gamma(n/2)} y^{n/2-1} e^{-y/2\tau} & \text{if } x = 0 \end{cases}$$

and

$$I_\nu(z) = \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{\nu+2k} \frac{1}{k!\Gamma(\nu+k+1)}.$$

(c) The measure μ such that $\mu([0, J]) = J^{\mathbf{n}/2\alpha}$ is invariant under $\mathcal{B}_{\alpha,\mathbf{n}}$.

Remark. Since in Theorem 1 $\mathbf{n} = \frac{2d}{3}$ one can expect that for mechanical systems $v(t) \to \infty$ if $d > 3$ and $v(t)$ oscillates if $d \leq 3$.

In our case it is intuitively clear why the limiting process satisfies properties (1)–(3) of Theorem 2. Indeed since $F(x, t)$ has very short memory we can expect the limiting process to satisfy (1). Property (2) is little bit more difficult to justify but still it is not surprising that after a correct scaling the limiting process does not have large oscillations at small scales. Finally the meaning of (3) is the following. To derive Theorem 1 we make change of variables $s = t/\sqrt{\varepsilon}$ where $\varepsilon = c^{-2/3}$. However if we let $\varepsilon = 2c^{-2/3}$ we should get the same limit process up to rescaling.

The case of time independent force field is much more difficult because in that case it is not true that the asymptotic behavior does not depend on initial conditions. Indeed whatever (x_0, v_0) we take there it is possible that the field near x_0 is atypical so that the trajectory starting from v_0 is periodic and explores only small region in the phase space. However in high dimension where we expect that $v(t) \to \infty$ and so diffusion approximation becomes better and better, it possible to show that if v_0 is large the event that atypical field changes the character of the dynamics has small probability.

Theorem 3. [17] *Consider the equation $\ddot{x} = F(x)$ with*

$$F(x, t) = \sum_j \phi_j(x - x_j)$$

where (x_j) *is a Poisson process on* $\mathbb{R}^d \times \mathbb{R}$ *and* $\phi_j(x)$ *are independent functions supported on a cube of side 1 with rotation invariant distribution. We assume that for any* j $||\phi_j||_{C^2} \leq K$. *Suppose that* $d > 3$. *Then*

(a) The probability of the following events tends to 1 *as* $v_0 \to \infty$

(i) $||v(t)|| \to \infty$.

(ii) $x(t)$ *has no near selfintersections*[1].

(b) There is a set Ω_{v_0} *such that* $\mathbb{P}(\Omega_{v_0}) \to 1$ *and conditioned on* Ω_{v_0} *the process* $K(c\tau)/c^{2/3}$ *converges weakly to* $\mathcal{B}_{2/3,2d/3}$ *started from* 0 *as* $c \to \infty$.

Here (aii) is the crucial property. Indeed it means that our particle always explores new regions of the space so its dynamics is effectively the same as in the space-time random case considered above. To derive (aii) we use rotation invariance in an essential way. A typical result we use is the following.

Proposition 2.2. *Fix* $R > 0$. *Then*

$$\mathbb{P}_{x_0,v_0}\left(d(x(t), A)\right) \leq R) \leq \mathrm{Const}(R)\left[|v_0| \sin(Ax_0v_0)\right]^{2-d}.$$

Proof. Foliate $\mathbb{R}^d$ by planes orthogonal to v_0, foliate each plane by spheres centered at projection of x_0 and observe that the conditional distribution of $x(t)$ on each sphere is uniform. □

On the other hand Theorem 3 should be true without the assumption of rotation invariance. However without this assumption the proof of property (aii) should be much more complicated requiring a local limit type results for diffusive approximation, extending the results of [16, 33]. Thus removing rotation invariance assumption in Theorem 3 would make the result more general but it would not add much to our understanding of this system. Below I list several open questions which seem to require new ideas.

Problem 1. Estimate the probability of the following events

(a) $x(t)$ is bounded;

(b) $v(t)$ is bounded.

The argument of [17] gives polynomial in $1/v_0$ estimate for probability of these events but it well may be much smaller.

---

[1]Property (ii) means the following. Fix $R > 0$. Then the probability that

$$\forall t > 0 \quad d(x(s), x(t)) > R \text{ for } s > t + \frac{10}{||v(t)||}$$

tends to 1 as v_0 tends to ∞.

Problem 2. What happens for $d \leq 3$? It is easy to see that for $d = 1$ every trajectory is periodic but it is likely that convergence to the Bessel process still holds if $d = 2$ and 3.

Related results are obtained in [29, 21, 34] but they appear to fall short of answering this question.

Problem 3. Describe the limiting motion in the following cases
(a) Force is unbounded (and has a heavy tail);
(b) The interaction is long range (e.g. $\phi_j(x) = c_j \frac{x}{||x||^\alpha}$).

In case (a) the limiting process need not be continuous so the classification of [36] can be useful. Some results about the motion in the field with long-range correlations is obtained in [35].

3. **One and a half degree of freedom.**

In the previous section we saw that simple heuristic arguments based on dimensional analysis and decorrelation estimates allow to obtain very precise predictions about energy evolution. Unfortunately very often mechanical systems do not conform with those predictions. To see what can go wrong we consider several examples.

The first example is so-called Ulam ping-pong. Consider two periodic infinitely heavy walls and a ball bouncing between them with elastic collisions. We may assume that the wall moves perpendicularly to the walls since the ball's parallel velocity stays constant. This model has been proposed by Ulam as a limiting case of the motion of the charged particle in the strong potential.

Theorem 4. [50, 52] *All ping-pong trajectories have bounded velocities.*

In the hindsight we see that the highway analogy described in the introduction works poorly in this case because the ball keeps colliding with the same walls all the time rather than encounter different walls.

Of course one can argue that the assumption that the walls move periodically does not reflect the dynamics of cosmic particles.

Problem 4. What happens for more general, in particular random motion of the walls?

The proof of Theorem 4 relies on the famous KAM theory about the stability of quasiperiodic motions (see [44, 45, 24, 55, 56, 46]). In particular the result relevant for our problem is Moser's Small Twist Theorem.

Proposition 3.1 (Moser Small Twist Theorem)**.** *Let $Q:\mathbb{R}_+\to\mathbb{R}_+$ be a C^5-function. Then for any numbers a, b such that $Q'(r)\neq 0$ for $r\in[a,b]$ for any K there is ε_0 such that if F_ε are exact mappings of the annulus $\mathbb{R}_+\times\mathbb{S}^1$ of the form*

$$F_\varepsilon(r,\phi)=(r+\varepsilon^{1+\delta}P(r,\phi),\phi+\varepsilon Q(r)+\varepsilon^{1+\delta}R(r,\phi))$$

where

$$||P||_{C^5([a,b]\times\mathbb{S}^1)}\leq K,\quad ||R||_{C^5([a,b]\times\mathbb{S}^1)}\leq K$$

then for $\varepsilon\leq\varepsilon_0$ F_ε has (many) invariant curve(s) separating $[a,b]\times\mathbb{S}^1$ into two parts.

For the ping-pong system in the case velocity is large is a small perturbation of the system where the walls stay fixed and the ball moves periodically between them with period proportional to $1/v$. This explains the twist condition required by Moser's theorem.

The second example is a modification of the previous one but the results are different. We have only one wall and ball's return is ensured by gravity force. That is, the motion between collisions with the wall is given by $\ddot{h}=-g$. Let

$$\mathcal{E}=\{(v_0,t_0):v_n\to\infty\}$$

denotes the set of points whose energy tends to ∞.

Theorem 5. *([50]) There is an open set of wall motions $f(t)$ (in the space of analytic functions admitting an analytic continuation to to a given strip $|\Im t|\leq\varepsilon$) such that* $\mathrm{mes}(\mathcal{E})=\infty$.

Thus in this case ball can accelerate but this does not happen for all trajectories because the system is not ergodic, different orbits explore different regions in the phase space and so the averaged calculations of the Section 2 do not work.

Quite surprisingly the proof of this result also relies on KAM theory. Namely it is easy to arrange parameters so that there is an orbit where the ball always hits the wall at the same height and the wall moves up at the moments of collisions. To prove Theorem 5 one linearizes around this orbit and uses a non-stationary version of KAM theorem.

Problem 5. How large is the set of parameters where $\mathrm{mes}(\mathcal{E})=\infty$? In particular, is it open and dense?

Some related results are obtained in [20].

Contrasting the last two examples shows how subtle this problem is. In fact, despite a large number of papers devoted to related problems where are still several modifications for which expected behavior is unclear.

Problem 6. What happens if the wall is not straight but is a graph of periodic function $h(x,t) = \phi(x - g(t)) + f(t)$ (x can be multidimensional)?

The second question what happens for non-constant force fields. Assume that between the collision the particle moves in the potential $U(h)$. The case $U(h) = gh$ is covered by Theorem 5. Another interesting example is *impact oscillator* $U = \frac{gh^2}{2}$ (it corresponds to the case a ball is attached to some point by an ideal spring). This system has been investigated by Ortega.

Theorem 6. *([47, 48]) (a) If* $2\sqrt{g} \notin \mathbb{Q}$ *and* $\int_0^{2\pi} f(t)dt \neq 0$ *then* $\mathcal{E} = \emptyset$.

(b) If $2\sqrt{g} = p/q$, *let* $\Phi(\tau) = \sum_{j=0}^{2q-1} f(\tau+\pi j)$. *Then if* Φ *changes sign and zeroes of* Φ *are non-degenerate then all solutions with sufficiently large energy belong to* $\mathcal{E}$. *If* Φ *does not change sign, then* $\mathcal{E} = \emptyset$.

There is a beautiful relation between impact oscillators and outer billiards which we describe next following [3]. Recall that an *outer billiard map* is defined outside a closed convex curve Γ. If z is a point outside of Γ then its image under outer billiard map $\Phi(z)$ is the reflection of z about an orienting supporting line to Γ. We now consider an auxiliary system. Let $z = (x, v)$ be such that x lies to the right of the projection of Γ to x-axis. Rotate both z and Γ counter clockwise with angular velocity $\sqrt{g}$ until the time t_1 when the projections of the images of z and Γ meet. Then we reflect the image of $z(t_1)$ around the rightmost point of $\Gamma(t_1)$ to get a new point z_1. Since rotations preserve supporting lines $(z_1, \Gamma(t_1))$ is obtained from $(\Phi(z), \Gamma)$ by a rotation by angle $\sqrt{g}t_1$. Continue this procedure inductively we obtain $(z_n, \Gamma(t_n))$ which differ by rotation from $(\Phi^n z, \Gamma)$. On the other hand between collisions the motion of z is given by $\ddot{x} + gx = 0$ and at the point of collision $x^+ = x^-$, $v^+ = 2\mathbf{v}(t_1) - v^-$ where $(\mathbf{x}(t), \mathbf{v}(t))$ is the rightmost point of the image of Γ. It turns out that the motion of $\mathbf{x}$ is given by

$$\dot{\mathbf{x}} = \mathbf{v}, \quad \dot{\mathbf{v}} + g\mathbf{x} = r(\mathbf{x}) \tag{2}$$

where r is the radius of curvature. Hence this system is equivalent to impact oscillator where f is given by $\ddot{f} + gf = r$. Thus to every outer billiard there corresponds an impact oscillator but the converse is not true. Indeed given f we can try to define curvature $\kappa(t) = 1/(\ddot{f} + f)$ but we can not ensure that the curve with this curvature will be closed and convex.

In fact for outer billiards with (C^7-)smooth boundary all trajectories are bounded [19]. An interesting question what happens if Γ is only piecewise smooth. The most degenerate case is then Γ is a polygon in which case $\Phi(z)$ is obtained from z by reflection around a vertex. [61, 32, 23] prove boundedness of trajectories for so called *quasi-rational* polygons, a class including rational polygons as well as regular n-gones. Since affine equivalent curves have conjugated outer billiards all triangular outer billiards have bounded (in fact, periodic) orbits. [57, 58] consider kites–quadrangles with vertices $(-1,0),(0,1),(A,0)$ and $(0,-1)$ and establishes that for all irrational A there exists an unbounded orbit (if $A \in \mathbb{Q}$ then all orbits are periodic by the results cited above).

Problem 7. Is it true what almost any n-gone, $n > 3$ has unbounded orbits for outer billiard? The same question can be asked about typical piecewise smooth non-smooth curves and typical curves consisting of smooth arcs and line segments.

For example, numerical simulations given in [60] indicate that there are unbounded orbits for the semicircle.

Problem 7 is a special case of the following more general question.

Problem 8. Make a general theory for small piecewise smooth perturbations of integrable systems.

Apart from impact oscillators and outer billiards another well studied example is a nonlinear oscillator with piecewise smooth forcing [39, 37, 64]. The fact that the above mentioned systems exhibit a rich array of different behaviors shows that the theory requested in problem 8 could be interesting.

We now return to the bouncing ball in a nonlinear potential. Consider a general power potential $U(h) = h^\alpha$. For simplicity we assume that the wall's motion is harmonic $h(t) = B + A\sin t$ where $0 < A < B$. KAM approach extends with little difficulty to the case of strong potentials.

Theorem 7. [15] *If $\alpha > 1$, $\alpha \neq 2$ then all trajectories are bounded.*

By contrast, in case of weak potentials, it is easy to construct accelerating orbits. In fact it is shown in [12] that if $\alpha < 1$ then $\mathcal{E}$ has full Hausdorff dimension. However this set is likely to be small from the measure theoretic point of view.

Theorem 8. [15] *If $\alpha < 1/3$ then* $\mathrm{mes}(\mathcal{E}) = 0$.

Problem 9. Is the same true for all $\alpha < 1$?

A more difficult question is the following. Call an orbit *oscillatory* if $\limsup v_n = \infty$, $\liminf v_n < \infty$.

Problem 10. Is it true that for $\alpha < 1$ the set of oscillatory trajectories has infinite measure?

Let us describe the idea behind the proof of Theorem 8. Consider a successor map $(v_n, t_n) \to (v_{n+1}, t_{n+1})$. One defines a *critical set* $\mathbf{C}$ as a small (of order $v^-\beta$ where $\beta = \alpha^{-1} - 1$) neighborhood of the set where the collides with the wall during the the time the latter has zero acceleration. If α is small then

$$\text{mes}(\mathbf{C}) < \infty. \tag{3}$$

The key technical tool in proving Theorem 8 is the following.

Proposition 3.2. *Almost every orbit passes through* $\mathbf{C}$.

Proof of Theorem 8. By Proposition 3.2 it suffices to show that $\text{mes}(\mathcal{E} \bigcap \mathbf{C}) = 0$. Let $T : \mathbf{C} \to \mathbf{C}$ denote the first return map (which is well defined due to Lemma 3.2). Denote $(v^{(j)}, t^{(j)}) = T^j(v, t)$. Due to (3) we can apply Poincare Recurrence Theorem which tells us that for almost every point there are infinitely many j such that $v^{(j)} < v + 1$. In particular $v^{(j)} \not\to \infty$. □

The idea of the proof of Proposition 3.2 is the following. Assume the set A of orbits avoiding $\mathbf{C}$ has positive measure. One constructs a family of invariant cones outside $\mathbf{C}$. This implies that A is hyperbolic. Using the theory of chaotic systems one shows that the trajectory of v_n is well approximated by a one-dimensional Brownian Motion. Since the one-dimensional Brownian Motion is recurrent one obtains the recurrence of the velocity process. In particular almost every orbit in A infinitely many times visits the region of moderate height where $\mathbf{C}$ has a definite size and so during every visit an orbit has a definite probability to fall into $\mathbf{C}$. This allows to obtain a contradiction with the assumption that A has positive measure.

We observe that the behavior of orbits inside $\mathbf{C}$ can be quite complicated. [12] considers the so called *static wall approximation* commonly used in physics literature. He is interested in elliptic islands –the regions around elliptic periodic points filled with the invariant curve (the existence of such regions near elliptic periodic points is guaranteed by the KAM theory [45]). Let $I_2 = \{(A, B)$ such that there are infinitely many elliptic islands of period $2\}$.

Theorem 9. *(a) For all $\alpha < 1$, I_2 contains a G_δ subset of parameters.*

(b) For all $\alpha < 1$ the complement of I_2 has zero area.

(c) If $\frac{2}{3} < \alpha < 1$ then all allowable parameters belong to I_2.

Thus for a typical parameter the motion inside $\mathbf{C}$ is likely to be very complex with elliptic islands separated by chaotic sea. However this complexity does not enter the proof of the fact that $\mathcal{E}$ has zero measure since $\mathbf{C}$ is small (finite measure). However for answering problem 10 the study of the dynamics inside $\mathbf{C}$ is probably needed making problem 10 much less accessible.

4. **Several degrees of freedom.**

The previous section shows that for one and a half degrees of freedom even the question of existence of orbits with unbounded energy is quite non-trivial. Now we are going to consider the same question for higher dimensional systems. A model problem here is Mather acceleration problem.

Let (M, g) be a compact Riemannian manifold. Consider a particle moving in M with periodic potential $U(q, t)$.

$$H(q, v) = \frac{1}{2} g(q)(v, v) + U(q, t).$$

Theorem 10 (Geodesic Acceleration Theorem.)**.** [42, 2, 11] *Let* $\mathcal{E} = \{E(t) \to \infty\}$. *If* g *has a hyperbolic closed geodesic* γ_1 *and transverse geodesic* γ_2 *homoclinic to* γ_1 $(\lim_{t\to\pm\infty} d(\gamma_2(t), \gamma_1) = 0)$ *then for generic* $U(q, t)$ $\mathcal{E} \neq \emptyset$.

The assumptions of Theorem 10 hold in the following cases (see e.g. [31])

- Surfaces of genus greater than 1;
- Generic metrics on $\mathbb{T}^2$ and $\mathbb{S}^2$;
- Manifolds of negative sectional curvature.

Under the assumptions of Theorem 10 [49] shows that there exists an orbit such that $E(t) \geq c_1 t - c_2$.

Problem 11. Prove Geodesic Acceleration Theorem for generic perturbation of generic metric on any manifold.

There are several approaches to Theorem 10. Variational method was developed by Mather. It is based on the fact that our system is Lagrangian and the change of Lagrangian

$$L \to L + < \omega, v >$$

where $d\omega = 0$ does not change the dynamics whereas it does change the set of minimal trajectories. A very sophisticated variational problem is constructed whose solutions lie in $\mathcal{E}$.

By contrast a starting point of geometric methods [2, 11] is to observe that for $U = 0$ the system has an invariant set which is a product of

Smale horseshoe and $\mathbb{R}^2$. The horseshoe has symbolic dynamics. Now normal hyperbolicity theory [25] guarantees that the product structure persist for large energies even then $U \neq 0$. One then uses the symbolic dynamics to prescribe the orbits with energy gain.

Problem 12. Does the set of the accelerating orbits have positive (=infinite) measure?

We observe that both variational and geometric methods produce orbits with fast energy growth. While in many applications it is important to have an optimal growth, for the problem at hand it is actually a drawback since the typical (in the measure theory) sense orbit probably enjoys a slower growth.

We now restrict our attention to the case where the geodesic flow is chaotic. Namely, following [18] we consider metric of negative curvature. Here we encounter first examples of deterministic systems exhibiting Fermi acceleration.

Before presenting our results in the potential force let us analyze an easier case of time independent non-potential forces.

Let μ_R be the uniform measure on the set $\{R \leq K \leq R+1\}$.

Theorem 11. *Let x be distributed according to μ_R. Then as $R \to \infty$ $K(x, \tau R^{3/2})/R$ converges to $\mathcal{B}_{\frac{2}{3},\frac{2d}{3}}$.*

Recall Proposition 2.1. Denote $\nu = \frac{d}{3} - 1$.

Corollary 4.1. *(a)If $d \geq 4$ then $\mu_R(K(x,t) \to \infty) \to 1$. Moreover define $I(x) = \min_{t \geq 0} K(x,t)$ then*

$$\mu_R\left(\frac{I}{R} < z\right) \to z^{3\nu/2}.$$

(b)If $d \geq 4$ then $\forall \zeta < \frac{3\nu}{2}$

$$\mu_R(I < R_0) \leq \frac{C_{R_0}}{R^\zeta}.$$

(c) Let $d \geq 4$. Take $t_R \gg R^{3/2}$. Then

$$\mu_R\left(\frac{K(x,t_R)^{3/2}}{\sigma^2 t_R} < z\right) \to \frac{1}{2^{d/3}\Gamma(d/3)} \int_0^z y^\nu e^{-y/2} dy.$$

(d) Fix $R_0 \gg 1$. Let τ be the first moment $K(x,\tau) = R_0$. If $d = 2$ then as $R \to \infty$ the μ_R distribution of $\sigma^2\tau/R^{3/2}$ converges to the distribution of the time it takes $\mathcal{B}_{1,4/3}$ started at 1 to reach 0.

Let

$$\mathbf{B}_\pm = \{x : \lim \sup_{t \to \pm\infty} K(x,t) < \infty\},$$

$$\mathbf{O}_\pm = \{x : \lim \inf_{t\to\pm\infty} K(x,t) < \infty, \quad \lim \sup_{t\to\pm\infty} K(x,t) = \infty\},$$

$$\mathbf{E}_\pm = \{x : \lim \inf_{t\to\pm\infty} K(x,t) = \infty\}^2.$$

Denote $\mathbf{B} = \mathbf{B}_- \bigcap \mathbf{B}_+$, $\mathbf{O} = \mathbf{O}_- \bigcap \mathbf{O}_+$, $\mathbf{E} = \mathbf{E}_- \bigcap \mathbf{E}_+$. Using specification property of the geodesic flow it is not difficult to show that $\mathbf{X}_- \bigcap \mathbf{Y}_+ \neq \emptyset$. On the other hand Poincare recurrence theorem implies (see [38]) that

$$\mathrm{mes}(\mathbf{E}_\alpha \bigcap \mathbf{B}_\beta) = 0 \quad \mathrm{mes}(\mathbf{O}_\alpha \bigcap \mathbf{B}_\beta) = 0.$$

Theorem 11 allows to obtain several new estimates.

Theorem 12. *(a) If $d = 2$ then* $\mathrm{mes}(\mathbf{E}_\pm) = 0$.
(b) If $d \geq 4$ then $\mathrm{mes}(\mathbf{E}) = \infty$.
(c) If $d \geq 16$ then $\mathrm{mes}(\mathbf{O}_\pm) < \infty$ *and* $\mathrm{mes}(\mathbf{B}_\pm) < \infty$.

Consider now a particle moving on a negatively curved manifold in the presence of periodic potential $U(x, \tau)$.

Theorem 13. *Let x be distributed according to μ_R. (a) As $R \to \infty$ $K(x, \tau R^{5/2})/cR$ converges to $\mathcal{B}_{\frac{2}{5}, \frac{2d}{5}}$.*
(b) $\mu_R(\mathcal{E}) \to 1$ as $R \to \infty$ if $d > 5$ and $\mu_R(\mathcal{E}) = 0$ if $d < 5$.

5. **Galton board.**

Here we describe a simple mechanical system exhibiting Fermi acceleration.

Galton board, also known as quincunx or bean machine, is one of the simplest mechanical devices exhibiting stochastic behavior. It consists of a vertical (or inclined) board with interleaved rows of pegs. A ball thrown into the Galton board moves under gravitation and bounces off the pegs on its way down. If many balls are thrown into the quincunx, then one can observe a normal distribution of balls coming to rest on the machine floor.

Consider an idealized infinite Galton board, that is, a ball moving in a bean machine of infinite length under a constant external field. We neglect friction and the spin of the ball. Our pegs are convex obstacles (scatterers) positioned periodically on the board and satisfying the finite horizon condition (the latter means that the ball cannot move in any direction indefinitely without meeting a scatterer).

---

[2] $\mathbf{E}_+$ is denoted by $\mathcal{E}$ elsewhere in the paper

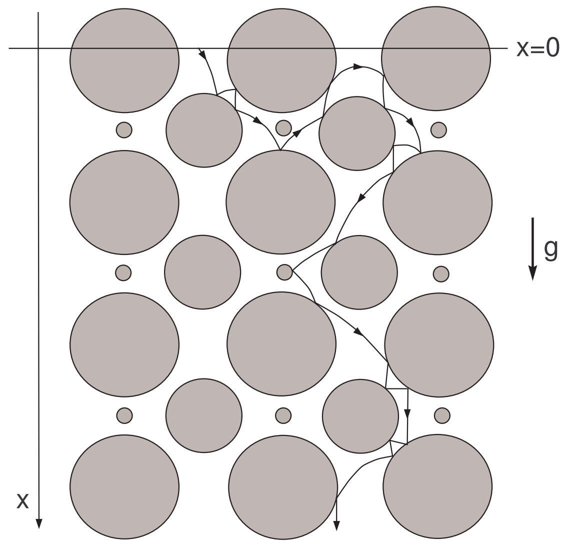

FIGURE 1. A trajectory of the Galton particle under an external field $\mathbf{g}$.

Our ball starts on the line $x = 0$, its y coordinate has a smooth distribution with a compact support, and its initial velocity is uniformly distributed in a sector

$$S_{V,\alpha} = \{\mathbf{v}(0) \colon c_1 V \le \|\mathbf{v}(0)\| \le c_2 V, \quad |\angle(\mathbf{v}(0), \mathbf{g})| \le \alpha\}. \tag{4}$$

Here $0 < c_1 < c_2$ are two constants whose values are irrelevant, and we assume that V is large enough and α is small enough.

We distinguish between the 'open' board $\mathcal{D}$ where the ball coming back to the line $x = 0$ escapes from $\mathcal{D}$ and the 'closed' one where the line $x = 0$ acts as a mirror reflecting the ball back into $\mathcal{D}$.

Theorem 14. *(a) In the open board the ball escapes from the board with probability one.*

(b) For the closed board there are constants $\tilde{c}, V_0$ such that if $V \ge V_0$, then $\tilde{c}\, t^{-1/3} v(t)$ converges, as $t \to \infty$, to a random variable with density

$$\frac{3z}{\Gamma(2/3)} \exp\left[-z^3\right], \qquad z \ge 0.$$

Accordingly, $2g\tilde{c}^2 t^{-2/3} x(t)$ converges to a random variable with density

$$\frac{3}{2\Gamma(2/3)} \exp\left[-z^{3/2}\right], \qquad z \ge 0.$$

Furthermore, the rescaled kinetic energy $K(\tau R^{3/2})/\tilde{c}R$ weakly converges, as $R \to \infty$, to $\mathcal{B}_{2/3,4/3}$.

(c) There are constants a, b, c such that the vector $\left(\frac{x(t)}{ct^{2/3}}, \frac{y(t)}{ct^{2/3}}\right)$ converges weakly to a random vector

$$(\mathbb{X}, \mathbb{Y}) = \left(\mathcal{K}(1),\ a\mathcal{K}(1) + b\mathcal{N}\sqrt{\int_0^1 \sqrt{\mathcal{K}(\chi)}d\chi}\right)$$

where $\mathcal{K} \sim \mathcal{B}_{2/3,4/3}$ and $\mathcal{N}$ be a Gaussian random variable with zero mean and unit variance independent of $\mathcal{K}$.

It is interesting to compare the Galton board with two related systems. First one is *Lorentz gas* corresponding to the case where $\mathbf{g} = 0$. The second is Lorentz gas with *Gaussian thermostat* introduced in [43]. Here we add to the Galton system an extra term modeling the dissipation of energy. Thus the motion between collisions is

$$\ddot{\mathbf{q}} = \mathbf{g} - \frac{\langle \mathbf{g}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\mathbf{v}.$$

In both cases the unit speed surface is preserved. For Lorentz gas the motion is diffusive and for the thermostated particle it is ballistic.

Theorem 15. *(a)* [4, 5] *For Lorentz gas there exists a matrix $\mathbf{D}$ such that*

$$\frac{\mathbf{q}(t)}{\sqrt{t}} \Rightarrow \mathcal{N}(0, \mathbf{D}).$$

(b) [6] *Consider the thermostated system with small field εg then there exist $\mathbf{a}(\varepsilon), \mathbf{D}(\varepsilon)$ such that*

$$\frac{\mathbf{q}(t) - \mathbf{a}(\varepsilon)t}{\sqrt{t}} \Rightarrow \mathcal{N}(0, \mathbf{D}(\varepsilon)).$$

(c) [9] *As $\varepsilon \to 0$, $\mathbf{D}(\varepsilon) \to \mathbf{D}$ and*

$$\frac{\mathbf{a}(\varepsilon)}{\varepsilon} \to \mathbf{D}\mathbf{g} \quad \textit{(Ohm law).}$$

Thus, as expected, the Galton particle moves faster than if we do not pump the energy into the system, however, quite surprisingly, it moves even faster if it shares the energy with the environment!

Part (c) of Theorem 15 is a special case of *Kawasaki formula* which we explain next. Kawasaki formula allows to determine the dimension of the limiting Bessel process. Other applications of this formula can be found in [54].

Consider a one-parameter family of flows $\phi_\varepsilon(t)$ having SRB measures μ_ε. That is, we assume that for any smooth measure ν

$$\nu(A \circ \phi_\varepsilon(t)) \to \mu_\varepsilon(A)$$

sufficiently quickly. Suppose further that μ_0 is smooth. Let $X(x) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{d}{dt}\Big|_{t=0}\phi_\varepsilon(t)x$. We wish to compute $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mu_\varepsilon(A)$. To this end observe that

$$\mu_\varepsilon(A)-\mu_0(A) = \lim_{t\to\infty}\left[\mu_0(A\circ\phi_\varepsilon(t))-\mu_0(A)\right] = \lim_{t\to\infty}\int_0^t \frac{d}{ds}\mu_\varepsilon(A\circ\phi_\varepsilon(s))ds.$$

To compare $\displaystyle\int A(\phi_\varepsilon(s+h)x)d\mu_0(x)$ with $\displaystyle\int A(\phi_\varepsilon(s)x)d\mu_0(x)$ make the change of variables $y = \phi_\varepsilon(h)x$ in the first integral. Since ϕ_0 preserves μ_0

$$\frac{d\mu_0(y)}{d\mu_0(x)} = 1+\varepsilon h\mathrm{div}_{\mu_0}X+\dots$$

Thus

$$\int A(\phi_\varepsilon(s+h)x)d\mu_0(x)-\int A(\phi_\varepsilon(s)x)d\mu_0(x) = -\varepsilon h\int A(\phi_\varepsilon(s)x)[\mathrm{div}_{\mu_0}X](x)d\mu_0(x)+\dots$$

Letting $\varepsilon\to 0$, $h\to 0$ we obtain

$$\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mu_\varepsilon(A) = -\int_0^\infty\left[\int A(\phi_\varepsilon(s)x)[\mathrm{div}_{\mu_0}X](x)d\mu_0(x)\right]ds$$

To derive the Ohm law observe that $A = v$, $\mathrm{div}X(\mathbf{q},\mathbf{v}) = -\langle\mathbf{g},\mathbf{v}\rangle$. Hence

$$\int v d\mu_\varepsilon \sim \varepsilon\int_0^\infty \mu_0\langle\mathbf{v},\mathbf{g}\rangle(x)\mathbf{v}(\phi_0(s)x)ds = \frac{\varepsilon}{2}\int_{-\infty}^\infty \mu_0\langle\mathbf{v},\mathbf{g}\rangle(x)\mathbf{v}(\phi_0(s)x)ds.$$

The last integral gives a well-known expression for the diffusion matrix.

Next we give an informal derivation of Theorem 14 which has much in common with proof of other Fermi acceleration results mentioned in the survey.

Pick a moment $t_0 > 0$ and a small $\varepsilon > 0$. We will use a new (fast) time variable $s = (t-t_0)/\sqrt{\varepsilon}$, in which the particle's velocity is

$$\tilde{\mathbf{v}}(s) = d\mathbf{q}/ds = \varepsilon^{1/2}\mathbf{v}(t(s)) \tag{5}$$

and its kinetic energy is

$$\tilde{K}(s) = \tfrac{1}{2}\|\tilde{\mathbf{v}}(s)\|^2 = \varepsilon K(t(s)). \tag{6}$$

We will call this ε-*rescaled dynamics*. The equations of motion now read

$$d\mathbf{q}/ds = \tilde{\mathbf{v}}, \quad d\tilde{\mathbf{v}}/ds = \varepsilon\mathbf{g} \tag{7}$$

i.e. the particle moves with slower speed in a weaker field. In particular, choosing $\varepsilon\sim K^{-1}$ brings our system to the form in which the speed $\tilde{v} = \|\tilde{\mathbf{v}}\|$, and hence the times between collisions, are of order one.

In other words, we get a so called slow-fast system, with a slow variable $\tilde{K}$ and a pair of fast variables $X = (\mathbf{q}, \boldsymbol{\omega})$, where $\boldsymbol{\omega} = \mathbf{v}/v$ denotes the particle direction. In these variables, equations (7), to the leading order, read

$$\dot{\mathbf{q}} = \sqrt{2\tilde{K}}\boldsymbol{\omega}, \quad \dot{\boldsymbol{\omega}} = \frac{\varepsilon}{\sqrt{2\tilde{K}}}\big[\mathbf{g} - \langle \mathbf{g}, \boldsymbol{\omega}\rangle\, \boldsymbol{\omega}\big] + \mathcal{O}(\varepsilon^2) \tag{8}$$

$$\dot{\tilde{K}} = \varepsilon\sqrt{2\tilde{K}}\langle \mathbf{g}, \boldsymbol{\omega}\rangle.$$

Now we approximate (8) by the system[3]

$$\dot{\mathbf{q}} = \sqrt{2K}\boldsymbol{\omega}, \quad \dot{\boldsymbol{\omega}} = \frac{\varepsilon}{\sqrt{2K}}\big[\mathbf{g} - \langle \mathbf{g}, \boldsymbol{\omega}\rangle\, \boldsymbol{\omega}\big], \quad \dot{K} = 0. \tag{9}$$

The advantage of this approximation is that the particle moves at constant speed. Furthermore, the dynamics on any energy surfaces can be reduced to that on the unit speed surface. Namely, the solution to (9) with initial condition $(\mathbf{q}_0, \boldsymbol{\omega}_0, K_0)$ takes the form

$$K(t) = K_0, \quad (\mathbf{q}, \boldsymbol{\omega})(t, \varepsilon, \mathbf{q}_0, \boldsymbol{\omega}_0, K_0) = (\hat{\mathbf{q}}, \hat{\boldsymbol{\omega}})(t\sqrt{2K_0}, \varepsilon/2K_0, \mathbf{q}_0, \boldsymbol{\omega}_0)$$

where $(\hat{\mathbf{q}}, \hat{\boldsymbol{\omega}})(t, \varepsilon, \mathbf{q}_0, \boldsymbol{\omega}_0)$ denotes the solution of

$$\dot{\hat{\mathbf{q}}} = \hat{\boldsymbol{\omega}}, \quad \dot{\hat{\boldsymbol{\omega}}} = \varepsilon\left[\mathbf{g} - \langle \mathbf{g}, \hat{\boldsymbol{\omega}}\rangle\, \hat{\boldsymbol{\omega}}\right]. \tag{10}$$

with initial condition $(\mathbf{q}_0, \boldsymbol{\omega}_0)$. Equations (10) describe a particle in a periodic Lorentz gas under a constant external field $\varepsilon\mathbf{g}$ moving at unit speed due to a Gaussian thermostat.

Let $T = \delta\varepsilon^{-2}$ with a small $\delta > 0$; then approximations (8)–(10) give

$$\tilde{K}(T) - \tilde{K}(0) \approx \varepsilon\sqrt{2\bar{K}}\int_0^T \langle \mathbf{g}, \boldsymbol{\omega}\rangle\, dt \approx \varepsilon\int_0^{\hat{T}} \langle \mathbf{g}, \hat{\boldsymbol{\omega}}\rangle\, dt,$$

where $\hat{T} = T\sqrt{2\bar{K}}$. Using parts (b) and (c) of Theorem 15 we obtain

$$\begin{aligned}\tilde{K}(T) - \tilde{K}(0) &\approx \frac{\langle \mathbf{g}, \mathbf{D}\mathbf{g}\rangle\delta}{2\sqrt{2\bar{K}}} + (2\bar{K})^{1/4}\sqrt{\delta}\,\langle \mathbf{g}, \sigma_0(\hat{\boldsymbol{\omega}})\mathcal{Z}^{(2)}\rangle,\\ &= \frac{\langle \mathbf{g}, \mathbf{D}\mathbf{g}\rangle\delta}{2\sqrt{2\bar{K}}} + (2\bar{K})^{1/4}\sqrt{\delta}\,\langle \mathbf{g}, \mathbf{D}\mathbf{g}\rangle^{1/2}\mathcal{Z},\end{aligned}$$

where $\mathcal{Z}, \mathcal{Z}^{(2)}$ denote standard 1D and 2D normal vectors. Likewise, if we divide a longer time interval $(0, \tau\varepsilon^{-2})$ into segments of size $\delta\varepsilon^{-2}$, we obtain

$$\tilde{K}_{j+1} - \tilde{K}_j \approx \frac{\sigma^2\delta}{2\sqrt{2\tilde{K}_j}} + (2\tilde{K}_j)^{1/4}\sigma\sqrt{\delta}\mathcal{Z}_j \tag{11}$$

[3]A rigorous justification of this approximation relies on Shadowing Lemma, see [13, 14] for more details.

where
$$\sigma^2 = \langle \mathbf{D}\mathbf{g}, \mathbf{g} \rangle,$$
$\tilde{K}_j = \tilde{K}(j\delta\varepsilon^{-2})$ and $\mathcal{Z}_j$ are independent. (11) is a discrete approximation to
$$d\mathcal{K} = \frac{\sigma^2}{2\sqrt{2\mathcal{K}}}\, d\tau + (2\mathcal{K})^{1/4}\sigma\, dW_\tau,$$
whose solution is $\mathcal{B}_{2/3,4/3}$, up to rescaling.

We note that if we want just to determine a dimension of the Bessel process than a simpler derivation is available. Namely, by Proposition 2.1, $\mathcal{B}_{2/3,\mathbf{n}}$ has invariant measure $\mu([0,\mathcal{K}]) = c\mathcal{K}^{\frac{3\mathbf{n}}{4}}$. On the other hand our system is Hamiltonian, it preserves a Liouville measure $\mathbf{m}$ and
$$\lim_{\varepsilon\to 0} \varepsilon \mathbf{m}\left(\frac{\varepsilon v^2}{2} \leq \mathcal{K}\right) = c\mathcal{K}.$$
Thus $3\mathbf{n}/4 = 1/2$ so $\mathbf{n} = 4/3$.

We observe that the fact that the associated billiard system has finite horizon plays an important role here. Indeed for the infinite horizon Galton board requires a nonstandard normalization in the Central Limit Theorem [1, 59]. Hence the derivation given above does not work. In other words we expect a different limit process since condition (2) of Theorem 2 probably fails.

Problem 13. (a) Obtain an analogue of the Ohm Law for infinite horizon Lorentz gas.

(b) Obtain an analogue of Theorem 14 for infinite horizon Lorentz gas.

So far in this section we considered perturbations of the billiard motion by a force. Another interesting class of perturbations is given by billiards with moving boundaries (breathing billiards). One example of the breathing billiard is given by Ulam ping-pong where all orbits are bounded. [28] considers billiard in a moving circle $x^2+y^2 = R^2(t)$. They show that if R is a C^7 periodic function then all orbits are bounded. [26] extends this result to a moving elliptic billiard where the axis are moving with constant angular velocity ω and the sizes of the axis change periodically so that the eccentricity remains constant.

Problem 14. Show that in a typical periodic elliptic billiard there are accelerating orbits.

[26] gives an example of resonant orbits where the energy increases by a constant factor but the possibility of larger oscillations remains open. [27] contains a similar study of perturbation of another integrable system–oscillating rectangular billiards.

On the other end of the spectrum we have the following question.

Problem 15. Describe the motion in oscillating dispersing billiards.

Heuristic arguments similar to those presented above for Galton board suggest that the energy behaves like $\mathcal{B}_{2,4}$ but the rigorous proof of this result has to overcome a number of technical difficulties (cf. the discussion in [7], Section 5).

We refer the reader to [41, 40] for numerical studies of various classes of breathing billiards.

6. **Conclusions.**

For mechanical systems there are two well understood regimes. One is KAM regime where the phase space is filled with invariant curves preventing acceleration. Another is hyperbolic regime where Lyapunov exponents are positive and Fermi acceleration mechanism works.

An outstanding open problem is to understand systems with mixed behavior (like bouncing ball in a gravity field). There are some beautiful partial results, however, the general problem is but little understood and the author hopes that specific problems mentioned in this survey can lead to a progress in this field.

References

[1] Bleher P. M. *Statistical properties of two-dimensional periodic Lorentz gas with infinite horizon,* J. Statist. Phys. 66 (1992), 315–373.

[2] Bolotin S.,Treschev D. *Unbounded growth of energy in nonautonomous Hamiltonian systems,* Nonlinearity **12** (1999) 365–388.

[3] Boyland, P. *Dual billiards, twist maps and impact oscillators,* Nonlinearity **9** (1996) 1411–1438.

[4] Bunimovich L. A. and Sinai Ya. G. *Statistical properties of Lorentz gas with periodic configuration of scatterers,* Comm. Math. Phys. **78** (1980/81), 479–497.

[5] Bunimovich L. A., Sinai Ya. G., and Chernov N. I. *Statistical properties of two-dimensional hyperbolic billiards,* Russ. Math. Surv. **46** (1991), 47–106.

[6] Chernov N. I. *Sinai billiards under small external forces,* Ann. Henri Poincare **2** (2001) 197–236.

[7] Chernov N., Dolgopyat D. *Hyperbolic billiards and statistical physics,* Proc. ICM. Vol. II, 1679–1704, Eur. Math. Soc., Zurich, 2006.

[8] Chernov N., Dolgopyat D. *Galton board: limit theorems and recurrence,* preprint.

[9] Chernov N. I., Eyink G. L., Lebowitz J. L. and Sinai Ya. G., *Steady-state electrical conduction in the periodic Lorentz gas*, Comm. Math. Phys. **154** (1993), 569–601.

[10] Cogburn R. and Ellison J. A. *A stochastic theory of adiabatic invariance,* Comm. Math. Phys. **149** (1992), 97–126.

[11] Delshams A., de la Llave R. and Seara T. M. *A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model,* Mem. AMS **179** (2006), no. 844, viii+141 pp.

[12] de Simoi J. *Stability and Instability results in a model of Fermi acceleration,* preprint.

[13] Dolgopyat D. *On differentiability of SRB states for partially hyperbolic systems,* Invent. Math. **155** (2004) 389–449.

[14] Dolgopyat D. *Averaging and invariant measures,* Mosc. Math. J. **5** (2005), no. 3, 537–576.

[15] Dolgopyat D. *Bouncing balls in non-linear potentials,* to appaer in Discrete and Continuous Dyn. Sys.–A.

[16] Dolgopyat D., Koralov L. *Avergaging of Hamiltonian flows with an ergodic component,* to appear in Ann. Prob.

[17] Dolgopyat D., Koralov L. *Motion in a random force field,* preprint.

[18] Dolgopyat D., de la Llave R. *Accelerating particles on negatively curved manifolds,* in preparation.

[19] Douady R. *These de 3-eme cycle* Universite de Paris 7, 1982.

[20] Duarte, P. *Plenty of elliptic islands for the standard family of area preserving maps,* Ann. Inst. H. Poincar Anal. Non Lineaire **11** (1994) 359–409.

[21] Durr D., Goldstein S., Lebowitz J., *Asymptotic Motion of a Classical Particle in a Random Potential in Two Dimensions: Landau Model*, Comm. Math. Phys. **113** (1987) 209-230.

[22] Fermi E. *On the origin of cosmic radiation,* Phys. Rev. **75** (1949), 1169–1174.

[23] Gutkin E., Simanyi N. *Dual polygonal billiards and necklace dynamics,* Comm. Math. Phys. **143** (1992) 431–449.

[24] Herman M. *Sur les courbes invariantes par les diffeomorphismes de l'anneau,* Asterisque 1**103–104** (1983).

[25] Hirsch M. W., Pugh C. C. and Shub M. *Invariant manifolds,* Lecture Notes in Math. **583** (1977) Springer, Berlin-New York.

[26] Itin, A. P., Neishtadt A. I. *Resonant phenomena in slowly perturbed elliptic billiards,* Regul. Chaotic Dyn. **8** (2003) 59–66.

[27] Itin, A. P., Neishtadt A. I., Vasiliev A. A. *Resonant phenomena in slowly perturbed rectangular billiards,* Phys. Lett. A **291** no. 2-3 (2001) 133-138.

[28] Kamphorst, S. O., de Carvalho, S. P. *Bounded gain of energy on the breathing circle billiard,* Nonlinearity **12** (1999) 1363–1371.

[29] Kesten H., Papanicolaou G. *A Limit Theorem for Stochastic Acceleration,* Comm. Math. Phys. **78** (1980/81) 19-63.

[30] Khasminskii R. Z. *Diffusion processes with a small parameter,* Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1281–1300.

[31] Knieper G. *Hyperbolic dynamics and Riemannian geometry,* in Handbook of dynamical systems, **1A** (2002) 453–545, North-Holland, Amsterdam/

[32] Kolodziej R. *The antibilliard outside a polygon,* Bull. Polish Acad. Sci. Math. **37** (1989)163–168.

[33] Komorowski T., Ryzhik L. *Diffusion in a Weakly Random Hamiltonian Flow*, Comm. Math. Phys. **263** (2006) 277-323.

[34] Komorowski T., Ryzhik L. *The Stochastic Acceleration Problem in Two Dimensions*, Israel Journal of Mathematics, **155** (2006) 157–204.

[35] Komorowski T., Ryzhik L. *Passive tracer in a slowly decorrelating random flow with a large mean,* Nonlinearity **20** (2007) 1215–1239.

[36] Lamperti J. *Semi-stable stochastic processes* Trans. AMS **104** (1962), 62–78.

[37] Levi M. *On Littlewood's counterexample of unbounded motions in superquadratic potentials,* in *Dynamics reported-1* Edited by C. K. R. T. Jones, U. Kirchgraber and H.-O. Walther. Springer, Berlin, 1992, 113–124.

[38] Littlewood J. E. *On the problem of n bodies,* Comm. Sem. Math. Unvi. Lund Tome Supplementaire (1952), 143–151.

[39] Littlewood J. E. *Unbounded solutions of an equation* $\ddot{y} + g(y) = p(t)$*, with* $p(t)$ *periodic and bounded, and* $g(y)/y \to \infty$ *as* $y \to \infty$*,* J. London Math. Soc. **41** (1966) 497–507.

[40] Loskutov A. Yu., Ryabov A.B. *Particle dynamics in time-dependent stadium-like billiards,* J. Statist. Phys. **108** (2002) 995–1014.

[41] Loskutov A. Yu., Ryabov A.B. and Akinshin L. G. *Mechanism of Fermi acceleration in dispersing billiards with time-dependent boundaries,* J. Exp. Theor. Phys. **89** (1999) 966–974.

[42] Mather J. Graduate course at Princeton, 95–96.

[43] Moran B. and Hoover W., *Diffusion in a periodic Lorentz gas*, J. Stat. Phys. **48** (1987), 709–726.

[44] Moser J. *On invariant curves of area-preserving mappings of an annulus,* Nachr. Akad. Wiss. Gttingen Math.-Phys. Kl. II **1962** (1962) 1–20.

[45] Moser J. Stable and random motions in dynamical systems, Ann. of Math. Studies **77** 1973, Princeton University Press, Princeton.

[46] Ortega R. *Asymmetric oscillators and twist mappings,* J. London Math. Soc. **53** (1996), no. 2, 325–342.

[47] Ortega R. *Boundedness in a piecewise linear oscillator and a variant of the small twist theorem,* Proc. London Math. Soc. **79** (1999) 381–413.

[48] Ortega R. *Dynamics of a forced oscillator having an obstacle,* in Variational and topological methods in the study of nonlinear phenomena (Pisa, 2000), Progr. Nonlinear Diff. Equations Appl., **49** (2002) 75–87, Birkhauser, Boston.

[49] Piftankin G. N. *Diffusion speed in the Mather problem,* Nonlinearity **19** (2006) 2617–2644.

[50] Pustylnikov, L. D. *Stable and oscillating motions in nonautonomous dynamical systems–II,* (Russian) Proc. Moscow Math. Soc. **34** (1977), 3–103

[51] Pustylnikov, L. D. A problem of Ulam. (Russian) Teoret. Mat. Fiz. **57** (1983) 128–132.

[52] Pustylnikov, L. D. *Poincar models, rigorous justification of the second law of thermodynamics from mechanics, and the Fermi acceleration mechanism,* Russian Math. Surveys **50** (1995) 145–189.

[53] Revuz D. and Yor M. *Continuous martingales and Brownian motion,* 3d edition. Grundlehren der Mathematischen Wissenschaften, **293** (1999) Springer-Verlag, Berlin.

[54] Ruelle D. *Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics,* J. Statist. Phys. **95** (1999) 393–468.

[55] Russmann, H. *Uber invariante Kurven differenzierbarer Abbildungen eines Kreisringes,* (German) Nachr. Akad. Wiss. Gottingen Math.-Phys. Kl. **II** **1970** (1970) 67–105.

[56] Russmann, H. *On the existence of invariant curves of twist mappings of an annulus,* in Geometric dynamics (Rio de Janeiro, 1981), Lecture Notes in Math., **1007** (1983) 677-718, Springer, Berlin.
[57] R. Schwartz *Unbounded Orbits for Outer Billiards-1,* J. of Modern Dyn. **1** (2007) 371–424.
[58] R. Schwartz *Outer Billiards on Kites*, preprint.
[59] Szasz D., Varju T. *Limit laws and recurrence for the planar Lorentz process with infinite horizon,* preprint.
[60] Tabachnikov S. *Outer billiards,* Russian Math. Surv. **48** (1993), no. 6, 81–109.
[61] Vivaldi F., Shaidenko A. V. *Global stability of a class of discontinuous dual billiards,* Comm. Math. Phys. **110** (1987) 625–640.
[62] Zaslavsky, G. M. *Chaos in dynamic systems,* Translated from the Russian by V. I. Kisin. Harwood Academic Publishers, Chur, 1985.
[63] Zaslavsky G. M., Sagdeev R. Z., Usikov, D. A., Chernikov, A. A. *Weak chaos and quasi-regular patterns,* Translated from the Russian by A. R. Sagdeeva. Cambridge Nonlin. Sci. Ser, **1** (1991), Cambridge Univ. Press, Cambridge.
[64] Zharnitsky V. *Breakdown of stability of motion in superquadratic potentials,* Comm. Math. Phys. **189** (1997) 165–204.

Department of Mathematics, University of Maryland, College Park, MD 20742