LIMIT THEOREMS FOR HYPERBOLIC SYSTEMS.

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1. Introduction

One of the major discoveries of the 20th century mathematics is the possibility of random behavior of deterministic systems. There is a hierarchy of chaotic properties for a dynamical systems but one of the strongest is when the smooth observables satisfy the same limit theorems as independent (or Markov related) random variables. In my lectures I will discuss how to prove limit theorems for hyperbolic systems with sufficiently strong mixing properties. In many application an appropriate notion of mixing for systems without smooth invariant measure is the following

$$\int B(x)A(f^n x)dx \approx \int B(x)dx \int Ad\mu_{SRB} \tag{1}$$

where μ_{SRB} is the measure defined the condition above. It is called Sinai-Ruelle-Bowen (SRB) measure. Here A and B are observable from suitable function spaces. For our technique it is convenient to assume that (1) holds when $A \in C^r(M)$ and B is smooth in the unstable direction.

One of the most powerful methods for proving limit theorems for random proceses is *martingale problem method* developed by Stroock-Varadhan and others (see [41]). In my lectures I will present this method and discuss ideas need to adapt it to the dynamics setting as well as mention several open problems.

2. Central Limit Theorem

2.1. **iid random variables.** In order to explain how the method work we start with simplest possible settings. Let X_n be independent identically distributed random variables which are uniformly bounded. (Of course the assumption that X_n are bounded is unnecessary. We impose it in order to simplify the exposition.) We assume that $\mathbb{E}(X) = 0, \mathbb{E}(X^2) = \sigma^2$. Denote $S_N = \sum_{n=1}^N X_n$. The classical Central Limit Theorem says that $\frac{S_N}{\sqrt{N}}$ converges weakly to the normal random variable with zero mean and variance σ^2. Our idea for proving this result is the following. We know the distribution of S_0 so we want to see how the distribution changes when we change N. To this end let $M \to \infty$ so that $M/N \to t$. Then

$$\frac{S_M}{\sqrt{N}} = \frac{\sqrt{M}}{\sqrt{N}} \frac{S_M}{\sqrt{M}} \approx \sqrt{t} \frac{S_M}{\sqrt{M}}.$$

The second factor here is normal with zero mean and variance $t\sigma^2$. Since multiplying normal random variable by a number has an effect of multiplying its variance by the square of this number the classical Central Limit Theorem can be restated as follows.

Theorem 1. *As $N \to \infty$ $\frac{S_{Nt}}{\sqrt{N}}$ converges weakly to the normal random variable with zero mean and variance $t\sigma^2$.*

Thus we wish to show that for large N our random variables behave like the random variables with density $p(t, x)$ whose Fourier transform satisfies

$$\hat{p}(t, \xi) = \exp\left(-\frac{t\sigma^2\xi^2}{2}\right).$$

Hence

$$\partial_t \hat{p} = (i\xi)^2 \frac{\sigma^2}{2} \hat{p} \text{ and so } \partial_t p = \frac{\sigma^2}{2} \partial_x^2 p.$$

Recall that any weak solution of the heat equation is also strong solution so we need show that if $v(t, x)$ is a smooth function of compact support in x then

$$\int v(T, x)p(T, x)dx - v(0, 0) = \iint p(t, x) \left[\partial_t v + \frac{\sigma^2}{2} \partial_x^2 v\right] dxdt. \tag{2}$$

In case $v(t, x) = u(x)$ is independent of t the last equation reduces to

$$\int u(x)p(T, x)dx - u(0) = \iint p(t, x)(\mathcal{L}u)(x)dxdt. \tag{3}$$

Conversely if (3) holds for each T and if $\mathbf{S}_t$ is any limit point of $\frac{S_{Nt}}{\sqrt{N}}$ then

$$\partial_t \mathbb{E}(u(\mathbf{S}_t)) = \mathbb{E}((\mathcal{L}u)(\mathbf{S}_t))$$

where $\mathcal{L} = \frac{\sigma_2}{2} \partial_x^2$. which implies (2) for functions of the form $v(t, x) = k(t)u(x)$ and hence for the dense family $\sum_j k_j(t)u_j(x)$. Thus p satisfies the heat equation as claimed. Thus we have to establish (3). For discrete system in amounts to showing that

$$\mathbb{E}\left(u\left(\frac{S_M}{\sqrt{N}}\right)\right) - u(0) - \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}\left((\mathcal{L}u)\left(\frac{S_n}{\sqrt{N}}\right)\right) = o(1).$$

where $M \sim tN$. Consider the Taylor expansion

(4)

$$u\left(\frac{S_{n+1}}{\sqrt{N}}\right)-u\left(\frac{S_n}{\sqrt{N}}\right) = (\partial_x u)\left(\frac{S_n}{\sqrt{N}}\right)\frac{X_n}{\sqrt{N}}+\frac{1}{2}\left(\partial_x^2 u\right)\left(\frac{S_n}{\sqrt{N}}\right)\frac{X_n^2}{N}+\mathcal{O}(N^{-3/2}).$$

Keeping the above example in mind we can summarize martingale problem approach as follows.

In order to describe the distribution of $\mathbf{S}_t$ we need to compute the averages $\mathbb{E}(u(\mathbf{S}_t))$ for a large class of test functions u. However rather than trying to compute the above averages directly we would like to split the problem in two two parts. First we find an equation which this average should satisfy. Secondly we show that this equation has unique solution. Only the first part involves the study of the system in question. The second part deals with a PDE question.

For the first step we need to compute the generator

$$(\mathcal{L}u)(x) = \lim_{N\to\infty}\lim_{h\to 0}\frac{\mathbb{E}(u(S_t^N)|S_0^N = x) - u(x)}{h}.$$

For the second step we need to establish the uniqueness for the equation

$$\partial_t u = \mathcal{L}u.$$

Once this is done we conclude that for a large class of test functions we have

$$\mathbb{E}(v(T,\mathbf{S}_t)) - \mathbb{E}(v(0),\mathbf{S}_0) = \int_0^T \mathbb{E}(\partial_t v + \mathcal{L}v)(t,\mathbf{S}_t)dt.$$

Choosing here v satisfying the final value problem

$$\partial_t v + \mathcal{L}v = 0, \quad v(T,\mathbf{S}) = u(\mathbf{S}) \tag{5}$$

we can achieve our goal of finding $\mathbb{E}(u(\mathbf{S}_T))$.

2.2. **Partially hyperbolic systems.** Now let us discuss how to extend this approach to the dynamics setting. namely we consider the case where

$$S_n = \sum_{j=0}^{n-1} A(f^j x)$$

where f is an Anosov diffeo, A is smooth and $\int A d\mu_{SRB} = 0$. Concerning x we assume that it is distributed on D with a smooth density ρ where D is a d_u-dimensional submanifold transversal to the stable direction. The difference with the previous example is that $A(f^n x)$

and S_n are no longer independent so a more careful analysis of (4) is needed. Take $L_N = N^{0.01}$ and let $\bar{n} = n - L_N$. We have

$$\mathbb{E}\left((\partial_x^2 u)\left(\frac{S_n(x)}{\sqrt{N}}\right)A(f^n x)\right) = \mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}(x)}{\sqrt{N}}\right)A(f^n x)\right)+\mathcal{O}\left(\frac{L_N}{\sqrt{N}}\right)$$

$$= \mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}(f^{-\bar{n}}y)}{\sqrt{N}}\right)A^2(f^{L_N} y)\right)+\mathcal{O}\left(\frac{L_N}{\sqrt{N}}\right).$$

The first factor is smooth in the unstable direction so

$$\mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}(f^{-\bar{n}}y)}{\sqrt{N}}\right)A^2(f^{L_N} y)\right) = \mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}(f^{-\bar{n}}y)}{\sqrt{N}}\right)\right)\mu_{SRB}\left(A^2\right)+\mathcal{O}\left(\theta^{L_N}\right)$$

$$= \mathbb{E}\left((\partial_x^2 u)\left(\frac{S_n(x)}{\sqrt{N}}\right)\right)\mu_{SRB}\left(A^2\right)+\mathcal{O}\left(\frac{L_N}{\sqrt{N}}\right).$$

This takes care about the second derivative. However the first derivative term is more difficult since it comes with smaller prefactor $\frac{1}{\sqrt{N}}$. We have

$$\mathbb{E}\left((\partial_x u)\left(\frac{S_n}{\sqrt{N}}\right)A(f^n x)\right)$$

$$= \mathbb{E}\left((\partial_x u)(\frac{S_{\bar{n}}}{\sqrt{N}})A(f^n x)\right)+\mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}}{\sqrt{N}}\right)A(f^n x))\sum_{k=\bar{n}}^{n-1}\frac{A(f^k x)}{\sqrt{N}}\right)+\mathcal{O}\left(\frac{L_N^2}{N}\right).$$

As before

$$\mathbb{E}\left((\partial_x u)(\frac{S_{\bar{n}}}{\sqrt{N}})A(f^n x)\right) = \mathcal{O}\left(\theta^{L_N}\right).$$

To address the second term fix a large M_0, let $m = n - k$ and consider two cases

(I) $m > M_0$. Then we let $y = f^k x$

$$\mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}}{\sqrt{N}}\right)A(f^k x))A(f^n x)\right) = \mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}(f^{-k}y)}{\sqrt{N}}\right)A(y))A(f^m y)\right)$$

$$= \mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}}{\sqrt{N}}\right)A(f^k x))\right)\mu_{SRB}(A)+\mathcal{O}(\theta^m) = \mathcal{O}(\theta^m).$$

(II) $m \leq M_0$. Denote $B_m(y) = A(y)A(f^m y)$. Then we have

$$\mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}}{\sqrt{N}}\right)A(f^k x))A(f^n x)\right) = \mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}(f^{-\mathbf{r}n}y}{\sqrt{N}}\right)B_m(f^k y))\right)$$

$$= \mathbb{E}\left((\partial_x^2 u)\left(\frac{S_{\bar{n}}}{\sqrt{N}}\right)\right)\mu_{SRB}\left(B_m(f^k y))\right).$$

Summation over m gives

$$\mathbb{E}\left(u\left(\frac{S_M}{\sqrt{N}}\right)\right)-u(0) = \frac{1}{N}\sum_{n=0}^{M-1}\frac{\sigma_{M_0}^2}{2}\mathbb{E}\left((\partial_x^2 u)\left(\frac{S_M}{\sqrt{N}}\right)\right)+\mathcal{O}\left(\theta^{M_0}\right)+o(1)$$

where

$$\sigma^2_{M_0} = \mu_{SRB}(A^2)+2\sum_{m=1}^{M_0}\mu_{SRB}(A(x)A(f^m x)) = \sum_{m=-M_0}^{M_0}\mu_{SRB}(A(x)A(f^m x)).$$

Letting $M_0 \to \infty$ we obtain that $\frac{S_N}{\sqrt{N}}$ is asymptotically normal with zero mean and variance given by the **Green-Kubo formula**

$$\sigma^2 = \sum_{m=-\infty}^{\infty}\mu_{SRB}(A(x)A(f^m x)).$$

We see that the Anosov property was not important in the proof. In fact the natural setting for the above results is the following

(1) There is an invariant cone family $df(K) \subset K$ and for each $v \in K$ we have

$$||df(v)|| \geq \Lambda||v||, \Lambda > 1.$$

(2) f is mixing in the following sense. Let D be a submanifold of the same dimension as the axis of the cone and such that

$$\mathrm{Vol}(D) > v_0, \quad \mathrm{Vol}(\partial_\varepsilon D) \leq C\varepsilon^\alpha \tag{6}$$

Let ρ be a Holder probability density on D then

$$\int_D \rho(x)A(f^n x) \leq \frac{C}{n^{1+\delta}}||A||_{C^r} \tag{7}$$

Concerning the initial distribution of x we assume that it is taken according to the measure

$$mu(A) = \int d\nu_\alpha \int_{D_\alpha} A(x)\rho_\alpha(x)dx \tag{8}$$

where ν is some factor measure. Roughly speaking μ is absolutely continuous with respect to the unstable foliation.

Theorem 2. *(*[24]*) Under the above assumption the CLT holds for Holder functions.*

Examples of systems satisfying the above assumptions include (generic elements of the) following

1. Anosov diffeomorphisms [9];
2. time one maps of Anosov flows [21, 36, 27];
3. partially hyperbolic translations on homogeneous spaces [34];
4. compact group extensions of Anosov diffeomorphisms [22];
5. partially hyperbolic toral automorphisms [32];
6. mostly contracting systems [23, 11, 12].

Problem 1. Can the CLT (and other results of these lectures) be extended to hyperbolic $\mathbb{Z}^k$-actions?

See [31] for the the definition and examples of hyperbolic $\mathbb{Z}^k$-actions.

2.3. **Systems with singularities.** The approach of the previous section can be extended to the systems with singularities. The results here are not as complete as for the partially hyperbolic setting so we discussing some ideas without making an attempt at completeness.

For the systems with singularities (7) is not sufficient. Indeed it claims the mixing for large pieces of unstable manifolds but we need another condition which tells that small pieces of unstable manifold grow under the dynamics. If D is smooth we can decompose

$$f^{\bar{n}} D = \bigcup_{\alpha} D_\alpha$$

where D_α are admissible for (7). (To see this one can introduce local coordinates so that D are given by graphs $z_s = \phi(z_u)$, pick a set of points $\{p_\beta\}$ in $\mathbb{R}^{d_u}$ and let

$$D_\alpha = \{x \in D : d(z(x), p_\alpha \leq d(z(x), p_\beta \forall \beta\}.)$$

However if f has singularities D_α can be arbitrary short failing (6). So (7) has to be supplemented by a **Growth Lemma.** Let μ be as in (8). Let

$$Z(\mu) = \int Z(D_\alpha) d\mu(\alpha) \text{ where } Z(D) = \sup_{\varepsilon>0} \frac{\mathbb{P}(r(x) < \varepsilon)}{\varepsilon}$$

and $r(x) = d(x, \partial D)$. We say that a system satisfies Growth Lemma if there is $C > 0, \theta < 1$ such that

$$Z(f\mu) \leq \theta Z(\mu) + C.$$

If (7) and the Growth Lemma hold then the CLT is satisfied for initial measures in the form (8) with $Z(\mu) < \infty$.

In order to check the Growth Lemma one typically has to verify a suitable **complexity bound**. For example suppose that $f : M \to M$, $\dim(M) = 2$ $M = \bigcup M_j$ f is smooth on each M_j and cone hyperbolic and $||df||$ is bounded. We also assume that there are numbers δ_0, K such that if $\mathrm{diam} D \leq \delta_0$ then D is cut by the discontinuity domain into at most K pieces where $K < \Lambda$ (the minimal expansion). Then the Growth Lemma holds. To see this we assume first that a more restrictive bound $K + 1 < \lambda$. Let μ satisfy (8). Consider two cases

(I) We have $\mathrm{diam}(D_\alpha) < \delta_0$ for all α. Noticing that $Z_\alpha = \frac{2}{\mathrm{length}(D_\alpha)}$ we obtain

$$\mathbb{P}_\alpha(r(fx) < \varepsilon) \leq \mathbb{P}_\alpha(r(x) < \frac{\varepsilon}{\Lambda}) + \mathbb{P}_\alpha(x \text{ is } \frac{\varepsilon}{\Lambda} - \text{ close to a singularity})$$
$$\leq Z_\alpha \varepsilon + K Z_\alpha \varepsilon,$$

so in this case $Z(f\mu) < \theta Z(\mu)$.

(II) (8) contains both short ($\leq \delta_0$) and long manifolds. Split $\mu = p\mu_1 + (1-p)\mu_2$ where μ_1 contains short and μ_2 long pieces. Then $Z(f\mu_1) < \theta Z(\mu_2)$ while the same consideration as in part (I) give $Z(f\mu_2) < C$. Thus

$$Z(f\mu) = pZ(f\mu_1) + (1-p)Z(\mu_2) \leq p\theta Z(\mu_1) + (1-p)C \leq \theta Z(\mu) + C.$$

Next if we replace $f \to f^n$ then $K \to K^n$ so if $K < \lambda$ then $K(f^n)+1 < \Lambda(f^n)$ for some n.

In fact one can farther relax the complexity bound as follows. Suppose that if $\mathrm{diam}(D) < \delta_0$ and if it is intersects the domains $M_1, M_2 \dots M_K$ then the corresponding expansion rates Λ_j satisfy

$$\sum_j \frac{1}{\Lambda_j} < 1.$$

This condition is in fact satisfied for Sinai billiards–billiards in $\mathbb{T}^2 - \bigcup_{j=1}^m S_j$ where S_j are disjoint strictly convex scatterers. For Sinai billiards the singularities are caused by grazing collisions (tangencies) and the expansion rates satisfy

$$\Lambda_1 > 1, \quad \Lambda_2 = \infty$$

so the Growth Lemma holds and the CLT follows ([10]). We refer the reader to [18] for a detailed exposition of the theory of hyperbolic billiards.

Problem 2. Extend the approach of this section to the multidimensional systems with singularities.

So far only partial results are available [5].

As an application of the CLT consider Lorentz gas–a billiard on the plane with periodic array of strictly convex scatters removed. A billiard map is Z^2-cover of a Sinai billiard on $\mathbb{T}^2$. The CLT theorem for bounded piecewise smooth observable is obtained in [14]. Let q_n be position of the particle after n collisions. Then

$$q_n - q_0 = \sum_{j=0}^{n-1} \Delta_j$$

where Δ_j is the free flight vector. Observe that Δ can be naturally regarded as a function of the Sinai billiard. Therefore the above results apply and we get

Corollary 3. *([10]) Suppose that the horizon is finite, that is the free flight vector is uniformly bounded. Then* $\frac{q(t)}{\sqrt{t}}$ *converges to the normal distribution.*

The theory presented above works for discrete time systems while Corollary 3 is stated for continuous time. However the passage from discrete to continuous time is quite standard (see e.g. [37]).

2.4. **CLT with non-standard normalization.** The assumption of the finite horizon in Corollary 3 is needed to ensure that mixing estimates of [45, 14] apply to Δ. This assumption is not merely technical however–the usual CLT fails for infinite horizon Lorentz gas. In fact the following result conjectured in [6] is proven in [42].

Theorem 4. *For infinite Lorentz gas* $\frac{q(t)}{\sqrt{t \ln t}}$ *converges to a normal distribution.*

Below we present an idea of the proof following [17]. The reason why the standard CLT fails is that

$$\mu(|\Delta| > H) \sim \frac{c}{H^3}$$

and so $\mu(|\Delta|^2) = \infty$ (in fact, it diverges logarithmically.

Lemma 1. *([17]) If $n \neq 0$ then*

$$\left|\mu\left(\Delta_0^{(\alpha)}\Delta_0^{(\beta)}\right)\right| \leq C\theta^{|n|}$$

where $\Delta_n^{(\alpha)}$ denote the components of $\Delta_n \in \mathbb{R}^2$.

This result is a consequence of the following statement. Let Ω_m denote the event that the particle crosses m fundamental domains before the next collision.

Lemma 2. *(a) $\mathbb{E}(|\Delta_n||\Omega_m) \leq C(\theta^n m + 1)$.*
(b) If n is fixed and $m \to \infty$ then $\mathbb{E}(|\Delta_n||\Omega_m) \leq Cm^{3/4}$.

Lemma 2 implies Lemma 1 by considering low and high values of m separately and using mixing bounds of [14] for low values of m and Lemma 2 for high values of m.

In order to prove Lemma 2 one verify by direct computations that if D is a curve inside Ω_m then

(a) $Z(D) \leq Cm^2$;

(b) $Z(fD) \leq Cm^{3/4}$.

Combining (b) with Growth Lemma we see that

$$Z(f^n D) \leq C(\theta^n m^{3/4} + 1).$$

Now (a) gives

$$\mathbb{P}_D(\Delta_n = k) \leq \frac{C(m^{3/4}\theta^n + 1)}{k^2}$$

proving both parts of Lemma 2.

Now we can proof Theorem 4 as follows. Split $q_N = q'_N + q''_N + q'''_N$ where q'''_N contains the sum of long flights ($|\Delta'''_n| \geq \sqrt{N} \ln^{100} N$), q''_N contains the sum of moderate flights ($\sqrt{N} \ln^{100} N \leq |\Delta''_n| \geq \frac{\sqrt{N}}{\ln^{100} N}$) and q'_N contains the sum of short flights ($|\Delta'_n| \leq \frac{\sqrt{N}}{\ln^{100} N}$). Now

$$\mathbb{P}(q'''_N \neq 0) \leq CN\mathbb{P}(|\Delta| > \sqrt{N} \ln^{100} N) \leq C \ln^{-200} N.$$

Moderate flights do occur but they tend to cancel each other. Indeed

$$\mathbb{E}(|q''_n|^2) = \sum_{n_1 n_2} \mathbb{E}(\langle \Delta''_{n_1}, \Delta''_{n_2} \rangle) \leq CN \ln \ln N$$

so $\frac{q''_N}{\sqrt{N \ln N}}$ can be disregarded. Finally q'_N can handled by the methods of the previous section giving CLT.

We observe that the full strength of Lemma 2(b) is not needed for the argument above to work. Namely it can be weakened to the requirement that the following limit exists

$$r_n = \lim_{m \to \infty} \frac{\mathbb{E}(\Delta_n | \Omega_m)}{m}.$$

In our case $r_n = 0$ for $n \neq 0$ but it is not the case for the billiard in the Bunimovich stadium. However our method can be adapted to prove the following result of Balint-Gouezel. They applied this criterion to a Bunimovich stadium bounded by two semicircles of radius 1 and two line segments Γ_1 and Γ_2 of length $L > 0$ each: given a Hölder continuous observable $A \in C^\alpha(\mathcal{M})$, denote by

$$I(A) = \frac{1}{2L} \int_{\Gamma_1 \cup \Gamma_2} A(s, \mathbf{n})\, ds$$

its average value on the set of normal vectors $\mathbf{n}$ attached to Γ_1 and Γ_2. (A slower decay of correlations for the stadium, compared to other Bunimovich billiards, is caused by trajectories bouncing between two flat sides of $\mathcal{D}$ and $I(A)$ represents the contribution of such trajectories.)

Theorem 5. [4] *The following results hold for Bunimovich stadia:*
(a) If $I(A) \neq 0$ then $S_n/\sqrt{n \ln n} \to \mathcal{N}(0, \sigma^2(A))$, where

$$\sigma^2(A) = \frac{4 + 3\ln 3}{4 - 3\ln 3} \times \frac{[I(A)]^2 L^2}{4(\pi + L)}.$$

(b) If $I(A) = 0$, then there is $\sigma_0^2 > 0$ such that $S_n/\sqrt{n} \to \mathcal{N}(0, \sigma_0^2)$.

The results presented above could be proven by several methods. Our method is perhaps the most direct and this allows an easier control of parameter dependence. For example consider infinite horizon Lorentz gas in the presence of small constant field and Gaussian thermostat. Namely, we assume that the motion between collisions is given by

$$\ddot{q} = \varepsilon \left(E - \frac{\langle \dot{q}, E \rangle}{|\dot{q}|^2} \dot{q} \right).$$

Then for $\varepsilon \neq 0$ the horizon becomes finite so the usual CLT applies. On the other hand for $\varepsilon = 0$ we have anomalous diffusion with an extra logarithmic factor. One can ask how the transition between two regimes occurs.

Theorem 6. [17] *As $t \to \infty, \varepsilon \to 0$*

$$\frac{q(t) - J_\varepsilon t}{\sqrt{t \ln \min(t, \varepsilon^{-1})}}$$

converges to a normal distribution. Here $J_\varepsilon = \mu_{SRB}^\varepsilon(\dot{q})$ is the average current of the thermostated system.

3. Law of Large Numbers and the first order averaging.

The Law of Large Numbers (Ergodic Theorem) is another basic limit law. Ergodic Theorem states that if X_n is an ergodic sequence and $S_n = \sum_{j=0}^{n-1}$ then $\frac{S_N}{N} \to \mathbb{E}(X)$. According to the philosophy of Section 2 we can restate this result as $\frac{S_{Nt}}{N} \to t\mathbb{E}(X)$. The generator for $\mathbf{S}_t = \mathbf{S}_0 + ct$ is $c\partial_{\mathbf{S}}$. The approach of Section 2 can be used to prove the following result

Theorem 7. [24, 7] *Suppose that f is a partially hyperbolic system having unique measure μ which is absolutely continuous with respect to the unstable foliation. Then for any $A \in C(M)$ and any admissible initial measure ν*

$$\frac{\sum_{j=0}^{N-1} A(f^j x)}{N} \to \mu(A) \quad \nu\text{-almost everywhere.}$$

The proof relies on the fact that the uniqueness of μ implies that

$$\int_D \rho(x) \frac{\sum_{j=0}^{N-1} A(f^j x)}{N} dx \to \mu(A) \tag{9}$$

since otherwise we could use the Krylov-Bogolyubov construction to obtain another invariant measure.

Theorem 7 can be extended to handle slow-fast systems

$$x_{n+1} = f_{S,\varepsilon}(x_n), \quad S_{n+1} = S_n + \varepsilon A(x_n, S_n, \varepsilon). \tag{10}$$

Theorem 8. *Suppose that for each S the map $x \to f_{S,0}(x)$ satisfies the conditions of Theorem 7 and let mu_S be the corresponding invariant measure. Then $S_{t/\varepsilon} \to \mathbf{S}_t$ satisfying*

$$\frac{d\mathbf{S}}{dt} = \bar{A}(\mathbf{S}) \textit{ where } \bar{A}(\mathbf{S}) = \int A(x, \mathbf{S}, 0) d\mu_{\mathbf{S}}.$$

Thus the generator of the limiting process is $\bar{A}(\mathbf{S})\partial_{\mathbf{S}}$.

For the proof take $N \gg 1$ and observe that

$$\mathbb{E}(u(S_{N(k+1)})) = \mathbb{E}(u(S_{Nk})) + \varepsilon \sum_{m=Nk}^{N(k+1)-1} \partial_S u(S_{Nk}) \mathbb{E}(A(f^m x)) + HOT.$$

After the change of variables $y = f^{Nk}x$ (9) can be used to handle the second term. We refer to [33, 26] for details.

Theorem 8 can be used to obtain the information about the perturbations of $f \times id$.

Theorem 9. [26] *Suppose that in* (10) *$f_{S,0} = f$ is an Anosov diffeo independent of S. Suppose further that the averaged vector field is Morse-Smale. Let γ_j be the periodic points for the averaged vectorfield and T_j be their periods. Suppose that*

$$\int_0^{T_j} \sigma^2(\mathbf{S}_t) dt \neq 0 \tag{11}$$

where

$$\sigma^2(\mathbf{S}) = \sum_{m=-\infty}^{\infty} [A(x, \mathbf{S}) - \bar{A}][A(f^m x, \mathbf{S}) - \bar{A}] d\mu_{SRB}(x).$$

Then for small ε the map $(x_n, S_n) \to (x_{n+1}, S_{n+1})$ is mostly contracting.

By Theorem 8 S_n spends most of the time near one of the periodic points of the averaged system. To prove Theorem 9 one has to show that in fact it spends most of the time near a sink. Intuitively it is unlikely that S_n stays a long time near an unstable orbit but to prove it rigorously one has go beyond the first order averaging analyzing the fluctuations about the averaged equation.

4. Diffusion processes and the second order averaging.

4.1. **CLT for small perturbations.** We begin with the following example. Suppose that f_ε is a smooth family of Anosov diffeos, μ_ε is the SRB measure for f_ε and A is a smooth observable such that $\mu_0(A) = 0$.

We need the following result

Theorem 10 (Linear response)**.** [29, 38] *The map $\varepsilon \to \mu_\varepsilon(A)$ is C^∞.*

In particular in our case we have $\mu_\varepsilon(A) = \varepsilon\omega(A)+o(\varepsilon)$. Let $N = t\varepsilon^{-2}$. By the CLT of the last section $\frac{S_{N,\varepsilon}-\mu_\varepsilon N}{\sqrt{N}}$ behaves as a normal random variable with variance $\sigma^2(A)$. A little algebra shows that this result can be restated as follows

Corollary 11. *$\varepsilon S_{N,\varepsilon}$ converges to a normal random variable with mean $\omega(A)t$ and variance $\sigma^2(A)t$.*

It turns out that this result is valid in a much more general setting. In particular we will not need to know much about the dynamics of f_ε for $\varepsilon \neq 0$. Concerning f_0 we assume that it is an **Anosov element inan abelian Anosov action**. That is f_0 is partially hyperbolic and its central direction is spanned by an action of the group $\mathbf{a}_t$ which commutes with f_0. We say that f_0 is **rapidly mixing** if the RHS of (7) is less than n^{-k} provided that $A \in C^{r(k)}$.

Theorem 12 (Local Linear Response-I)**.** [25] *Let $N \geq \varepsilon^{-0.001}$. Then*

$$\int_D \rho(x)A(f_\varepsilon^N)dx = \mu_0(A) + \varepsilon\omega(A) + o(\varepsilon).$$

One can check this estimate is sufficient for the argument of the previous section to work. Thus we get

Corollary 13. [25] *The result of Corollary 11 remains valid if f_0 is rapidly mixing Anosov element in an abelian Anosov action.*

Observe that no assumptions are imposed on the perturbation f_ε.

Hence the limiting random variable has density with Fourier transform

$$\hat{p}(t,\xi) = \exp\left(-i\xi t\omega(A) - \frac{t^2\xi^2\sigma^2(A)}{2}\right).$$

so that its generator is $\omega\partial_x + \frac{\sigma^2}{2}\partial_x^2$.

We say that $\mathbf{S}_t$ is a diffusion process if its small scale increments are approximately normal

$$\mathbf{S}_{t+h} - \mathbf{S}_t \approx a(\mathbf{S}_t)h + \sigma(\mathbf{S}_t)\sqrt{h}\mathcal{N}.$$

In other words the generator is

$$\mathcal{L} = a(\mathbf{S})\partial_{\mathbf{S}} + \frac{\sigma^2(\mathbf{S})}{2}\partial^2_{\mathbf{S}}.$$

4.2. **Example. Skew products near identity.** Consider the system

$$S_{n+1} - S_n = \varepsilon A(f^n x, S_n) + \varepsilon^2 B(f^n x, S_n).$$

Then

$$u(S_{n+1}) - u(S_n) = (\partial_x u)(S_n)\left(\varepsilon A + \varepsilon^2 B\right) + (\partial_x^2 u)(S_n)\frac{\varepsilon^2}{2}A^2 + \mathcal{O}(\varepsilon^3).$$

As before

$$(\partial_x u B) \to (\partial_x u)\mu_{SRB}(B),$$

$$(\partial_x^2 u A^2) \to (\partial_x u)\mu_{SRB}(A^2). \tag{12}$$

On the other hand $(\partial_x u)(S_n)A(f^n x, S_n) =$

$$(\partial_x u)(S_{\bar{n}})A(f^n x, S_{\bar{n}}) + \varepsilon(\partial_x^2 u)(S_{\bar{n}})\sum_{k=\bar{n}}^{n-1} A(f^k x, S_{\bar{n}})A(f^n x, S_{\bar{n}}) + \varepsilon\partial_S A(f^n x, \bar{S}_n)\sum_{k=\bar{n}}^{n-1} A(f^k x, S_{\bar{n}}).$$

As before the first term is negligible while second together with (12) adds up to

$$\frac{1}{2}\sum_{m=-\infty}^{\infty} \mu_{SRB}(A(x,S)A(f^m x, S)).$$

The third term is new but it can be analyzed similarly to others giving rise to

$$\sum_{m=1}^{\infty} \mu_{SRB}(A(x,S)\partial_S A(f^m x, S)).$$

Therefore we obtain the following result

Theorem 14. [24] *As $\varepsilon \to 0$ $S_{t\varepsilon^{-2}}$ converges to the diffusion process with generator*

$$(\mathcal{L}u)(S) = \left[\mu_{SRB}(B) + \sum_{m=1}^{\infty} \mu_{SRB}(A(x,S)\partial_S A(f^m x, S))\right]\partial_S u$$
$$+\frac{1}{2}\left[\sum_{m=-\infty}^{\infty} \mu_{SRB}(A(x,S)A(f^m x, S))\right]\partial_S^2 u.$$

4.3. **Fully coupled averaging.** We wish to extend Theorem 14 to allow feedback between fast and slow variables. Consider the system

$$x_{n+1} = f_{S,\varepsilon}(x_n) \quad S_{n+1} = S_n + \varepsilon A(x_n, S_n, \varepsilon). \tag{13}$$

Assume that for each S the map $x \to f_{S,0}x$ is an Anosov element in an abelian Anosov action enjoying stretched exponential decay of correlations.

Theorem 15. [26] *Any limit process of the family* $\mathbf{S}_t = S_{t\varepsilon^{-2}}$ *is a diffusion process with generator*

$$Lu = a\partial_S u + \frac{\sigma^2}{2}\partial_S^2 u$$

where as before

$$\sigma_{\alpha\beta}(S) = \sum_{m=0}^{\infty} \mu_{SRB}(A(x,S)A(f^m x, S))$$

while a *is given by a more complicated expression*

$$a(S) = \mu_{SRB}\left(\frac{\partial A}{\partial \varepsilon}\right) + \sum_{m>0} \mu_{SRB}(A(x,S)\partial_S A(f^m x, S)) + \omega\left(A, \frac{\partial f}{\partial \varepsilon}\right) + \sum_{n=0}^{\infty} \omega\left(A, Y_n\right)$$

where $Y_n = \frac{\partial f}{\partial S}(x,S)A(f^{-n}x,S)$.

Observe that this theorem does not claim to give a complete description of the limiting system. We only claim that $\mathbb{E}(u(\mathbf{S}_T))$ can be found using (5) but without extra assumptions we do not know if this equation is well-posed. Below we list some special cases where the well-posedness can be verified.

(1) For each S, e the function $x \to \sigma^2(S)$ is strictly positively definite. In this case well-posedness follows from [40]. The assumption is not very restrictive. Indeed if for some e $\sigma^2(A)e = 0$ then denoting $B = Ae$ we get that

$$\sigma^2(B) = \sum_{m=-\infty}^{\infty} \mu_{SRB}(B(x)B(f^m x)) = 0.$$

Then

$$\mu_{SRB}\left(\left(\sum_{n=0}^{N} B(f^n x)\right)^2\right) = N\sigma^2(B) + \mathcal{O}(\sum_m |m|\mu_{SRB}(B(x)B(f^m x))$$

is bounded. Therefore the sequence $\Phi_N = \sum_{m=0}^{N} B(f^n x)$ is weakly compact in $L^2(\mu_{SRB}$ and so it has a limiting point Φ. As $B(f^{N+1}x)$ converges to 0 weakly due to mixing we have $\Phi(x) - \Phi(fx) = B(x)$. Thus

B is measurable coboundary. Then [44] implies that B is a smooth coboundary. Then [30] allows to conclude that positivity of σ^2 fails on a codimension infinity subspace of functions.

(2) $x \to f_S$ is Anosov. In this case the smoothness of a can be established by the transfer operator method [28] and then well-posedness follows from [41].

The proof of Theorem 15 is similar to the proof of Corollary 13 but Theorem 12 has to be modified. Let F denote the map

$$F_\varepsilon(x, S) = (f_{S,\varepsilon}(x), S_n + \varepsilon A(x, S, \varepsilon).$$

Observe that F_0 is partially hyperbolic and hence so is F_ε for small ε. Let D be a disc satisfying (6) which is approximately horizontal in the sense that

$$||\pi_S v|| \leq C||\pi_x v|| \forall v \in TD$$

and let ρ be a probability density on such that $||\rho||_{C^\alpha} \leq K$.

Theorem 16 (Local Linear Response-II)**.** *(a) Let $\bar{N} = \varepsilon^{0.001}$. Pick some $z^* = (x^*, S^*) \in D$*

$$\int_D \rho(z) A(F_\varepsilon^{\bar{N}}, \varepsilon) dz = \varepsilon a(S^*) + o(\varepsilon). \tag{14}$$

(b) Consequently for all $N \geq \bar{N}$

$$\int_D \rho(z) A(F_\varepsilon^n, \varepsilon) dx = \varepsilon \int_D \rho(z) a(S_{N-\bar{N}}) dz + o(\varepsilon).$$

Problem 3. Obtain diffusion limit theorem for the case where the fast motion is an Anosov flow.

Currently Theorem 15 is proven under stretched exponential mixing assumptions which are not known for Anosov flow (with the exception of the contact flows, see [36, 43]). The assumption of strong mixing is used in particular to establish the convergence of the series appearing in the formula for the drift a. However in the case of Anosov flows one can hope to utilize the fact that central direction is particularly simple to overcome this difficulty.

4.4. **On the proof of Linear Response.** Since the Linear Response Theorem plays the central role in this section we discuss the idea behinds its proof. We consider two special cases where one can understand how the SRB measure depends on parameters.

(a) f_ε are contractions of $\mathbb{R}^d$. In this case the SRB measures are δ-measures concentrated at the fixed points $\mathbf{x}_\varepsilon$ of F_ε. By Implicit Function

Theorem

$$\frac{d\mathbf{x}}{d\varepsilon} = (1 - df)^{-1} \frac{\partial f}{\partial \varepsilon}. \tag{15}$$

This formula can be understood as follows. Pick $y_0 \in \mathbb{R}^d$ and consider the recurrence $y_{n,\varepsilon} = f_\varepsilon(y_{n-1,\varepsilon})$. We have

$$\frac{dy_n}{d\varepsilon} = df_\varepsilon \frac{dy_n}{d\varepsilon} + \frac{\partial f}{\partial \varepsilon}(y_{n-1,\varepsilon}) = \sum_{j=0}^{n-1} df_\varepsilon^j \frac{\partial f}{\partial \varepsilon}(y_{n-1-j,\varepsilon}).$$

If $n \gg j$ then $y_{n-j,\varepsilon}$ is close to $\mathbf{x}_\varepsilon$ proving (15). Thus while μ_ε are mutually singular, the map $\varepsilon \to \mu_\varepsilon(A)$ is smooth and

$$\mu_\varepsilon(A) = DA \frac{d\mathbf{x}}{d\varepsilon} = \sum_{j=0}^{\infty} \partial_{df_\varepsilon^j \frac{\partial f}{\partial \varepsilon}(\mathbf{x}_\varepsilon)} A. \tag{16}$$

(b) μ_0 has smooth density and $\mu_\varepsilon(A) = \lim_{n\to\infty} \mu_0(A \circ f_\varepsilon^n)$ where the convergence is sufficiently fast. This is the case, for example, if f_ε are expanding maps. Then

$$\mu_\varepsilon(A) - \mu_0(A) = \sum_{n=0}^{\infty} \left[\mu_0(A(f_\varepsilon^{n+1}x)) - \mu_0(A(f_\varepsilon^n x)) \right].$$

To understand the individual term in this sum make a change of variables in the first term $y = f_\varepsilon x$. Observe that since f_0 preserves μ_0 we have

$$\frac{d\mu_0(y)}{d\mu_0(x)} = 1 + \varepsilon \mathrm{div}_{\frac{df}{d\varepsilon}} + \mathcal{O}(\varepsilon^2).$$

Therefore

$$\mu_0(A(f_\varepsilon^{n+1}x)) - \mu_0(A(f_\varepsilon^n x)) = \int A(f_\varepsilon^n y) \left[\frac{d\mu_0(x)}{d\mu_0(y)} - 1 \right] d\mu_0(y)$$

$$= -\varepsilon \int A(f_\varepsilon^n y) \mathrm{div}_{\frac{df}{d\varepsilon}} d\mu_0(y) + \mathcal{O}(\varepsilon^2).$$

So if the convergence of the series is fast enough we get **Kawasaki formula**

$$\frac{d\mu_\varepsilon(A)}{d\varepsilon}\Big|_{\varepsilon=0} = -\varepsilon \sum_{n=0}^{\infty} \int A(f_0^n y) \mathrm{div}_{\frac{df}{d\varepsilon}} d\mu_0(y). \tag{17}$$

The proof of the Linear Response Theorem in general case combines the ideas from two cases considered above. Namely, in the center-stable direction we use standard perturbation theory similarly to case (a) above which is manageable because $df^n|E_{cs}$ does not grow too fast. In order to handle the perturbation in the unstable direction we use a change of variables similar to case (b). The formula for ω the derivative contains two terms. One term is similar to (16) the other is similar to (17).

Problem 4. Investigate the validity of Linear Response Theorem for other classes of partially hyperbolic systems. For example, does Linear Response Formula hold for robustly hyperbolic examples of [1, 8]?

Problem 5. What are error term in Linear Response Formula? In particular, find conditions for validity of higher order expansions of $\mu_\varepsilon(A)$. See [39] for more discussion of this topic.

4.5. **Systems with singularities.** At present little is known about the applicability of the above results to systems with singularities. The reason is that the Local Linear Response Formula is not know is not known for systems with singularities. In fact, examples of [2] show that even global Linear Response Formula may fail!

Problem 6. Investigate the validity of Linear Response Formula for systems with singularities. In particular does it hold for one-to-one maps?

For dispersing billiards [19] prove Linear Response Formula using one-to-one property as well as the fact that μ_0 is absolutely continuous. They rely on Kawasaki argument.

Problem 7. Is local Linear Response Formula valid for the perturbations of billiards?

Because of this problem diffusion limit theorem is only known under special circumstances.

(1) [16] considers a particle moving in a finite horizon Lorentz array with small field. That is the motion between collisions is given by

$$\ddot{q} = \varepsilon E. \tag{18}$$

This system is a slow-fast system with the slow variable being the kinetic energy and the fast variable being the pair (q, ω) where $\omega = \frac{v}{|v|}$ is the particle's direction. Observe that the evolution of the fast variable is $\mathcal{O}(\varepsilon^2)$ perturbation of the Gaussian thermostat (by the very definition of the thermostat!). Therefore one can use the strong mixing properties of the thermostat dynamics (coupling!) to deduce Local Linear Response Formula from the global one. Using this fact [16] obtain diffusion approximation for (18).

(2) [15] considers a system of two particles moving in a finite horizon Sinai billiard. Particles collide with the scatterers and with each other elastically. The first particle is a heavy disk of mass $M \gg 1$ and radius $R \sim 1$. The second particle called is a dimensionless point of unit mass.

Thus if V and v are particles velocities then the interparticle collision rule gives

$$V^+ = V^- + \frac{2}{M} v^\perp.$$

Suppose now that initially the first particle is at rest. Then its velocity after n collisions equals

$$V_n = \sum_{j=1}^{n} v_j^\perp.$$

Since the average value of $v^\perp$ is zero we expect that

$$V_n \sim \frac{\sqrt{n}}{M} \text{ and } Q_n \sim \frac{n^{3/2}}{M}.$$

[15] shows that $(Q_{tM^{2/3}}, M^{2/3} V_{tM^{2/3}}$ converges to the diffusion process with generator

$$(\mathcal{L}\phi)(Q,V) = \sum_{i=1}^{2} V_i \partial_{Q_i} \phi + \frac{1}{2} \sum_{ij=1}^{2} \sigma_{ij}^2 \partial_{V_i} \partial_{V_j} \phi.$$

Observe that this system is NOT of the form considered before because the first order averaged system is

$$\dot{Q} = V, \quad \dot{V} = 0$$

(that is if the energies of the particles are comparable than the heavy particle moves without noticing the light one) rather than

$$\dot{Q} = 0, \quad \dot{V} = 0.$$

We are only interested in what happens near the fixed points of the averaged system. At a fixed the linearized system is nilpotent. Because of this the time needed to get a non-trivial evolution is much shorter than ε^{-2} required section 4.3. Namely, since the maximal possible velocity of the heavy particle is $\mathcal{O}(M^{-1/2})$ due to the energy preservation the natural slow variable is $\hat{V} = \sqrt{M} V$ and so the natural scale separation parameter is $\varepsilon = M^{-1/2}$ so we are dealing with times of order $\varepsilon^{-4/3}$. For this reason the full strength of the Linear Response Formula is not needed, one only needs to show that the RHS of (14) is $o(\varepsilon^{1/3})$.

Problem 8. Extend Theorem 15 to dispersing billiards.

4.6. **Einstein relation.** In this section we discuss how the symmetries of the slow-fast system (13) are reflected in the limiting process. To simplify the exposition we suppose in this section that there is unique process with generator $\mathcal{L}$.

Theorem 17. *Assume that F_ε admits the following symmetry*

$$F_{c\varepsilon} = \mathcal{G}_c^{-1} F_\varepsilon \mathcal{G}_c \text{ where}$$
$$\mathcal{G}_c(x, S) = (G(x, S, c), g_c(S)).$$

Then $\mathbf{S}_t$ and $g_c(\mathbf{S}_{c^2 t})$ have the same distribution.

Proof. Consider our system for parameter $c\varepsilon$ and the initial distribution having a smooth density on $S = \bar{S}$. This is an admissible initial distribution so Theorem 15 tells that $S^{c\varepsilon}_{t(c\varepsilon)^{-2}}$ converges to $\mathbf{S}_t$ (started from $\bar{S}$). On the other hand

$$S^{c\varepsilon}_{t(c\varepsilon)^{-2}} = g_c^{-1} S^{\varepsilon}_{(tc^{-2})\varepsilon^{-2}},$$

so it converges to $g_c^{-1}\mathbf{S}_{tc^{-2}}$ (with $\mathbf{S}_0 = g_c\bar{S}$). The result follows. □

Theorem 18. [26] *Assume that F_ε preserves an admissible measure μ_ε. Let ν_ε be the projection of this measure to the S-component. If $\nu_\varepsilon \to \nu$ as $\varepsilon \to 0$ then ν is invariant measure for the process $\mathbf{S}_t$.*

Proof.

$$\begin{aligned}\nu(u(\mathbf{S}_t)) &= \lim_{\varepsilon\to 0} \nu_\varepsilon(u(\mathbf{S}_t)) = \lim_{\varepsilon\to 0} \nu_\varepsilon(u(S^\varepsilon_{t\varepsilon^{-2}})) = \lim_{\varepsilon\to 0} \mu_\varepsilon(u(S^\varepsilon_{t\varepsilon^{-2}})) \\ &= \lim_{\varepsilon\to 0} \mu_\varepsilon(u(S^\varepsilon_0)) = \lim_{\varepsilon\to 0} \nu_\varepsilon(u(S^\varepsilon_0)) = \nu(u(\mathbf{S}_0)).\end{aligned}$$

□

If ν has density ρ then the invariance condition means that $\langle \rho, \mathcal{L}u\rangle = 0$ for all u so that $\mathcal{L}^*\rho = 0$. That is, we have

$$\partial_S \left(a + \frac{\sigma^2}{2}\partial_S\right)\rho = 0 \tag{19}$$

If ρ is known (19) gives a relation between a and σ^2.

As an illustration of above results consider (18). The time change $s = \frac{t}{\sqrt{c}}$ reduces

$$\frac{d^2q}{dt^2} = c\varepsilon E \quad \text{to} \quad \frac{d^2q}{ds^2} = \varepsilon E$$

and the kinetic energy is rescaled by $K_s = \frac{K_t}{c}$. By Theorem 17 the limiting process should be invariant under

$$\mathbf{K} \to c\mathbf{K}, \quad t \to c^{3/2}t$$

(an extra factor of $c^{-1/2}$ in the last formula is due to the time change).

On the other hand our system is Hamiltonian so it preserves the Liouville measure and so we can apply Theorem 18 with $d\nu = dK$. It turns out that the scaling invariance property together with the given invariant measure determine the limiting process up to choosing the time unit. Namely the scaling symmetry allows to relate the drift at diffusion at any point to drift and diffusion at $\mathbf{K}=1$. Choosing time units fixes the value of the diffusion at 1 and (19) allows to determine the drift at 1. So the limiting process for (18) has generator

$$\sigma^2\left[\frac{1}{2\sqrt{2\mathbf{K}}}\partial_{\mathbf{K}}+\sqrt{2\mathbf{K}}\partial^2_{\mathbf{K}}\right].$$

5. Poisson Limit Theorem.

The third basic limit theorem in probability theory is Poisson Limit Theorem. It says that if $S_N=\sum_{n=0}^N X_{n,N}$ where for each N $X_{n,N}$ are independent identically distributed taking values 0 with probability $1-p_N$ and 1 with probability p_N and $p_N N\to\lambda$ then S_N converges to a Poisson distribution with intensity λ, that is,

$$\mathbb{P}(S_N=l)=e^{-\lambda}\frac{\lambda^l}{l!}.$$

Thus one can expect Poisson Limit Theorem to hold if only few terms are different from 0 and each term makes a contribution of order 1. For example let $f:M\to M$ be Anosov diffeo preserving a smooth measure μ. Take a point x_0 then one can expect that $S_{tr^{-d}}\to\mathcal{P}(\mathbf{c}t)$ where

$$S_N=\mathrm{Card}(n\leq N-1: f^n(x)\in B(x_0,r))=\sum_{n=0}^{N-1} I_{B(x_0,r)}(f^n x) \tag{20}$$

and $B(x_0,r)\sim\mathbf{c}r^d$. To see if (20) holds recall that the Poisson process with intensity $\mathbf{c}$ has generator

$$(\mathcal{L}u)(\mathbf{S})=\mathbf{c}[u(\mathbf{S}+1)-u(\mathbf{S})].$$

Let $\bar N=\delta r^{-d}$ We have

$$\mathbb{E}(u(S_{\bar N}))=u(0)\mathbb{P}(S_{\bar N}=0)+u(1)\mathbb{P}(S_{\bar N}=1)+\mathcal{O}(\mathbb{P}(S_{\bar N}>1)).$$

Since

$$\sum_n \mathbb{P}(I_{B(x_0,r)}(f^n x))=\delta r^{-d}\mu(B(x_0),r)(1+o(1))=\mathbf{c}\delta(1+o(1))$$

we need to show in particular that

$$\mathbb{P}(S_{\bar N}>1))=o(\delta). \tag{21}$$

However (21) fails if x_0 is fixed (or periodic point) of f. Indeed if it was valid (for all r) then we would have

$$\mathrm{Card}(n\leq\bar N: f^n x\in B(x_0,r/L))\sim\frac{\mathbf{c}\delta}{L^d}$$

however if $f^n x\in B(x_0,r/L)$ and L is larger than the maximal expansion of f then $f^{n+1}x\in B(x_0,r)$.

Theorem 19. [24] *Suppose that f is a partially hyperbolic diffeo preserving a smooth measure μ such that*

$$\left|\int_D \rho(x)A(f^n x)dx - \mu(A)\right| \leq \frac{C}{n^p}||A||_{C^1(M)}, \quad p = p(d).$$

Let x_0 be non-periodic point then

$$\mathrm{Card}(n \leq tr^{-d} : f^n \in B(x_0, r)) \to \mathcal{P}(\mathbf{c}t).$$

If x_0 is aperiodic we can establish (21) as follows. We need to prove

$$\mathbb{P}(\exists n \leq \bar{N} : f^n x \in B(x_0, r) | x \in B(x_0, r)) = o(1). \tag{22}$$

To show this we distinguish five cases:

(I) $n \leq M_0$ there $M_0 \to 0$ as $r \to 0$ very slowly. Then

$$\mathbb{P}(\exists n \leq \bar{N} : f^n x \in B(x_0, r) | x \in B(x_0, r)) = 0$$

since x_0 is aperiodic.

(II) $M_0 \leq n \leq \delta |\ln r|$. Then $f^n B(x_0, r) \cap B(x_0, r)$ has at most one component and since $f^n B(x_0, r)$ has length at least Λ^n in the unstable direction we have

$$\mathbb{P}(\exists n \leq \bar{N} : f^n x \in B(x_0, r) | x \in B(x_0, r)) \leq \frac{C}{\Lambda^{d_u n}}.$$

(III) $\delta |\ln r| \leq n \leq K \ln r$. Then we can divide $f^n B(x_0, r)$ into components each of which has unstable length $r^{1-\kappa}$ for some $\kappa > 0$ and since only the set of diameter $2r_0$ can belong to $B(x_0, r)$ we get

$$\mathbb{P}(\exists n \leq \bar{N} : f^n x \in B(x_0, r) | x \in B(x_0, r)) \leq C r^{\kappa d_u}.$$

(IV) $K |\ln r| \leq n \leq \left(\frac{1}{r}\right)^{1/p+\delta}$. Then we can divide $f^n B(x_0, r)$ into components each of which has unstable length $\mathcal{O}(1)$ for some $\kappa > 0$ and since only the set of diameter $2r_0$ can belong to $B(x_0, r)$ we get

$$\mathbb{P}(\exists n \leq \bar{N} : f^n x \in B(x_0, r) | x \in B(x_0, r)) \leq C r^{d_u}.$$

(V) $n \geq \left(\frac{1}{r}\right)^{1/p+\delta}$. Then

$$\mathbb{P}(\exists n \leq \bar{N} : f^n x \in B(x_0, r) | x \in B(x_0, r)) = \mu(B(x_0, r))(1 + o(1))$$

due to mixing.

Summing all the cases we obtain (22) and hence (21).

(21) shows that

$$\mathbb{E}(u(S_{n+\bar{N}}) - u(S_n) | S_n = 0) = \mathbf{c}\delta[u(1) - u(0)].$$

A similar argument shows that

$$\mathbb{E}(u(S_{n+\bar{N}}) - u(S_n) | S_n = k) = \mathbf{c}\delta[u(k+1) - u(k)]$$

proving Theorem 19.

Problem 9. Is Theorem 19 valid without the assumption that μ is smooth?

The foregoing discussion illustrates that in order to prove that

$$\mathrm{Card}(n \leq \frac{t}{\mu(B_r)} : f^n x \in B_r) \to \mathcal{P}(t)$$

for some family of sets B_ε such that $\mu(B_r) \to 0$ as $r \to 0$ two ingredients are needed.

(I) Mixing (used to estimate $\mu(B_r \cap f^n B_r)$ for small n).

(II) Geometric analysis (used to estimate $\mu(B_r \cap f^n B_r)$ for large n).

For example let V be a compact manifold of negative curvature.

Theorem 20.

$$\mathbb{P}\left(\text{A geodesic } \gamma(t) \text{ visits } B(q_0, r) \quad l \text{ times for } t \leq Tr^{-(d-1)}\right) \to e^{-c(q_0)T} \frac{(c(q_0)T)^l}{l!}.$$

To prove this theorem we let κ to be much smaller than the injectivity radius of V, let f be time κ map of the geodesic flow and

$$B_r = \{(q, v) : \gamma_{q,v}([0, \kappa]) \bigcap B(q_0, r) \neq \emptyset\}.$$

In this case mixing comes from [36, 43] while $\mu(B_r \cap f^n B_r)$ is small for for small n since this set corresponds to orbits which are near a geodesic passing q_0 twice.

Finally let us describe an application of Theorem 19.

Corollary 21. *Let $m_N(x) = \min_{n \leq N} d(f^n x, x_0)$. Under the conditions of Theorem 19*

$$\mathbb{P}(N^{1/d} m_N(x) \geq z) \to \exp(-\mathbf{c} z^d).$$

Proof.

$$\mathbb{P}(N^{1/d} m_N(x) \geq z) = \mathrm{Card}\left(n \leq N : f^n x \in B\left(x_0, \frac{z}{N^{1/d}}\right) = 0\right)$$
$$\sim \exp(-\mu\left(B\left(x_0, \frac{z}{N^{1/d}}\right)\right) N) = \exp(-\mathbf{c} z^d).$$

□

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