Local Limit Theorems for Random Walks in a 1D Random Environment

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Abstract. We consider random walks (RW) in a one-dimensional i.i.d. random environment with jumps to the nearest neighbours. For almost all environments, we prove a quenched Local Limit Theorem (LLT) for the position of the walk if the diffusivity condition is satisfied. As a corollary, we obtain the annealed version of the LLT and a new proof of the theorem of Lalley which states that the distribution of the environment viewed from the particle (EVFP) has a limit for a. e. environment.

Mathematics Subject Classification (2010). Primary 60K37; Secondary 60F05.

Keywords. RWRE, quenched random environments, Local Limit Theorem, environment viewed from the particle.

1. Introduction.

Transient random walks in random environments (RWRE) on a one-dimensional (1D) lattice with jumps to the nearest neighbours were analyzed in the annealed setting in [10] in 1975. The authors found that, depending on the randomness, the walk can exhibit either diffusive behavior where the Central Limit Theorem (CLT) is valid or have subdiffusive fluctuations. The quenched CLT for this model in the diffusive regime was proved in [6] in 2007 for a wide class of environments (including the iid case) and, independently, for iid environments in [13]. For RWRE on a strip, the quenched CLT was established in [7] and the annealed case was considered in [14].

Unlike the CLT, the Local Limit Theorem (LLT) had not so far been proved for any natural classes of 1D walks. The only result we are aware of is concerned with a walk which either jumps one step to the right or stays where it is ([12]).

The aim of this work is two-fold. First, to fill this gap for the so called simple RW in 1D RE and secondly, to apply the obtained LLT to the investigation of the limiting behaviour of the distribution of the environment viewed from the particle (EVFP). We note that the latter is in fact one more aspect of a local limiting behaviour of a RW which is specific to random environments.

A very important feature of our method is that it extends to the RWRE on a strip allowing us to obtain the analogues of all the results proved below for this case. This in particular means that we have a full control over the RWRE with bounded jumps in 1D. However, for the sake of transparency of the proofs, here we restrict our attention to the classical 1D walks with jumps to the nearest neighbours. The strip model will be discussed elsewhere.

We thus prove that the quenched LLT holds for almost every (a.e.) environment (Theorems 3.2). It should be emphasized that, unlike in the CLT (Theorem 3.4), one has to have an additional random factor ρ_n in front of the exponent which is due to the randomness of the environment.

Naturally, the quenched LLT implies the annealed one (Theorems 3.3).

We then prove that in the diffusive regime the limit of the distribution of the environment viewed from the particle exists for a. e. environment (Theorems 3.4). Originally, the existence of the limit for the simple 1D walk was proved by S. Lalley in [11]. Our proof is completely different from the one explained in [11].

2. Definition of the model.

Let $\mathcal{S} = \{p, q, r : p \geq 0, q \geq 0, r \geq 0, p + q + r = 1\}$ and σ be a distribution on $\mathcal{S}$ such that

$$\text{for some } \kappa > 0 \quad \sigma(p \geq \kappa, q \geq \kappa) = 1 \text{ and } \sigma(r > 0) > 0. \tag{2.1}$$

Denote $\Omega = \mathcal{S}^{\mathbb{Z}}$. An element $\omega = \{(p_k, q_k, r_k)\} \in \Omega$ will be called an *environment*. We assume that (p_k, q_k, r_k) are iid random vectors with distribution σ which defines the measure $\mathbf{P} = \sigma^{\mathbb{Z}}$ on the space of environments. We thus have the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ describing random environments, where $\mathcal{F}$ is the natural sigma-algebra of subsets of Ω.

A random walk $X_n,\ n \geq 0$ in a given random environment ω is a Markov chain with a starting point $X_0 = 0$ and a transition kernel given by

$$\mathbb{P}_\omega(X_{n+1} = k + \Delta | X_n = k) = \begin{cases} p_k & \text{if } \Delta = 1, \\ r_k & \text{if } \Delta = 0, \\ q_k & \text{if } \Delta = -1. \end{cases} \tag{2.2}$$

Denote by $\mathfrak{X}$ the set of all trajectories starting from 0. Formulae (2.2) define the measure $\mathbb{P}_\omega$ on $\mathfrak{X}$ with a natural sigma-algebra $\mathcal{F}_{\mathfrak{X}}$.

Similarly, let $\mathfrak{X}_z = \{X(\cdot) : X(0) = z\}$ be the space of trajectories starting from $z \in \mathbb{Z}$. A fixed ω provides us with a conditional (or *quenched*) probability measure $\mathbb{P}_{\omega,z}$ on $\mathfrak{X}_z$ with a naturally defined probability space $(\mathfrak{X}_z, \mathcal{F}_{\mathfrak{X}_z}, \mathbb{P}_{\omega,z})$.

Finally the semi-direct product measure $\mathrm{P}_z(d(\omega, X)) := \mathbf{P}(d\omega)\mathbb{P}_{\omega,z}(dX)$ is the *annealed* probability measure on $(\Omega \times \mathfrak{X}_z, \mathcal{F} \times \mathcal{F}_{\mathfrak{X}_z})$.

The expectations with respect to $\mathbb{P}_{\omega,z}$, $\mathbf{P}$, and P_z will be denoted by $\mathbb{E}_{\omega,z}$, $\mathbf{E}$, and E_z respectively.

Remark 2.1. The condition $\sigma(r > 0) > 0$ is needed to make the above Markov chain aperiodic and thus to avoid the necessity to consider odd and even moments of time separately.

Remark 2.2. The notations $\mathfrak{X}_z$, $\mathbb{P}_{\omega,z}$, E_z etc. emphasize the dependence of these objects on the starting point z of the walk. We use the simplified version of these notations such as $\mathbb{P}_\omega$, $\mathbb{E}_\omega$, E if the RW starts from 0 or if it is clear from the context what is the starting point of the walk.

3. Results.

Throughout the paper we assume that $\sigma(\ln(p/q)) > 0$ so that due to [15] $X_n \to +\infty$ almost surely. We also assume that

$$\sigma((p/q)^2) < 1. \tag{3.1}$$

In this case the walk satisfies a quenched Central Limit Theorem. Namely let T_k be the first time the walk visits site k. Let

$$b_n = b_n(\omega) = \min(k : \mathbb{E}_\omega(T_k) \geq n). \tag{3.2}$$

Theorem 3.1. [6, 13]. *There is a constant $D > 0$ such that $\mathbf{P}$-almost surely*

$$\frac{X_n - b_n(\omega)}{\sqrt{n}D} \Rightarrow F.$$

Here and below $F(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$. For a sequence of random variables Ξ_n, we write $\Xi_n \Rightarrow F$ if $\lim_{n\to\infty} P(\Xi_n < x) = F(x)$ for all $x \in \mathbb{R}$.

The following result is a local version of Theorem 3.1. Denote

$$\rho_k = \mathbb{E}_{\omega,k}\mathrm{Card}(n : X_n = k)$$

the expectation of the number of visits to k and let $a = \mathbf{E}\rho$, where ρ has the same distribution as ρ_k. (For completeness, let us mention that $\rho_k = p_k^{-1}(1 + \alpha_{k+1} + \alpha_{k+1}\alpha_{k+2} + \ldots)$, where $\alpha_j = q_j/p_j$; see [4] for a derivation of this formula.)

Theorem 3.2. $\mathbf{P}$*-almost surely the following holds. For each $\varepsilon, R > 0$ there exists $n_0 = n_0(\omega)$ such that for $n \geq n_0$ uniformly for*

$$|k - b_n| \leq R\sqrt{n} \tag{3.3}$$

we have

$$\left| \frac{\sqrt{2\pi n}Da}{\rho_k} \exp\left[\frac{(k - b_n)^2}{2D^2 n}\right] \mathbb{P}_\omega(X_n = k) - 1 \right| < \varepsilon.$$

Let us present two corollaries of this result.

The first application is the annealed local limit theorem. Recall ([8], Corollary 1(c)) that there is a constant $\hat{D}$ such that

$$\frac{b_n - \frac{n}{a}}{\sqrt{n}\hat{D}} \Rightarrow F. \tag{3.4}$$

Let $\mathbf{D} = \sqrt{D^2 + \hat{D}^2}$.

Theorem 3.3. *For each $\varepsilon, R > 0$ there exists $n_0 = n_0(\omega)$ such that for $n \geq n_0$ uniformly for*

$$\left|k - \frac{n}{a}\right| \leq R\sqrt{n} \tag{3.5}$$

we have

$$\left|\sqrt{2\pi n}\mathbf{D} \exp\left[\frac{(k-\frac{n}{a})^2}{2\mathbf{D}^2 n}\right] \mathbf{P}(X_n = k) - 1\right| < \varepsilon.$$

The other application is a direct proof of the following theorem of Lalley [11].

Theorem 3.4. *Let $\mathcal{T}$ be the natural shift on the space of environments. Then for almost every ω and for every continuous function $\Phi : \Omega \to \mathbb{R}$ it holds that*

$$\mathbb{E}_\omega(\Phi(\mathcal{T}_{X_n}\omega)) \to \frac{\mathbf{E}(\rho_0\Phi)}{a} \text{ as } n \to \infty.$$

4. Preliminaries.

4.1. LLT for sums of independent random variables.

The following result from [3] provides very general sufficient conditions under which the Local Limit Theorem for sums of independent random variables holds.

Theorem 4.1. *([3]) Let ξ_i, $i \geq 1$, be independent integer valued random variables and let $d_i = \sum_j \min[P(\xi_i = j), P(\xi_i = j+1)]$, $\mathfrak{d}_n = \sum_{i=1}^n d_i$. Denote $\Xi_n = \sum_{i=1}^n \xi_i$. Suppose that there are numbers $\mathfrak{c}_n > 0$, $\mathfrak{a}_n$, $n \geq 1$, such that $(\Xi_n - \mathfrak{a}_n)/\mathfrak{c}_n \Rightarrow F$, where $\mathfrak{c}_n \to \infty$ and $\limsup \mathfrak{c}_n^2/\mathfrak{d}_n < \infty$. Then*

$$\sup_k \left|\mathfrak{c}_n P\left(\Xi_n = k\right) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(k-\mathfrak{a}_n)^2}{2\mathfrak{c}_n^2}\right)\right| \to 0 \text{ as } n \to \infty. \tag{4.1}$$

Remark 4.2. The requirement $\mathfrak{d}_n \to \infty$ implies that sufficiently many d_i's are positive. Had this not been the case, then it could happen that Ξ_n would be taking, say, only even values as n becomes large. In our applications the role of Ξ_n is played by T_n and all the corresponding d_i's are uniformly separated from 0.

To apply Theorem 4.1, we have to verify the asymptotic normality of T_n; the latter results from the following lemma.

Lemma 4.3. [6, 13] *There exists* $\bar{D} > 0$ *such that for almost every* ω

$$\lim_{n\to\infty} \mathbb{P}_\omega\left(\frac{T_n - \mathbb{E}_\omega T_n}{\sqrt{n}\bar{D}} < x\right) = F(x) \text{ for all } x \in \mathbb{R}.$$

4.2. Several useful estimates.

Let s denote the positive solution of $\sigma(p^s/q^s) = 1$. Due to (3.1) $s > 2$ (s can be equal to $+\infty$ if the walker has positive drift with probability 1). In the proofs of Theorems 3.2–3.4 we assume that $s < \infty$. Proofs become easier if $s = \infty$ and we leave the corresponding modifications to the reader.

Lemma 4.4. *(a)* $\mathbf{P}(\rho > t) \le Ct^{-s}$.
(b) For any $\hat{u} > \frac{1}{s}$ *for almost every* ω *there is a constant* $C(\omega)$ *such that* $\rho_k < C(\omega)k^{\hat{u}}$.

Remark 4.5. In the case $s = \infty$ the statements of Lemma 4.4 read as follows. For any $u, \hat{u} > 0$ we have $\mathbf{P}(\rho > t) \le Ct^{-u}$ and $\rho_k < C(\omega)k^{\hat{u}}$.

Proof. Part (a) is proven in [[5], equation (8.3)]. (We note that if the distribution of $\ln p - \ln q$ is non arithmetic then [9] gives a result which is stronger than (a), namely $\mathbf{P}(\rho > t) \sim \bar{c}t^{-s}$.) Part (b) follows from part (a) and the Borel-Cantelli Lemma. □

We finish this section by stating two technical results.

Lemma 4.6. [2] *(a) There exists* $C > 0$ *and* $\theta < 1$ *such that*

$$\mathrm{P}(X \text{ visits } k \text{ after visiting } k+m) \le C\theta^m.$$

(b) Accordingly, for almost every ω *there is a constant* $K(\omega)$ *such that*

$$\mathbb{P}_\omega(\exists k < n : X \text{ visits } k \text{ after } T_{k+\ln^2 n}) \le K(\omega)n^{-100}.$$

Lemma 4.7. *([6],Lemma 5) There exists* $\varepsilon_0 > 0$ *such that almost surely*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \max_{l \le n^{\frac{1+\varepsilon_0}{2}}} |\mathbb{E}_\omega(T_{n+l} - T_n - la)| = 0.$$

5. Proof of the Quenched LLT.

Proof of Theorem 3.2. Take $\frac{1}{s} < u < \frac{1}{2}$. We claim that for $\mathbf{P}$-almost all ω

$$\mathbb{P}_\omega(\exists k \le n \;\; \exists m \in \mathbb{N} : X_m = k \text{ and } T_k < m - n^u) \le \frac{\bar{C}(\omega)}{n^{100}}. \tag{5.1}$$

Indeed, if $X_m = k$ and $m > T_k + n^u$ then one of the following events takes place:

$$A_1 = \{X_t \in [k - \ln^2 n, k + \ln^2 n] \text{ for all } t \in [T_k, T_k + n^u]\};$$

$$A_2 = \{\exists t \in [T_k, T_k + n^u] \text{ such that } X_t < m - \ln^2 n\};$$

$$A_3 = \{\exists t \in [T_k, T_k + n^u] \text{ s. t. } X_t > m + \ln^2 n \text{ and then } X \text{ backtracks to } k\}.$$

$\mathbb{P}_\omega(A_2)$ and $\mathbb{P}_\omega(A_3)$ are $O(n^{-100})$ by Lemma 4.6. Take $\frac{1}{s} < u' < u'' < u$. If A_1 happens then there exists $\bar{k} \in [k - \ln^2 n, k + \ln^2 n]$ which is visited more than $n^{u''}$ times. However the number of visits to $\bar{k}$ has geometric distribution with mean $\rho_{\bar{k}} < C(\omega)n^{u'}$. Thus $\mathbb{P}_\omega(A_1) \ll n^{-100}$ proving (5.1).

Remember the following often used equality: $T_n = \sum_{i=0}^{n-1} \tau_i$, where τ_i is the time the walk takes to reach (hit) $i+1$ after having reached i. The random variables τ_i are independent if the environment ω is fixed. It follows from (2.1) that

$$d_i = \sum_j \min[P(\tau_i = j), P(\tau_i = j+1)] \geq \min[P(\tau_i = 1), P(\tau_i = 2)] \geq \kappa r_i.$$

Therefore $\mathfrak{d}_n \geq \kappa \sum_{i=1}^n r_i$. By the Strong Law of Large Numbers $\frac{1}{n}\sum_{i=1}^n r_i$ converges for almost every ω to a positive limit. These two remarks and Lemma 4.3 imply that T_n satisfies the conditions of Theorem 4.1 and hence there is a $\bar{D} > 0$ such that for almost all ω and a given $\bar{R} > 0$

$$\mathbb{P}_\omega(T_k = l)\sqrt{2\pi k}\bar{D}\exp\left(\frac{(l - \mathbb{E}_\omega T_k)^2}{2\bar{D}^2 k}\right) \to 1 \text{ as } k \to \infty \tag{5.2}$$

uniformly for $|l - \mathbb{E}_\omega T_k| \leq \bar{R}\sqrt{k}$. Taking into account (5.1) we see that if

$$|n - \mathbb{E}_\omega T_k| \leq \bar{R}\sqrt{k} \tag{5.3}$$

then

$$\mathbb{P}_\omega(X_n = k) = \left[\sum_{j=0}^{n^u} \mathbb{P}_\omega(T_k = n-j)\mathbb{P}_\omega(X_j = k|X_0 = k)\right] + O(n^{-100}).$$

For $j \in [0, n^u]$ we have due to (5.2)

$$\mathbb{P}_\omega(T_k = n-j) \sim \frac{1}{\sqrt{2\pi k}\bar{D}} \exp -\left(\frac{(n - \mathbb{E}_\omega T_k)^2}{2\bar{D}^2 k}\right).$$

On the other hand

$$\sum_{j=0}^{n^u} \mathbb{P}_\omega(X_j = k|X_0 = k) = \rho_k + O(n^{-100}\rho_k) = \rho_k + O(n^{-99}).$$

Thus

$$\mathbb{P}_\omega(X_n = k) \sim \frac{\rho_k}{\sqrt{2\pi k}\bar{D}} \exp -\left(\frac{(n - \mathbb{E}_\omega T_k)^2}{2\bar{D}^2 k}\right). \tag{5.4}$$

Next we claim that given R we can take $\bar{R}$ so large that (3.3) implies (5.3). Indeed we have

$$n - \mathbb{E}_\omega T_k = (n - \mathbb{E}_\omega T_{b_n}) + (\mathbb{E}_\omega T_{b_n} - \mathbb{E}_\omega T_k). \tag{5.5}$$

Observe that by definition $\mathbb{E}_\omega T_{b_n - 1} < n \leq \mathbb{E}_\omega T_{b_n}$ and by Lemma 4.7 $\mathbb{E}_\omega(T_{b_n} - T_{b_n-1}) = o(\sqrt{n})$ so that the first term in (5.5) is $o(\sqrt{n})$. Next, by Ergodic Theorem $\frac{b_n}{n} \to \frac{1}{a}$ so Lemma 4.7 implies that

$$\mathbb{E}_\omega(T_{b_n} - T_k) = a(b_n - k) + o(\sqrt{n}).$$

This implies (5.3) and shows moreover that

$$\frac{(n - \mathbb{E}_\omega T_k)^2}{k} \sim \frac{a^3(b_n - k)^2}{n}.$$

Combining this with (5.4) we get

$$\begin{aligned}\mathbb{P}_\omega(X_n = k) &\sim \frac{\sqrt{a}\rho_k}{\sqrt{2\pi n}\bar{D}} \exp - \left(\frac{(k-b_n)^2 a^3}{2\bar{D}^2 n}\right) \\ &= \frac{\rho_k}{\sqrt{2\pi n}Da} \exp - \left(\frac{(k-b_n)^2}{2D^2 n}\right),\end{aligned}$$

where $D = \bar{D}/a^{3/2}$. □

6. Annealed LLT.

Proof of Theorem 3.3. The result would be immediate if b_n and ρ_k were independent. This is not the case. However, they are almost independent. Namely by Lemma 4.3

$$b_n = b_{n-n^{3/4}} + \frac{n^{3/4}}{a} + \varepsilon_n$$

where $\mathbf{P}(|\varepsilon_n| \geq n^{(3/8)+\eta}) \to 0$ for each $\eta > 0$. Also by (3.4) we have

$$\mathbf{P}\left(\left|b_{n-n^{3/4}} - \frac{n-n^{3/4}}{a}\right| \geq n^{(1/2)+\eta}\right) \to 0 \text{ for each } \eta > 0.$$

Hence we can approximate b_n by $\tilde{b}_n = \min(b_{n-n^{3/4}}, \frac{n}{a} - n^{5/8}) + \frac{n^{3/4}}{a}$. Note that if (3.5) holds then $\tilde{b}_n$ and ρ_k are independent since the former depends only on the environment to the left of $\frac{n}{a} - n^{5/8}$ while the latter depends only on environment to the right of k. (Indeed if the walker is at $k-1$ then he visits k with probability 1 so ρ_k is determined by the probability that the walker starting from $k+1$ does not return to k.) Thus

$$\sqrt{2\pi n}\mathbf{P}(X_n = k) = \sqrt{2\pi n}\mathbf{E}(\mathbb{P}_\omega(X_n = k)) \sim \mathbf{E}\left(\frac{1}{D}\exp\left(-\frac{(k-\tilde{b}_n)^2}{2nD^2}\frac{\rho_k}{a}\right)\right)$$

$$= \mathbf{E}\left(\frac{\rho_k}{a}\right)\mathbf{E}\left(\frac{1}{D}\exp\left(-\frac{(k-\tilde{b}_n)^2}{2nD^2}\right)\right).$$

The first factor equals to 1 while due to (3.4) the second factor is asymptotic to

$$\frac{1}{\mathbf{D}}\exp\left(-\frac{(k-\frac{n}{a})^2}{2\mathbf{D}^2 n}\right).$$

The result follows. □

7. Environment as seen from the particle.

Proof of Theorem 3.4. Due to the properties of the product topology it suffices to consider the case where Φ depends only on $\{(p_j, q_j, r_j)\}_{|j|\leq M}$. We consider the case where $M = 0$, the general case is completely similar except for notational complications. So we assume that $\Phi(\omega) = \phi(p_0, q_0, r_0)$. Denote $\phi_k = \phi(p_k, q_k, r_k)$. We have

$$\mathbb{E}_\omega(\Phi(\mathcal{T}_{X_n}\omega)) = \sum_{k=-\infty}^{\infty} \mathbb{P}_\omega(X_n = k)\phi_k.$$

By Theorem 3.2 given $\varepsilon > 0$ we can find R and $n_0 = n_0(\omega)$ such that for $n \geq n(\omega)$

$$\left| \mathbb{E}_\omega(\Phi(\mathcal{T}_{X_n}\omega)) - \sum_{|k-b_n| \leq R\sqrt{n}} \mathbb{P}_\omega(X_n = k)\phi_k \right| \leq \varepsilon.$$

Divide $[b_n - R\sqrt{n}, b_n + R\sqrt{n}]$ into intervals I_j of length n^u for some $\frac{1}{s} < u < \frac{1}{2}$. Let k_j be the center of I_j. Theorem 3.2 allows us to approximate $\mathbb{E}_\omega(\Phi(\mathcal{T}_x\omega))$ by

$$\frac{1}{\sqrt{2\pi n}Da} \sum_j \left[\exp\left(-\frac{(k_j - n)^2}{2D^2 n} \right) \right] \sum_{k \in I_j} \rho_k \phi_k.$$

Lemma 4.4 allows us to cutoff the last expression as follows

$$\mathbb{E}_\omega(\Phi(\mathcal{T}_x\omega)) \sim \frac{1}{\sqrt{2\pi n}Da} \sum_j \left[\exp\left(-\frac{(k_j - n)^2}{2D^2 n} \right) \right] \sum_{k \in I_j} \bar{\rho}_k \phi_k$$

where $\bar{\rho}_k = \rho_k 1_{\rho_k < n^{\bar{u}}}$ for some $\frac{1}{s} < \bar{u} < u$. Let $A = \mathbf{E}(\rho_0 \phi_0)$. By Borel-Cantelli Lemma it suffices to prove that

$$\mathbf{P}\left(\left| \sum_{k=m}^{m+N} [\bar{\rho}_k \phi_k - A] \right| \geq \varepsilon N \right) \leq \frac{C}{N^{100}}$$

where N is of order n^u. By stationarity we may assume that $m = 0$. Pick v such that $2v < u < sv$ and split $\bar{\rho}_k = \rho_k^h + \rho_k^l$ where $\rho_k^h = \rho_k 1_{n^v < \rho_k \leq n^{\bar{u}}}$, $\rho_k^l = \rho_k 1_{\rho_k \leq n^v}$. Denote $Z = \sum_{k=0}^N [\bar{\rho}_k \phi_k - A]$ then $Z = Z^l + Z^h + N\tilde{A}$ where

$$Z^l = \sum_{k=0}^N \left[\rho_k^l \phi_k - A^l \right], \quad Z^h = \sum_{k=0}^N \rho_k^h \phi_k, \quad A^l = \mathbf{E}(\rho_0^l \phi_0), \quad \tilde{A} = A - A^l.$$

Note that $\tilde{A} \to 0$ so it is enough to show that

$$\mathbf{P}(|Z^l| > \varepsilon N) = O(N^{-100}), \quad \mathbf{P}(|Z^h| > \varepsilon N) = O(N^{-100}).$$

We need

Lemma 7.1. *(cf [5], Lemma 3.3) For each d there is K such that if $k_1, k_2 \dots k_d$ satisfy*

$$|k_{i_1} - k_{i_2}| > K \ln N \tag{7.1}$$

then we have

$$\mathbf{P}(\rho_{k_i} > n^v \text{ for } i = 1 \dots d) \leq \frac{C}{n^{vsd}}.$$

Note that if $|Z^h| > \varepsilon N$ then there are $k_1, k_2 \dots k_d$ satisfying (7.1) such that $\rho_{k_i} > n^v$. By Lemma 7.1 the probability of such an event is $O(n^{ud-svd})$ which can be made less than N^{-100} if d is large enough since $sv > u$.

It remains to handle Z^l. Split $[0, N]$ into segments J_j of length n^w where $w \ll 1$. Let

$$Z_j^l = \sum_{k\in J_j} \left[\rho_k^l \phi_k - A^l\right], \quad Z_{odd} = \sum_{j-odd} Z_j^l, \quad Z_{even} = \sum_{j-even} Z_j^l.$$

It suffices to show that

$$\mathbf{P}(|Z_{odd}| > \varepsilon N) = O(N^{-100}), \quad \mathbf{P}(|Z_{even}| > \varepsilon N) = O(N^{-100}). \tag{7.2}$$

We shall prove the first inequality, the second one is similar. Lemma 4.6 easily implies that

$$\mathbf{P}\left(Z_j^l - Z_j^* > \frac{1}{N}\right) = O(N^{-100})$$

where

$$Z_j^* = \sum_{k\in J_j} \left[\rho_k^* \phi_k - A^*\right], \quad \rho_k^* = \hat{\rho}_k 1_{\hat{\rho}_k < n^v},$$

$\hat{\rho}_k$ is expected number to visits to k before $T_{k_{j+1}}$ and $A^* = \mathbf{E}(\rho_0^* \phi_0)$. Since $\{Z_j^*\}_{j-odd}$ are iid random variables satisfying

$$\mathbf{E}(Z_j^*) = 0, \quad |Z_j^*| \leq C n^{v+w}$$

we have

$$\mathbf{E}\left(\left(\sum_{j-odd} Z_j^*\right)^{2d}\right) \leq C n^{2vd+2wd} n^{ud} = C n^{(2v+u+2w)d}.$$

By the Markov inequality

$$\mathbf{P}\left(\left|\sum_{j-odd} Z_j^*\right| > \varepsilon N\right) \leq C(\varepsilon) n^{(2v+2w-u)d}$$

which is less than N^{-100} if w is small enough and d is large enough since $2v < u$. (7.2) follows and hence Theorem 3.4 is proven. □

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