# BOUNDED ORBITS OF ANOSOV FLOWS

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## 1. Introduction.

In this paper we develop a symbolic dynamics approach to a problem of studing of dimensional characteristics of the set of bounded geodesics on manifolds of negative curvature.

Recall that an orbit of a flow on a non-compact manifold is called bounded if it is confined to a compact set. A bounded geodesic is a bounded orbit of the geodesic flow.

The problem under consideration goes back to the classical theorem of Jarnik and Besicovitch which states that the set of badly approximable numbers on the segment $[0, 1]$ has Hausdorff dimension equal to 1. Recall that a real number $x$ is called badly approximable if for any rational $\frac{p}{q}$ one has $|x - \frac{p}{q}| > \frac{C(x)}{q^2}$. The equivalent definitions are the following: $x$ is badly approximable if the partial convergents of its continued fraction $k_j(x)$ are bounded or if the closure of the x-orbit by the Kuzmin-Gauss map $x \to \{\frac{1}{x}\}$ does not contain 0.

The above-mentioned result can be reformulated in another way. Consider the modular surface $Q$ on the Poincare model of the Lobachevsky plane. A geodesic on $Q$ is bounded if and only if both its endpoints are badly approximable. Thus the Jarnik-Besicovitch theorem is equivalent to the fact that the Hausdorff dimension of the set of bounded geodesics on the unit tangent bundle to $Q$ equals 3.

This result was generalized to the manifolds of the constant negative curvature having finitely generated fundamental groups by Patterson [Pt], Stramann [St], Fernandez & Melian [FM] and Bishop & Jones [BJ] and to cofinite manifolds of (variable) non-positive curvature by Dani ([D1], [D2]). In all cases considered the Hausdorff dimension of the set of bounded geodesics is equal to the Hausdorff dimension of the set of recurrent geodesics, i.e. those whose forward and backward rays spend each an infinite time in some compact region.

Note that the set of bounded geodesics has zero Liouville-Patterson measure by the ergodicity of the geodesic flow. However, the above mentioned statement is not so surprising because geodesic flows on manifolds of negative curvature have an abundance of invariant measures so that the set of non-typical points for any of them is quite large.

Another generalization of the Jarnik-Besicovitch theorem comes from the third definition of badly approximable number. Namely, one may ask whether, for an arbitrary Anosov system on a compact manifold it is true that the dimension of the set of orbits whose closure does not contain some point (or a small set of points $A$) equals the dimension of the phase space. This question also makes sense in the non-compact case if we restrict ourselves to the set of recurrent points (since a non-recurrent trajectory has no either backward or forward limit points). The affirmative answer was given in [D3] for the algebraic automorphism of the torus in the case of a countable set $A$.

It is interesting, therefore, to know how large the set $A$ so that this theorem remains true. Two of the simplest cases are described in this paper. We start with the analysis of one-dimensional piecewise expanding maps.

THEOREM 2. *Let $f$ be a piecewise expanding map of an interval $I$ and suppose $A \subset I$ has Hausdorff dimension less than* 1. *Then the Hausdorff dimension of the set of points whose $f$-orbits do not have limit points belonging to $A$ equals* 1.

As was observed by Manning ([M]), for studying such kinds of sets it is useful to have a nice symbolic representation because then the orbits not visiting some region have simple enough symbolic description. However in our case even the construction of an infinite Markovian partition is quite difficult if we want the exceptional set to have zero Lesbegue measure. The situation changes completely if we require that the set in which our dynamics is well-defined be large only in the dimensional sense (cf [K],[MS]). Namely, we can find an invariant subset of the dimension close to 1, so that the restriction of our map to this set is a subshift of finite type with a finite number of states. This is proven in Section 4. This fact allows us to derive theorem 2 from the following statement.

THEOREM 1. *Let $(\Sigma_+, \sigma)$ be an one-sided subshift of finite type and let $A \subset \Sigma_+$ be a subset such that the topological entropy of $A$ with respect to $\sigma$ is less than the topological entropy of the whole space $h(\Sigma_+)$. Then the topological entropy of the set of points whose $\sigma$-orbits do not have limit points belonging to $A$ equals $h(\Sigma_+)$.*

Theorem 1 is proven in Section 3. The proof requires only simple combinatorial estimations.

In Section 5 we study the case of an Anosov diffeomorphism $f$ on a compact surface. The result is the following.

THEOREM 3. *Let $f$ be an Anosov diffeomorphism of the two-dimensional torus $M$ and denote by $H$ the Hausdorff dimension of SBR-measure. Suppose that $A \subset M$ has the Hausdorff dimension less than $H$. Then the Hausdorff dimension of the set of points whose $f$-orbits do not have limit points belonging to $A$ equals 2. Conversely, for any $s > H$ one can find a set $A$ of the Hausdorff dimension less than $s$ for which the above statement fails.*

Roughly speaking, if the intersection of the set $A$ with the set of the typical points with respect to the SBR measure is small, then there are quite a lot orbits avoiding $A$, but on the other hand a majority (in the dimensional sense) of orbits possess limit points with 'almost typical' behaviour. Theorem 3 is proven in paragraphs $1-5$. The optimality of the estimate given there is proven in paragraphs $6-9$. In Section 6 we consider a uniformly $C^2$ Anosov flow $g^t$ on a connected (non-compact) Riemannian manifold $M$ of bounded sectional curvature. It turns out that the most convenient dimensional characteristic to work with is the dimension with respect to a dynamical system introduced by Pesin in [P2], because this dimension takes into account the distortion properties of $g^t$. In paragraph 2.5 we recall this definition adapted to our situation.

In paragraphs $2-4$ of Section 6 we prove the following main theorem:

THEOREM 4. *The dimension with respect to $g^t$ of the intersection of any unstable manifold with the set of forward bounded orbits equals the dimension with respect to $g^t$ of the intersection of this manifold with the set of forward recurrent orbits.*

Our arguments provide also the following statement.

COROLLARY 1. *The dimension with respect to $g^t$ of the intersection of any unstable manifold with the set of forward bounded orbits depends lower semicontinuously on $g^t$ in the $C^2$ metric.*

The proof of theorem 4 is very similar to that of theorem 2. It depends on the theory of self-similar sets. Necessary facts are given in paragraphs $5, 6$.

Theorem 4 allows to treat most of the results from above mentioned papers from a unified point of view, as well as to obtain some new facts. In paragraphs $7, 8$ we derive from theorem 4 the following corollary.

THEOREM 5. *Let either 1)* $\dim M = 3$ *or*
*2)* $g^t$ *be the geodesic flow on a manifold of the constant negative curvature or*
*3)* $g^t$ *have a finite smooth invariant measure.*
*Then the Hausdorff dimension of the set of forward bounded orbits equals the Hausdorff dimension of the set of forward recurrent orbits.*

The orbits whose limit points do not belong to a countable set are discussed in paragraph 9 where we show the following:

THEOREM 6. *The dimension with respect to* $g^t$ *of the intersection of any unstable manifold with the set of bounded orbits whose limit set does not intersect a countable set* $\{b_i\}$ *equals the dimension with respect to* $g^t$ *of the intersection of this manifold with the set of recurrent orbits.*

COROLLARY 2. *Under the assumptions of theorem 3, the corresponding result holds for the Hausdorff dimension.* The arguments of Section 6 or the construction of a suspended flow give

COROLLARY 3. *The results of theorems* $4-6$ *and corollaries* $1$ *and* $2$ *hold also for uniformly* $C^2$ *Anosov diffeomorphisms.*

We do not give the detailed proof of these statements because after we obtain the symbolic dynamics the proof can be completed by various methods, for example, by one we use in Section 3.

In the case of the existence of a finite smooth invariant measure the geometric approach developed by Schmidt, Dani, Aravinda and Leuzinger is also very powerful. For example, the author does not know a dynamical proof of the following theorem of Dani [D3]:

PROPOSITION 1. *Denote by* $\mathbf{T^n}$ *the* $n-$*dimensional torus and let* $A$ *be the set* $\{x \in \mathbf{T^n}$ : *for all semisimple surjective endomorphisms* $f$ *of* $\mathbf{T^n}$ *the closure of the* $(x, f)-$ *orbit does not intersect* $\mathbf{Q^n}/\mathbf{Z^n}\}$. *Then the Hausdorff dimension of* $A$ *equals* $n$.

Some interesting results for orbits of flows on finite volume homogenious spaces were obtained recently by Kleinbock & Margulis in [KM].

We use a separate enumeration of lemmas, propositions,formulae and constants in each section. We refer to, for example, lemma 1 of Section 2 as lemma 2.1. However the enumeration of theorems and corollaries agrees with that given in Section 1. In Section 2 we introduce the notation used in this paper and provide the reader with some background from ergodic theory.

**2. Background.**

1) In this section we introduce our notation and collect some facts from the general theory of dynamical systems which will be used throughout this paper.

We study some dimensional characteristics which are defined as follows. Given collections of sets $V_\varepsilon$ ($\varepsilon \in \mathbf{R}$) such that $V_{\varepsilon_1} \subset V_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$ and functions $w_s : \bigcup_\varepsilon V_\varepsilon \to \mathbf{R}_+$ ($s \in \mathbf{R}_+$) such that for fixed $v \subset V_\varepsilon$ ($\varepsilon \leq \varepsilon_0$) $w_s$ is decreasing and $\forall s_1,\ s_2 : s_2 > s_1 \lim\limits_{\varepsilon \to 0} \sup\limits_{v \in V_\varepsilon} \frac{w_{s_1}(v)}{w_{s_2}(v)} = 0$, we define for every set $A$ a measure

$$m^s_\varepsilon(A) = \inf_{\substack{v_i \in V_\varepsilon \\ A \subset \cup v_i}} \sum w_s(v_i).$$

Clearly, $m^s_\varepsilon(A)$ increases as $\varepsilon$ decreases and hence there exists the limit $m^s(A) = \lim\limits_{\varepsilon \to 0} m^s_\varepsilon(A)$ which can be a positive number, zero or infinity. Moreover, under our assumptions it holds that if $m^{s_0}(A) > 0$ then $\forall s < s_0$ $m^s(A) = +\infty$ and if $m^{s_0}(A) < +\infty$ then $\forall s > s_0$ $m^s(A) = 0$. So, we can define the magnitude $dm(A) = \inf\{s : m^s(A) = 0\} = \sup\{s : m^s(A) = +\infty\}$ which is called the dimension of A corresponding to the measure $m^s_\varepsilon(A)$.

For example, the Hausdorff dimension $\mathrm{HD}(A)$ is the dimension corresponding to the measure

$$h^s_\varepsilon(A) = \inf_{\substack{|V_i| < \varepsilon \\ A \subset \cup V_i}} \sum_i |V_i|^s,$$

where the infimum is taken over all coverings of $A$ by balls $V_i$ with diameters $|V_i|$ less than $\varepsilon$.

Another magnitude we shall deal with is the topological entropy ([B1]). Given a continuous map $f$ of a compact set $X$ to itself, and a subset $Y \subset X$ we consider for any open covering $U = (U_1, U_2, \dots U_n)$ of $X$ the dimension $h(Y, U)$, corresponding to the measure

$$h^s_\varepsilon(Y, U) = \inf_{\substack{N(V_i) > \frac{1}{\varepsilon} \\ Y \subset \bigcup\limits_i V_i}} \sum e^{-N(V_i)s},$$

where the $V_i$ are sets of the form $V_i = \bigcap_{j=0}^{N(V_i)-1} f^{-j}U_{k_j(i)}$. The topological entropy of $Y$ with respect to $f$ is $h(Y,f) = \sup_U h(Y,U)$, where the supremum is taken over all open coverings of $X$. Finally, the topological entropy of $f$ $h_{top}(f)$ is the topological entropy of $X$ with respect to $f$.

The dimension with respect to a dynamical system ([P2]) is defined in paragraph 4.

2) In this paragraph we introduce the fractals we deal with.

Given a semigroup $\{g^t\}$ ($t \in \mathbf{R}_+$ *or* $t \in \mathbf{Z}_+$) acting on a metric space M we denote by $\mathrm{Lim}^+(\mathrm{m})$ the set of limit points of the $g^t-$orbit of the point $m$. In Section 4 we consider different semigroups $\{f^n\}$ $n \in \mathbf{Z}_+$ and write $\mathrm{Lim}^+(\mathrm{m},\mathrm{f})$ when the second argument is not clear from the context. The set of forward bounded orbits $B^+$ can be defined as $B^+ = \{m \in M\ \exists a \in M,\ \ \rho > 0 : \forall t \geq 0\ \ g^t(m) \in B(a,\rho)\}$ and the set of forward recurrent trajectories $R^+$ as $R^+ = \{m \in M\ \exists a \in M,\ \ \rho > 0 : \forall T \geq 0\ \exists t > T\ \ g^t(m) \in B(a,\rho)\}$. If $X$ is a subset of $M$ we denote by $R^+(X)$ the set $\{m \in M : \forall \varepsilon > 0\ \ \forall T \geq 0\ \ \exists t > T : \mathrm{dist}(g^t m, X) \leq \varepsilon\}$ (such an awkward definition is needed to satisfy the condition (1) below). If $a$ is a point in $M$ we use the notation $Ig^+(a)$ to mean the set $\{m :\ \ a \notin \mathrm{Lim}^+(\mathrm{m})\}$ and if $A$ is a subset of $M$ then $Ig^+(A)$ denotes $Ig^+(A) = \bigcap_{a \in A} Ig^+(a)$.

The main tool for studying these sets is self-similar set theory. Given a collection $\Phi$ of maps $\{\phi_i\}$ whose domains of definition $D(\phi_i)$ and ranges of values $E(\phi_i)$ belong to a set $X$ we can define for every subset $Y \subset X$ the set $\Phi^{-1}(Y) = \bigcup_i \phi_i^{-1}(Y \bigcap E(\phi_i))$. Define by induction $\Phi^{-n}(Y) = \Phi^{-1}(\Phi^{-(n-1)}(Y))$. Let $\omega(\Phi) = \bigcap_{n=0}^{\infty} \Phi^{-n}(Y)$. We use the notation $\omega_f(\Phi)$ for the union of the limit sets for all finite subcollections of $\Phi$ : $\ \omega_f(\Phi) = \bigcup_{k=1}^{\infty} \omega(\{\phi_i\}_{i=1}^k)$. In estimations of a dimension of such sets we shall use the following notation. Let $\mathbf{A} = \{\alpha_i\}$ be a sequence of real numbers satisfying $0 < \alpha_i < 1$. Consider $\zeta_{\mathbf{A}}(s) = \sum_{i=1}^{\infty} \alpha_i^s$. Denote by $r(\mathbf{A})$ the root of $\zeta_{\mathbf{A}}(r) = 1$ if this equation has a root and the bound of convergence of $\zeta_{\mathbf{A}}$ otherwise. In other words $r(\mathbf{A}) = \inf\{s : \zeta_{\mathbf{A}}(s) \leq 1\} = \sup\{s : \zeta_{\mathbf{A}}(s) \geq 1\}$, if $\zeta$ is viewed as a function with values in $\mathbf{R}_+ \bigcup\{+\infty\}$.

3) In this paragrath we recall the notion of the an Anosov system. The

standard reference for this subject is [A].

In Section 6 we consider an Anosov flow $g^t$ on a complete (non-compact) Riemannian manifold $M$ of bounded sectional curvature. Recall that a flow $g^t$ is call Anosov if the tangent space $TM$ is decomposed into a continuous $dg^t-$invariant sum $TM = E_u + E_0 + E_s$, where
1) $E_0$ is generated by tangent vectors to $g^t-$orbits;
2) there exist constants $C_1 > 0,\ \lambda > 0$ such that
for every $v \in E_s,\ t > 0\ \|dg^t v\| \leq C_1 e^{-\lambda t}\|v\|$ and
for every $v \in E_u,\ t < 0\ \|dg^t v\| \leq C_1 e^{\lambda t}\|v\|$.
3) There is a $\gamma > 0$ such that the angle between any two of $E_u, E_0, E_s$ is at least $\gamma$.

The main example of Anosow flow is the geodesic flow on a negatively curved manifold ([A]).

We require also that $g^t$ be uniformly $C^2$ : that is, for $t$ lying in a bounded interval covariant derivatives of $g^t$ up to second order are uniformly bounded in spite of the non-compactness of $M$.

In the discrete time case the same definition applies with $E_0 = 0$. We denote by $W^{(u)}(m)\ \ (W^{(s)}(m), W^{(su)}(m), W^{(ss)}(m))$ the unstable (resp. stable, strong unstable, strong stable) manifold of the point $m$, that is the integral surface of the field $E_u + E_0$ ($E_s + E_0,\ E_u,\ E_s$ respectively) containing the point $m$. Under our assumptions these are $C^2-$immersed submanifolds. For Anosov diffeomorphisms $W^{(u)} = W^{(su)},\ W^{(s)} = W^{(ss)}$, so that we will use either notation for these manifolds to treat simultaneously both discrete and continuous time cases.

We write $B(m, \rho)$ for the ball with center $m$ and radius $\rho$. The notation $W^{(u)}_\varepsilon(m), W^{(s)}_\varepsilon(m)$ etc. mean the ball in the corresponding manifold, where the distance is defined with the help of the induced Riemannian metric.

Recall that a set $\Pi$ is called a parallelogram if for any $x, y \in \Pi\ [x, y] = W^{(su)}_{\varepsilon_0}(x) \bigcap W^{(s)}_{\varepsilon_0}(y) \in \Pi$ for some $\varepsilon_0$ so small that this intersection is a singleton. In particular, if $X$ and $Y$ are subsets of $M$ lying in a small enough ball then the set $[X, Y] = \{[x, y]\ x \in X,\ \ y \in Y\}$ is a parallelogram. We shall deal with parallelograms of the form $\Pi(a, \varepsilon_1, \varepsilon_2) = [W^{(ss)}_{\varepsilon_1}(a), W^{(su)}_{\varepsilon_2}(a)]$.

By the definition of $\Pi(a, \varepsilon_1, \varepsilon_2)$ it has the natural partition into pieces of strong unstable manifolds (resp. stable manifolds). The element of this partition containing a point $m$ is denoted by $W^{(u)}_{\Pi(a,\varepsilon_1,\varepsilon_2)}(m)$ $(W^{(s)}_{\Pi(a,\varepsilon_1,\varepsilon_2)}(m))$.

We would like to introduce coordinates $(u, s)$ on $\Pi(a, \varepsilon_1, \varepsilon_2)$ using the following procedure. Choose some smooth coordinates $u$ on $W^{(su)}_{\varepsilon_1}(a)$ and $s$ on $W^{(ss)}_{\varepsilon_2}(a)$. For a point $p \in \Pi$ take its coordinates to be the coordinates of its projections along $W^{(s)}(p)$ and $W^{(su)}(p)$ respectively. So, $[(u_1, s_1), (u_2, s_2)] = (u_1, s_2)$. This coordinate system makes $\Pi$ a Hölder submanifold of $M$.

Similarly, in the continuous time case we can introduce the coordinates $(u, , s, t)$ on $g^{[-\varepsilon,\varepsilon]}\Pi$ in such a way that a point $m$ has coordinates $(u_0, s_0, t_0)$ if $g^{-t_0}$ belongs to $\Pi$ and has coordinates $(u_0, s_0)$ there.

Given two parallelograms $\Pi_1$ and $\Pi_2$ a point $a \in \Pi_1$ and a positive $t_0$ such that $g^{t_0}a \in \Pi_2$, we denote by $\sigma_{a,t_0,\Pi_1,\Pi_2}$ the corresponding Poincare map. More precisely, if $g^{t_0}a \in \Pi_2$ we can define in some neighbourhood of $a$ on $\Pi_1$ a continuous function $\tau$ such that $g^{\tau(m)}m \in \Pi_2$ and $\tau(a) = t_0$. If $\bar{\tau}(m)$ is such a function with the largest domain of definition we put $\sigma_{a,t_0,\Pi_1,\Pi_2}(m) = g^{\bar{\tau}(m)}m$. We shall refer to $\bar{\tau}(m)$ as $\tau(\sigma_{a,t_0,\Pi_1,\Pi_2}, m)$. If $(u_1, s_1)$ are the coordinates on $\Pi_1$, and $(u_2, s_2)$ those on $\Pi_2$ and $(u_2, s_2) = \sigma_{a,t_0,\Pi_1,\Pi_2}(u_1, s_1)$ then by the invariance of $W^{(su)}$ and $W^{(s)}$ the map $\sigma_{a,t_0,\Pi_1,\Pi_2}$ splits into the product $u_2 = U(u_1),\ \ s_2 = S(s_1)$.

It is convenient to use the following notation:
$N_u = \dim E_u,\ \ N_s = \dim E_s$;
$D(\sigma)-$the domain of $\sigma(= \sigma_{a,t_0,\Pi_1,\Pi_2})$;
$E(\sigma)-$the image of $\sigma$;
$t(\sigma) = \min\limits_{D(\sigma)} \tau(\sigma, m)$;
$R^{(u)}(m, t) = |\det(dg^t|E_u)_m|,\ R^{(s)}(m, t) = |\det(dg^t|E_s)_m|$;
$Q^{(u)}(m, t) = \ln R^{(u)}(m, t),\ Q^{(s)}(m, t) = |\ln R^{(s)}(m, t)|$;
$R_\sigma(m)$-the expansion rate of $\sigma$ in $m$ $\ R_\sigma(m) = R(m, \tau(\sigma, m))$;
$\bar{\chi}_+ = \max\limits_m Q^{(u)}(m, 1), \underline{\chi}_+ = \min\limits_m Q^{(u)}(m, 1), \bar{\chi}_- = \max\limits_m Q^{(s)}(m, 1), \underline{\chi}_- = \min\limits_m Q^{(s)}(m, 1)$;
$\underline{C}(\sigma) = \inf\limits_{D(\sigma)} R_\sigma(m),\ \ \bar{C}(\sigma) = \sup\limits_{D(\sigma)} R_\sigma(m)$;
$t(\Pi_1, \Pi_2) = \sup\limits_{a,t}(\max\limits_m \tau(\sigma_{a,t,\Pi_1,\Pi_2}, m) - \min\limits_m \tau(\sigma_{a,t,\Pi_1,\Pi_1}, m))$.

We write $t(\Pi)$ for $t(\Pi, \Pi)$.

We also benefit from the absolute continuity of the stable foliation, which can be formulated as follows. Let $V_1$ and $V_2$ be manifolds of dimensions $N_u$ lying in a small ball and transversal to $E_s + E_0$. Let $l_1$ and $l_2$ be Lesbegue measures on $V_1$ and $V_2$ respectively calculated with the help of the induced Riemannian metric. Denote by $p$ the projection from $V_1$ to $V_2$ along the leaves of $W^{(s)}$. Then $p_* l_2$ has a density with respect to $l_1$ on $p^{-1}V_1$ which is bounded away from zero and infinity.

5) Now we are going to recall the definition of the dimension with respect to a dynamical system for our case. A more general definition as well as motivation, can be found in [P2]. So, let $A$ be a set such that

$$\forall a \in A \Rightarrow W^{(s)}(a) \subset A. \tag{1}$$

Take some $m_0$ and let $\bar{A}$ be $A \cap W^{(su)}(m_0)$. Consider the dimension $\Delta(\bar{A})$ corresponding to the measure

$$dh^s_\varepsilon(\bar{A}) = \inf \sum_i \mathrm{Vol^s(U_i)},$$

where the infimum is taken over all coverings of $\bar{A}$ by sets $U_i$ with diameters less than $\varepsilon$ such that each $U_i$ is a preimage of a ball $U_i = g^{-t_i} W^{(su)}_{\varepsilon_0}(a_i)$, where $\varepsilon_0$ is a fixed constant. The magnitude $d_u(\bar{A}) = \dim(E_u)\Delta(\bar{A})$ is called the dimension of $\bar{A}$ with respect to the dynamical system $g^t$. (The rescaling is done in order to make the dimension of $W^{(su)}$ equal to its topological dimension.) Since the value of $d_u(A)$ is independent on the choice of $m_0$ (by the absolute continuity of the stable and unstable foliations) we use the notation $\dim_{g^t}(A)$ instead of $d_u(\bar{A})$ and $\Delta(A)$ instead of $\Delta(\bar{A})$.

6) Here we recall some notions from the theory of symbolic dynamical systems.

Given a $m \times m$ matrix $Q$ whose entries are zeroes and ones, we denote by $\Sigma$ the set of two-sided infinite sequences $\vec{x} = \dots x_{-1}x_0x_1x_2\dots x_k\dots$ such that $x_i \in \{1, 2, \dots m\}$ and $Q_{x_ix_{i+1}} = 1$. We consider also the spaces $\Sigma_+$ of one-sided sequences, and $\Sigma_n$ of sequences of length $n$ satisfying the conditions above. We denote $\Sigma_f = \bigcup\limits_n \Sigma_n$. Elements of $\Sigma_f$ will be called words. The subshift of finite type with the transition matrix $Q$ is the map $\sigma : \Sigma \to \Sigma$ $(\Sigma_+ \to \Sigma_+)$ such that $(\sigma(\vec{x}))_i = x_{i+1}$. $\sigma$ is a continuous map in the topology induced from the product of discrete topologies of $\{1, 2, \dots m\}$.

If $w = i_1i_2\dots i_n$ is a word we denote by $C_w$ the cylinder in $\Sigma$ ($\Sigma_+$ respectively):

$$C_w = \{\vec{x} \in \Sigma(\Sigma_+) : x_j = i_j \ for \ 1 \le j \le n\}.$$

We say that $C_w$ is a cylinder of width $n$ if $w \in \Sigma_n$. We also use the notation $C^{m,n}(\vec{x}) = \{\vec{y} : y_j = x_j \ for \ m \le j \le n\}$. We write simply $C^n(\vec{x})$ instead of $C^{0,n}(\vec{x})$.

For a subshift of finite type $(\Sigma_+, \sigma)$ the following formulae for the calculation of the topological entropy are useful:

$$h(Y,\sigma) = h(Y,W),$$

where $W$ is the covering of $X$ by the sets $C_i$, and

$$h_{top}(\sigma) = \lim_{n\to\infty} \frac{\ln \operatorname{Card} \Sigma_n}{n}.$$

If $Q$ is irreducible: that is, for any $i, j < m$ there exists $n$ such that $Q^n_{ij} > 0$, then also for any $i$

$$h_{top} = \lim_{n\to\infty} \frac{\ln \operatorname{Card}(\vec{x} \in \Sigma_n : \ x_1 = i)}{n}. \tag{2}$$

7) To use results from the theory of subshifts of finite type for the study of differential dynamical systems one needs the notion of a Markovian partition [Sn1]. Here we consider the case of an Anosov diffeomorphism $g$. The partition $M = \bigcup\limits_{i=1}^{n} \Pi_i$ of $M$ into parallelograms is called Markovian if

*a*) $\operatorname{Int}(\Pi_i) \bigcap \operatorname{Int}(\Pi_j) = \emptyset$ for $i \neq j$;

*b*) If $m \in \Pi_i$ and $gm \in \Pi_j$ then $gW^{(s)}_{\Pi_i}(m) \subset \Pi_j$;

*c*) If $m \in \Pi_i$ and $g^{-1}m \in \Pi_j$ then $g^{-1}W^{(u)}_{\Pi_i}(m) \subset \Pi_j$.

Consider the matrix $Q : Q_{ij} = 1$ if $g\Pi_i \bigcap \Pi_j \neq \emptyset$ and $Q_{ij} = 0$ otherwise. Let $(\Sigma, \sigma)$ be the subshift of finite type with transition matrix $Q$. The map $p : \Sigma \to M : p(\vec{x}) = \bigcap\limits_{j=-\infty}^{+\infty} g^{-j}\Pi_{x_j}$ conjugates $\sigma$ and $g$ that is $p \circ \sigma = g \circ p$. Using the map $p$ we can identify $M$ and $\Sigma$. In Section 5 we write $C^{n_1,n_2}$ instead of $p^{-1}C^{n_1,n_2}$ etc.

In the case $\dim M = 2$ one can find a Markovian partition such that set of the points which either have no inverse images with respect to $p$ or have more than 1, has Hausdorff dimension 1 (See [Sn2]).

Markovian partitions for more complicated systems were constructed in [Rt], [B2] and others. However, it is clear that in the case of a non-compact phase space one cannot find a finite partition. Nevertheless, as we show in Section 6 one still can define a good symbolic dynamics on a set of large dimension.

8) One can use the existence of Markovian partitions to analyse the statistical behaviour of the orbits for Anosov diffeomorphisms. Here we mention

the results we need in this paper. Let $f$ be an Anosov diffeomorphism. Then there exist a measure $\mu$ such that for any continuous function $g$

$$\frac{1}{n}\sum_{k=0}^{n-1} g(f^k m) \to \int g(m)\, d\mu(m) \;\; Vol \; a.e. \tag{3}$$

This measure is called the Sinai-Bowen-Ruelle (SBR) measure. The existence of this measure is proven in [Sn1]. One can also estimate the deviations from this law.

Proposition 1. ([OP]) *For any continuous function $g$ and for any closed set $A \subset \mathbf{R}$*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathrm{Vol}\{\mathrm{m} : \frac{1}{\mathrm{n}}\sum_{\mathrm{k}=0}^{\mathrm{n}-1} \mathrm{g}(\mathrm{f^k m}) \in \mathrm{A}\} \leq \sup_{\substack{\nu-\mathrm{f}-\mathrm{invariant} \\ \int \mathrm{g(m)\, d\nu(m)} \in \mathrm{A}}} \mathrm{h}_\nu(\mathrm{f}) - \int \mathrm{Q}^{(\mathrm{u})}(\mathrm{m}, 1)\, \mathrm{d}\nu(\mathrm{m})$$

*where $h_\nu(f)$ denotes the metric entropy of the measure $\nu$.*

9) In this paragraph we recall formulae for calculation of the Hausdorff dimension of certain invariant sets of Anosov diffeomorphisms. So, let $f$ be an Anosov diffeomorphism of a compact surface $M$ and $\nu$ be an ergodic $f$-invariant measure. Then

$$\mathrm{HD}(\nu) = h_\nu(f)\left(\frac{1}{\chi_+(\nu)} + \frac{1}{\chi_-(\nu)}\right), \tag{4}$$

where $\mathrm{HD}(\nu)$ stands for $\mathrm{HD}(\mathrm{supp}\ \nu)$, and $\chi_\pm(\nu)$ are the Lyapunov exponents $\chi_+(\nu) = \int Q^{(u)}(m,1)\, d\nu(m), \chi_-(\nu) = \int Q^{(s)}(m,1)\, d\nu(m)$ (see [Y]).

Recall that a point $m$ is called $\nu$-forward typical if for any continuous function $g$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} g(f^k m) = \int f(m)\, d\nu(m).$$

The Hausdorff dimension of the set of $\nu-$forward typical points can be calculated by means of the Pesin-Manning-McClusky formula ([MM])

$$\mathrm{HD}^+(\nu) = \frac{h_\nu(f)}{\chi_+(\nu)} + 1. \tag{5}$$

10) We conclude this section with one property of Anosov systems we use in Sections 5 and 6.

LEMMA 1. *For every two parallelograms* $\Pi_1$ *and* $\Pi_2$ *there is a constant* $K(\Pi_1, \Pi_2)$ *depending only on the size of the parallelograms, such that for every* $a_0 \in \Pi_1$ *and* $t_0 > 0$ *for which* $g^{t_0}a_0 \in \Pi_2$ $\sigma = \sigma_{a_0,t_0,\Pi_1,\Pi_2}$ *satisfies the following condition: for every* $m_1, m_2 \in D(\sigma)$

$$\frac{1}{K(\Pi_1, \Pi_2)} R_\sigma(m_1) \leq R_\sigma(m_2) \leq K(\Pi_1, \Pi_2) R_\sigma(m_1).$$

PROOF: Let $n_1 = \sigma(m_1)$, $n_2 = \sigma_{m_2}$. Consider three possibilities.

1) $m_1 \in W^{(ss)}(m_2)$. The statement follows from the fact that the limit $\lim\limits_{t\to+\infty} \frac{R^{(u)}(m_1,t)}{R^{(u)}(m_2,t)}$ exists and is uniformly bounded as $\mathrm{dist}(m_1, m_2)$ is bounded.

2) $m_1 \in W^{(u)}_{\Pi_1}(m_2)$ and hence $n_1 \in W^{(u)}_{\Pi_2}(n_2)$. The statement follows from the fact that the limit $\lim\limits_{t\to-\infty} \frac{R^{(u)}(n_1,t)}{R^{(u)}(n_2,t)}$ exists and is uniformly bounded as $\mathrm{dist}(n_1, n_2)$ is bounded.

3) In the general case one can find $m_3$ and $m_4$ such that $m_1 \in W^{(u)}_{\Pi_1}(m_3)$, $m_2 \in W^{(u)}_{\Pi_1}(m_4)$ and $m_3$, $m_4 \in W^{(s)}_{\Pi_1}(a)$. ■

**3. Zero-dimensional dynamics.**

1) In this section we deal with one-sided subshifts of finite type. The main theorem proven here allows us to derive our results for the Hausdorff dimension in Sections $4-6$ using the relationship between the topological entropy and the Hausdorff dimension revealed in [B], [M], [MM], [P2], [P3] and others.

We will omit $\sigma$ in $h(A, \sigma) and\, h_{top}(\sigma)$.

In this section we prove the following

THEOREM 1. *Let* $A \subset \Sigma_+$ *be such that* $h(A) < h_{top}$ *. Then* $h(Ig^+(A)) = h_{top}$ *.*

REMARK. Actually, the same arguments lead to the following result.

PROPOSITION 1. *Fix* $s < h_{top}$ *and let* $U = \{U_k\}$ *be a subset of* $\bigcup\limits_{n>N_0} \Sigma_n$*. Let* $l(U_k)$ *the length of* $U_k$ *and assume that* $\sum\limits_k e^{-sl(U_k)} < 1$*. Denote by* $Ig^+(U) = \{\vec{x} \in \Sigma_+ : \forall k, m_1, m_2\ x_{m_1}x_{m_1+1}\ldots x_{m_2} \notin U\}$*. Then* $h(Ig^+(U)) \geq c_s(N_0)$*, where* $\lim\limits_{N_0\to\infty} c_s(N_0) = h_{top}$ *.*

The proof of the theorem is divided into several steps.

Without loss of generality we may assume that the transition matrix is irreducible.

2) In this paragraph we introduce another dimensional characteristic which is easier to handle than the topological entropy. Denote by $\mathrm{h_{bin}}(Y)$

the dimension corresponding to the measure

$$h^{\varepsilon}_{bin,\,s}(Y) = \inf_{\substack{l(V_i)>\frac{1}{\varepsilon},\, Y\subset\bigcup\limits_i V_i \\ l(V_i)=2^{k_i}}} e^{-sl(V_i)}.$$

So, the difference with the topological entropy is that now we consider only coverings by cylinders whose widths are powers of 2.

Since the infimum is now taken over a smaller set we get

$$\mathrm{h}_{\mathrm{bin}}(Y) \geq h_{top}(Y),$$

but in view of (2.2) we still have $\mathrm{h}_{\mathrm{bin}}(\Sigma_+) = h_{top}$ .

In the general case we have the following

LEMMA 1. $\mathrm{h}_{\mathrm{bin}}(Y) \leq \frac{h(Y)+h_{top}}{2}$.

PROOF: Let $\{V_i\}$ be a covering of $Y$ with $N(V_i) > N_0$ and $\sum\limits_i e^{-sl(V_i)} \leq 1$. Let $2^{k_i} < l(V_i) \leq 2^{k_i+1}$. Denote by $V_i^j$ the elements of $\Sigma_{2^{k_i+1}}$, such that $\bigcup\limits_j V_i^j = V_i$, and let $n_i = \mathrm{Card}\{V_i^j\}$. By (2.2)

$$n_i \leq C_1 e^{(h_{top}+\varepsilon)(2^{k_i+1}-l(V_i))}.$$

Hence,

$$\sum_{i,j} e^{-l(V_i^j)(\frac{s+h_{top}+\varepsilon}{2})} \leq \sum_i n_i e^{-2^{k_i+1}(\frac{s+h_{top}+\varepsilon}{2})} \leq C_1 \sum_i e^{-\phi_i},$$

where

$$\phi_i = (h_{top}+\varepsilon)(2^{k_i+1}-l(V_I))-2^{k_i}(s+h_{top}+\varepsilon) = (h_{top}+\varepsilon)(2^{k_i}-l(V_i))-2^{k_i}s \leq -l(V_i)s.$$

Therefore, $\mathrm{h}_{\mathrm{bin}}(Y) \leq \frac{h(Y)+h_{top}+\varepsilon}{2}$ and since $\varepsilon$ can be chosen arbitrary small the lemma is proven. ■

So it suffices to show that for all $\varepsilon$ $\mathrm{h}_{\mathrm{bin}}(Ig^+(A)) > h_{top} - \varepsilon$, provided that $\mathrm{h}_{\mathrm{bin}}(A) < h_{top}$ .

3) Here we define the set of prohibited words. Take a real $\theta_1$ such that $\mathrm{h}_{\mathrm{bin}}(A) < \theta_1 < h_{top}$ . Then for all $N_0$ we can find a covering of $A$ by cylinders $A \subset \bigcup\limits_i U_i$ such that $l(U_i) = 2^{k_i}, k_i > N_0$ and

$$\sum_i e^{-\theta_1 l(U_i)} \leq 1.$$

So we have $Ig^+(A) \supset Ig^+(U)$. Now we construct a smaller set, denoted by $Ig^+_{bin}(U)$ as follows.

Let $P = \bigcup_i U_i^{bin}$, where $U_i^{bin} = \{U_{ij} : l(U_{ij}) = 2^{k_i+1}$ and $U_{ij}$ contains $U_i$ as a subword$\}$. We call elements of $P$ prohibited words.

By (2.2) the number of prohibited words of length $2^{k+1}$ does not exceed $C_2 2^k e^{2^k(\theta_1 + h_{top} + \varepsilon)}$, which for large enough $k$ (that can be guaranteed by choosing $N_0$ large) is less then $e^{\theta_2 2^{k+1}}$, where $\theta_2$ is a constant such that $\frac{h_{top} + \theta_1}{2} < \theta_2 < h_{top}$ . Form the set

$$Ig^+_{bin}(U) = \{\vec{x} \in \Sigma_+ : \forall p \in P \ \forall m \leq 0 \ x_{m\frac{l(p)}{2}+1} x_{m\frac{l(p)}{2}+2} \dots x_{(m+2)\frac{l(p)}{2}} \neq P\}.$$

(The number $\frac{l(p)}{2}$ appears since $p$ contains an element of $U$ of length $\frac{l(p)}{2}$). We call words, that is finite sequences, satisfying the same condition, admissible.

Since $Ig^+_{bin}(U) \subset Ig^+(U) \subset Ig^+(A)$, it is enough to estimate $h(\ Ig^+_{bin}(U))$.

4) Here we divide all admissible words of length $2^k$ into good and bad ones.

We call the first $2^{k-1}$ symbols of the word of length $2^k$ its prefix and the last $2^{k-1}$ symbols its suffix.

The admissible word $W_k$ of length $2^k$ is said to be good if
$a)$ both its prefix and suffix are good and
$b)$ it forms the prefix of at most $e^{\frac{(h_{top} + \theta_2)2^k}{2}}$ prohibited words and the suffix of at most $e^{\frac{(h_{top} + \theta_2)2^k}{2}}$ prohibited words,
and said to be bad otherwise.

Note that the number of words not satisfying the condition $b)$ does not exceed $2e^{2^k(\frac{3}{2}\theta_2 - \frac{h_{top}}{2})}$, which, if $N_0$ is large enough, is less than $e^{2^k\theta_3}$, where $\theta_3$ is a constant such that $\frac{3}{2}\theta_2 - h_{top} < \theta_3 < h_{top}$ .

Define $G_k$ as

$$G_k = \min_{W_k - good} \#\{\tilde{W}_k - good : \ W_k\tilde{W}_k - admissible\}.$$

Note that, by its very definition, the number of good words of length $2^k$ is at least $G_k$.

Let $W_k^{(p)}$, $W_k^{(s)}$($\tilde{W}_k^{(p)}$, $\tilde{W}_k^{(s)}$ respectively) be the prefix and the suffix of $W_k$ ($\tilde{W}_k$ respectively).

$W_k\tilde{W}_k$ is admissible and $\tilde{W}_k$ is good provided that

1) $W_k^{(s)}\tilde{W}_k^{(p)}$ and $\tilde{W}_k^{(p)}\tilde{W}_k^{(s)}$ are admissible;
2) $W_k\tilde{W}_k \notin P$;
3)$\tilde{W}_k^{(p)}\tilde{W}_k^{(s)}$ is not bad.

There exist at least $G_{k-1}^2$ possibilities to satisfy the first condition, among which at most $e^{(\frac{h_{top}+\theta_2}{2})2^k}$ violate the second condition by property $b)$ of the good word and at most $e^{\theta_3 2^k}$ violate the third condition by the above estimate.

Therefore, we get

$$G_k \geq G_{k-1}^2 - 2e^{\theta_4 2^k},$$

where $\theta_4 = \max(\frac{\theta_2+h_{top}}{2}, \theta_3)$. We claim that for any $\alpha < h_{top}$ $G_k \geq e^{\alpha 2^k}$, provided that $N_0$ is large enough. Indeed without loss of generality we may assume that $\alpha > \theta_4$. Denote $g_k = \frac{G_k}{e^{\alpha 2^k}}$. Then $g_k \geq g_{k-1}^2 - 2e^{(\theta_4-\alpha)2^k}$ for $k > N_0$ and $g_{N_0} \geq e^{(h_{top}-\alpha-\varepsilon)2^{N_0}}$. Clearly $g_k$ increases to infinity if $g_{N_0}$ is large enough.

5) Here we introduce a measure on $\Sigma_+$ satisfying the uniform mass distribution principee for $\mathrm{h_{bin}}$ .

As the first step put $\mu(\Sigma_+) = 1$. Let us assume that we have already defined $\mu$ for cylinders of width $2^k$. Let $C_{(k)}$ be such a cylinder. If $C_{(k)}$ is good denote by $C_{(k+1)}^i$ admissible cylinders of width $2^{k+1}$ inside $C_{(k)}$ and put $\mu(C_{(k+1)}^i) = \frac{\mu(C_{(k)})}{\mathrm{Card}\{C_{(k+1)}^i\}}$. In this case $\mu(C_{(k+1)}^i) \leq \frac{\mu(C_{(k)})}{G_k}$ and by induction it follows that

$$\mu(C_{(k+1)}^i) \leq C_2 e^{-\alpha 2^k}. \tag{1}$$

If $C_{(k)}$ is bad we require that $\mu|_{C_{(k)}}$ be concentrated at a single point in $C_{(k)}$.

If $C_{(k)}$ is not even admissible we claim that $\mu|_{C_{(k)}}$ was completely specified in the previous steps. Indeed, if a cylinder of width $2^{(k-1)}$ containing $C_{(k)} - C_{(k-1)}$ is admissible then $\mu|_{C_{(k)}}$ is either zero or $\delta-$measure. If already $C_{(k-1)}$ is not admissible then $\mu|C_{(k-1)}$ was completely described earlier by induction.

By the construction of $\mu$ it follows that $\mu = \mu_d + \mu_c$, where $\mu_d$ is concentrated on a countable set of points and $\mu_c$ is supported on $Ig_{bin}^+(U)$.

(1) implies that

$$\mu_d(\Sigma_+) \leq \sum_{k=N_0}^{\infty} (number\ of\ bad\ intervals\ of\ range\ 2^k) C_2 e^{-\alpha 2^k} \leq$$

$$\leq C_2 \sum_{k=N_0}^{\infty} e^{-(\alpha-\theta_3)2^k}.$$

So, if $N_0$ is large enough, $\mu_d(\Sigma_+) \leq 1$ and hence $\mu_c(\Sigma_+) > 0$. It follows from (1) that for every cylinder $C$ of width $2^k$

$$\mu_c(C) \leq C_2 e^{-\alpha 2^k}.$$

Therefore if $\{V_i^{(2^{k_i})}\}$ is any covering of $Ig^+_{bin}(U)$ we obtain

$$\sum_i e^{-\alpha l(V_i)} \geq \frac{1}{C_2} \sum_i \mu_c(V_i^{(2^k_i)}) \geq \frac{\mu_c(\, Ig^+_{bin}(U))}{C_2} = \frac{\mu_c(\Sigma_+)}{C_2} > 0.$$

So, $\mathrm{h_{bin}}(Ig^+(A)) \geq \mathrm{h_{bin}}(\, Ig^+_{bin}(U)) \geq \alpha$. By the remark at the end of paragraph 2 this completes the proof of theorem 1.

**4. One-dimensional dynamics.**

1) In this section we study a piecewise expanding map $f$ of an interval $I$ into itself. More precisely, we assume that there exists a finite partition of $I$ into intervals $I = \bigcup_{i=1}^{l} I_i$, such that if $f_i$ denotes $f|_{I_i}$, then the following conditions are satisfied:

*a)* $f_i \in C^2(I_i)$;

*b)*$\exists \lambda > 1 : \; |f_i'| \geq \lambda$, therefore

*c)*$f_i$ is strictly monotone.

Recall the notion of an interval of range $n$ for $f$. Intervals of range 1 are precisely $\{I_i\}$. If we have already defined intervals $\{I_j^{(n)}\}$ of range n we say that $\hat{I}$ is an interval of range $n+1$ if it has the form $\hat{I} = f_i^{-1} I_{j_1}^{(n)} \cap I_{j_2}^{(n)}$ for some $i$ and intervals $I_{j_1}^{(n)}$, $I_{j_2}^{(n)}$ of range $n$. Hence, $f^n$ is a $C^2-$continuous function on any interval of range $n$. We denote by $I^n(x)$ the interval of range $n$ containing $x$.

The aim of this section is to prove the following statement:

Theorem 2. *Let $A \subset I$ be a subset of $I$ such that* $\mathrm{HD}(A) < 1$. *Then* $\mathrm{HD}(Ig^+(A)) = 1$.

The proof consists of several steps.

2) Firstly, we want to reduce our problem to the case when our map is almost uniformly expanding in order to make the Hausdorff dimension of an invariant set proportional to the topological entropy.

Fix natural $N$. We consider the map $g(x) = f^{n_1(x)}(x)$, where $n_1(x) = \min\{n : \min_{I^{(n)}(x)} |f^n(x)'| > N\}$. Clearly,

$$n \leq M = [\frac{\ln N}{\ln \lambda}] + 1. \tag{1}$$

We denote by $J_j$ the intervals of range 1 for $g$ so that each $J_j = I^{n_1(x)}(x)$ for some $x$. We shall use the distortion inequality (see, for example [CFS]) which states that there exists a constant $C_1$ (which does not depend on $n$), such that

$$\max_{I_i^{(n)}} |f^n(x)'| \leq C_1 \min_{I_i^{(n)}} |f^n(x)'|.$$

This inequality implies that

$$N \leq |g'(x)| \leq C_2 N, \tag{2}$$

where $C_2 = C_1 \max_x |f'(x)|$.

Denote by $A_k = \bigcup_{j=0}^{k} f^{-j}(A)$. It is straightforward to check that $Ig^+(A, f) = Ig^+(A_M, g)$ and $\mathrm{HD}(A) = \mathrm{HD}(A_M)$ (where $M$ is defined by (1)). So it suffices to prove that $\mathrm{HD}(Ig^+(A_M, g)) \geq 1 - \varepsilon(C_2, N)$, where $\varepsilon \to 0$ as $N \to +\infty$. In the sequel we often omit the index $M$ in $A_M$.

3) In this paragraph we define an invariant subset of the interval $K$ such that $g|_K$ is a subshift of finite type and $\mathrm{HD}(K)$ is close to 1.

Lemma 1. *(cf [MS].) There exists a $g-$invariant set $K$ such that $g|_K$ is a subshift of finite type and* $h(K, g) \geq \ln(N-4) - \ln 2$, *where* $h(g|K)$ *denotes the topological entropy of* $g$ *with respect to* $K$.

Proof: Without loss of generality we can assume that

$$\max_i |J_i| \leq 2 \min_i |J_i|, \tag{3}$$

considering, if necessary, a finer partition of $I$. Denote by $J_{i_1 i_2 \dots i_k}$ the set $(g_{i_k} \ldots \circ g_{i_2} \circ g_{i_1})^{-1} I$ and set $\pi(i_1 i_2 \dots i_k \dots) = \bigcap_{k=1}^{\infty} J_{i_1 i_2 \dots i_k}$. This set can either be empty or a singleton, so we shall write $x = \pi(\vec{i})$ instead of $\{x\} = \pi(\vec{i})$. We say that $J_i$ covers $J_j$ iff $J_j \subset g(J_i)$. Put $K = \{\pi(\vec{i}) : J_{i_{k-1}} \text{ covers } J_{i_k}\}$. Note that this condition guarantees that $\pi(\vec{i}) \neq \emptyset$. By (2) and (3) every interval $J_i$ covers at least $\frac{N}{2} - 2$ intervals of range 1 for $g$, so $h(g|K) \geq \ln(\frac{N}{2} - 2)$. ■

4) In this paragraph we derive theorem 1 from theorem 2.

Lemma 2. $\forall Y \subset K$

$$\frac{h(Y, g)}{\ln C_2 + \ln N} \leq \mathrm{HD}(Y) \leq \frac{h(y, g)}{\ln N}.$$

Proof: 1) Let $s > h(Y, g)$ and $\{J_i^{(n_i)}\}$ be a covering of $Y$ by intervals of range $n_i$ such that $\sum\limits_i e^{-sn_i} \leq 1$. Since $|J_i^{(n_i)}|N^{n_i} \leq 1$, we obtain

$$\sum_i |J_i^{(n_i)}|^{\frac{s}{\ln N}} \leq \sum_i N^{-\frac{sn_i}{\ln N}} \leq 1.$$

Hence $\frac{s}{\ln N} > \mathrm{HD}(Y)$.

2) Let $s > \mathrm{HD}(Y)$ and $\{V_i\}$ be a covering of $Y$, so that $\sum\limits_i |V_i| \leq 1$. Consider $n_i = \min\{n : \exists y_i \in Y\ J^{(n)}(y_i) \subset V_i\}$. Since, by the definition of $K$, $f_i^n J^{(n_i)}$ contains some interval of range 1 we have

$$|V_i|(C_2 N)^n \geq C_3,$$

where $C_3 = \min\limits_K |J^{(1)}(y)|$. On the other hand $V_i \bigcap Y$ is contained in the union of at most two intervals of range $(n_i - 1)$, say $J_{i1}^{(n_i-1)}$ and $J_{i2}^{(n_i-1)}$. For the covering $Y \subset \bigcup\limits_i (J_{i1}^{(n_i-1)} \bigcup J_{i2}^{(n_1-1)})$ we obtain

$$\sum_i e^{-s(\ln N + \ln C_2)(n_i-1)} \leq (C_2 N)^s \sum_i (C_2 N)^{-n_i s} \leq \frac{(C_2 N)^s}{C_3} \sum_i |V_i|^s \leq \frac{(C_2 n)^s}{C_3}.$$

Hence, $h(g, Y) < s(\ln N + \ln C_2)$ and the lemma is proven. ∎

Lemma 2 implies that for $N$ large enough $h(A, g) < h(K, g)$. Therefore

$$\mathrm{HD}(Ig^+(A)) \geq \frac{h(Ig^+(A), g)}{\ln N + \ln C_2} = \frac{h(K, g)}{\ln N + \ln C_2} \geq \frac{\ln(N-4) - \ln 2}{\ln N + \ln C_2}.$$

Since N can be chosen arbitrary large, theorem 2 is proven.

## 5. Two-dimensional dynamics.

1) In this section we deal with a $C^2$ Anosov diffeomorphism $f$ of two-dimensional torus $M$. We denote by $\mu$ the SBR measure for $f$.

Set $es(f) = \sup\{s : \forall A\ \mathrm{HD}(A) < s \Rightarrow \mathrm{HD}(Ig^+(A) = 2\}$. In this section we prove the following.

Theorem 3. *$es(f)$ equals to the dimension of SBR measure.*

For the SBR measure formulae (2.4) and (2.5) imply

$$\mathrm{HD}(\mu) = 1 + \frac{\chi_+}{\chi_-}, \tag{1}$$

where $\chi_+ = \chi_+(\mu) = \chi_+(Vol)$, $\chi_- = \chi_-(\mu) = \chi_-(Vol)$. It is formula (1) that we use in the proof. We needed $\mathrm{HD}(\mu)$ only to explain the meaning of the statement in the introduction.

2) Here we begin the proof of the lower bound for $es(f)$. Consider a Markovian partition $\Pi = (\Pi_1, \Pi_2 \dots \Pi_P)$ so that $\mathrm{HD}(M \setminus \bigcup_{i=1}^{P} \mathrm{Int}\, \Pi_i) = 1$. Fix a small constant $\varepsilon_1$. We call a cylinder $C_j^n$ of the width n typical if it satisfies the following conditions:

$a)$ $\forall m \in C_j^n$ $|\frac{Q^{(u)}(m,n)}{n} - \chi_+| < \varepsilon_1$;

$b)$ $\forall m \in C_j^n$ $|\frac{Q^{(s)}(m,n)}{n} - \chi_-| < \varepsilon_1$;

By (2.3) we can find $n$ so large that

$\alpha)$ the Volume of the typical cylinders of the width $n$ is greater than $\frac{1}{2}$;

$\beta)$ for any typical cylinder $C_j^n$ $\frac{\mathrm{Vol}(\mathrm{C_j^n} \bigcap \bigcup_{\mathrm{k}} \mathrm{f}^{-1}(\mathrm{C_k^n}))}{\mathrm{Vol}(\mathrm{C_j^n})} > \frac{1}{2}$, where the union is taken over all typical cylinders.

Denote by $N$ the set $\{m : \forall k \in \mathbf{Z}$ $f^{kn}m$ lies in a typical cylinder of the width $n\}$. It suffices to prove that for given $\varepsilon$, $A$ such that $\mathrm{HD}(A) < es(f) - \varepsilon$ we can choose $\varepsilon_1$ so small that $\mathrm{HD}(Ig^+(A, f^n) \bigcap N) \geq 2 - \varepsilon$.

3) In this paragraph we estimate the topological entropy of $N$ with respect to $f^n$. Since in the two dimensional case the stable and unstable foliations are smooth([HP]), the Volume on $M$ is equivalent to $du\, ds$, where $du$ is the induced metric on $W^{(u)}$ and $ds$ that on $W^{(s)}$. Therefore there is a constant $C_1$ such that

$$(\frac{1}{C_1})e^{-n(\chi_+ + \varepsilon_1)} < Vol(C_j^n) < C_1 e^{-n(\chi_+ - \varepsilon_1)} \tag{2}$$

and

$$(\frac{1}{C_1})e^{-2n(\chi_+ + \varepsilon_1)} < Vol(C_j^n \bigcap f^{-n} C_k^n) < C_1 e^{-2n(\chi_+ - \varepsilon_1)}.$$

By the property $\beta$ $C_j^n$ intersects at least $C_2 e^{n(\chi_+ - 2\varepsilon_1)}$ inverse images of typical cylinders. Hence by (2.2)

$$h_{top}(N, f^n) \geq n(\chi_+ - 2\varepsilon) - \ln C_2.$$

So, for any fixed $\varepsilon_2$ we can choose $\varepsilon_1$ so small and $n$ so large that

$$h_{top}(N, f^n) \geq n(\chi_+ - \varepsilon_2). \tag{3}$$

4) In this paragraph we point out another way of calculating of $\mathrm{HD}(A \bigcap N)$. We will consider forward cylinders of the form

$$\Pi^{+} = \bigcap_{j=0}^{k^{+}} f^{-ni}(C^{n}_{i^{+}_{j}})$$

and backward cylinders of the form

$$\Pi^{-} = \bigcap_{j=0}^{k^{-}} f^{ni}(C^{n}_{i^{-}_{j}}),$$

where $n$ was defined in paragraph 2 and the $C^{n}_{i^{\pm}_{j}}$ are typical.

Set $Q_{+}(P^{+}) = \min_{m \in P^{+}} Q^{(u)}(m, k^{+}n)$, $Q_{-}(P^{-}) = \min_{m \in P^{-}} Q^{(s)}(m, k^{-}n)$. The cylinder $P = P^{+} \bigcap P^{-}$ will be called a square if $C^{n}_{j^{+}_{0}} = C^{n}_{j^{-}_{0}}$, $Q_{-}(P^{-}) \geq Q_{+}(P^{+})$ but $Q_{-}(\bigcap_{j=0}^{k^{-}-1} f^{ni}(C^{n}_{j^{-}_{i}})) < Q_{+}(P^{+})$. It is easy to see (cf. the proof of lemma 4.2 or [F]) that $\mathrm{HD}(A \bigcap N)$ can be calculated using only coverings of $A \bigcap N$ by squares.

So given $\theta > \mathrm{HD}(A)$, we can find such a covering $A \subset \bigcup_{i} P_i$ such that $\sum_{i} |P_i|^{\theta} \leq 1$. On the other hand $|P_i| \geq C_3 e^{-(\chi_{+}+\varepsilon_1)k^{+}_{i}n}$

Therefore

$$\sum_{i} e^{-(\chi_{+}+\varepsilon_1)k^{+}_{i}n\theta} \leq (\frac{1}{C_3})^{\theta}. \tag{4}$$

Denote by $U$ the set $\{U_i\}$, where $U_i = f^{-k^{-}_{i}n} P_i$. From the definition of the typical cylinder it follows that, if the diameters of the $P_i$ are small enough, then $|k^{-}_{i} - k^{+}_{i}\frac{\chi_{+}}{\chi_{-}}| < \varepsilon_3 k^{+}_{i}$, where $\varepsilon_3 \to 0$ as $\varepsilon_1 \to 0$. So, for $l(U_i) = k^{+}_{i} + k^{-}_{i} + 1$, we obtain the estimate $|l(U_i) - k^{+}_{i}\,\mathrm{HD}(\mu) + 1| < \varepsilon_3 k^{+}_{i}$. Substituting this into (4) we get, provided that $|P_i|$ are small enough,

$$\sum_{i} e^{-(\frac{\chi_{+}}{\mathrm{HD}(\mu)}+\varepsilon_4)l(U_I)n\theta} \leq 1,$$

where $\varepsilon_4 \to 0$ as $\varepsilon_1 \to 0$. Using formula (3) and the fact that $\theta < \mathrm{HD}(\mu)$ we obtain that, if $\varepsilon_2$ and $\varepsilon_4$ are small enough,

$$(\frac{\chi_{+}}{\mathrm{HD}(\mu)} + \varepsilon_4)n\theta < h_{top}(N, f^{n}).$$

Now by Proposition 3.1 $h_{top}(Ig^+(U), f^n) > h_{top}(N, f^n) - \varepsilon_5 n$, where $\varepsilon_5 \to 0$ as $\sup\limits_i |P_i| \to 0$.

5) To finish the proof of the lower bound it remains to apply the inequality

$$\mathrm{HD}(Ig^+(U, f^n)) \geq \frac{h_{top}(Ig^+(U, f^n))}{\max\limits_{m \in Ig^+(U,f^n)} Q^{(u)}(m, n)} + 1,$$

which is proven in [M] (cf. also the proof of lemma 4.2). Indeed, in our case

$$\mathrm{HD}(Ig^+(U, f^n)) \geq \frac{n(\chi_+ - \varepsilon_2 - \varepsilon_5)}{n(\chi_+ + \varepsilon_1)} + 1 = 2 - \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_5}{\chi_+ + \varepsilon_1},$$

as was to be demonstrated.

6) Now we pass to the proof of the upper bound for $es(f)$. Namely, we consider we set $T_{\varepsilon_1} = \{m :$ for infinitely many $n$ $f^n C^{-n,0}(m)$ is $\varepsilon_1$-typical$\}$. We want to prove two facts. The first is that

$$\lim_{\varepsilon_1 \to 0} \mathrm{HD}(T_{\varepsilon_1}) \leq \mathrm{HD}(\mu). \tag{5}$$

The second claim is that for any positive $\varepsilon_1$

$$\mathrm{HD}(Ig^+(T_{\varepsilon_1})) < 2. \tag{6}$$

In this paragraph we prove (5). It is enough to show that for any parallelogram $\Pi_{i_0}$ from our partition the inequality

$$\lim_{\varepsilon_1 \to 0} \mathrm{HD}(T_{\varepsilon_1} \bigcap \Pi_{i_0}) \leq \mathrm{HD}(\mu)$$

holds. Note that if $m \in T_{\varepsilon_1}$ then $W^{(u)}_{\Pi_{i_0}}(m) \subset T_{\varepsilon 1}$. Therefore we have to prove that

$$\lim_{\varepsilon \to 0} \mathrm{HD}(W^{(s)}_{\Pi_{i_0}} \bigcap T_{\varepsilon 1}) \leq \frac{\chi_+}{\chi_-}.$$

Indeed for any $n_0$

$$T_{\varepsilon_1} \subset \{\exists n > n_0 : f^n C^{-n,0}(m) \; is \; \varepsilon_1 - typical\}.$$

So, we can consider the covering of $W^{(s)}_{\Pi_{i_0}}(m) \bigcap T_{\varepsilon_1}$ by segments

$$\bigcup_{n=n_0}^{\infty} \bigcup_{C^n_j - \varepsilon_1 - typical} (f^{-n} C^n_j \bigcap W^{(s)}_{\Pi_{i_0}}(m)).$$

By property $a)$ of $\varepsilon_1$-typical cylinders for fixed $n$

$$N_n = Card\{C_j^n - \varepsilon_1 - typical\} \leq C_1 e^{(\chi_+ + \varepsilon_1)n}.$$

On the other hand the lengths of the sets $C_j^n \bigcap W^{(s)}_{\Pi_{i_0}}(m)$ do not exceed the magnitude $d_n = C_4 e^{(\chi_- - \varepsilon_1)n}$. Hence for any $s > \frac{\chi_+ + \varepsilon_1}{\chi_-}$

$$\sum_n \sum_j |C_j^n \bigcap W^{(s)}_{\Pi_{i_0}}(m)|^s \leq \sum_{n=1}^{+\infty} N_n d_n^s < +\infty,$$

which proves (5).

7) It remains to prove the second part of the upper bound: that is, formula (6).

Moreover we prove that there exists a sequence $\{n_j\},\ n_j \to \infty$ as $j \to \infty$ such that if $\tilde{T}_{\varepsilon_1} = \{m :\ f^{n_j} C^{-n_j,0}(m)\ is\ \varepsilon_1 - typical\}$, then $\mathrm{HD}(Ig^+(\tilde{T}_{\varepsilon_1})) < 2$. The plan of the proof is the following. Denote by $\tilde{T}^J_{\varepsilon_1} = \{m :\ \forall j\, 1 \leq j \leq J\ f^{n_j} C^{-n_j,0}(m)\ is\ \varepsilon_1 - typical\}$. We show that

$$\mathrm{HD}(Ig^+(\tilde{T}^J_{\varepsilon_1})) \leq 2 - \gamma, \tag{7}$$

where the constant $\gamma$ depends only on $\varepsilon_1$ and $f$ but not on $J$. Since $\mathrm{Lim}^+(\mathrm{m})$ is a closed set we conclude that $\mathrm{HD}(Ig^+(\tilde{T}_{\varepsilon_1})) \leq 2 - \gamma$.

8) This paragraph contains some technical results we need to prove the existence of the sequence $\{n_j\}$, declared in the previous section. Without loss of generality we may assume that the Riemannian metric on $M$ is the Lyapunov one, so $\underline{\chi}_+ > 0$. We shall use the following

Proposition 1. *For any $\varepsilon_1,\ \varepsilon_2$ there exists a constant $C_5 = C_5(\varepsilon_1, \varepsilon_2)$, such that for every $n$*

$$\mathrm{Vol(m : C^n(m)\ is\ not\ } \varepsilon_1 - \mathrm{typical)} \leq \mathrm{C_5 e^{-n(\alpha(\varepsilon_1) - \varepsilon_2)}},$$

*where $\alpha(\varepsilon_1) > 0$ if $\varepsilon_1 > 0$.*

This proposition follows immediately from proposition 2.1 and lemma 2.1 (see also (2.5)).

Denote by $\underline{K}^n_j = \exp(-\min_{m \in C^n_j} Q^{(u)}(m,n)), \bar{K}^n_j = \exp(-\max_{m \in C^n_j} Q^{(u)}(m,n))$.

Now we define $\gamma$ in (7). Without loss of generality we may assume that $\alpha(\frac{\varepsilon_1}{2}) \leq \bar{\chi}_+$. Put $\gamma = \frac{\alpha(\frac{\varepsilon_1}{2})}{4\bar{\chi}_+}$. Denote by $\mathbf{A}_{n,l} = \{\underline{K}^{(n+l)}_j\}_{C^n_j\ is\ not \frac{\varepsilon_1}{2} - typical}$. Let $\mathbf{B}_{n_0} = \bigcup_{n \geq n_0} \bigcup_{l=0}^{n} \mathbf{A}_{n,l}$.

LEMMA 1. $\lim_{n\to\infty} \zeta_{\mathbf{B}_n}(1-\gamma) = 0$.

PROOF: We claim that $\zeta_{\mathbf{B}_1}(1-\gamma) < +\infty$. Indeed, since for $l \leq n$ $\underline{K}_j^{(n+l)} \geq e^{-2\bar{\chi}+n}$, we obtain

$$\sum_{\mathbf{A}_{n,l}} (\underline{K}_j^{(n+l)})^{1-\gamma} \leq e^{2\bar{\chi}+n\gamma} \sum_{\mathbf{A}_{n,l}} \underline{K}_j^{n+l} \leq C_6 e^{2\bar{\chi}+n\gamma} \sum_{\mathbf{A}_{n,l}} \mathrm{Vol}(\mathrm{C}_\mathrm{j}^{(\mathrm{n+l})}) \leq \mathrm{C}_6 \mathrm{e}^{-\alpha(\frac{\varepsilon_1}{2})\frac{\mathrm{n}}{2}}$$

which completes the proof (the middle inequality follows by lemma 2.1). ■

9) In this paragraph we define the sequence $\{n_j\}$.

By lemma 1 there exists $n_1(\varepsilon_1)$ such that $\zeta_{\mathbf{B}_{n_1(\varepsilon_1)}(\varepsilon_1)}(1-\gamma) < 1$. Put $n_1 = n_1(\varepsilon_1)$.

LEMMA 2. *$\forall n \exists N(n)$ : if $C^N(m)$ is not $\varepsilon_1$-typical then $f^n C^{-n,N}(m)$ is not $\frac{\varepsilon}{2}$-typical.*

PROOF: $n\underline{\chi}_+ \leq Q^{(u)}(f^{-n}m, N+n) - Q^{(u)}(m,N) \leq n\bar{\chi}_+$. ■

Put $n_k = N(\sum_{j=1}^{k-1} n_j)$. We call a word marked if it corresponds to a cylinder $C_j^{(n+l)}$ such that $n \geq n_1,\ 0 \leq l \leq n$ and $C_j^n$ is not $\frac{\varepsilon_1}{2}$-typical. We want to estimate $\mathrm{HD}(IG^+(\tilde{T}_{\varepsilon_1}^J))$. It suffices to consider the set $CT_{\varepsilon_1}^J = \{m : \forall n\ \exists j \leq J : C^{n_j}(f^{n-n_j}m) \textit{ is not } \varepsilon_1 - \textit{typical}\}$. Again it is enough to estimate the dimension of $CT_{\varepsilon_1}^J \bigcap W_{\Pi_{i_0}}^{(u)}$.

LEMMA 3. *Let $\vec{x} = x_0x_1\dots x_k\dots$ be the future symbolic representation of a point from $CT_{\varepsilon_1}^J$. Then $\vec{x}$ can be decomposed as $\vec{x} = S_0w_1w_2\dots w_k\dots$, where the length of the word $S_0$ is at most $n_J$ and the words $w_k$ are marked.*

PROOF: Denote by $\mathbf{N}_J = \mathbf{N} \setminus \{1, 2, \dots n_j\}$. By the definition of $CT_\varepsilon^J$ there exist intervals $L_k = [M_k, N_k]$, with an integer andpoints, such that

1) $\mathbf{N}_J \subset \bigcup_k L_k$;

2) $\forall k \exists j : N_k - M_k = n_j - 1$;

3) $C_{x_{M_k}x_{M_k+1}\dots x_{N_k}}$ is not $\varepsilon_1$-typical.

We can throw away some of $L_k'$s in such a way that the remaining set still satisfies 1) and covers $\mathbf{N}_J$ with multiplicity at most 2. Indeed if some natural number $n$ is covered by more than 2 intervals we throw away all but the one with the smallest $M_k$ and that with the largest $N_k$. We still denote the refined covering by $\{L_k\}$. We assume that the $L_k$ are ordered in such a way that $M_k < M_{k+1}$. Now we modify this covering to obtain a partition of $\mathbf{N}_J$. We proceed by induction. As the first step put $L_k^{(1)} = L_k$. We will resolve intersections from left to right. So let us assume that on the $l-$th step we

get $L_k^{(l)} = [M_k^{(l)}, N_k^{(l)}]$ and $M_k^{(l)} < M_{k+1}^{(l)}$. Let $k(l)$ be the smallest number such that $L_k^{(l)} \bigcap L_{k+1}^{(l)} \neq \emptyset$. Since in the previous steps we dealt with the intersections from the left of $L_{k(l)}^{(l)}$ all intervals $L_k^{(l)}$ with $k > k(l)$ are elements of $\{L_k^{(1)}\}$. In particular there exist $j(l)$ such that $N_{k(l)+1}^{(l)} - N_{k(l)+1}^{(l)} = n_{j(l)}$. We also assume by induction that $\exists \tilde{j}(l)$ so that $n_{\tilde{j}(l)} - 1 \leq N_{k(l)}^{(l+1)} - M_{k(l)}^{(l)} < \sum_{m=1}^{\tilde{j}(l)} n_m$. We put $L_k^{(l+1)}$ equal to $L_k^{(l)}$ if $k < k(l)$ and to $L_{k+1}^{(l)}$ if $k > k(l)$. To define $L_{k(l)}^{(l+1)}$ consider two possibilities.
If $\tilde{j}(l) < j(l)$ we assign $L_{k(l)}^{(l)} = L_{k(l)}^{(l)} \bigcup L_{k(l)+1}^{(l)}$.
If $\tilde{j}(l) \geq \tilde{j}(l)$ we assign $L_{k(l)}^{(l+1)} = [M_{k(l)}^{(l)}, M_{k(l)+2}^{(l)} - 1]$.

It is clear that, for fixed $k$, $L_k^{(l)}$ stabilize after a finite number of steps. We denote the resulting partition by $\tilde{L}_k = [\tilde{M}_k, \tilde{N}_k]$. By induction we get that $\forall k\ C_{x_{\tilde{M}_k} x_{\tilde{M}_k+1} \dots x_{\tilde{N}_k}}$ is marked.

So, fix $n$ and consider the covering of $W_{\Pi_{i_0}}^{(u)} \bigcap CT_{\varepsilon_1}^J$ by sets $W_{\Pi_{i_0}}^{(u)} \bigcap C_{S_0 w_1 w_2 \dots w_n}$ for possible $S_0$, $w_1 \dots w_n$ satisfying the condition of lemma 2.

Since $|W_{\Pi_{i_0}}^{(u)} \bigcap C_{S_0 w_1 w_2 \dots w_n}| \leq C_7 \prod_{j=1}^{n} \underline{K}(C_{w_{i_j}})$, we obtain

$$\sum_{S_0,\, w_1 \dots w_n} |W_{\Pi_{i_0}}^{(u)} \bigcap C_{S_0 w_1 w_2 \dots w_n}|^{1-\gamma} \leq C_7^{1-\gamma} \zeta_{\mathbf{B}_{n_1}} (1-\gamma)^n \to 0$$

as $n \to \infty$. So, the proof of (7) and hence that of theorem 3 is complete.

**6. Multidimensional dynamics.**

1) In this section we study an Anosov flow $g^t$ on a complete (non-compact) Riemannian manifold $M$ of bounded sectional curvature. We prove the following theorems.

THEOREM 4. $\dim_{g^t}(B^+) = \dim_{g^t}(R^+)$.

THEOREM 5. *If either*
1) $\dim(M) = 3$ *or*
2) $g^t$ *is the geodesic flow on a manifold of constant negative curvature or*
3) $g^t$ *has a finite smooth invariant measure,*
*then* $\mathrm{HD}(B^+) = \mathrm{HD}(R^+)$.

THEOREM 6. *For any countable set $A$* $\dim_{g^t}(B^+ \bigcap I g^+(A)) = \dim_{g^t}(R^+)$.

COROLLARY 2. *Under assumptions of theorem* 5

$$\mathrm{HD}(B^+ \bigcap I g^+(A)) = \mathrm{HD}(R^+).$$

The proof of theorem 4 implies

COROLLARY 1. $\dim_{g^t}(B^+)$ *depends lower semicontinuously on* $g^t$ *in the* $C^2$ *metric.*

2). In the next three sections we carry out the proof of theorem 4. Recall the notation of paragraph 2.3.

We want to reduce our global problem to a local analysis. The first step towards this aim is the following statement.

PROPOSITION 2. *For every* $\delta > 0$ *there is a point* $a$ *such that for any its neighbourhood* $U(a)$ $\Delta(R^+(U(a))) \geq \Delta(R^+) - \delta$. (Such points will be called $\delta-$hospitable.)

PROOF: Take some $m_0 \in M$. Since $R^+ = \bigcup_{n=1}^{\infty} R^+(B(m_0, n))$ and $d_u(\bigcup_{n=1}^{\infty} A_n) = \sup_n d_u(A_n)$ there is an $n_0$ such that $\Delta(R^+(B(m_0, n_0))) \geq \Delta(R^+) - \delta$. By the same argument, for every $n$ there is a point $m_n \in B(m_0, n_0)$ such that $\Delta(R^+(B(m_n, \frac{1}{n}))) \geq \Delta(R^+) - \delta$. Any limit point of the sequence $\{m_n\}$ is $\delta-$hospitable. ■

3) In this section we construct an invariant set of large dimension admitting a simple symbolic description.

Take some $\delta-$hospitable point $a$. From now on we make the simplifying assumption that $a$ is not a periodic point. The case when this assertion does not hold requires few modifications of the proof which are explained at the end of paragraph 4.

Given an arbitrary small constant $\theta$ one can find $T = T(\theta)$ such that for all $t \geq T, m \in M$ $R(m, t) \geq \frac{1}{\theta}$. Knowing this $T$, by non-periodicity of $a$ we can find $\varepsilon$ so small that for points from $\Pi(a, \varepsilon, \varepsilon)$ the return time to $\Pi(a, \varepsilon, 2\varepsilon)$ exceeds $T + t(\Pi(a, \varepsilon, 2\varepsilon))$. For the sake of brevity we denote $\Pi(a, \varepsilon, 2\varepsilon)$ by $\Pi$ and $\Pi(a, \varepsilon, \varepsilon)$ by $\tilde{\Pi}$.

Now we are going to consider some sets not satisfying the condition (2.1). For such a set $X$ the notation $\Delta(X)$ means $\Delta(X \bigcap W_{\Pi}^{(u)}(a))$.

Define the collection $\tilde{\Phi}$ of maps $\{\tilde{\phi}_i\}$ of $\tilde{\Pi}$ to itself as follows. If $a_i \in W_{\varepsilon}^{(u)}(a)$ and $t_i > 0$ are such that $g^{t_i} a_i \in \Pi(a, \frac{\varepsilon}{2}, \varepsilon)$ and, moreover, $t_i$ is the time of the first return of $a_i$ to $\Pi(a, \frac{\varepsilon}{2}, \varepsilon)$, then we add the map $\tilde{\phi}_i = \sigma_{a_i, t_i, \tilde{\Pi}, \tilde{\Pi}}$ to $\tilde{\Phi}$. Here we impose the restriction of returning to the smaller parallelogram $\Pi(a, \frac{\varepsilon}{2}, \varepsilon)$ to guarantee that if $m \in D(\phi_i)$ then $W_{\tilde{\Pi}}^{(s)}(m) \subset D(\phi_i)$. It is easy to see that $\omega_f(\tilde{\Phi}) \subset B^+$.

We want to estimate $\Delta(\omega_f(\tilde{\Phi}))$ from below in terms of $\{\bar{C}(\tilde{\phi}_i)\}$. One has a chance to do this because the worst that can happen is that $R_{\tilde{\phi}_i} \equiv \bar{C}(\tilde{\phi}_i)$, because then $D(\tilde{\phi}_i)$, $D(\tilde{\phi}_j \circ \tilde{\phi}_i)$ and so on have the least possible volume, so it is easy to find an economic covering of $\bigcup_{(i_1 i_2 \dots i_n)} D(\tilde{\phi}_{i_n} \circ \dots \tilde{\phi}_{i_2} \circ \tilde{\phi}_{i_1})$ and hence a covering of $\omega_f(\tilde{\Phi})$.

Unfortunately, there are two obstacles to carrying out such an estimate.

1) It might happen that $E(\tilde{U}_i) \neq W_\varepsilon^{(su)}(a)$ (because $a_i$ is very close to $\partial W_\varepsilon^{(su)}(a)$). Then $\mathrm{Vol}(\mathrm{D}(\tilde{\phi}_\mathrm{i}))$ could be small even though $\bar{C}(\tilde{\phi})$ were small.

2) $D(\tilde{U}_i)$ could intersect $D(\tilde{U}_j)$ for $i \neq j$. Then one covers $D(\tilde{U}_i) \bigcup D(\tilde{U}_j)$ more economically than if they were disjoint.

To overcome the first difficulty put $\phi_i = \sigma_{a_i, t_i, \Pi, \Pi}$. Now $E(U_i) = W_{2\varepsilon}^{(u)}(a)$ if $\theta$ is small since the $a_i$ are far enough from the boundary of $\Pi$.

To deal with the second difficulty consider $\Phi_n = \{\phi_i : \ nT \leq t(\phi_i) \leq (n+1)T\}$.

4) In this section we prove that, for some $n$, $\Delta(\omega(\Phi_n))$ is close to $\Delta(\omega(\tilde{\Phi}))$ which completes the proof of theorem 2 since $\omega(\Phi_n) \subset B^+$.

Since $D(\phi_i) \bigcap D(\phi_j) = \emptyset$ for $\phi_i \neq \phi_j \in \Phi_n$ we are able to establish the following result. Denote $r_n = r(\{\frac{1}{\bar{C}(\phi_i)}\}_{\phi_i \in \Phi_n})$.

Lemma 1. $\Delta(\omega(\Phi_n)) \geq r_n$.

This is a standard result from the theory of self-similar sets. The proof is given in paragraph 5.

To estimate $r_n$ from below we need the following statement. Denote $r_0 = r(\{\frac{1}{\underline{C}(\tilde{\phi}_i)}\})$.

Lemma 2. $\Delta(\omega(\tilde{\Phi})) \leq r_0$.

This proposition is proven in paragraph 6.

Hence $r_0 \geq \Delta(R^+) - \delta$ (we have used what $\omega(\tilde{\Phi}) \supset R^+(\Pi(a, \frac{\varepsilon}{4}, \varepsilon)) \bigcap \Pi(a, \frac{\varepsilon}{4}, \varepsilon)$ and the equalities $d_u(\bigcup_{n=1}^{\infty} A_n) = \sup_n d_u(A_n)$ and $d_u(g^t \bar{A}) = d_u(\bar{A})$).

It remains to compare $r_n$ and $r_0$. We claim that, for $\theta$ small enough, there is some $n$ such that $r_n \geq \Delta(R^+) - 3\delta$. Indeed, let us assume that the contrary is true. Note that if $\phi_i \in \Phi_n$ then $\underline{C}(\tilde{\phi}_i) \geq (\frac{1}{\theta})^n$. By lemma 2.2 there exists $C_1 = C_1(\varepsilon, \delta)$ such that for $\phi_i \in \Phi_n$

$$(\frac{1}{\underline{C}(\tilde{\phi}_i)})^{\Delta(R^+) - 2\delta} \leq C_1 \theta^{n\delta} (\frac{1}{\bar{C}(\phi_i)})^{\Delta(R^+) - 3\delta}.$$

Summation over $\phi_i \in \Phi_n$ gives

$$\sum_{\phi_i \in \Phi_n} (\frac{1}{\underline{C}(\tilde{\phi}_i)})^{\Delta(R^+)-2\delta} \leq C_1 \theta^{n\delta}.$$

Sumating over $n$ we obtain

$$\sum_i (\frac{1}{\underline{C}(\tilde{\phi}_i)})^{\Delta(R^+)-2\delta} \leq \frac{C_1 \theta^\delta}{1-\theta^\delta}, \tag{1}$$

which contradicts lemma 2 if $\frac{C_1\theta^\delta}{1-\theta^\delta} < 1$.

Since $\delta$ can be arbitrarily small the theorem is proven.

The case in which there exists $T_0$ such that $g^{T_0}a = a$ requires few changes. In this case we can achieve $t(\phi_i) \geq T + t(\Pi)$ for all $\phi_i$ except $\phi_1 = \sigma_{a,T_0,\Pi,\Pi}$. Proceeding along the same line as in the proof of (1), we obtain

$$\sum_i (\frac{1}{\underline{C}(\tilde{\phi}_i)})^{\Delta(R^+)-2\delta} \leq (\frac{1}{\underline{C}(\tilde{\phi}_1)})^{\Delta(R^+)-2\delta} + \frac{C_1\theta^\delta}{1-\theta^\delta},$$

which also contradicts lemma 2 for $\theta$ small enough. ■

Remark. Since $r_n$ depends continuously on $g^t$ corollary 1 is proven. The meaning of this result is that in the general case the set of non-recurrent orbits has non-empty interior, so one can enlarge $\dim_{g^t}(R^+)$ and hence $\dim_{g^t}(B^+)$ by making them recurrent with the help of a small change of the flow in this interior. However, the set $B^+(B(m,n))$ is much more persistent to such perturbations (see also [D2]).

5) In the next two sections we discuss the theory of self-similar sets. The estimates presented here for the dimension with respect to a dynamical system are completely analogous to those for the Hausdorff dimension (see [F]). Moreover the former can be derived from the latter if one notes that $\Delta(\bar{A})$ is the Hausdorff dimension with respect to the metric on $W^{(su)}(m_0)$

$$\rho(m_1, m_2) = \exp -(\min\{\tau > 0 : d(g_s^\tau(m_1), g_s^\tau(m_2)) \geq \varepsilon_0\}),$$

where the distance $d$ is measured by means of the induced Riemannian metric on $W^{(su)}(m_0)$ and $g_s^\tau$ denotes the synchronised flow obtained from $g^t$ through the change of time

$$\frac{d\tau}{dt}(m) = \frac{d}{dt}|_{t=0} Q^{(u)}(t,m) = \frac{d}{dt}|_{t=0} R^{(u)}(t,m).$$

(For synchronized flows see [Pr]; the metric $\rho$ is discussed in [H], [Hs].)

We present the proof in this paper because we shall use the arguments given here in the succeeding paragraphs.

PROOF OF LEMMA 1. Let us numerate the elements of $\Phi_n : \phi_1, \phi_2 \dots \phi_{l_n}$. Throughout the proof we denote for the sake of brevity $C(i) = \bar{C}(\phi_i)$, $T_n = \max_{i=1\dots l_n}(t(\phi_i) + t(\Pi))$, $D_{i_1\dots i_k} = D(\phi_{i_k} \circ \dots \circ \phi_{i_1}) \bigcap W_{\Pi}^{(u)}(a)$. Consider the measure $\mu_n$ on $\omega(\Phi_n) \bigcap W_{\Pi}^{(u)}(a)$ defined by $\mu(D_{i_1\dots i_k}) = \prod_{j=1}^{k}(\frac{1}{C(i_j)})^{r_n}$.

LEMMA 3. *There exists $C_2 = C_2(n)$ such that for every $U_0 = g^{-t_0}W_{\varepsilon_0}^{(su)}(a_0)$ we have $\mu_n(U_0) \leq C_2 \mathrm{Vol}^{\mathrm{r_n}}(\mathrm{U_0})$.*

PROOF: Note that for $D_{i_1\dots i_k}$ one has $\mu_n(D_{i_1\dots i_k}) \leq C_3 \mathrm{Vol}^{\mathrm{r_n}}(\mathrm{D_{i_1\dots i_k}})$ by the definition of $C(i)$. Set $I$ $I = \{(i_1 \dots i_k) : t_0 \leq t(\phi_{i_k} \circ \dots \circ \phi_{i_1}) \leq t_0 + T_n\}$. Then $\{D_{i_1\dots i_k}\}_{(i_1\dots i_k)\in I}$ forms a covering of $\omega(\Phi_n)$ and each point belongs to at most $\frac{T_n+t(\Pi)}{T}$ elements of the covering. There are two constants $C_4(n)$ and $C_5(n)$ such that for every $(i_1 \dots i_k) \in I$ $C_4 \leq \mathrm{Vol}(\mathrm{g_0^t D_{i_1\dots i_k}}) \leq \mathrm{C_5}$, since $g^t D_{i_1\dots i_k} = W_{\Pi}^{(u)}(m)$ for some $t$ $t_0 \leq t \leq t_0 + T_n + t(\Pi)$. Set $J$ $J = \{(i_1 \dots i_k) \in I : D_{i_1\dots i_k} \bigcap U_0 \neq \emptyset\}$. Note that if $J \neq \emptyset$ then $a_0$ lies in some compact part of $M$ (which depends on $n$) and hence by volume comparison arguments one can find $C_6(n)$ such that $\mathrm{Card}(J) \leq C_6$. Lemma 2.1 gives, for $(i_1 \dots i_k) \in J$, $\frac{\mathrm{Vol(D_{i_1\dots i_k})}}{\mathrm{Vol(U_0)}} \leq C_7$ and so the constant $C_2 = \frac{C_3C_6}{C_7^{r_n}}$satisfies the condition of the lemma. ■

Let $\{U_i\}$ be a covering of $\omega(\Phi_n) \bigcap W_{\Pi}^{(u)}(a)$. Then

$$\sum_i Vol^{r_n}(U_i) \geq \sum_i \frac{1}{C_2}\mu(U_i) \geq \frac{1}{C_2}.$$

Hence, $dh^{r_n}(\omega(\Phi_n) \bigcap W_{\Pi}^{(u)}(a)) > 0$ and $\Delta(\omega(\Phi_n)) \geq r_n$. ■

6) PROOF OF LEMMA 2: Now we denote

$$\tilde{D}_{i_1\dots i_k} = D(\tilde{\phi}_{i_k} \circ \dots \circ \tilde{\phi}_{i_k}) \bigcap W_{\tilde{\Pi}}^{(u)}(a).$$

For fixed $k$ such sets form a covering of $\omega(\Phi) \bigcap W_{\tilde{\Pi}}^{(u)}(a)$. Given $\Delta r$ we have

$$\sum_{(i_1\dots i_k)} \mathrm{Vol}^{\mathrm{r_0+\Delta r}}(\mathrm{D_{i_1\dots i_k}}) \leq \sum_{(\mathrm{i_1\dots i_k})} \mathrm{C_8}(\prod_{\mathrm{j=1}}^{\mathrm{k}} \frac{1}{\underline{\mathrm{C}}(\tilde{\phi}_{\mathrm{i_j}})})^{\mathrm{r_0+\Delta r}} \leq \mathrm{C_8}\theta^{\mathrm{k\Delta r}}.$$

Hence $\Delta(\omega(\tilde{\Phi})) \leq r_0 + \Delta r$. Since $\Delta r$ can be arbitrary small the lemma is proven. ■

7) PROOF OF THEOREM 5.1)&2) : Under these assumptions $\Pi$ is a smooth submanifold and coordinates $(u, s)$ are smooth on it (see [HP]). So are the coordinates $(u, s, t)$ on $g^{[-\varepsilon,\varepsilon]}\Pi$ introduced in Section 2. In these coordinates $B^+$ and $R^+$ have the local product structure $B^+ = B_u \times W_{2\varepsilon}^{(ss)}(a) \times [-\varepsilon, \varepsilon]$ and $R^+ = R_u \times W_{2\varepsilon}^{(ss)}(a) \times [-\varepsilon, \varepsilon]$. Hence, $\mathrm{HD}(B) = \mathrm{HD}(B_u) + N_s + 1$ and $\mathrm{HD}(R) = \mathrm{HD}(R_u) + N_s + 1$. The statement follows from the fact that under our assumptions

$$\mathrm{HD}(B_u) = \mathrm{HD}(B^+ \bigcap W_\Pi^{(u)}(a)) = d_u(B^+ \bigcap W_\Pi^{(u)}(a)) =$$

$$= d_u(R^+ \bigcap W_\Pi^{(u)}(a)) = \mathrm{HD}(R^+ \bigcap W_\Pi^{(u)}(a)) = \mathrm{HD}(R_u). ■$$

8) To prove theorem 5 for flows with a smooth invariant measure we need the following

LEMMA 4. *Suppose $V$ with $\dim V = N_u$ lies in a small neighbourhood of $a$ and is transversal to the leaves of the stable foliation. Then $\mathrm{HD}(V \bigcap B^+) = N_u$. (Here we assume that $a$ is $\delta-$hospitable for any $\delta$. Such points exist by ergodicity).*

PROOF: Under the conditions of the lemma we can choose $\Pi$ so small that the projection $\pi$ of $V$ along the leaves of the stable foliation are absolutely continuous and if $l$ denotes the Lebesgue measure on $V$ and $\hat{l}$ is the image by $\pi_*$ of Vol on $W^{(su)}(a)$ then

$$\frac{1}{C_9}\hat{l} \leq l \leq C_9\hat{l}.$$

Take a ball $U$ of radius $\rho$ on $V$ and let $\tilde{U}$ be the ball with the same center and radius $2\rho$. Since $V$ lies in a bounded part of $M$ there exists a constant $C_{10}$ such that

$$\frac{1}{C_{10}}\rho^{N_u} < \mathrm{Vol}(\tilde{U}) < C_{10}\rho^{N_u}.$$

As was proven in paragraph 5, if $\Pi$ is small enough there exists $n$ such that $\Delta(\Phi_n) > 1 - \delta$ and moreover $\mu_n(D_{i_1 i_2 \ldots i_k}) \leq C_2 \mathrm{Vol}^{1-\delta}(D_{i_1 i_2 \ldots i_k})$, where $D_{i_1 i_2 \ldots i_k} = D(\phi_{i_k} \circ \ldots \circ \phi_{i_2} \circ \phi_{i_1}) \bigcap W_\Pi^{(u)}(a)$. We want to project $\mu_n$ down to $V$ and prove the analogue of lemma 3. So, let $Y = \pi(U)$, $\tilde{Y} = \pi(\tilde{U})$. Since the strong unstable foliation is Hölder continuous there exist a constant $\beta_1$ such that $\mathrm{dist}(Y, \partial\tilde{Y}) > \rho^{\beta_1}$. Consider

$$I = \{(i_1 i_2 \dots i_k) : \mathrm{diam}(D_{i_1 \dots i_{k-1}}) > \rho^{\beta_1} \mathrm{but}\, \mathrm{diam}(D_{i_1 \dots i_{k-1} i_k}) \leq \rho^{\beta_1}\}.$$

Since $\Phi_n$ is finite there exist a constant $C_{11}$ such that $\mathrm{diam}(D_{i_1 \dots i_{k-1} i_k}) \geq C_{11}\,\mathrm{diam}(D_{i_1 \dots i_{k-1}})$. So for all $(i_1 \dots i_k) \in I$ we have $\mathrm{diam}(D_{i_1 \dots i_k}) > \frac{\rho^{\beta_1}}{C_{11}}$. By the uniformity condition for $g^t$ there exist constants $C_{12}$ and $\beta_2$ such that $\mathrm{Vol(D_{i_1 \dots i_k}) \geq C_{12}\, diam(D_{i_1 \dots i_k})^{\beta_2}}$.

Let $J = \{(i_1 \dots i_k) \in I : \; D_{i_1 \dots i_k} \bigcap Y \neq \emptyset\}$. So if $(i_1 \dots i_k) \in J$ then $D_{i_1 \dots i_k} \subset \tilde{Y}$. Further $\{D_{i_1 \dots i_k}\}_{(i_1 \dots i_k) \in J}$ form a covering of $\omega(\Phi_n) \bigcap Y$. of multiplicity bounded by some constant $C_{13} = C_{13}(n)$. Therefore we obtain

$$\mu_n(Y) \leq \sum_{(i_1 \dots i_k) \in J} \mu_n(D_{i_1 \dots i_k}) \leq C_2 \sum \mathrm{Vol}^{1-\delta}(\mathrm{D_{i_1 \dots i_k}}) \leq$$

$$\leq \frac{C_2 \sum \mathrm{Vol(D_{i_1 \dots i_k})}}{C_{12} \rho^{\beta_1 \beta_2 \delta}} \leq C_2 C_{13} (C_{12})^{-1} \mathrm{Vol}(\tilde{\mathrm{Y}}) \rho^{-\beta_1 \beta_2 \delta} \leq$$

$$\leq C_2 C_{13} (C_{12})^{-1} C_9 \mathrm{Vol}(\tilde{\mathrm{U}}) \rho^{-\beta_1 \beta_2 \delta} \leq \mathrm{C_{14}} \rho^{\mathrm{N} - \beta_1 \beta_2 \delta},$$

where $C_{14} = \frac{C_2 C_{13} C_9}{C_{11} 2^N}$. Hence the mass distribution principee gives

$$\mathrm{HD}(\mathrm{supp}\ \pi_* \mu_n) \geq N - \beta_1 \beta_2 \delta.$$

Since $\delta$ can be arbitrarily small and supp $\pi_* \mu_n \supset B^+$ the lemma is proven.■

PROOF OF THEOREM 5.3): Choose a smooth coordinate system $(\tilde{u}, \tilde{s}, \tilde{t})$ in a neighbourhood of $a$ so that the manifolds $V_{\tilde{s}_0, \tilde{t}_0} = \{\tilde{s} = \tilde{s}_0, \; \tilde{t} = \tilde{t}_0\}$ are transversal to the leaves of the stable foliation. Then by the previous lemma $\mathrm{HD}(V_{\tilde{s}_0, \; \tilde{t}_0} \bigcap B^+) = N_u$ which implies that $\mathrm{HD}(B^+) = \dim M$ (see, for example [F]). ■

9) Theorem 4 asserts that the set of points not-visiting some neighbourhood of infinity has a large dimension. The "finite" counterpart of this result is theorem 6 (see the introduction).

The proof of theorem 6 does not differ too much from that of theorem 4 but one should sharpen lemma 1 as follows.

LEMMA 5. $\Delta(\omega(\Phi_n) \bigcap I g^+(\{b_i\})) \geq r_n$.

This lemma can be proven by the arguments of Section 3 Another approach is given in [Sc]. After proving theorem 6 one can use the arguments of paragraphs $7-8$ to derive Corollary 2.

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