MULTIFRACTAL PROPERTIES OF THE SETS OF ZEROES OF BROWNIAN PATHS

Dmitry Dolgopyat[1)*] and Vadim Sidorov[2)].

[1)] Mathematics Department Princeton University Fine Hall-Washington Road Princeton, NJ 08540
e-mail: dolgopit@math.princeton.edu
[2)] Moscow State University

Fundamenta Mathematicae **147** (1995) 157–171

ABSTRACT

In this paper we study Brownian zeroes in the neighborhood of which one can observe non-typical growth rate of Brownian excursions. We interprete the multifractal curve for the Brownian zeroes calculated in [6)] as the Hausdorff dimension of certain sets.

Key words: independent random variables, Brownian motion, local time, Hausdorff dimension, self-similarity.

1. INTRODUCTION

1.1. Notations. In this article we deal with multifractal structure of zeroes of Brownian path. A Brownian path, denoted by $\omega(t)$, is a point of the space C[0,1] equipped with the Wiener measure denoted by P. Recall, that this measure is specified by the condition that for disjoint intervals $[t_1^1, t_2^1]$, $[t_1^2, t_2^2], ..., [t_1^n, t_2^n]$ the corresponding increments of Brownian curve $\omega(t_2^1) - \omega(t_1^1)$, $\omega(t_2^2) - \omega(t_1^2)$,...,$\omega(t_2^n) - \omega(t_1^n)$ are independent normal variables with mean values 0 and variances $t_2^k - t_1^k$.

The set of zeroes $\{t : \omega(t) = 0\}$ is denoted by Z[0,1]. It is random as soon as ω is random. It's also well-known that Z[0,1] is closed, nonwhere dense and its Hausdorff dimension h-dim(Z[0,1])=$\frac{1}{2}$ for a.e. ω. The purpose of this paper is to study the fine structure of Z[0,1]. Denote by Cm[0,1] the complement to Z[0,1]. It is an open set consisting of a countable set of intervals. Take $\varepsilon > 0$ and delete from [0,1] all intervals whose length is not less than ε. The connected components of remaining set will be called ε-clusters. We denote them by $K_i(\varepsilon)$ (counting from the left to the right). Sometimes it will be convenient to consider ε-clusters on the whole semiline, assuming that Wiener measure is considered on the space C[0,∞). We denote ε- cluster, containing $t \in$[0,1] by $K(\varepsilon, t)$. We use also the following notation:

$L(t)$ is the local time on Z[0,1] (the definition and basic properties of the local time see in [5)]); in the subsection 1.2 we discuss some properties of local time connected with fractal structure of Z[0,1];

$l_i(\varepsilon), l(\varepsilon, t)$ are the increments of the local time on $K_i(\varepsilon)$ and $K(\varepsilon, t)$ respectively;

$\delta_i(\varepsilon), \delta(\varepsilon, t)$ are the lengths of $K_i(\varepsilon)$ and $K(\varepsilon, t)$;

$\Delta_i(\varepsilon)$ is the distance between $K_i(\varepsilon)$ and $K_{i+1}(\varepsilon)$;

H^s is the s-dimensional Hausdorff measure; H^s_ε is the corresponding ε-measure ($H^s_\varepsilon(A) = \inf \sum_i |I_i|^s$, where infimum is taken over all coverings of the set A with diameter less than ε and $|\cdot|$ denotes the diameter);

ε_m=$(\frac{1}{2})^m$;

$A_m(\gamma) = \{t : \frac{\ln l(\varepsilon_m,t)}{\ln \delta(\varepsilon_m,t)} \geq \frac{1}{2} + \gamma\}$;

$B_m(\gamma) = \{t : \delta(\varepsilon_m, t) \leq \varepsilon_m^{1+\gamma}\}$;

$C_m(\gamma) = \{t : l(\varepsilon_m, t) \leq \varepsilon_m^{\frac{1}{2}+\gamma}\}$;

ν_m([a,b]) is the number of ε_m-clusters , intersecting the segment [a,b].

During the proofs we omit some indices if it does not lead to misunderstanding (for example we usually write $\delta(t)$ and A_m). All statements about Z[0,1] hold only for a subset of probability 1 even in cases we do not mention this explicitly.

1.2. Fractal geometry of Z[0,1] and local time. As it was already mentioned, h-dim(Z[0,1])=$\frac{1}{2}$. This fact follows from a more strong theorem. Denote $\phi(s) = \sqrt{s \ln \ln s}$ and put

$$\phi - m(A) = \lim_{\varepsilon \to 0} \inf \sum_i \phi(|I_i|),$$

* The work was done in Moscow State University

where the infimum is taken over all coverings of the set A by segments , whose length is less than ε .Then

$$\phi - m(Z[0,1]) = \text{const} \cdot L(1), \tag{1.1}$$

(See [9]).

There is another curious property of the local time. According to the Frostman's lemma (see [3]), for a given set A, for any s, $s < h-\dim(A)$ one can find a measure $\mu(s)$ and a constant $c(s)$ with the following property: for any x, y the measure $\mu([x,y]) < c(s)\,|x-y|^s$. For A=Z[0,1] we can describe this measure explicitly. Indeed, for sufficiently small ε the following inequality holds for the set of Wiener measure 1:

$$L(t+\varepsilon) - L(t) < \sqrt{3\varepsilon \ln\left(\frac{1}{\varepsilon}\right)} \tag{1.2}$$

(see [5]).

1.3. The main result. (1.2) implies that for all points of Z[0,1]

$$\lim_{\varepsilon\to 0}\inf \frac{\ln(L(t+\varepsilon)-L(t))}{\ln\varepsilon} \geq \frac{1}{2}.$$

This statement can be reformulated in the following way: for all points of Z[0,1]

$$\lim_{m\to\infty}\inf \frac{\ln l(\varepsilon_m, t)}{\ln \delta(\varepsilon_m, t)} \geq \frac{1}{2}.$$

The goal of this paper is to strengthen the last inequality.

Theorem 1. *For any $\gamma : 0 < \gamma < \frac{1}{4}$ for a.e. ω*

$$h - \dim\{t : \lim_{m\to\infty}\inf \frac{\ln l(\varepsilon_m, t)}{\ln \delta(\varepsilon_m, t)} \geq \frac{1}{2} + \gamma\} = 0,$$

$$h - \dim\{t : \lim_{m\to\infty}\sup \frac{\ln l(\varepsilon_m, t)}{\ln \delta(\varepsilon_m, t)} \geq \frac{1}{2} + \gamma\} = \frac{1}{2} - 2\gamma.$$

(*an equivalent form is following* :

$$h - \dim \lim_{m\to\infty}\inf A_m(\gamma) = 0$$

$$h - \dim \lim_{m\to\infty}\sup A_m(\gamma) = \frac{1}{2} - 2\gamma.$$

This theorem imply that $H^{\frac{1}{2}}\{t : \lim_{m\to\infty} \frac{\ln\ l(\varepsilon_m,t)}{\ln\delta(\varepsilon_m,t)} \neq \frac{1}{2}$ or the limit of this ratio fails to exist$\} = 0$, while in virtue of (1.1) $H^{\frac{1}{2}}$(Z[0,1])=$+\infty$.

1.4. The dimension of other singularities. The method used to prove theorem 1 is also applicable to the investigation of $\lim_{m\to\infty}\inf B_m$, $\lim_{m\to\infty}\inf C_m$, $\lim_{m\to\infty}\sup B_m$, $\lim_{m\to\infty}\sup C_m$, that is , respectively, $\{t : \lim_{m\to\infty}\inf \frac{\ln\delta(\varepsilon_m,t)}{\ln\varepsilon_m} \geq 1+\gamma\}$, $\{t : \lim_{m\to\infty}\inf \frac{\ln l(\varepsilon_m,t)}{\ln\varepsilon_m} \geq \frac{1}{2}+\gamma\}$, $\{t : \lim_{m\to\infty}\sup \frac{\ln\delta(\varepsilon_m,t)}{\ln\varepsilon_m} \geq 1+\gamma\}$, $\{t : \lim_{m\to\infty}\sup \frac{\ln l(\varepsilon_m,t)}{\ln\varepsilon_m} \geq \frac{1}{2}+\gamma\}$.

Roughly speaking, the structure of these sets is the following. X_m consists of about $(\frac{1}{\varepsilon_m})^\rho$ segments (any of A, B or C may be substituted instead of X), which are "almost equidistributed" on the segment [0,1], and most of these segments have the length of order of magnitude ε_m^θ . In this case $h-\dim(\liminf X_m) = 0$, $h-\dim(\limsup X_m) = \frac{\rho}{\theta}$.

Moreover, one can extract from the proof, that if we replace $\varepsilon_m = \left(\frac{1}{2}\right)^m$ by an arbitrary sequence $\widetilde{\varepsilon}_m$, then the following statements hold with the probability 1:

— if $\lim_{m\to\infty} \frac{\widetilde{\varepsilon}_{m-1}}{\widetilde{\varepsilon}_m} = +\infty$, then $h-\dim(\liminf X_m) = \frac{\rho}{\theta}$;

— if $\frac{\widetilde{\varepsilon}_{m-1}}{\widetilde{\varepsilon}_m}$ remains bounded then $h-\dim(\liminf X_m) = 0$; (of course, the exceptional sets of the measure 0, there any of the statements above doesn't hold may differ for different sequences).

The plan of our paper is the following. In the subsection 1.5 we explain our results using the notion of multifractality applied to the set Z[0,1] (equipped with $L(t)$). In section 2 we present some facts about the distribution of $l_i(\varepsilon)$, $\delta_i(\varepsilon)$, and $\Delta_i(\varepsilon)$. The proof of theorem 1 is contained in sections 3 and 4 . In section 3 we describe the set of Wiener measure 1 for which the statement of theorem 1 is true. In section 4 we give the proof of the main statement for this set. Essentially it doesn't differ too much from one in the case when X_m is the union of $(\frac{1}{\varepsilon_m})^\rho$ equidistributed segments of the length ε_m^θ . At last in the section 5 we calculate the above formulated dimensions for B_m and C_m. Since the proof in this case almost completely coincides with the proof of theorem 1, we restrict ourselves by the calculations of ρ and θ . The answer is following:

Proposition 1. *With probability* 1 *for* $0 < \gamma \le 1$

$$h - \dim\left(\liminf B_m\right) = 0,$$

$$h - \dim\left(\limsup B_m\right) = \frac{1}{2}\cdot\frac{1-\gamma}{1+\gamma}.$$

Proposition 2. *With probability* 1 *for* $0 < \gamma \le \frac{1}{2}$

$$h - \dim\left(\liminf C_m\right) = 0,$$

$$h - \dim\left(\limsup C_m\right) = \frac{1}{2}\cdot\frac{1-2\gamma}{1+2\gamma}.$$

1.5. Singular points and the multifractal structure of Z[0,1]. As one will see in the section 2, typical ε_m-clusters have the size of order ε_m and for majority of them $\frac{\ln l_i}{\ln \delta_i} \approx \frac{1}{2}$. At the same time there exist few ε_m-clusters for which $\alpha < \frac{\ln l_i}{\ln \delta_i} \le \alpha + \Delta\alpha$, where $\alpha \ne \frac{1}{2}$. For some α the share of such clusters varies polynomially with ε_m , i.e. it approximately equals ${\varepsilon_m}^{f(\alpha)}$$(f(\alpha) = \frac{3}{2} - 2\alpha$, where $\alpha \in [\frac{1}{2}, \frac{3}{4}]$ (cf [6])). In this case one says that $f(\alpha)$ lay in multifractal spectrum of Z[0,1] . The multifractal structure of Z[0,1] is studied in a different way in [6].

In our paper we interprete $f(\alpha)$ as the Hausdorff dimension of the $\limsup A_m$. It is quite clear why we use lim sup (instead of lim inf). Indeed, it reflects very complicated behavior of $r_t(\varepsilon) = \frac{\ln l(\varepsilon,t)}{\ln \delta(\varepsilon,t)}$ as the function of ε. Really, $r_t(\varepsilon)$ is piecewise constant, and it grows up in points, whose coordinates are equal to the length of the intervals from $Cm[0,1]$, laying close to t. The quite complicated structure of $Cm[0,1]$ as being compared with the complement to Cantor dust, for example, explains the chaotic behavior of $r_t(\varepsilon)$.

2. BASIC DISTRIBUTIONS, RELATED TO ε-CLUSTERS

Here we give some properties of the distributions, related to ε-clusters, which will be used in the following sections. The proofs can be found in [6].

Proposition 3.

a) *The triples* $(\Delta_i, \delta_i, l_i)$ *are independent and identically distributed.*

b) *the pair of random variables* (δ_i, l_i) *does not depend on* Δ_i *for any* i.

c) *introduce new random variables* ξ_i^-, ξ_i^+, *and* η_i, *where* $\delta_i(\varepsilon) = \varepsilon\xi_i^-$, $\Delta_i(\varepsilon) = \varepsilon\xi_i^+$, and $l_i(\varepsilon) = \sqrt{\frac{\pi\varepsilon}{2}}\eta_i$. *Then the distribution of* ξ_i^+ , ξ_i^- and η_i *does not depend on* ε *and*

d) η_i *has the exponential distribution with the mean value* 1, *i.e.* $F_{\eta_i}(x) = 1 - \exp\{-x\}$.

e) *The distribution function of* ξ_i^+ *is* $F_{\xi_i^+}(x) = 1 - \frac{1}{\sqrt{x}}$, $x > 1$.

f) ξ_i^- *has all positive moments and*

$$F_{\xi_i^-}(x) = \frac{1}{\sqrt{\pi}\sqrt{x}}(1 + O(1)),\ x \to 0.$$

g) $P(\xi_i^- > s^\beta, \eta < s^\gamma) \sim \mathrm{const}(\gamma,\beta)\cdot s^{2\gamma-\frac{\beta}{2}}$, $s \to 0$, $0 < \frac{\beta}{2} < \gamma < \frac{1}{2}$.

h) *Fix* γ , $0 < \gamma < 1$. *We shall say that an* ε_m*-cluster is poor of zeroes if it belongs to* $A_m(\gamma)$. *The probability of the event "*$K_i(\varepsilon)$ *is poor of zeroes" has the asymptotics* $\mathrm{const}\cdot\varepsilon^{2\gamma}$ as $\varepsilon \to 0$, *when* $\gamma \ne \frac{1}{2}$, $\gamma < 1$ (*we are interested only in a dense set of* γ).

i) *a cluster which is poor of zeroes and satisfies the inequality* $\frac{\varepsilon}{2} < \delta_i < \varepsilon$ *will be called standard* ε_m*-cluster. Then the probability of standard cluster has the same asymtotics as in the item* h), i.e. $\widetilde{\mathrm{const}}\cdot\varepsilon^{2\gamma}$.

3. THE DESCRIPTION OF THE SET OF THE FULL MEASURE,

3.1. Number of ε-clusters. Let us recall that $\nu_m([a,b])$ is the number of ε_m- clusters, intersecting the segment [a,b].

Lemma 1. *For any given δ and a.e. ω for almost all m, (i.e. all m except a finite set)*

$$\left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-\delta} < \nu_m([0,1]) < \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}+\delta}.$$

Proof.

$$\nu_m([0,1]) = \min\left\{n : \sum_{i=1}^{n} l_i(\varepsilon) \geq L(1)\right\} = \min\left\{n : \sum_{i=1}^{n} \eta_i \geq \sqrt{\frac{2}{\pi\varepsilon}} L(1)\right\}$$

We shall use Bernstein's inequality in the following form: let Z_i be independent and exponentially distributed random variables with mean value 1; then there are positive constants c_1, c_2, c_3, c_4, such that

$$\mathrm{P}\left(c_1 n < \sum_{i=1}^{n} Z_i < c_2 n\right) > 1 - c_3 \exp\{-c_4 n\}. \tag{3.1}$$

In particular,

$$\mathrm{P}\left(\sum_{i=1}^{(\frac{1}{\varepsilon_m})^{\frac{1}{2}-\delta}} \eta_i(\varepsilon) > c_2 \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-\delta}\right) < c_3 \exp\left\{-c_4 \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-\delta}\right\}. \tag{3.2}$$

and

$$\mathrm{P}\left(\sum_{i=1}^{(\frac{1}{\varepsilon_m})^{\frac{1}{2}+\delta}} \eta_i(\varepsilon) < c_1 \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}+\delta}\right) < c_3 \exp\left\{-c_4 \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}+\delta}\right\} \tag{3.3}$$

In virtue of Borel-Cantelli lemma, inequalities in (3.2) and (3.3) with probability 1 take place only a finite number of times.

Since

$$c_2 \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-\delta} < L(1)\sqrt{\frac{1}{\varepsilon_m}} < c_1 \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-\delta},$$

if m is large enough, the lemma is proven.

Lemma 2. *For any given δ with probability 1 for all m large enough the number of ε_m-clusters, poor of the zeroes, $\nu_m[0,1]$ is confined between $\left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-2\gamma-\delta}$ and $\left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-2\gamma+\delta}$.*

Proof. Let us prove, for example, the lower estimation. In virtue of lemma 1 it is sufficient to show that for all m, except for a finite number of them not less than $(\frac{1}{\varepsilon_m})^{\frac{1}{2}-2\gamma-\delta}$ ε_m-clusters among the first of $(\frac{1}{\varepsilon_m})^{\frac{1-\delta}{2}}$ of them are poor of zeroes. The probability of the complementary event is

$$\underline{\mathrm{P}}(m) = \sum_{k=0}^{(\frac{1}{\varepsilon_m})^{\frac{1}{2}-2\gamma-\delta}} b\left(\left(\frac{1}{\varepsilon_m}\right)^{\frac{1-\delta}{2}}, k, p_m\right),$$

where $b(n,k,p) = C_n^k p^k (1-p)^{n-k}$ and $p_m = P\{\varepsilon_m\text{-cluster with the given number is poor of zeroes }\} \sim \text{const}\cdot\varepsilon^{2\gamma}$ (see proposition 3.h). The inequality

$$\sum_{k=0}^{l} b(n,k,p) < b(n,l,p)\ \frac{np-k}{np-kp},$$

which is valid for $k < np$ in our case gives

$$\underline{\mathrm{P}}(m) < \frac{c(\frac{1}{\varepsilon_m})^{\frac{1}{2}-2\gamma-\frac{\gamma}{2}} - (\frac{1}{\varepsilon_m})^{\frac{1}{2}-2\gamma-\delta}}{c((\frac{1}{\varepsilon_m})^{\frac{1}{2}-2\gamma-\frac{\delta}{2}} - (\frac{1}{\varepsilon_m})^{\frac{1}{2}-4\gamma-\delta})}\ \text{const}\ b\left(\left(\frac{1}{\varepsilon_m}\right)^{\frac{1-\delta}{2}}, \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-2\gamma-\delta}, p_m\right). \tag{3.4}$$

It is easy to show (using Stirling's formula) that

$$\underline{\mathrm{P}}(m) < c_5(\delta)\left(\frac{1}{\varepsilon_m}\right)^{c_6(\delta)} \exp\left\{-c_7\left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-\frac{\delta}{2}-2\gamma}\right\}.$$

Hence,

$$\sum_{m=1}^{\infty} \underline{\mathrm{P}}(m) < +\infty.$$

In the same way, if we define $\overline{\mathrm{P}}(m)$ as

$$\overline{\mathrm{P}}(m) = \mathrm{P}\Big\{\text{not less than}\left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-2\gamma+\delta}\text{among the first }\left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}+\delta 2}\quad \varepsilon_m\text{−clusters are poor of zeroes}\Big\},$$

we obtain, using again the Stirling's formula and the inequality

$$\sum_{k=l}^{n} b(n,k,p) < b(n,l,p)\,\frac{k-np}{k-kp}$$

(which is valid for $k > np$), that

$$\overline{\mathrm{P}}(m) < c_8(\delta)\left(\frac{1}{\varepsilon_m}\right)^{c_9(\delta)} \exp\left\{-c_{10}\left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-\frac{\delta}{2}-2\gamma}\right\},$$

and therefore

$$\sum_{m=1}^{\infty} \underline{\mathrm{P}}(m) < +\infty.$$

Lemma 2 is proven.

Proposition 4. *The statement of lemma* 2 *holds for standard ε_m-clusters too.*

Proof. P{ certain ε_m−cluster is a standard one} has the same asymtotics as p_m.

3.2. The decay of $\varepsilon_{\mathbf{m}}$-clusters' size. Denote by r_m the maximal size of ε_m-clusters on the segment [0,1].

Lemma 3. *With probability* 1 $\lim_{m\to\infty}\sup \dfrac{\ln r_m}{\ln \varepsilon_m} \geq 1$ (*and, therefore,* $\lim_{m\to\infty}\sup \dfrac{\ln r_m}{\ln \varepsilon_m} = 1$).

Proof. Fix $\delta > 0$. It is sufficient to show that $r_m \leq \varepsilon^{1-\delta}$ for all m large enough .
In virtue of lemma 1 it is sufficient to prove that for all large enough m the inequality holds:

$$(\max_{j\leq(\frac{1}{\varepsilon_m})^{\frac{1}{2}+\delta}} \delta_j(\varepsilon_m)) \leq \varepsilon_m^{1-\delta} \qquad \textit{or}$$

$$(\max_{j\leq(\frac{1}{\varepsilon_m})^{\frac{1}{2}+\delta}} \xi_j^-) \leq \left(\frac{1}{\varepsilon_m}\right)^{\delta}.$$

$$\mathrm{P}\Big((\max_{j\leq(\frac{1}{\varepsilon_m})^{\frac{1}{2}+\delta}} \xi_j^-) \leq \left(\frac{1}{\varepsilon_m}\right)^{\delta}\Big) \leq \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}+\delta} \mathrm{P}\left(\xi_j^- \geq \left(\frac{1}{\varepsilon_m}\right)^{\delta}\right) \leq \varepsilon^{\frac{3}{2}-\delta}\mathrm{E}\Big(\left(\xi_j^-\right)^{\frac{2}{\delta}}\Big).$$

So,

$$\sum_{m=1}^{\infty} \mathrm{P}((\max_{j\leq(\frac{1}{\varepsilon_m})^{\frac{1}{2}+\delta}} \delta_j(\varepsilon_m)) \leq \varepsilon_m^{1-\delta}) < \infty$$

and lemma is proven.

Now we introduce two sequences of numbers:
$k_n = \left(\dfrac{\frac{1}{2}+\gamma+\delta}{\frac{1}{2}-2\gamma-\delta}\right)^{n}$ and $\varepsilon(n) = 2^{-2^{n^2}}$.

3.3. The equidistribution of ε-clusters.

Lemma 4. *Fix $\delta > 0$. Then a.e. the following statements hold:*

a) *fix natural n. Then for almost all m for any j $(1 \le j \le n)$ any poor of zeroes $(\varepsilon_m^{k_{j-1}})$-cluster contains not more than*
$\left(\frac{1}{\varepsilon_m}\right)^{\left[\frac{k_j-k_{j-1}}{2}-\gamma k_j-2\gamma k_{j-1}+\delta(k_j+k_{j-1})\right]}$ $\varepsilon_m^{k_j}$*-clusters, which are poor of zeroes;*

b) *for almost all n the number of standard $\varepsilon(n)$-clusters falling inside any standard $\varepsilon(n)$-cluster is contained between*
$2^{\left[\frac{2^{n^2}-2^{(n-1)^2}}{2}-\gamma 2^{(n-1)^2}-2\gamma 2^{n^2}-\delta 2^{n^2}\right]}$ *and* $2^{\left[\frac{2^{n^2}-2^{(n-1)^2}}{2}-\gamma 2^{(n-1)^2}-2\gamma 2^{n^2}+\delta 2^{n^2}\right]}$.

Proof. We shall prove only the statement a) (the statement b) is similar).

In virtue of lemma 3, if m is large enough, the increment of the local time on any poor of zeroes $\varepsilon_m^{k_{j-1}}$-cluster is less than $\left(\varepsilon_m^{k_{j-1}}\right)^{\frac{1}{2}+\gamma-\delta}$ (since, otherwise, $r_{mk_{j-1}} > \varepsilon_m^{k_{j-1}\frac{\frac{1}{2}+\gamma-\delta}{\frac{1}{2}+\gamma}}$) and the number of these clusters is less than $\left(\frac{1}{\varepsilon_m}\right)^{k_{j-1}(\frac{1}{2}-2\gamma+\delta)}$. Let $l(K)$ be the number of $\varepsilon_m^{k_j}$-clusters, the left boundary of which coincides with the left boundary of $\varepsilon_m^{k_{j-1}}$-cluster K. Then, the number $n(K)$ of the $\varepsilon_m^{k_j}$-clusters being inside K is less then

$$\min\left\{n : \sum_{j=l}^{n+l} l_i\left(\varepsilon_m^{k_j}\right) \ge \left(\varepsilon_m^{k_{j-1}}\right)^{\frac{1}{2}+\gamma-\delta}\right\} = \min\left\{\sum_{j=l}^{n+l}\eta_i \ge \left(\frac{1}{\varepsilon_m}\right)^{\left[\frac{k_j-k_{j-1}}{2}-\gamma k_{j-1}+\delta_{k_{j-1}}\right]}\right\}.$$

By the estimation (3.2)

$$\sum_{m=1}^{\infty}\left(\frac{1}{\varepsilon_m}\right)^{k_j(\frac{1}{2}-2\gamma+\delta)} \mathrm{P}\left\{n(K) > \left(\frac{1}{\varepsilon_m}\right)^{\left[\frac{k_j-k_{j-1}}{2}-\gamma k_{j-1}+\delta_{k_{j-1}}+\frac{\delta}{2}k_j\right]}\right\} < \infty\ .$$

So, for all m large enough

$n(K) < \left(\frac{1}{\varepsilon_m}\right)^{\left[\frac{k_j-k_{j-1}}{2}-\gamma k_{j-1}+\delta_{k_{j-1}}+\frac{\delta}{2}k_j\right]}$ and estimation (3.5) implies that

$$\sum_{m=1}^{\infty}\left(\frac{1}{\varepsilon_m}\right)^{k_j(\frac{1}{2}-2\gamma+\delta)} \mathrm{P}\Big\{\text{ the number of poor of zeroes } \varepsilon_m^{k_j}\text{-clusters among the } \varepsilon_m^{k_j}\text{-clusters with the numbers from}$$

$$l(K) \text{ to } l(K) + \left(\frac{1}{\varepsilon_m}\right)^{\left[\frac{k_j-k_{j-1}}{2}-\gamma k_{j-1}-\delta_{k_{j-1}}-\frac{\delta}{2}k_j\right]} \text{ is more than } \left(\frac{1}{\varepsilon_m}\right)^{\left[\frac{k_j-k_{j-1}}{2}-\gamma k_{j-1}-2\gamma k_j-\delta_{k_{j-1}}-\frac{\delta}{2}k_j\right]}\Big\} < \infty.$$

The lemma is proven.

Lemma 5. *For any positive δ there is a constant $c(\delta)$ such that for a.e. ω for almost all n any interval on the segment $[0,1]$ containing $c(\delta)$ of standard $\varepsilon(n)$-clusters contains not less than $\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}$ $\varepsilon(n)$-clusters.*

Proof. In virtue of lemma 2 it is sufficient to consider the case when the number of standard $\varepsilon(n)$-clusters does not exceed $\left(\frac{1}{\varepsilon(n)}\right)^{\frac{1}{2}-2\gamma+\delta}$.

Therefore,

$$\mathrm{P}\{\text{there exists a segment containing less than } \left(\frac{1}{\varepsilon(n)}\right)^{\frac{1}{2}-2\gamma+\delta} \varepsilon(n)\text{-clusters among which } c(\delta) \text{ are standard ones}\} \le$$

$$\le \mathrm{P}\{\text{the segment beginning from the given standard } \varepsilon\text{-cluster and containing } c(\delta) \text{ of them, does not cover}$$

$$\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta} \varepsilon(n)\text{-clusters}\} \times \left(\frac{1}{\varepsilon(n)}\right)^{\frac{1}{2}-2\gamma+\delta} \le$$

$$\le [\mathrm{P}\{\text{there are less than } \left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta} \varepsilon(n)\text{-clusters falling between two neighboring standard}$$

$$\text{ones}\}]^{c(\delta)}\left(\frac{1}{\varepsilon(n)}\right)^{\frac{1}{2}-2\gamma+\delta} \le \left(c_{11}\varepsilon^{2\gamma}(n)\right)^{c(\delta)-1}\left(\frac{1}{\varepsilon(n)}\right)^{\frac{1}{2}-2\gamma+\delta} \le c_{11}^{c(\delta)-1}(\varepsilon(n))^{2\gamma c(\delta)-\frac{1}{2}+\delta},$$

i.e. for example, $\frac{1}{\gamma}+1$ is the possible value for $c(\delta)$ and the lemma is proven.

Lemma 6. *Fix $\delta > 0$. A.e. for all n except for a finite number of them, any segment I on the t-axis, containing k standard $\varepsilon(n)$-clusters, has the length exceeding $\varepsilon(n)\left\{\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}\right\}^{2-\delta}$.*

Proof. In virtue of lemma 5 it is sufficient to give the proof in the case when the number of the $\varepsilon(n)$-clusters on the segment I is more than $\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}$.

Let us numerate standard $\varepsilon(n)$-clusters and denote by $p_{jk}(n)$ the probability of the event that the maximal distance between neighboring $\varepsilon(n)$-clusters on the segment, beginning from the j-th standard $\varepsilon(n)$-cluster and containing $\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}$ $\varepsilon(n)$-clusters, is less than $\varepsilon(n)\left\{\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}\right\}^{2-\delta}$.

It is sufficient to check the convergence of the series

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\left(\frac{1}{\varepsilon(n)}\right)^{\frac{1}{2}-2\gamma+\delta}}\sum_{k=1}^{\left(\frac{1}{\varepsilon(n)}\right)^{\frac{1}{2}-2\gamma+\delta}} p_{kj}(n) \le \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\left(\frac{1}{\varepsilon(n)}\right)^{\frac{1}{2}-2\gamma+\delta}} p_{1k}(n)\right)\left(\frac{1}{\varepsilon(n)}\right)^{\frac{1}{2}-2\gamma+\delta}.$$

$$p_{1k}(n) = \left[\mathrm{P}\left\{\Delta < \varepsilon(n)\left\{\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}\right\}^{2-\delta}\right\}\right]^{\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}} \sim$$

$$\sim\left(1-\frac{1}{\sqrt{\left\{\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}\right\}^{2-\delta}}}\right)^{\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}} < \exp\left\{-c_{12}\left(\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}\right)^{\frac{\delta}{2}-\frac{\delta}{4}}\right\} =$$

$$= \exp\left\{-c_{12}\left(\left[\frac{k}{c(\delta)}\right]\left(\frac{1}{\varepsilon(n)}\right)^{2\gamma-\delta}\right)^{\frac{\delta}{4}}\right\}.$$

The last inequality is valid when n is large enough (we have used the asymptotics $\ln p_{1k}(n) \sim \sim ([\frac{k}{c(\delta)}](\frac{1}{\varepsilon(n)})^{2\gamma-\delta})^{\frac{\delta}{4}}$).

QED.

4. GEOMETRICAL CONSIDERATIONS

In this part we consider those Brownian paths where the statements of lemmas 1-6 and proposition 4 are valid for all positive δ.

4.1. The dimension of the lower limit of $\mathbf{A_m}$.

Lemma 7. h-dim$(\liminf_{m\to\infty} A_m)$=0.

Proof. Fix n. Denote by $A(m)$ the set $\{$ t: for any $j: 0 \le j \le n$ $K(\varepsilon_m^{k_j}, t)$ is poor of zeroes $\}$. Denote by $N_m(n)$ the number of $\varepsilon_m^{k_n}$-clusters involved in $A(m)$.

$N(m)$ $\le$(the number of poor of zeroes ε_m-clusters)$\prod_{j=1}^{n}$(the maximal number of poor of zeroes $\varepsilon_m^{k_j}$-clusters inside $\varepsilon_m^{k_{j-1}}$-cluster)$\le \left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-2\gamma+\delta}\prod_{j=1}^{n}\left(\frac{1}{\varepsilon_m}\right)^{k_j(\frac{1}{2}-2\gamma+\delta)-k_{j-1}(\frac{1}{2}+\gamma-\delta)} = \left(\frac{1}{\varepsilon_m}\right)^{\varphi}$, where

$$\varphi = (\frac{1}{2}-2\gamma+\delta)\frac{\left(\frac{\frac{1}{2}+\gamma+\delta}{\frac{1}{2}-2\gamma-2\delta}\right)^{n+1}-1}{\left(\frac{\frac{1}{2}+\gamma+\delta}{\frac{1}{2}-2\gamma-2\delta}\right)-1} - (\frac{1}{2}+\gamma-\delta).$$

In virtue of lemma 3 $\quad H^s_{(\varepsilon_m^{k_n})^{1-\delta}}(A(m)) \le \varepsilon_m^{k_n(1-\delta)\delta-\varphi}$.

Since δ is arbitrary small, we obtain that h-dim$(\bigcap\limits_{k=m}^{\infty} A(k)) = 0$. But $\liminf A \subset \bigcup\limits_{m=1}^{\infty}\bigcap\limits_{k=m}^{\infty} A(k)$, and the lemma is proven.

4.2. The upper estimation of the upper limit dimension. To obtain the upper estimation of h-dim(lim supA_m) we need the following lemma.

Lemma 8. *Let sets X_n and sequence $\varepsilon_n \to 0$ are such that*

$$\sum_{n=1}^{\infty} H^s_{\varepsilon_n}(X_n) < +\infty,$$

then h-dim (lim sup $X_n) \le s$.

Proof.

$$H^s_{\varepsilon_n}(\lim\ \sup X_n) \le H^s_{\varepsilon_n}\left(\bigcup_{k=n}^{\infty} X_k\right) \le \sum_{k=n}^{\infty} H^s_{\varepsilon_k}(X_k) \to 0.$$

QED.

Corollary 1. h-dim (lim supA_m)$\le \frac{1}{2} - 2\gamma$.

Proof. For any positive s $\quad H^s_{\varepsilon_m}(A_m) \le \text{const}(s)\left(\frac{1}{\varepsilon_m}\right)^{\frac{1}{2}-2\gamma+\delta}(\varepsilon_m)^{(1-\delta)s}$, i.e. if $s > \frac{\frac{1}{2}-2\gamma+\delta}{1-\delta}$, then h-dim (lim sup $A_m) \le s$.

Since δ is arbitrary small, the proof is completed.

4.3. The lower estimation for the upper limit dimension. Let us now consider n, starting from which the statement of the item b) of lemma 4 is true. Take an arbitrary standard $\varepsilon(n)$-cluster. We'd like to introduce the probability measure μ on the set $\{\ t$: for any $k \ge n\ \ K(\varepsilon(k), t)$ is standard $\varepsilon(k)$-cluster and $K(\varepsilon(n), t) = K\ \}$ according to the following condition: all standard $\varepsilon(l+1)$-clusters falling inside the same $\varepsilon(l)$-cluster have the equal measure.

In virtue of lemma 4.b) for any standard $\varepsilon(l)$-cluster ($l > n$) it holds that

$$\mu(K_l) \le \sum_{k=n+1}^{l}\left(\frac{1}{\varepsilon(n)}\right)^{[\frac{1}{2}(2^{k^2}-2^{(k-1)^2})-\gamma 2^{(k-1)^2}-2\gamma 2^{k^2}-\delta 2^{k^2}]} \le \text{const}\cdot\left(\frac{1}{2}\right)^{2^{l^2}\left(\frac{1}{2}-2\gamma-2\delta\right)}.$$

Lemma 9. *Given interval I, then* $|I|^{\frac{1}{2}-2\gamma-2\delta} \ge \text{const}\,\mu(I)$.

Proof. Let j be the minimal natural number such that I covers the entire standard $\varepsilon(j)$-cluster with positive measure, and k is the number of $\varepsilon(j)$-clusters inside I. There are two alternatives:

a) $k < c(\delta)$. Then $\mu(I) < (c(\delta)+2)\cdot$ const $\left(\frac{1}{2}\right)^{2^{j^2}(\frac{1}{2}-2\gamma-2\delta)}$ ($c(\delta)+2$ takes account of more fine clusters as well) and $|I| > \frac{1}{2}\cdot\left(\frac{1}{2}\right)^{2^{j^2}}$.

b) $k > c(\delta)$. In this case the statement follows from lemma 6 and the estimation

$$\mu(I) \le (k+2)\,\text{const}\left(\frac{1}{2}\right)^{2^{j^2}(\frac{1}{2}-2\gamma-2\delta)}.$$

The lemma is proven.

Corollary 2. h-dim(lim sup$A_m) \ge \frac{1}{2} - 2\gamma$.

Proof. The implication (lemma 9) $\to$ (corollary 2) is well-known in the fractal geometry. We are presenting the proof here, because it is short enough.

Let I_j be the ε-cover of $(\bigcap\limits_{k=n}^{\infty} A_k)\bigcap K$, then

$$\sum_{j=1}^{\infty}|I_j|^{\frac{1}{2}-2\gamma-2\delta} \ge \text{const}\cdot\sum_{j=1}^{\infty}\mu(I_j) \ge \text{const}\cdot\mu\left(\left(\bigcap_{k=n}^{\infty} A_k\right)\bigcap K\right) = \text{const}\quad,$$

i.e. for any $\varepsilon > 0$ $\quad H_{\varepsilon}^{\frac{1}{2}-2\gamma-2\delta}((\bigcap\limits_{k=n}^{\infty} A_k)\bigcap K) \geq$ const or h-dim$((\bigcap\limits_{k=n}^{\infty} A_k)\bigcap K) \geq \frac{1}{2} - 2\gamma - 2\delta$ and the corollary is proven.

This completes the proof of theorem 1.

5. THE DIMENSION OF OTHER SETS OF SINGULAR POINTS OF THE BROWNIAN ZEROES

5.1. Small size clusters. The proof of $h - \dim(\liminf B_m) = 0$ is similar to the proof of lemma 7.

The probability of small size clusters (belonging to B_m) has the asymptotics const $\cdot\varepsilon^{\frac{\gamma}{2}}$. Hence, the number of those clusters has the order $\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2}-\frac{\gamma}{2}}$ in the sense of lemma 2 . The length of clusters is bounded by $\varepsilon^{1+\gamma}$, therefore

$$h - \dim(\limsup B_m) \leq \frac{1}{2}\cdot\frac{1-\gamma}{1+\gamma}$$

To prove the inverse inequality one should define the standard small cluster the one having the length confined between $\frac{1}{2}\varepsilon^{1+\gamma}$ and $\varepsilon^{1+\gamma}$ and act in proving in a similar way as we did with lemmas 4–6, 9 and corollary 2.

5.2. Small local time increment clusters. In the same way as in subsections 4.1 and 5.1 we get $h - \dim(\liminf C_m) = 0$.

To study the upper limit let us divide the segment $[0, 2\gamma]$ into subsegments of the length $\frac{1}{n}$. Denote $\beta_i^{(n)} = \frac{i}{n}$ and

$$C_m(i,n) = \{t :\ \varepsilon_m^{1+\beta_{i+1}^{(n)}} < \delta(\varepsilon_m, t) \leq \varepsilon_m^{1+\beta_i^{(n)}}\ ;\ l_i(\varepsilon_m, t) < \varepsilon^{\frac{1}{2}+\gamma}\}.$$

The probability of clusters from $C_m(i,n)$ has an asymptotics const $\cdot\varepsilon^{2\gamma-\frac{\beta_i^{(n)}}{2}}$.

Similarly to lemma 3, with probability 1 $\quad C_m \subset \left(\bigcup\limits_{i=1}^{[2(\frac{1}{2}-\gamma)n]+1} C_m(i,n)\right)$, if m is large enough . So

$$h - \dim(\limsup C_m) = \max_i (h - \dim(\limsup C_m(i,n)))$$

Similarly to subsections 4.2–3 and 5.1 we have the inequality

$$\frac{\frac{1}{2} - 2\gamma - \frac{\beta_i^{(n)}}{2}}{1 + \beta_{i+1}^{(n)}} \leq h - \dim(\limsup C_m(i,n)) \leq \frac{\frac{1}{2} - 2\gamma - \frac{\beta_i^{(n)}}{2}}{1 + \beta_i^{(n)}}. \tag{5.1}$$

Since n is arbitrary large (5.1) implies that

$$h - \dim(\limsup C_m) = \sup_{\beta\in[0,2\gamma]} \frac{\frac{1}{2} - 2\gamma - \frac{\beta}{2}}{1+\beta} = \frac{1}{2}\cdot\frac{1-2\gamma}{1+2\gamma}.$$

6. ACKNOWLEDGMENTS

This work have been fulfilled under the guidance of K. M. Khanin, G. M. Molchan, and Ya. G. Sinai. We thank all of them and D. V. Kosygin for useful discussions. We also are grateful to all participants of the Dynamical Systems Seminar of Moscow State University and Burgers Equation Seminar in the International Institution of Earthquake Prediction Theory & Math. Geophysics,where the multifractality of the Brownian zeroes was discussed many times.

REFERENCES

1) Evertsz K. "Laplacian fractals." Ph.D. thesises Yale University, 1989.

2) Falkoner K.J. "The geometry of fractal sets." Cambridge University Press, 1985.

3) Feller W. "An introduction to probability theory and its application.", v.2. John Wiley&Sons, 1970.

4) Ibragimov I.A., Linnik Yu.V. "Independent and stationary sequances of random variables.",Groningen, Wolters-Noordhoff, 1971.

5) Ito K., Mc Kean H. "Diffusion processes and there sample paths.", Springer, Berlin, 1965.

[6] Moltchan G.M. "Multi-mono-fractal properties of Brownian zeroes", Proc. Russian Academy of Science ("Doclady...") vol. 335, 6 (1994).

[7] Taylor S.J. "The α-dimensional measure on the graph and set of zeroes of a Brownian path.", Proc Cambridge Phil. Soc. v.51, pp.31-39 (1953).

[8] Taylor S.J. "The measure theory of random fractals", Proc Cambridge Phil. Soc. v.100, pp.383-406 (1986).

[9] Taylor S.J. and Wendel J.G. "The exact Hausdorff measure of the zero set of a stable process.", Z. Wahrscheinlichkeitctheoric v.6, pp.170-180 (1966)