LECTURES ON U-GIBBS STATES.

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1. SRB states and u-Gibbs sates.

An important problem in smooth ergodic theory is to understand an appearance of chaotic behavior in systems governed by deterministic laws. Now it is understood that chaotic behavior is caused by the exponential divergence of nearby trajectories. However hyperbolic systems usually have many invariant measures with quite different properties. Thus an important question is which measures should be studied. If the system preserves a smooth invariant measure then it is natural to investigate this measure first. In the dissipative setup when there are no smooth invariant measures it is natural to start with some smooth measure and look how it evolves in time. There are at least two approaches

(a) Take a smooth measure μ and consider weak limits of $\frac{1}{n}\sum_{j=0}^{n-1} f^j(\mu)$;

(b) (SRB states.) Consider Birkhoff averages $S_n(A)(x) = \frac{1}{n}\sum_{j=0}^{n-1} A(f^j x)$. Given an f invariant measure μ define *basin of* μ as follows

$$B(\mu) = \{x : \forall A \in C(M) S_n(A)(x) \to \mu(A) \quad \text{as} \quad n \to +\infty\}.$$

μ is called *SRB* measure if the Lebesgue measure of its basin is positive.

SRB states are named after Sinai, Ruelle and Bowen who proved that topologically transitive diffeomorphisms and flows have unique SRB state whose basin of attraction has total Lebesgue measure. This result gives us the first example of the situation when there are any invariant measures but only one describes the dynamics of Lebesgue measure. In general for partially hyperbolic systems either (1) or (2) impose certain restrictions on the class of invariant measures which can appear in the limit. To explain this let me recall some definitions and set the notation.

A diffeomorphism f of a smooth manifold M is called *partially hyperbolic* if there is an f invariant splitting

$$TM = E^u \oplus E^c \oplus E^s$$

and constants

$$\lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \lambda_5 \le \lambda_6 \quad \lambda_2 < 1, \quad \lambda_5 > 1$$

such that

$$\begin{aligned} \forall v \in E^s \quad & \lambda_1 ||v|| \leq ||df(v)|| \leq \lambda_2 ||v||, \\ \forall v \in E^c \quad & \lambda_3 ||v|| \leq ||df(v)|| \leq \lambda_4 ||v||, \\ \forall v \in E^u \quad & \lambda_5 ||v|| \leq ||df(v)|| \leq \lambda_6 ||v||. \end{aligned}$$

A standard reference for partially hyperbolic systems is [22]. We need the following facts:

– There are foliations W^u and W^s tangent to E^u and E^s respectively. These foliations can be characterized as follows. Take $\delta > 0$ then

$$W^s(x) = \{y : \frac{d(f^j x, f^j y)}{(\lambda_2 + \delta)^j} \to 0 \text{ as } j \to +\infty\}$$

$$W^s(x) = \{y : \frac{d(f^{-j} x, f^{-j} y)}{(1/\lambda_5 + \delta)^j} \to 0 \text{ as } j \to +\infty\} \tag{1}$$

– W^u and W^s are absolutely continuous. Let V_1 and V_2 be smooth manifolds with

$$\dim(V_1) = \dim(V_1) = \dim(E^c) + \dim(E^s)$$

transversal to E^u. Let $\pi : V_1 \to V_2$ be the holonomy map along the leaves of W^u then π is absolutely continuous and

$$\det(\pi)(x) = \prod_{j=0}^{\infty} \frac{\det(df^{-1}|T(f^{-j}V_1))(f^{-j}x)}{\det(df^{-1}|T(f^{-j}V_2))(f^{-j}\pi x)} \tag{2}$$

(That is $\forall A \subset V_1$ $\text{mes}(\pi(A)) = \int_A \det(\pi)(x) dx$.) The convergence of (2) follows from the fact that $f^{-j}x$ and $f^{-j}\pi x$ are exponentially close by (1). This implies also that π is Holder continuous.

Remark. π *is usually not Lipschitz.*

Sometimes it is more convenient to express this property differently. To this end let us introduce a collection $\mathcal{P}$ of subset of leaves of W^u. Fix constants K_1, K_2, K_3, γ_1. Let S be a subset of a leave of W^u. $S \in \mathcal{P}$ if it satisfies the following conditions:

- $\text{diam}(S) \leq K_1$
- $\text{mes}(S) \leq K_2$
- Let $\partial_\varepsilon S = \{y \in S \text{ such that } d(y, \partial S) \leq \varepsilon\}$ then

$$\text{mes}(\partial_\varepsilon S) \leq K_3 \varepsilon^{\gamma_1}.$$

Given K_4 and γ_2 let E_1 be the set of probability measures of the form

$$l(A) = \int_S A(x)\rho(x) dx$$

where $S \in \mathcal{P}$, $\ln \rho \in C^{\gamma_2}(S)$ and $||\ln \rho||_{\gamma_2} \leq K_4$. Let E_2 be convex hall of E_1 and E_3 be weak closure of E_2. Thus E_3 is the set of measures absolutely continuous with respect to W^u with nice conditional densities. Let $\mathcal{F}$ be a smooth foliation transversal to E^u. Let D be a topological disc in some leaf of $\mathcal{F}$ and let $S \in \mathcal{P}$. Let V be the local product of S and D.

Lemma 1. *The restriction of Lebesgue measure to V belongs to E_3 if K_j and γ_j are chosen appropriately.*

Proof. Decompose D into small cubes $D = \bigcup_j D_j$. Let $V_j = [D_j, S]$. Let S_j be the piece of W^u inside V passing through the center of D_j. If $A \in C(M)$ then

$$\int_{V_j} A(x)dx \approx \int_{S_j} A(x)\mathrm{Vol}(D_j(x))\frac{\mathrm{Vol}(TM)}{\mathrm{Vol}(TS_j)\mathrm{Vol}(TD_j)}(x)dx$$

where $D_j(x)$ is the piece of $\mathcal{F}$ inside V_j passing through x. But $\mathrm{Vol}(D_j(x)) \approx \mathrm{Vol}(D_j)\det(\pi_x)$ where π_x is the holonomy map $D_j \to D_j(x)$. Thus

$$\int_{V_j} A(x)dx \approx \mathrm{Vol}(D_j)\int_{S_j} A(x)\rho_j(x),$$

where

$$\rho_j(x) = \frac{\det(\pi)\mathrm{Vol}(TM)}{\mathrm{Vol}(TS_j)Vol(TD_j)}(x)dx$$

as claimed. $\square$

Let $\bar{E}$ be the set of measures obtained similar to E_3 but with restriction $||\ln \rho||_{\gamma_2} \leq K_4$ replaced by

$$||\rho||_{\gamma_2} \leq K_5.$$

Since any function can be represented as a difference of two functions each of each is less than say 10 it follows that $\forall K_5\ \exists K_4$ such that

$$\bar{E}(K_5) \subset E_3(K_4) - E_3(K_4).$$

Lemma 2. *If K_j, γ_j are chosen appropriately then Lebesgue measure belongs to $\bar{E}$.*

Proof. We cover M by a finite number of cylinders $M = \bigcup_j V_j$ where each V_j is as in Lemma 1. Take a partition of unity based on $\{V_j\}$ and argue as in Lemma 1. $\square$

Lemma 3. *For appropriate choice of constants the following holds. If $l \in E_3$ and μ is a limit point of $\frac{1}{n}\sum_{j=0}^{n-1} f^j(l)$ then $\mu \in E_3$.*

Proof. It is enough to show this for $l \in E_1$. Let

$$l(A) = \int_S A(x)\rho(x)dx.$$

We first show that $f^j S$ can be well approximated by the element of $\mathcal{P}$. Fix some r and let $Q = \{q_l\}$ be a maximal r-separated set in a leaf of W^u containing $f^j S$. Set

$$K_l = \{z : d(z, q_l) = \min_m d(z, q_m)\}.$$

We will call K_l's *Dirichlet cells.* We have

$$B(q_m, \frac{r}{2}) \subset K_l \subset B(q_l, r)$$

and ∂K_l consist of a finite number of smooth hypersurfaces $\{d(z, q_l) = d(z, q_m)\}$. Thus $K_l \in \mathcal{P}$. Let T_j be the union of cells lying strictly inside $f^j S$. Then $T_j \in f^j S$ and $f^j S - T_j \subset \partial_r(f^j S)$. Thus

$$\mathrm{mes}(S - f^{-j}T_j) \leq \mathrm{mes}(\partial_{r/\lambda_5^j} S) \leq K_3 \left(\frac{r}{\lambda_5^j}\right)^{\gamma_1}.$$

Let

$$l_l(A) = \frac{\int_{f^{-j}K_l} \rho(x)A(f^j x)dx}{\mathrm{mes}(f^{-j}K_l)}.$$

Then

$$l_l(A) = c_l \int_{f^{-j}K_l} \rho(f^{-j}y)\det(df^{-j}|E^u)(y)A(y)dy.$$

We need to obtain a uniform bound on the Holder norm of $\ln[\rho(f^{-j}y)\det(df^{-j}|E^u)(y)]$. Since f^{-j} is a strong contraction on W^u we obtain

$$\left|\ln\rho(f^{-j}y_1) - \ln\rho(f^{-j}y_2)\right| \leq K_4 d(f^{-j}y_1, f^{-j}y_2)^{\gamma_2} \leq \frac{K_4 d(y_1, y_2)^{\gamma_2}}{\lambda_5^{\gamma_2 j}}$$

and

$$\left|\ln\det(df^{-j}|E^u)(y_1) - \ln\det(df^{-j}|E^u)(y_2)\right| \leq$$

$$\sum_p \left|\ln\det(df^{-1}|E^u)(f^{-p}y_1) - \ln\det(df^{-1}|E^u)(f^{-p}y_2)\right| \leq$$

$$\sum_p \mathrm{Const}\frac{d(y_1, y_2)}{\lambda_5^p}.$$

Thus $f^j l = l_j^I + l_j^{II}$ where $l_j^I \in E^3$ and $||l_j^{II}|| \leq \mathrm{Const}\theta^j$ for some $\theta < 1$. This implies the desired result. $\square$

A slight modification of the proof shows that this result remains true if the initial measure belongs to $\bar{E}$. Thus we get

Corollary 1. *(a) Any limit point of $\frac{1}{n}\sum_{j=0}^{n-1} f^j$ (Lebesgue) belongs to E_3.*

(b) There is at least one invariant measure in E_3.

Definition. *Invariant measures in E_3 are called u-Gibbs states.*

Thus if we want to study the iterations of Lebesgue measures we have to deal with u-Gibbs states. Before we show that the same is true for SRB measures let us make a few remarks. Namely we note that for u-Gibbs states it is enough to consider very special densities instead of arbitrary Holder ones. Namely we saw in the proof of Lemma 3 that if ρ is a density which is the image of a smooth density ρ^* under f^j then

$$\frac{\rho(y_1)}{\rho(y_2)} = \frac{\rho^*(f^{-j}y_1)}{\rho^*(f^{-j}y_2)} \prod_{p=0}^{j-1} \frac{\det(df^{-1}|E^u)(f^p y_1)}{\det(df^{-1}|E^u)(f^p y_1)}$$

Thus as $j \to \infty$

$$\frac{\rho(y_1)}{\rho(y_2)} \to \prod_{p=0}^{\infty} \frac{\det(df^{-1}|E^u)(f^p y_1)}{\det(df^{-1}|E^u)(f^p y_1)}$$

Definition. *For $S \in \mathcal{P}$ canonical density ρ_{can} is defined by two conditions*

$$(I) \quad \frac{\rho(y_1)}{\rho(y_2)} = \prod_{p=0}^{\infty} \frac{\det(df^{-1}|E^u)(f^p y_1)}{\det(df^{-1}|E^u)(f^p y_1)}$$

and

$$(II) \quad \int_S \rho_{can}(y)dy = 1.$$

We call $\rho_{can}dy$ canonical volume form.

It follows that ρ_{can} is defined uniquely since if we now ρ_{can} at one point then we can find it at any other point using (I) and then (II) allows to compute the value at the reference point. Also ρ_{can} depends on S only via the normalization constant. Canonical density allows to identify u-Gibbs states in many examples.

Example. *$M = \mathbb{T}^d$, $Q \in SL_d(\mathbb{Z})$, $fx = Qx$ mod 1. Suppose that* $\mathrm{Sp}(Q)$ *is not contained in the unit circle. Let Γ_u be the sum of eigenspaces with eigenvalues larger than 1. Then the leaves of W^u are planes parallel to Γ_u and $(df|E^u)$ is multiplication by Q. Thus df transfers Lebesgue measure to its multiple and so canonical density with respect to Lebesgue measure is 1. Thus u-Gibbs states are measures invariant with respect to f and Γ_u considered as a subgroup of $\mathbb{T}^d$.*

Example. *$M = \mathrm{SL}_d(\mathbb{R})/\Gamma$ where Γ is a cocompact lattice in $\mathrm{SL}_d(\mathbb{R})$ and $fx = \mathrm{diag}(\lambda_1, \lambda_2 \dots \lambda_d)$ where $\{\lambda_j\}$ is a decreasing subsequance. Then the leaves of W^u are the orbits of the group N of upper triangular matrices and f acts on the leaves by conjugation. In particular f transfers the Haar measure on N to its multiple so the canonical density with respect to Haar measure is one. So again the u-Gibbs states are measures invariant with respect to N and f.*

Now we discuss the relation between u-Gibbs states and SRB states.

Proposition 1. *Let $A \in C^\gamma(M)$ and $I = \{\int A d\mu\}_{\mu u\text{-}Gibbs}$. Then $\forall \varepsilon > 0$ $\exists \delta > 0, C > 0$ such that $\forall l \in E_3$*

$$l(d(\frac{1}{n}S_n(A), I) \geq \varepsilon) \leq Ce^{-\delta n}.$$

Proof. We need to bound the probabilities of two events: $\frac{1}{n}S_n(A)$ is greater than the maximal average $+\varepsilon$ and it is less than the maximal average $-\varepsilon$. It suffices to estimate the probability of the first event the second one can be bounded similarly. So suppose that the integral of A with respect to any u-Gibbs state is less than $-\varepsilon$ and let us estimate the probability that $S_n(A) \geq 0$. Note first that there exists n_0 such that $\forall l \in E_3$ $\forall n \geq n_0$

$$l\left(\frac{1}{n}S_n(A)\right) \leq -\frac{\varepsilon}{2}.$$

(For if there existed sequences $\{l^{(j)}\}$, $\{n_j\}$ violating this inequality then taking a limit point of $\frac{1}{n_j}\sum_{p=0}^{n-1}(f^p l^{(j)})$ we would get a u-Gibbs state with a large average of A.) To simplify the notation let us assume that $n_0 = 1$. It first consider the measures of the form $l(A) = \int_K \rho_{can}(x)A(x)dx$, where K is a Dirichlet cell.

Lemma 4. *There exist constants K_6 and $\theta_1 < 1$ which may depend on A but not on K such that for any K there is a countable partition $K = \bigcup_j K_j$ and numbers n_j such that*

(a) $f^{n_j}K_j$ is a Dirichlet cell;

$$(b) \quad \mathrm{mes}(\bigcup_{n_j > N} K_j) \leq K_6\theta_1^N$$

$$(c) \quad \sum_j \mathrm{mes}(K_j) \sup_{K_j}[S_{n_j}(A) + \frac{\varepsilon n_j}{4}] \leq 0$$

The proof of the lemma is given in the appendix. Let us now deduce the proposition from it. Consider

$$\phi_K(\delta) = \sum_j \mathrm{mes}(K_j) \sup_{K_j} \exp\left[\delta S_{n_j}(A)\right]$$

It follows from Lemma 4 that $\exists \delta_0, C_1, C_2 > 0$ such that uniformly in K ϕ_K is analytic in $|\delta| < \delta_0$, $\phi_K(0) = 0$, $\phi_K'(0) \leq -C_1$ and $|\phi_K''(0)| \leq C_2$. Hence $\exists \bar{\delta}, \bar{\theta} < 1$ such that $\forall K$ $\phi_K(\bar{\delta}) < \bar{\theta}$. Now given $m > 0$ let us define inductively the partition $K = \bigcup_j K_{j,m}$ and numbers $n_{j,m}$ as follows. Let $K_{j,1} = K_j$ and if for some m $K_{j,m}$ are already defined apply Lemma 4 to obtain the partition $f^{n_{j,m}} K_{j,m} = \bigcup_l Q_l$ and numbers $n(Q_l)$ satisfying the conclusion of the lemma. Set $n_{j,m,l} = n(Q_l) + n_{j,m}$ and $K_{j,m,m} = f^{-n_{j,m,l}} K_{j.m,l}$ and reindex $\{K_{j,m,l}\}$ to obtain $\{K_{j,m+1}\}$. Let

$$\phi_{K,m} = \sum_j \mathrm{mes}(K_{j,m}) \exp\left[\delta \sup_{K_{j,m}} S_{n_{j,m}}\right].$$

We claim that

$$\phi_{K,m}(\bar{\delta}) \leq \bar{\theta}^m \tag{3}$$

Indeed, suppose that (3) is verified up to some m. Then

$$\phi_{K,m+1}(\bar{\delta}) = \sum_j \sum_l \mathrm{mes}(K_{j,m,l}) exp\left[\delta \sup_{K_{j,m,l}} S_{n_{j,m,l}}(A)\right] \leq$$

$$\sum_j \sum_l \mathrm{mes}(K_{j,m,l}) exp\left[\delta(\sup_{K_{j,m,}} S_{n_{j,m}}(A) + \sup_{f^{-n(Q_l)}Q_l} S_{n(Q_l)}(A)\right].$$

Summation over l for fixed j gives

$$\phi_{K,m+1} \leq \bar{\theta} \phi_{K,m}(\bar{\delta}) \leq \bar{\theta}^{m+1}$$

as claimed. Since (3) is defined in terms of supremum we get

$$\int_K \rho_{can}(x) \exp\left[\delta S_{n_m(x)}(A)(x)\right] dx \leq \bar{\theta}^m,$$

where $n_m(x) = n_{j,m}$ if $x \in K_{j,m}$. This implies

$$l(S_{n_m(x)}(A) \geq -m\bar{\varepsilon}) \leq (e^{\bar{\varepsilon}} \bar{\theta})^m.$$

Using similar argument for Laplace transform of $n(x)$ we get that there exists $C, \tilde{\theta} < 1$ such that

$$l(n_m(x) \geq Cm) \leq \tilde{\theta}^m.$$

From (b) we obtain

$$l(n_{m+1}(x) - n_m(x) \geq \epsilon) \leq K_6 \theta_1^{\epsilon m}.$$

Now let $m(n,x)$ be the largest number such that $n_m(x) \leq n$. If $S_n(A)(x) \geq 0$ then one of three events should happen.

Either (A) $m > \frac{n}{C}$ or

(B) $S_{n_{m(n,x)}}(A)(x) \geq -m\bar{\varepsilon}$ or

$$(C) \quad n_{m+1} - n_m \geq \frac{m\bar{\varepsilon}}{||A||_0}$$

but each of them has exponentially small probability. This proves the proposition for Dirichlet cells with canonical densities. If instead of canonical density we have a density ρ such that $c\rho_{can} \leq \rho \leq C\rho_{can}$ then the same result is true with larger constant so the conclusion is true for Dirichlet cells with arbitrary density satisfying $||\rho|| \leq K_4$. Now take arbitrary $S \in \mathcal{P}$, let $\tilde{n} = \varepsilon n$ and decompose $f^{\tilde{n}} S = (\bigcup K_l) \bigcup Z$ where K_l are Dirichlet cells and $\mathrm{mes}(f^{-\tilde{n}} Z) \leq \bar{\theta}_2^{\tilde{n}}$. Applying our result to each cell K_l we obtain the statement in full generality. □

Theorem 1. *(a) Any SRB measure is u-Gibbs.*

(b) If there is only one u-Gibbs state then it is SRB measure and its basin has total measure in M.

Thus to find SRB states we have to look among u-Gibbs states and there is a way to prove existence of SRB states.

Proof. Let μ be an SRB state. Let $\{A_j\}$ be a sequence of functions whose linear span is dense in $C(M)$. By proposition 1 $\forall m$ there exists u-Gibbs state ν_m such that $\nu_m(A_j) = \mu(A_j)$ for $j = 1 \dots m$. (Indeed $\{\mu(\vec{A}_j)\} \in \{\{\nu(\vec{A}_j)\}\}_{\nu-\text{u-Gibbs}}$ since otherwise there would exist $\{c_j\}$ such that $\mu(\sum_j c_j A_j) \notin \{\nu(\sum_j c_j A_j)\}_{\nu-\text{u-Gibbs}}$ which would contradict Proposition 1. Then $\nu_m \to \mu$ and so μ is a u-Gibbs state as claimed.

(b) By Proposition 1 $\frac{1}{n} S_n(A)(x) \to \mu(A)$ for Lebesgue almost all x. □

Exercise 1. *Deduce from Lemma 1 that if Ω is a set such that for all x $\mathrm{mes}(\Omega \bigcap W^u(x)) = 0$ then Ω has zero Lebesgue measure.*

Exercise 2. *Let S be a compact submanifold (with boundary) transversal to $E^c \oplus E^s$ and ρ be continuous probability density on S. Prove that any limit point of*

$$l_n(A) = \frac{1}{n} \sum_{j=0}^{n-1} \int_S A(f^j x) \rho(x) dx$$

is u-Gibbs.

Exercise 3. $M = \mathbb{T}^d$, $Q \in \mathrm{SL}_d(\mathbb{Z})$, $fx = Qx$ *mod 1.*

(a) Prove that f is ergodic iff $\mathrm{Sp}(Q)$ does not contain roots of unity.

(b) If f is ergodic show that it has unique u-Gibbs state (Lebesgue measure on $\mathbb{T}^d$.)

Hint. Let Γ be the sum of eigenspaces with eigenvalues larger than 1. Unique ergodicity of Γ is equivalent to projection of Γ to $\mathbb{T}^d$ being dense. Let $T = \bar{\Gamma}$, then T is a torus and f can be projected to $\tilde{f} : \mathbb{T}^d/T \to \mathbb{T}^d/T$. Now $\mathrm{Sp}(\tilde{f}) \subset S^1$ and so all eigenvalues of $\tilde{f}$ are roots of unity since overwise $\mathrm{Sp}(\tilde{f}^m)$ are different for different m but $\det(tf^m - \lambda)$ is an integer polynomial and since its roots are on the unit circle this polynomial can assume only finitely many different values.

Exercise 4. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism and G be a compact connected Lie group. Let $\tau : M \to G$ be a smooth function. Define $F : M \times G \to M \times G$ by $F(x, g) = (fx, \tau(x)g)$.*

(a) Prove that F is partially hyperbolic. Relate $W^u(F)$ to $W^u(f)$. What can be said about canonical densities?

(b) Prove that for any u-Gibbs state μ_f for f there is at least one u-Gibbs state μ_F for F which projects down to μ_f.

Exercise 5. *Give an example of a diffeomorphism having unique SRB measure but many u-Gibbs measures.*

Exercise 6. *Let $f_j \to f$ in C^2 and $\mu_j \to \mu$. If μ_j are u-Gibbs for f_j then μ is u-Gibbs for f.*

Exercise 7. *Let $A \in C^\gamma(M)$ and $I = \{\int A d\mu\}_{\mu u\text{-Gibbs}}$. Then $\forall \varepsilon\ \exists C, \delta$ and a neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}^2(M)$ such that $\forall f_j \in \mathcal{U}$ if $F_j = f_j \circ f_{j-1} \cdots \circ f_1$ then*

$$\mathrm{mes}\left(x : d(\frac{1}{n}\sum_{j=1}^{n} A(F_j x), I) \geq \varepsilon \right) \leq Ce^{-\delta n}.$$

References for Section 1: General information about partially hyperbolic systems could be found in [9, 22]. The first result about the absolute continuity of W^u was proven in [2]. u-Gibbs states are introduced in [35]. Further large deviation type bounds for partially hyperbolic systems can be found in [42]. Our proofs are motivated by [43].

Appendix A. Proof of Lemma 4.

We first show how to construct a partition satisfying (a) and (b) and slightly modify the construction to ensure (c). We follow [43]. Let n_0 be sufficiently large number. Set $F = f^{n_0}$. n_j will be multilples of n_0.

Let $\lambda=\lambda_5^{n_0}$ be the minimal expansion of F on W^u. Let

$$J_k=\{y: \frac{r_0}{\lambda^{k+1}}\leq d(y,\partial K)\leq \frac{r_0}{\lambda^k}\}.$$

Define $t_0(y)$ to equal k on J_k and 0 elsewhere. We proceed by induction. Let $D_n=\bigcup_{n_j\leq n_0 n}K_j$, $D_0=\emptyset$. We suppose that D_n is already defined and that there is a function $t_n: K-D_n\to\mathbb{N}$. Let $A_n=\{t_n=0\}$, $B_n=\{t_n\geq 0\}$. (The meaning of t_n is that we will not try to add a point to our partition for next t_n iterations.) Take $F^{n+1}A_n$ and let $\{Q_l\}$ be the partition of the leaf containing $F^{n+1}K$ into Dirichlet cells. Let $Q_1, Q_2\dots Q_l$ be the cells such that $Q_j\subset \mathrm{Int}(F^{n+1}A_n)$ and $d(Q_j,\partial F^{n+1}A_n)\geq r_0$. Add $F^{-n-1}Q_j$ to D_{n+1}. Set $t_{n+1}=k$ on

$$\{y\in A_n: \frac{r_0}{\lambda^{k+1}}\leq d(F^{n+1}y, F^{n+1}D_{n+1})\leq \frac{r_0}{\lambda^k}\}$$

and $t_{n+1}=0$ elsewhere on A_n. On B_n set $t_{n+1}=t_n-1$. Our goal is to prove (b). We first establish three estimates.

$$(I)\quad \frac{\mathrm{mes}(D_{n+1})}{\mathrm{mes}(A_n)}\geq c_1,$$

$$(II)\quad \frac{\mathrm{mes}(B_{n+1}\bigcap A_n)}{\mathrm{mes}(A_{n+1})}\leq c_2,$$

$$(III)\quad \frac{\mathrm{mes}(A_{n+1})}{\mathrm{mes}(B_n)}\geq c_3.$$

The proofs of all three are similar. To establish (I) note that if $y\in A_n-D_{n+1}$ then

$$d(F^n y, F^n B_n)\leq \frac{2r_0}{\lambda}.$$

Let z be a point such that

$$d(F^n y, F^n z)\leq \frac{2r_0}{\lambda}.$$

Let $m<n$ be the last time z was transfered from A_{m-1} to B_m. Then

$$d(F^m y, F^m z)\leq \frac{2r_0}{\lambda^{n-m}}.$$

Let

$$T=F^{n+1}(A_n-D_{n+1})\bigcap B(z,\frac{2r_0}{\lambda}),$$

$\tilde{T}=F^{m-n}T$ and $\tilde{U}$ be the union of geodesic segments passing through $\tilde{z}=F^{n-m}z$ such that the length of each segment is twice the length from $\tilde{z}$ to $\tilde{T}$ along the corresponding ray. Then $\frac{\mathrm{mes}(\tilde{U})}{\mathrm{mes}(\tilde{T})}\geq\tilde{c}_1$ and all points in $F^{n-m}(\tilde{U}-\tilde{T}) \subset F^{n+1}D_{n+1}$. Using bounded distortion properties along the orbit of F we obtain (I). (*II*) and (*III*) can be verified in a similar fashion.

(*II*) and (*III*) imply that $\mathrm{mes}(A_n)/\mathrm{mes}(B_n)$ is uniformly bounded from below (since if $\mathrm{mes}(A_n) \leq \delta \mathrm{mes}(B_n)$, then

$$\mathrm{mes}(A_{n+1}) \geq c_3 \mathrm{mes}(B_n) \geq c_3 \frac{\mathrm{mes}(A_n \bigcup B_n)}{1-\delta},$$

$$\mathrm{mes}(B_{n+1} \leq (1-c_3)\mathrm{mes}B_n \leq (1-c_3)(1+\delta)\mathrm{mes}(A_n \bigcup B_n),$$

so

$$\frac{\mathrm{mes}(A_{n+1})}{\mathrm{mes}(B_{n+1})} \geq \frac{c_3}{1-c_3}\frac{1+\delta}{1-\delta} \geq \delta$$

if δ is sufficiently small. Thus for all n either $\mathrm{mes}(A_n) \geq \delta \mathrm{mes}(B_n)$ or $\mathrm{mes}(A_{n+1}) \geq \delta \mathrm{mes}(B_{n+1})$. So the claim follows from (**I**).)

Let q be the constant such that $\forall n$ $\mathrm{mes}(A_n) \geq q\mathrm{mes}(B_n)$, then

$$\frac{\mathrm{mes}(D_{n+1})}{\mathrm{mes}(A_n \bigcup B_n)} \geq \frac{c_1}{1+1/q}$$

and so

$$\mathrm{mes}(A_{n+1} \bigcup B_{n+1}) \leq 1 - \frac{c_1}{1+1/q}\mathrm{mes}(A_n \bigcup B_n).$$

This proves (b). Thus we have constructed a partition satisfying (a) and (b). To ensure (c) we make the above construction but at the first step wait not n_0 but N iterations where $N \gg n_0$. Then for most of K $n_j = N$ and so

$$\int \rho(x)\left[S_{n_j(x)}(A)(x) + \frac{\varepsilon n_j(x)}{4}\right]dx \leq -\frac{N\varepsilon}{4} + \int_{n_j > N} \rho(x)\left[\sum_{p=N}^{n_j(x)-1}\left(A(f^p x) + \frac{\varepsilon}{4}\right)\right]dx$$

By (b) the second part is bounded uniformly in N so if N is sufficiently large

$$\int \rho(x)\left[S_{n_j(x)}(A)(x) + \frac{\varepsilon n_j(x)}{4}\right]dx \leq -\frac{N\varepsilon}{8}.$$

On the other hand the oscillations of $S_{n_j}(A)$ on $f^{n_j}K_j$ are of order 1 so replacing the integral by the supremum increases it by at most a constant amount. This completes the proof of Lemma 4. $\square$

2. Uniqueness.

2.1. **Coupling argument.** Let us now explain how to demonstrate the uniqueness of u-Gibbs state. Let us begin with the simplest example: $M = \mathbb{T}^2$ and f is a linear Anosov automorphism $fx = Qx \mod 1$ where $Q \in SL_2(\mathbb{Z})$, $\mathrm{Sp}(Q) \bigcap S^1 = \emptyset$. One way to examine this system is in term of Fourier analysis but we will explain a method which works in a more general setting. Take two measures $l_1, l_2 \in \bar{E}$. We want to show that $f^n l_1 - f^n l_2 \to 0$, then taking l_2 to be a u-Gibbs state μ we get $f^n l \to \mu$ as needed. Of course it is enough to consider the case when $l_j \in E_1$. Moreover we can suppose that

$$l_j(A) = \int_{\gamma_j} A(x)dx, \tag{4}$$

where γ_j are unstable curves of length 1. Indeed for any

$$l(A) = \int_\gamma \rho(x)A(x)dx$$

$$(f^n l)(A) = \frac{1}{\lambda^n} \int_{f^n\gamma} \rho(Q^{-n}y)A(y)dy$$

where λ is the largest eigenvalue of Q. Decomposing $Q^m\gamma = \bigcup_{j=1}^m \sigma_j$ where all σ_j except the last have length 1 and approximating $\rho \circ Q^{-n}$ by constants on each σ_j we approximate $f^n l$ by a convex combination of measures of type (4). So let γ_j satisfy (1). Lift γ_j to $\mathbb{R}^2$. There is an integer translate $\tilde{\gamma}_2$ of $Q^{\frac{n}{2}}\gamma_2$ such that the distance between the endpoints of $\tilde{\gamma}_2$ and $Q^{\frac{n}{2}}\gamma_1$ is less than 2. Thus we can cut the ends of $\tilde{\gamma}_2$ and $Q^{\frac{n}{2}}\gamma_1$ to obtain the curves $\bar{\gamma}_1$ and $\bar{\gamma}_2$ such that $\bar{\gamma}_1 \subset Q^{\frac{n}{2}}\gamma_1$, $\bar{\gamma}_2 \subset \tilde{\gamma}_2$ $\mathrm{length}(\tilde{\gamma}_2 - \bar{\gamma}_2) \leq 1$, $\mathrm{length}(Q^{\frac{n}{2}}\gamma_1 - \bar{\gamma}_1) \leq 1$, and $\bar{\gamma}_2$ is obtained from $\bar{\gamma}_1$ by projection π along the leaves of W^s and $d(x, \pi x) \leq 1$. Now if $A \in C^\gamma(M)$ then

$$f^n(l_1)(A) - f^n(l_2)(A) = \frac{1}{\lambda^{\frac{n}{2}}}\left[\int_{Q^{\frac{n}{2}}\bar{\gamma}_1} A(y)dy - \int_{Q^{\frac{n}{2}}\bar{\gamma}_2} A(y)dy + O(1)\right]$$

(the second term corresponds to $Q^{\frac{n}{2}}\gamma_1 - \bar{\gamma}_1$ and $\tilde{\gamma}_2 - \bar{\gamma}_2$.) Now

$$\int_{Q^{\frac{n}{2}}\bar{\gamma}_2} A(y)dy = \int_{Q^{\frac{n}{2}}\bar{\gamma}_1} A(\pi y)dy,$$

so

$$\frac{1}{\lambda^{\frac{n}{2}}}\left|\int_{Q^{\frac{n}{2}}\bar{\gamma}_1} A(y)dy - \int_{Q^{\frac{n}{2}}\bar{\gamma}_2} A(y)dy\right| =$$

$$\frac{1}{\lambda^{\frac{n}{2}}}\int_{Q^{\frac{n}{2}}\bar{\gamma}_1} |A(y) - A(\pi y)|dy \leq ||A|| d^\gamma(Q^{\frac{n}{2}}\bar{\gamma}_1, Q^{\frac{n}{2}}\bar{\gamma}_2) = ||A|| \frac{d^\gamma(\bar{\gamma}_1, \bar{\gamma}_2)}{\lambda^{\frac{n}{2}}}.$$

This show the uniqueness of u-Gibbs state for linear Anosov automorphism.

The same approach to divide $f^n S_1$ and $f^n S_2$ into parts so that the elements of $f^n S_1$ are close to elements of $f^n S_2$ work in a more general setting. Additional difficulty is that in a more general situation projection along the leaves of a complimentary foliation need not to be measure preserving but this could be overcomed by coupling 'thick' parts of $f^n S_1$ to several 'thin' parts of $f^n S_2$ and vice verse.

Let us give the precise statement. We consider partially hyperbolic diffeomorphisms $f : M \to M$ such that the central distribution is integrable and $df|_{W^c}$ is an isometry. We also assume that the non-wandering set of f is all of M.

Definition. *f is called* topologically transitive *if $\forall$ open U_1, U_2 $\exists n$ such that $f^n U_1 \bigcap U_2 \neq \emptyset$. f is called* topologically mixing *if $\forall$ open U_1, U_2 $\exists n_0$ such that $\forall n \geq n_0$ $f^n U_1 \bigcap U_2 \neq \emptyset$.*

Theorem 2. *Let f be as above. If f is topologically mixing then it has unique u-Gibbs state μ and $\forall l \in \bar{E}$*

$$f^n l \to \mu. \tag{5}$$

Corollary 2.

$$(a) \quad \forall A, B \in C^\gamma(M) \quad \int B(x) A(f^n x) d\mu(x) \to \mu(B)\mu(A).$$

$$(b) \quad \forall A, B \in C^\gamma(M) \quad \int B(x) A(f^n x) dx \to \int B(x) dx \mu(A).$$

Proof. Apply Theorem 2 to $l_1(A) = \mu(BA)$ and to $l_2(A) = \int B(x)A(x)dx$. □

Sketch of proof of Theorem 2. Again we want to show that $\forall l_1, l_2 \in \bar{E}$ $f^n l_1 - f^n l_2 \to 0$. The key step is the following lemma.

Lemma 5. *$\forall \varepsilon$ $\exists n_0, c$ such that $\forall l_1, l_2 \in \bar{E}$*

$$f^{n_0} l_j = c l_j^{I} + (1-c) l_j^{II},$$

where $\forall A \in C^\gamma(M)$ $\forall n$

$$\left| f^n(l_1^I)(A) - f^n(l_2^I)(A) \right| \leq \varepsilon ||A||_\gamma.$$

Theorem 2 is obtained by repeatedly applying Lemma 5 to l_j, l_j^{II} etc.

Proof of Lemma 5. We claim that topological mixing implies that $\forall$ open U $\forall S \in \mathcal{P}$ there exists n_0 such that $\forall n \geq n_0$

$$f^n S \bigcap U \neq \emptyset. \tag{6}$$

In fact $\exists y, r$ such that $B(y,r) \subset U$. Let $\tilde{S} = \bigcup_{x \in S} B_{cs}(x, \frac{r}{2})$. Since f is topologically mixing $\exists n_0$ such that $\forall n \geq n_0$ $f^n \tilde{S} \bigcap B(y, \frac{r}{2}) \neq \emptyset$. But then $f^n S \bigcap U \neq \emptyset$. It suffices to prove Lemma 5. Let

$$l_j(A) = \int_{S_j} \rho_j(x) A(x) dx.$$

But by (6) $\forall \hat{\varepsilon}$ $\exists n_0$ $\exists \bar{S}_1, \bar{S}_2$ such that $\bar{S}_j \subset S_j$ and $f^{n_0} \bar{S}_2$ is obtained from $f^{n_0} \bar{S}_1$ by the projection π_{cs} along the leave of W^{cs} and $d(x, \pi_{cs} x) \leq \hat{\varepsilon}$. Take δ sufficiently small and let

$$l_1^I(A) = \delta \int_{\bar{S}_1} \rho_1(x) A(x) dx,$$

$$l_2^I(A) = \delta \int_{\bar{S}_2} \rho_1(Px) A(x) \overset{-1}{\det}(Px) dx,$$

where P denotes $f^{-n_0} \circ \pi_{cs} f^{n_0}$. $\square$

Exercise 8. *Prove that if in Theorem 2 we assume that f is topologically transitive (rather than topologically mixing) then f has unique u-Gibbs state (but (5) is not necessarily satisfied).*

In case non-wandering set of f is M topological mixing follows follows from accessibility [9].

Exercise 9. *Suppose that W^c are orbit of a group G which acts on fibers by isometries and $gf = fg$. Given x define accessibility class of $\mathcal{A}(x) = \{y : \exists$ chain $x = z_0, z_1 \dots z_n = y\}$ such that $z_{j+1} \in W^u(z_j) \bigcup W^s(z_j)$. Let $\mathcal{A}_c(x) = \overline{\mathcal{A}(x)} \bigcap W^c(x)$.*

(a) Prove that $\mathcal{A}_c(x)$ is an orbit of a subgroup $\Gamma(x)$ of G.

(b) Show that f is topologically mixing iff $\mathcal{A}_c(x) = W^c(x)$.

Hint. Consider a function $\phi(y) = d(\mathcal{A}(y) \bigcap W^c(x), A_c(x))$.

Exercise 10. *Let $\mathcal{A}_0(x) = \{y : \exists$ chain $x = z_0, z_1 \dots z_n = y\}$ such that $z_{j+1} \in W^u(z_j) \bigcup W^s(z_j) \bigcup \mathrm{Orb}(x)$. Let $\mathcal{A}_{0c}(x) = \overline{\mathcal{A}_0(x)} \bigcap W^c(x)$. Prove that*

(a) $\mathcal{A}_{0c}(x)$ is an orbit of a subgroup $\Gamma_0(x)$ of G;

(b) $\Gamma(x)$ is normal in $\Gamma_0(x)$ and Γ_0/Γ is abelian;

(c) f is topologically transitive iff $\mathcal{A}_{0c}(x) = W^c(x)$.

References to Subsection 2.1. Our exposition follows [32, 8, 44].

2.2. **Rates of convergence.** Here we review what is know about the rates of convergence. We say that f *is strongly u-transitive with exponential rate* if $\forall\gamma$

$$|l(A\circ f^n)-\mu(A)|\leq \mathrm{Const}(\gamma)||A||_{C^\gamma(M)}\theta^n$$

for some $\theta(\gamma)<1$. We say that f *is strongly u-transitive with superpolinomial rate* if $\forall m\ \exists k(m)$ such that $\forall l\in \bar{E}\ \forall A\in C^k(M)$

$$|l(A\circ f^n)-\mu(A)|\leq \mathrm{Const}||A||_{C^k(M)}\frac{1}{n^m}.$$

(a) Anosov diffeomorphisms. These are defined by the condition that $E_c=0$. This is perhaps the most studied class of partially hyperbolic systems.

Proposition 2. *(see e.g* [7]*.) Topologically transitive Anosov diffeomorphisms are strongly u-transitive with exponential rate.*

(b) Time one maps of Anosov flows.

Proposition 3. *(*[12, 13]*) Suppose that f is a time one map of topologically transitive Anosov flow whose stable and unstable foliations are jointly non-integrable, then f is strongly u-transitive with superpolinomial rate. If in addition E_u and E_s are C^1 then f is strongly u-transitive with exponential rate.*

(c) Compact skew extensions of Anosov diffeomorphisms. Let $h:N\to N$ be topologically transitive Anosov diffeomorphism, K be a compact connected Lie group, $M=N\times G$ and $\tau:N\to G$ be a smooth map. Let $f(x,y)=(hx,\tau(x)y)$.

Proposition 4. *(*[14]*) Generic skew extension is strongly u-transitive with superpolynomial rate. In particular if G is semisimple then all ergodic extensions are strongly u-transitive with superpolynomial rate. Also, if N is an infranilmanifold then all stably ergodic with superpolinomial rate.*

(d) Quasihyperbolic toral automorphisms. Here $M=\mathbb{T}^d$ and $f(x)=Qx \pmod 1$ where $Q\in \mathrm{SL}_d(\mathbb{Z})$, $sp(Q)\not\subset S^1$.

Proposition 5. *(*[26]*) Quasi-hyperbolic toral automorphisms are strongly u-transitive with exponential rate.*

(e) Translations on homogeneous spaces. Let $M=G/\Gamma$ where G is a connected semisimle group without compact factors and Γ is an irreducible compact lattice in G. Let $f(x)=gx$, $g=\exp(X)$.

Proposition 6. *([27]). Suppose that there is a factor G' of G which is not locally isomorphic to $SO(n,1)$ or $SU(n,1)$ and such that the projection g' of g to G is not quasiunipotent (i.e. $sp(\mathrm{ad}(g')) \not\subset (S^1)$) then f is strongly u-transitive with exponential rate.*

Exercise 11. *Prove Proposition 2.*

Hint. Improve Lemma 5 and show that for Anosov diffeomorpisms $\exists c, n_0$ such that $\forall l_1, l_2 \in E_1$

$$l_j = c l_j^{I} + (1-c) l_j^{II}$$

where

$$|l_1^I(A \circ f^N) - l_2^I(A \circ f^N)| \leq \mathrm{Const}\theta^N ||A||_\gamma$$

and

$$l_j^{II} = \sum_k c_{jk} l_{jk}$$

where $f^{n_{jk}} l_{jk} \in E_1$ for some n_{jk} and

$$\sum_{n_{jk}>N} c_{jk} \leq \mathrm{Const}\theta^N.$$

(Use Lemma 4.) Use the arguments of Proposition 1 to complete the proof. (This proof is taken from [43].)

Exercise 12. *([26]) (a) Let $R \in SL_d(\mathbb{Z})$ be such that $\mathrm{Sp}(R)$ does not contain roots of unity. Let Γ_u be the sum of expanding eigenspaces of R and Γ_{cs} be the sum of complimentary eigenspaces. Let $\pi_* : \mathbb{R}^d \Gamma_*$ denote the corresponding projections. Prove that $\forall \lambda \in \mathbb{Z}^d$*

$$||\pi_u(\lambda)|| \geq \frac{\mathrm{Const}}{||\lambda||^d}.$$

Hint. Let $P(x) = x^k + \sum_j a_j x^j$ be the minimal polynomial of $R|_{V_{cs}}$. $\forall Q\ \exists r_1 \dots r_{k-1}$, and $q < Q^k$ such that $|\frac{r_j}{q} - a_j| \leq \frac{1}{qQ}$. Let $P_Q(x) = x^k + \sum_j \frac{r_j}{q} x^j$, then $||P_Q(R)\lambda|| \geq \frac{1}{Q}$. Let $v = \pi_{cs}\lambda$ then

$$P_Q(R)\lambda = P_Q(R)(\lambda - v) + P_Q(R)(v).$$

Take $Q \sim \mathrm{Const}||\lambda|| \dots$

(b) Use (a) to prove Proposition 5.

Exercise 13. *Next exercise taken from [31] usually does not give an optimal bounds but in case it applies its conclusions are sufficient for all the applications described below.*

Suppose that E^c is generated by the action φ_a of $\mathbb{R}^d$ such that $f \circ \varphi_a = \varphi_a \circ f$. Suppose that $W^u, W^s \in C^\infty$ and that $\exists$ vectorfields $X_1 \dots X_m \in$

E^u, $Y_1 \dots Y_n \in E^s$ such that $\{X_j\}, \{Y_j\}$ and $\{\nabla_{X_j} Y_k\}$ generate TM. Let f preserve smooth measure dx. Let $V(A)(x) = A(fx)$ and

$$U(A) = \int_{0 \le u_j \le 1} A(\psi_{X_u} x) du, \quad S(A) = \int_{0 \le s_j \le 1} A(\psi_{Y_s} x) ds$$

where ψ_Z denote the flow generated by Z. Let $C^{cs}(M)$ denote the space of functions which are continuous with Lipschitz restrictions to W^{cs} and $C^u(M)$ denote the space of functions which are continuous with Lipschitz restrictions to W^u Denote $||A||_\infty = \sup_M |A(x)|$,

$$||A||_s = \limsup_{s \to 0} \frac{|A(\psi_{Y(s)} x) - A(x)|}{|s|},$$

$$||A||_u = \limsup_{u \to 0} \frac{|A(\psi_{X(u)} x) - A(x)|}{|u|},$$

$$||A||_0 = \limsup_{a \to 0} \frac{|A(\varphi_a x) - A(x)|}{|a|}.$$

Let $\mathbf{P}_n = V^n US$. (a) Prove that

$$||\mathbf{P}_n^N A||_\infty \le ||A||_\infty,$$

$$||\mathbf{P}_n^N A||_0 \le ||A||_0 + \mathrm{Const} N ||A||_\infty,$$

$$||\mathbf{P}_n^N A||_s \le \theta^N \left(||A||_s + N ||A||_\infty + ||A||_0 \right)$$

for some $\theta < 1$.

(b) Prove that

$$\left| \int_M \mathbf{P}_n^N (A)(x) B(x) dx - \int (V^{nN} A)(x) B(x) dx \right|$$

$$\le \mathrm{Const} N^2 \left(\theta^N (||A||_s + ||A||_0 + ||A||_\infty) ||B||_\infty + ||A||_\infty ||B||_u \right)$$

(c) Prove that $\exists n_0, c_1, c_2$, such that $\forall n \ge n_0$ $\mathbf{P}_n^2 = c_1 I_n + (1 - c_1) J_n$ where I_n and J_n are Markov operators (i.e. $A \ge 0$ implies $I(A) \ge 0$, $J(A) \ge 0$ and $I(1) = J(1) = 1$) and I_n is an integral operator with kernel bounded from below by c_2.

(d) Prove that $\exists \tilde{\theta} < 1$ such that

$$\left| \int A(f^N x) B(x) dx - \int A(x) dx \int B(x) dx \right| \le \mathrm{Const} ||A||_1 ||B||_1 \theta^{\sqrt{N}}.$$

Hint. Apply the previous estimates to the identity

$$A(f^N x) B(x) dx = \int \tilde{A}(f^{N/3} x) \tilde{B}(x) dx$$

where $\tilde{A} = A \circ f^{N/3}$, $\tilde{B} = B \circ f^{N/3}$.

(e) Deduce from (d) that $\forall l \in \bar{E}$

$$\left| l(A \circ f^N) - \int A(x)dx \right| \leq \text{Const}||A||_1 \theta^{\sqrt{N}}.$$

2.3. **Singular foliations.** As we saw above a crucial property of W^u is its absolute continuity. Here we show that W^c need not be absolutely continuous. We follow [40] with modifications of [15]. Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a skew product over Anosov diffeo of $\mathbb{T}^2$. We assume that f has accessibility property. Let φ be a diffeomorphism close to id and let $F_n = f^n \varphi f^n$.

Proposition 7. *F_n is partially hyperbolic, $E^c(F_n)$ is integrable and leaves of $W^c(F_n)$ are circles.*

Proof. f is partially hyperbolic and $W^c(f)$ is C^1. Therefore by [22] there exists a neighborhood $\mathcal{U}(f)$ such that if $\{f_j\}$ is any sequence with $f_j \in \mathcal{U}$, then

$$\{f_m \circ \cdots \circ f_2 \circ f_1\}$$

is partially hyperbolic sequence and $E^c(\{f_j\})$ is integrable. But

$$F_n = f \circ \cdots \circ f \circ \varphi \circ f \cdots \circ f. \quad \square$$

Let $d\varphi$ be given in the frame $\{e_u, e_c, e_s\}$ by the matrix $Q(x)$.

Theorem 3. *Let $\lambda_c(n, \nu)$ denote the central Lyapunov exponent for F_n invariant measure μ. Let*

$$L(\mu) = \int [\ln(Q_{uu}Q_{cc} - Q_{uc}Q_{cu}) - \ln Q_{uu}] \, d\mu(x).$$

(a) If f and φ preserve a smooth measure m then

$$\lim_{n \to \infty} \lambda_c(n, m) = L(m)$$

(b) In general if ν_n is any u-Gibbs state for F_n then $\lambda_c(n, \nu_n)$ converges uniformly to $L(\mu)$ where μ is the u-Gibbs state for f.

Proof. (a) F_n has unstable vector of the form

$$v_u(x) = e_u(x) + z_u(x)$$

and center-unstable bivector v_{uc} of the form

$$v_{uc}(x) = e_u(x) \wedge e_c + z_{uc}(x).$$

Let $\lambda_u(x, k) = \ln ||df^k(e_u)||(x)$. Then

$$\ln ||F_n(v_u)|| = \ln \lambda_u(x, n) + \ln Q_u u(f^n x) + \ln \lambda_u(\varphi f^n x) + O(\theta^n)$$

and

$$\ln ||F_n(v_{uc})|| = \ln \lambda_u(x, n) + \ln(Q_{uu}Q_{cc} - Q_{uc}Q_{cu})(f^n x) + \ln \lambda_u(\varphi f^n x) + O(\theta^n)$$

for some $\theta < 1$. Hence

$$\lambda_c(n,m) = \int [\ln(Q_{uu}Q_{cc} - Q_{uc}Q_{cu}) - \ln Q_{uu}]\,(f^n x)d\mu(x) = L(m)$$

since f preserves m;

The proof of (b) is similar taking into account Theorem 2 and Exercise 2. □

Exercise 14. *Show that* $\exists(f,\varphi)$ *such that* $L(m,f,\varphi) \neq 0$.

Hint. Take some $x_0 \in \mathbb{T}^3$ and choose a coordinate system ξ_1, ξ_2, ξ_3 so that

$$E^s(x_0) = \frac{\partial}{\partial \xi_1} \quad E^c(x_0) = \frac{\partial}{\partial \xi_2} \quad E^u(x_0) = \frac{\partial}{\partial \xi_3}.$$

Let $\beta : \mathbb{R} \to \mathbb{R}$ be a function of compact support. Define

$$\varphi_{\varepsilon,\delta}(\xi) = (R_{\delta\beta(||\xi||^2/\varepsilon^2)}(\xi_1,\xi_2),\xi_3)$$

where R_β denotes a rotaion on angle β. Show that

$$L(m,\varphi_{\varepsilon,\delta}) \sim -\varepsilon^3\delta^2 \int\int\int \xi_1^2\xi_2^2(\beta'(||\xi||^2))^2 d\xi_1 d\xi_2 d\xi_3.$$

(See [40, 38]) for other proofs, all proofs proceed by using Taylor series for sine and cosine etc.)

Applying Proposition 1 we obtain

Corollary 3. *If* $L(\mu) \neq 0$ *then for almost all* x

$$\lim_{N\to\infty} \frac{1}{N} \ln ||dF_n^N|E^c||(x) \neq 0.$$

Combining this corollary with [1, 4] we obtain

Corollary 4. *If* $L(\mu) \neq 0$ *then for large* n F_n *has unique u-Gibbs states and its basin of attraction has total Lebesgue measure in* M.

Lemma 6. *If* f,φ *preserve a smooth measure* m *and* $L(\mu) \neq 0$ *then* $W^c(F_n)$ *is not absolutely continuous for large* n.

Proof. Without the loss of generality we can assume that $L(\mu) > 0$. Let

$$\Lambda = \{x : \lambda_c(x,F_n) > 0\}.$$

Then $m(\Lambda) = 1$ but for any leaf W of W^c $\mathrm{mes}(W \bigcap \Lambda) = 0$. □

References to Subsection 2.3 Note that the construction of the partially hyperbolic systems with singular central foliation does not use anything beyond ergodic theorem and theory of invariant manifolds (see [22].) In particular the results of Sections 1 and 2 are not needed. (Cf. [3, 38] where non-ergodic examples with singular center are given.)

However to understand the dynamics of these examples theory given above is helpful. For more detailed description of this dynamics see [1, 4, 15, 39, 40].

2.4. **Fractional parts of linear forms.** Here we will describe an application of u-Gibbs states to number theory. This example is taken from [33]. It will use translation on $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$, which is has finite volume but is not compact. However Theorem 2 and Corollary 2 can be extended to this case with little difficulty.

Consider a linear form of $d-1$ variables:

$$L_\alpha(m) =< \alpha, m >= \sum_{j=1}^{d-1} \alpha_j m_j$$

with $1 \leq m_j \leq N$. All together we have $N^* = N^{d-1}$ points and we ask how the set of fractional parts looks at scale $\frac{1}{N^*}$. In this subsection we let $[x]$ **denote the fractional part of** x. More precisely choose some n and a set $V \subset \mathbb{R}^n$ with smooth boundary. Let $\Lambda_N(\alpha, V)$ be the number of $(n+1)$-tuples such that $m(1) \dots m(n+1)$

$$\{N^*[L_\alpha(m(j+1)) - L_\alpha(m(j))]\} \subset V.$$

Theorem 4. *([33]) Suppose that $h(\alpha)$ is chosen randomly from $\mathbb{T}^d$ with smooth probability density $h(\alpha)$. Then*

$$\exists \lim_{N\to\infty} \mathrm{Prob}\left(\frac{\Lambda_N(\alpha, V)}{N^*} < s\right) = \mu(s, V)$$

and this limit does not depend on h.

Proof. Let $k(j) = m(j+1) - m(j)$. We deal with the event $\{N^* L_\alpha(k(j))\} \in V$. For each $n-$tuple $\{k(j)\}$ let $\tau_N(\{k(j)\})$ denote the number of ways we can represent $k(j) = m(j+1) - m(j)$. Let

$$\mathbf{M}_s(\{k(j)\}) = \max_j m_s(j) - \min_j m_s(j).$$

$\mathbf{M}_s(\{k(j)\})$ depends only on $\{k(j)\}$:

$$\mathbf{M}_s(\{k(j)\}) = \max_{a,b} \sum_{j=a}^{b} k_s(j).$$

Then the number of ways we can represent $k_s(j) = m_s(j+1) - m_s(j)$ with $1 \leq m_s(j) \leq N$ equals $(N - \mathbf{M}_s(\{k(j)\}))_+$ where $x_+ = \max(x, 0)$. Thus

$$\tau_N(\{k(j)\}) = \prod_{s=1}^{d-1} (N - \mathbf{M}_s(\{k(j)\}))_+.$$

The condition that all $m(j)$ are different in terms of $\{k(j)\}$ reads

$$(DIF) \quad \forall a, b \quad \sum_{j=1}^{b} k(j) \neq 0.$$

Thus

$$\frac{\Lambda_N(\alpha, V)}{N} = \sum_{\nu(1)\ldots\nu(n)\in\mathbb{Z}} \sum_{k(1)\ldots k(n)\in\mathbb{Z}^{d-1}}^{DIF}$$

$$\int_V \frac{1}{N^*}\prod_{s=1}^{d-1}(N - \mathbf{M}_s(\{k(j)\}))_+ \prod_{j=1}^{n} \delta\left(x(j) - N^*\left(< \alpha, k(j) > +\nu(j)\right)\right) dx(1)\ldots dx(n) =$$

$$\sum_{\nu(1)\ldots\nu(n)\in\mathbb{Z}} \sum_{k(1)\ldots k(n)\in\mathbb{Z}^{d-1}}^{DIF}$$

$$\int_V \prod_{s=1}^{d-1}(1 - \mathbf{M}_s(\{\frac{k(j)}{N}\}))_+ \prod_{j=1}^{n} \delta\left(x(j) - N^*\left(< \alpha, k(j) > -\nu(j)\right)\right) dx(1)\ldots dx(n).$$

Now let $\bar{k}(j) = (k(j), \nu(j))$. Let

$$M(N, \alpha) = \begin{pmatrix} \frac{1}{N} & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \frac{1}{N} & 0 \\ 0 & \ldots & 0 & N^* \end{pmatrix} M(\alpha)$$

where

$$M(\alpha_1 \ldots \alpha_{d-1}) = \begin{pmatrix} 1 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 \\ \alpha_1 & \ldots & \alpha_{d-1} & 1 \end{pmatrix}$$

Let

$$(DIF*) \quad \forall a, b \quad \sum_{j=1}^{b} \bar{k}(j) \neq 0.$$

We claim that for large N $\sum^{DIF} = \sum^{DIF*}$. Consider for example the simplest case $a = b$, that is $k(a) = 0$ for some a. Then for large N

$$N^*(< \alpha, k(a) > +\nu(a)) = N^*\nu(a)$$

and this can not be coordinate of the point in V unless $\nu(a) = 0$. Thus

$$\frac{\Lambda_N(\alpha, V)}{N} \sim$$

$$\sum_{\bar{k}(1)\dots\bar{k}(n)\in\mathbb{Z}^d}^{DIF*}\int_V \prod_{s=1}^{d-1}(1-\mathbf{M}_s(\{M(N,\alpha)\bar{k}(j)\}))_+ \prod_{j=1}^{n}\delta(x(j)-(M(N,\alpha)\bar{k}(j))_d)dx(1)\dots dx(n).$$

Let $D(V,M) =$

$$\sum_{\bar{k}(1)\dots\bar{k}(n)\in\mathbb{Z}^d}^{DIF*}\int_V \prod_{s=1}^{d-1}(1-\mathbf{M}_s(\{M\bar{k}(j)\}))_+ \prod_{j=1}^{n}\delta(x(j)-(M\bar{k}(j))_d)dx(1)\dots dx(n).$$

Then $\forall \bar{M}\in \mathrm{SL}_d(\mathbb{Z})$ $D(V, M\bar{M}) = D(V,M)$ since (DIF*) is $\mathrm{SL}_d(\mathbb{Z})$ invariant. So $D(V,\cdot)$ can be considered as a function on $\mathrm{SL}_d(\mathbb{R})/SL_d(\mathbb{Z})$. Hence for large N

$$\frac{\Lambda_N(\alpha,V)}{N^*} = D_N(V, M(N,\alpha)).$$

Now $M(N,\alpha)$ lie on the $M(\alpha)$-orbit of

$$\Phi(t) = \mathrm{diag}(e^{-t},\dots e^{-t}, e^{(d-1)t}).$$

This flow is partially hyperbolic and W^u consist of orbits of $\{M(a)\}$, $a\in\mathbb{R}^{d-1}$. Thus Corollary 2 gives

$$\int_{\mathbb{T}^{d-1}} h(\alpha)1(D(V,\Phi(t)M(\alpha))\le s)d\alpha \to \int_{\mathrm{SL}_d(\mathbb{R})/SL_d(\mathbb{Z})} 1(D(V,M)\le s)dM. \quad \Box$$

References to Subsection 2.4. This example is taken from [33]. Other applications of u-Gibbs states to number theory are discussed in [19, 41].

3. Central Limit Theorem.

To give more application of uniqueness of u-Gibbs states we need to make some assumptions about the convergence rate. Namely we assume that f has unique u-Gibbs state μ and that there is a Banach algebra $\mathbb{B}$ of Holder continuous functions such that for any $A \in \mathbb{B}$ for any $l \in \bar{E}$

$$|l(A \circ f^n) - \nu(A)| \leq a(n)||A||_{\mathbb{B}}$$

where

$$\sum_n a(n) < \infty. \tag{7}$$

(It can be shown that (7) does not depend on the arbitrariness present in the definition of $\bar{E}$. Let $A \in \mathbb{B}$ be a function of zero mean ($\mu(A) = 0$) and let

$$D(A) = \sum_{n=-\infty}^{+\infty} \mu(A(A \circ f^n)). \tag{8}$$

Theorem 5. *Let x be chosen according to some $l \in \bar{E}$ then as $n \to +\infty$ $\frac{1}{\sqrt{n}} S_n(A)(x)$ converges weakly to a Gaussian random variable with zero mean and variance $D(A)$.*

Recall that a Gaussian random variable X has Laplace transform

$$\phi(\xi) = \mathbb{E}(e^{\xi X}) = e^{\frac{D\xi^2}{2}}.$$

Hence

$$\mathbb{E}(X^k) = \left[\left(\frac{d}{d\xi}\right)^k \phi\right](0) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{D^m (2m)!}{2^m m!} & \text{if } k = 2m \end{cases}$$

Let us compare this situation with the case of independent identically distributed random variables. Let $\zeta_1 \dots \zeta_j \dots$ be independent, $\mathbb{E}(\zeta_j) = 0$, $\mathbb{E}(\zeta_j^2) = D$. Let $S_n = \sum_{j=1}^n \zeta_j$. Then

$$\mathbb{E}(\left(\left(\frac{S_n}{\sqrt{n}}\right)^k\right) = \frac{1}{n^{\frac{k}{2}}} \sum_{(j_1 \dots j_s)(p_1 \dots p_s): p_1 + \dots + p_s = k} \mathbb{E}(\zeta_1^{p_1} \dots \zeta_s^{p_s}) = \sum_{\vec{j}, \vec{p}} \prod_{l=1}^{s} \mathbb{E}(\zeta_{j_l}^{p_l}).$$

Note that this product equals zero unless $p_l \geq 2$. From this it is easy to see that the main contribution comes from the terms where all $p_l = 2$. Thus $\mathbb{E}((\frac{S_n}{\sqrt{n}})^k) \to 0$ if k is odd and if $k = 2m$ then $\mathbb{E}((\frac{S_n}{\sqrt{n}})^{2m}) \sim \frac{D^m}{n^m} \times$ (number of terms with all $p_l = 2$). This number equals the number of ways to chose $2m$ elements out of n so that each element appears exactly twice. If the ordering is not important there would be about n^m possibilities. To take the ordering into account we need to multiply this by τ_m the number of ways to divide $2m$ elements into pairs. Recurrence relation $\tau_m = (2m-1)\tau_{m-1}$ implies $\tau_m = \prod_{j=1}^m (2j-1) = \frac{(2m)!}{2^m m!}$. Thus $\mathbb{E}((\frac{S_n}{\sqrt{n}})^{2m}) \sim \frac{(2m)! D^m}{2^m m!}$ as required. Thus for independent random variables the Central Limit Theorem is proved by showing that the main contribution to the moments comes from the terms where the elements are divided into pairs of coinciding elements. In our situation $A(f^j x)$ are weakly dependent rather then independent so the main contribution should come from the terms where the indices can be divided into pairs so that the indices in the same pair maybe not coincide but are close to each other.

To carry over the precise estimate we need a preliminary bound.

Lemma 7. *Let* $S_n(A)(x) = \sum_{j=0}^{n-1} A(f^j x)$. *Then*

$$|l(S_n^k)| \leq Const n^m \tag{9}$$

where $k = 2m$ *or* $k = 2m+1$.

Proof. We prove this result inductively. In fact, we establish slightly more general inequality. Namely we show that (9) is true if $S_n(A)(x) = \sum_{j=0}^{n-1} A_j(f^j x)$, where $\mu(A_j) = 0$ and $||A||_{\mathbb{B}}$ are uniformly bounded. We have

$$l(S_n^k) = \sum_{j_1 \dots j_k} l(\prod_q A_{j_q} \circ f^{j_q}).$$

In case two indices here coincide, say $j_{k-1} = j_k$ we have

$$I = \sum_{j_1 \dots j_{k-1}} l((\prod_{q=1}^{k-2} A_{j_q} \circ f^{j_q}) A^2_{j_{k-1}} \circ f^{j_{k-1}}) =$$

$$\sum_{j_1 \dots j_{k-1}} l((\prod_{q=1}^{k-2} A_{j_q} \circ f^{j_q}) \left[(A^2_{j_{k-1}} \circ f^{j_{k-1}} - \mu(A^2_{j_{k-1}})) + \mu(A^2_{j_{k-1}}) \right] =$$

$$\mu(A^2_{j_{k-1}}) \sum_{j_1 \dots j_{k-1}} l((\prod_{q=1}^{k-2} A_{j_q} \circ f^{j_q}) + \sum_{j_1 \dots j_{k-1}} l((\prod_{q=1}^{k-2} A_{j_q} \circ f^{j_q}) \left[(A^2_{j_{k-1}} \circ f^{j_{k-1}} - \mu(A^2_{j_{k-1}})) \right]$$

By induction the first term is at most $\sum_{j_{k-1}} \mathrm{Const} n^{m-1}$ and in the second term we have only $k-1$ indices so this term is less then either $\mathrm{Const} n^m$ or $\mathrm{Const} n^{m-1}$ depending on the parity of k by inductive hypothesis. Now we consider two cases.

(a) $k = 2m$ is even. We have

$$l(S_n^{2m}) = \sum_{j_1 \dots j_{2m}} l(\prod_q A_{j_q} \circ f^{j_q}) = \sum_r \Theta_r,$$

where Θ_r denotes the sum of the terms where the second largest index equals r. Since we do not have to worry about the term with coinciding indices we get

$$l(S_n^{2m}) = \sum_{r=1}^{n-1} l\left(S_r^{2m-2} A_r \circ f^r \left(\sum_{p=r+1}^{n-1} A_p \circ f^p\right)\right) + O(n^m) = \sum_r \bar{\Theta}_r + O(n^m).$$

Now it suffices to estimate this sum for $l \in E_1$ thus $l(A) = \int_S \rho(x) A(x) dx$. Divide $f^r S = (\bigcup_t K_t) \bigcup Z$ where K_t are Dirichlet cells and $Z \subset \partial_{r_0}(f^r S)$ so that $\mathrm{mes}(f^{-r} Z) \leq \mathrm{Const}\theta^r$ for some $\theta < 1$. Let $c_t = \int_{f^{-r}K_t} \rho(x) dx$, then

$$\int_{f^{-r}K_t} \rho(x) S_r^{2m-2} A_r(f^r x) \sum_p A_p(f^p x) dx =$$

$$\int_{K_t} \rho_t(y) S_r^{2m-2}(f^{-r} y) A_r(y) \sum_p A_p(f^{p-r} y) dy.$$

Let $\Gamma_t = \sup_{K_t} S_r^{2m-2} + 1$ and

$$\underline{\bar{\rho}_t(y) = \rho_t(y) S_r^{2m-2}(f^{-r} y) A_r(y)} \atop \Gamma_t.$$

Lemma 8. *$\bar{\rho}_t$ is uniformly Holder continuous.*

Proof. Since ρ_t and A_r are uniformly Holder continuous we only need to estimate

$$\left|S_r^{2m-2}(f^{-r} y_1) - S_r^{2m-2}(f^{-r} y_2)\right| =$$

$$\left|S_r(f^{-r} y_1) - S_r(f^{-r} y_2)\right| \left|\sum_j S_r^j(f^{-r} y_1) S_r^{2m-3-j}(f^{-r} y_2)\right| \leq$$

$$\mathrm{Const} \sup_{K_t} |S_r^{2m-3}| \left|S_r(f^{-r} y_1) - S_r(f^{-r} y_2)\right|$$

$$\leq \mathrm{Const}\Gamma_t \sum_{q=1}^{r} \left|A_q(f^{q-r} y_1) - A_q(f^{q-r} y_2)\right|$$

$$\leq \mathrm{Const}\Gamma_t \sum_{q=1}^{r} d^\gamma(f^{q-r} y_1, f^{q-r} y_2) \leq \mathrm{Const}\Gamma_t d^\gamma(y_1, y_2) \sum_{q=1}^{r} \frac{1}{\lambda_5^{\gamma(q-r)}}. \quad \square$$

By Lemma 8

$$\left|\sum_t c_t \int_{K_t} \sum_p \rho_t(y) A(y) S_r(y) A_p(f^{p-r} y) dy\right| \leq$$

$$\left|\sum_t c_t \Gamma_t \int_{K_t} \sum_p \bar{\rho}_t(y) A_p(f^{p-r} y) dy\right| \leq$$

$$\sum_t c_t\Gamma_t \sum_r a(p-r) \leq \text{Const} \sum_t c_t\Gamma_t.$$

Now

$$\Gamma_t = \int_{K_t} \rho_t(y) S_r^{2m-2}(f^{-r}y)dy + O\left(\left|S_r^{2m-3}(f^{-r}y)\right|\right).$$

Since $|S_r^{2m-3}| \leq (S_r^{2m-2}+1)$ we obtain

$$\sum_t c_t\Gamma_t = \int_S \rho(x) S_r^{2m-2}(x)dx + O(1) = O(n^{m-1})$$

by induction hypothesis. Hence

$$l(S_n^{2m}) \leq \text{Const} \sum_{r=0}^{n-1} n^{m-1} \leq \text{Const} n^m.$$

This completes the proof for even k.

In the case k the proof is odd is the same but now r should be the largest index. $\square$

Lemma 9. *Let $S_n = \sum_{j=0}^{n-1} A(f^j x)$, then $\forall l \in \bar{E}$*

$$\frac{l(S_n^{2m})}{n^m} \sim \frac{D^m (2m)!}{2^m m!}.$$

Proof.

$$l(S_n^{2m}) = \sum_{j_1 \dots j_{2m}} l(\prod_q A_q(f^q x)).$$

Let β_s be the sum of terms where the difference between the largest and the second largest term is exactly s. Thus

$$l(S_n^{2m}) = \sum_s \beta_s.$$

Lemma 10. $\forall \varepsilon$ $\exists n_0$ *such that* $\forall n$

$$\sum_{s \geq n_0} \beta_s \leq \varepsilon n^m.$$

Proof. In the proof of Lemma 9 we saw that

$$\sum_{s \geq n_0} \beta_s \leq \text{Const} \sum_{s \geq n_0} a(s) n^m. \quad \square$$

Let us now estimate β_s for fixed s. Let $\beta_{s,s'}$ denote the sum of the terms from β_s where the difference between the second and the third largest indices equals s'.

Lemma 11.

$$\forall s' \quad \lim_{n\to\infty} \frac{\beta_{s,s'}}{n^m} = 0.$$

Proof. $\beta_{s,s'}$ can be bounded by

$$l(\sum_{j_1\dots j_{2m-3}}\prod_q A_{j_q}(f^{j_q}x)\sum_{j_{2m-2}} B(f^{j_{2m-2}}x))$$

where $B(x)=A(x)A(f^{s'}x)A(f^{s+s'}x)$. Hence for fixed s' $\beta_{s'}$ is $O(n^{m-1})$ by Lemma 9. $\square$

Thus for any fixed n_0

$$\beta_s \sim \sum_{s'\geq n_0} \beta_{s,s'}.$$

Now let $\beta_{s,s'}(r)$ denote the sum of the terms where the second largest index is r. Since there are $2m(2m-1)$ ways to choose the largest and second largest indices we have for $s>0$

$$\sum_{s'\geq n_0}\sum_r \beta_{s,s'}(r) \sim 2m(2m-1)\sum_r l(S_r^{2m-2}(x)A(f^r x)A(f^{r+s})) \sim$$

$$2m(2m-1)\mu(A(A\circ f^s))\sum_r l(S_r^{2m-2}(x))+$$

$$2m(2m-1)\sum_r l(S_r^{2m-2}(x)\left[A(f^r x)A(f^{r+s})-\mu(A(A\circ f^s))\right]).$$

Now in the second sum we have $2m-1$ different functions so by Lemma 9 it is $O(n^{m-1})$. The first term can be computed by induction

$$2m(2m-1)\mu(A(A\circ f^s))\sum_r l(S_r^{2m-2}(x)) \sim$$

$$2m(2m-1)\mu(A(A\circ f^s))\sum_r \frac{D^{m-1}(2m-2)!}{2^{m-1}(m-1)!}r^{m-1} \sim$$

$$2(2m-1)\mu(A(A\circ f^s))n^m\frac{D^{m-1}(2m-2)!}{2^{m-1}(m-1)!} = 2n^m D^{m-1}\frac{(2m-1)!}{2^{m-1}2^{m-1}}\mu(A(A\circ f^s)).$$

Likewise if $s=0$ then the largest and second largest index coincide so we get

$$\beta_0 \sim n^m D^{m-1}\frac{(2m-1)!}{(m-1)!2^{m-1}}\mu(A^2).$$

Since

$$\frac{(2m-1)!}{(m-1)!2^{m-1}} = \frac{(2m)!}{m!2^m}$$

we obtain

$$l(S^{2m}) \sim \frac{(2m)!}{2^m m!}D^{m-1}\left[\mu(A)^2+\sum_{s=1}^{n_0}\mu(A(A\circ f^s))+o_{n_0\to\infty}(1)\right].$$

The term in brackets can be rewritten as

$$\sum_{|s|\leq n_0} \mu(A(A\circ f^s)) + o_{n_0\to\infty}(1).$$

Letting $n_0 \to \infty$ we obtain the statement required. $\square$

Exercise 15. *Let $w_n(t) = \frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} A(f^j x)$. Show that as $n \to \infty$ $w_n(t)$ converges to Brownian Motion $w(t)$. That is, for*

$$0 \leq t_1 \leq t_2 \leq \dots \leq t_n$$

$w(t_{j+1}) - w(t_j)$ and $w(t_{k+1}) - w(t_k)$ are independent Gaussian random variables, $w(t)$ has mean 0 and variance Dt.

Exercise 16. *Let M be a compact manifold of variable negative curvature, $\tilde{M}$ be a covering such that $M = \tilde{M}/\mathbb{Z}$. Choose a closed one form ω and a reference point q_0 and mark position of point $q \in \tilde{M}$ by $x(q) = \int_{q_0 q} \omega$. Let*

$$M_n = \{q : n \leq x(q) \leq n+1\}$$

and

$$Q_n = \{(q,v) : q \in M_n \quad \text{and} \quad ||v|| = 1\}.$$

(a) Suppose that $(q,v)(0)$ is chosen Lebesgue uniformly on Q_0. Let $q(t)$ be the geodesic defined by (q,v). Let

$$w_n(t) = \frac{1}{\sqrt{n}} x(q(tn)).$$

Show that as $n \to \infty$ $w_n(t)$ converges to Brownian Motion.

(b) Let $\rho(s)$ be a smooth positive function with compact support on $\mathbb{R}$. Suppose that we put on each Q_n $N\rho(\frac{n}{\sqrt{M}})$ points independently and Lebesgue uniformly. Let $\rho_{N,M}(t,x)$ be the number of points in $Q_{[x\sqrt{M}]}$ at the moment tM. Show that if $M, n \to \infty$ so that $\frac{N}{\sqrt{M}} \to \infty$ then $\frac{\rho_{N,M}(t,x)}{N} \to \rho(t,x)$ where

$$\partial_t \rho = D\Delta\rho, \quad \rho(0,x) = \rho(x).$$

References to Section 3. Our exposition is taken from [16] which follow [23]. Other approaches to Central Limit Theorem could be found in [30, 21, 34]. Applications to hydrodynamic equations (cf. Exercise 16) are discussed in [6].

Appendix A. Random partially hyperbolic systems.

Here we discuss what is analogue of partial hyperbolicity for systems with noise. Of course one can define uniform partial hyperboplicity in terms of existence of invariant cones. However, if we are interested in statistical properties when a weaker analogue of non-uniform partial hyperbolicity which we describe below. We follow [18]. Let M be a compact manifold and consider a system of stochastic differential equations

$$dx = Y(x)dt + \sum_{j=1}^{d} X_j(x) \circ dw_j(t) \tag{10}$$

where w_j are independent Brownian Motions. We impose some non-degeneracy conditions. Namely let

$$d(x,v) = \tilde{Y}(x,v)dt + \sum_{j=1}^{d} \tilde{X}_j(x,v) \circ dw_j(t)$$

be the induced flow on TM. We require

$$(A) \quad \forall (x,v) \quad \mathrm{Lie}(\{\tilde{X}_j\}) = T(TM)$$

and

$$(B) \quad \forall x \neq y \quad \mathrm{Lie}(\{(X_j(x), X_j(y))\}) = TM \times TM.$$

Let λ be the largest Lyapunov exponent of (10).

Proposition 8. *(*[10]*) For generic d-tuple* $\{X_j\}$ $\lambda \neq 0$.

Thus from now on we assume that

$$(C) \quad \lambda \neq 0.$$

As we will explain below (A)–(C) can serve as a substitute for partial hyperbolicity.

As in the deterministic case one can define SRB measures by considering the iterations of Lebesgue measure on a submanifold. In the deterministic partially hyperbolic case one can take any submanifold transversal to $E_c \oplus E_s$. However in the random case directions of the subleading growth are random so they will be transversal to a deterministic direction with probability one.

Proposition 9. $\exists \nu_t(\omega)$ *such that for any curve* γ *with probability one* $\forall A \in C(M)$

$$\lim_{s \to -\infty} \int_\gamma A(x_t) dx_s \to \nu_t(A).$$

In fact one has exponential convergence to this random SRB state.

Proposition 10. [18] $\forall A \in C^\gamma(M)$

$$\left|\int_\gamma A(x_t)dx_0 - \nu_t(A)\right| \leq C(\{w\})||A||_\gamma e^{-\delta t}.$$

Now we have to distinguish between $\lambda > 0$ and $\lambda < 0$ cases.

Proposition 11. [28] *If* $\lambda < 0$ *then* $\exists y(t,w)$ *such that* $\nu_t = \delta_{y(t)}$.

Consider $B(x,t,w) = A(x_t) - \nu_t(A)$.

Proposition 12. *Let* x_0 *be chosen uniformly from* γ*. Then for almost any realization of* $\{w_j\}$

$$\frac{\int_0^t B(x,s,w)ds}{\sqrt{t}}$$

converges weakly as $t \to \infty$ *to a normal random variable.*

The proof of this result is similar to the proof of Theorem 5 using Proposition 10. The proof of Proposition 10 works by computing the varaince of $\int_\gamma A(x_t)dx_0$. This variance involves two point process $(x,y) \to (x_t, y_t)$. One shows that (A)–(C) implies the exponential convergence of Lebesgue measure on $\gamma \times \gamma$. In case $\lambda < 0$ in converges to the ergodic invariant measure on diagonal and in case $\lambda > 0$ off-diagonal, since $\lambda > 0$ implies that if x_t is close to y_t they are likely to diverge again.

Hence even though partial hyperbolicity involves the strong topological restrictions to underlying manifold the same picture can be obtained for arbitrary system subject to a small random noise.

2. Dependence on parameters.

2.1. **Perturbation expansions.** Now we know several examples of open sets having unique u-Gibbs state, so the natural question is how they depend on parameters. One of the first results in this direction is the following.

Theorem 6. *([25]) In the space of Anosov diffeomorphisms $\forall A \in C^\infty(M)$ the map $f \to \mu_{SRB}(A)$ is C^∞.*

[25] also proves the similar result for Anosov flows. Let me explain the proof of a weaker statement that the map $f \to \mu_{SRB}(A)$ is C^1 and various generalizations of this. We know from subsection 2.2 that if K is Dirichlet cell then $\int_K \rho(x)A(f^n x)dx$ converges to $\mu_{SRB}(A)$ exponentially fast. It is easy to see that the same holds if instead of requiring that $K \in E^u$ we ask only that K is a submanifold transversal to E^s. So if f_ε is a one-parameter family of Anosov diffeos we can get a good approximation of $\mu_{SRB}(f_\varepsilon)(A)$ by looking at $\int_K \rho(x)A(f_\varepsilon^n x)dx$ where ρ is a density of compact support inside K. Now given x $f^n x$ and $f_\varepsilon^n x$ are far apart but by shadowing lemma $\forall x, \forall n\ \exists y_n \in K$ such that $f^n y_n$ is close to f_ε^n. To define such y_n uniquely choose a smooth distribution $\tilde{E}^s$ C^0-close to E^s and require that $f_\varepsilon^n x = \exp_{f^n y_n}(V_n)$ where $V_n \in \tilde{E}^s$. V_n's then satisfy

$$V_{n+1} = \pi_{\tilde{E}^s}(df(V_n) + \varepsilon X(f^{n+1}y_n)) + \text{H.O.T.} \tag{11}$$

where $\pi_{\tilde{E}^s}$ is the projection to $\tilde{E}^s$ along E^u and $X + \frac{df_\varepsilon}{d\varepsilon}$. Let $Q : \tilde{E}^s \to \tilde{E}^s$ denote $\pi_{\tilde{E}^s} \circ df$ and

$$Q_n = Q(f^{n-1}x) \dots Q(fx)Q(x).$$

Solving (11) we obtain

$$V_{n+1} = \varepsilon \sum_{j=1}^{n} Q_j(f^{-j} z_{n+1})[X_s] + \text{H.O.T.}$$

where $z_n = f^n y_n$, $X_s = \pi_{\tilde{E}_s} X$. Thus as $n \to \infty$ $V_{n+1} \sim \varepsilon V(z_{n+1})$ where

$$V(z) = \sum_{j=1}^{\infty} Q_j(f^{-j} z)[X_s]$$

Take A such that $\mu(A) = 0$ then

$$\int_K \rho(x)A(f_\varepsilon^n x)dx =$$

$$\int_K \rho(x)A(z_n)dx + \int_K \rho(x)[A(f_\varepsilon^n x) - A(z_n)]dx.$$

Now $dx = \frac{dx}{dy_n} dy_n$. But $y_0 = x$ so

$$\frac{dx}{dy_n} = \frac{dy_0}{dy_n} = \prod_{j=0}^{n-1} \left(\frac{dy_{j-1}}{dy_j} \right).$$

Now

$$f^{j+1} y_{j+1} \sim \exp_{f^{j+1} y_j} \pi_{E^u} \left(\varepsilon X + df(V_j) \right)$$

where π_{E^u} denotes projection to E^u along $\tilde{E}^s$. Thus

$$\frac{d(f^{j+1} y_{j+1})}{d(f^{j+1} y_j)} \sim 1 + \operatorname{div}[\varepsilon X(f^j y_j) + df(V_j)] \tag{12}$$

Now

$$\frac{dy_{j+1}}{dy_j} = \frac{d(f^{j+1} y_{j+1})}{d(f^{j+1} y_j)} \left[\frac{dy_{j+1}}{df^{j+1} y_{j+1}} : dy_{j+1} df^{j+1} y_j \right].$$

Note that the second term would be equal to one both K and f^{j+1} were equipped with canoniacal density. Then divergence in (12) also would be with respect to canoniacal density so that

$$\frac{dy_0}{dy_n} \sim 1 - \varepsilon \sum_{j=0}^{n-1} \operatorname{div}_{can} (\pi_{E^u} df(V) + X)(f^{-j} z_n)$$

from this we get

$$\int_K A(f_\varepsilon^n x) \rho(x) dx - \int_K A(f^n y_n) \rho(x(y_n)) dy_n = \varepsilon \int_K (\partial_V A) \circ f^n \rho(x(y_n)) dy_n -$$

$$\varepsilon \sum_j \int_K \operatorname{div}_{can} [X + df(V)] \circ f^{-j} A(f^n y_n) \rho(x(y_n)) dy_n + \text{H.O.T.}.$$

Choosing $n \sim \operatorname{Const} \ln(\frac{1}{\varepsilon})$ we get

$$\frac{d}{d\varepsilon} \mu_{SRB}(A) = \mu(\partial_V A) - \sum_{j=1}^{\infty} \mu(A \left[\operatorname{div}_{can} (X + df(V) \right] \circ f^{-j}).$$

This calculation can be extended to a more general situation giving some information about u-Gibbs states when we do not know uniqueness.

The example we consider is abelian Anosov actions. These are partially hyperbolic systems such that E^c is tangent to the orbits of $\mathbb{R}^d$-action $\varphi_a : M \to M$ such that $f \varphi_a = \varphi_a f$. f is called *Anosov element* of the action. One example of abelian Anosov actionis time one map of an Anosov flow.

Theorem 7. *([17]) Suppose that f is an Anosov element in an abelian Anosov action and assume that $\forall m\ \exists k(m)$ such that $\forall l \in \bar{E}\ \forall A \in C^k(M)$*

$$|l(A \circ f^n) - \mu(A)| \le \mathrm{Const}||A||_{C^k(M)} \frac{1}{n^m}$$

(cf. Subsection 2.2.) Then $\exists k$ and a linear functional $\omega : C^k(M) \to \mathbb{R}$ such that if μ_ε is any u-Gibbs state for f_ε then

$$\mu_\varepsilon(A) - \mu(A) = \varepsilon\omega(A) + o(\varepsilon||A||_k).$$

Corollary 5. *$\forall \delta > 0$ existsε_0 such that $\forall \varepsilon \le \varepsilon_0$ for Lebesgue almost all x $\exists n = n(x)$ such that for $n \ge n(x)$*

$$\left| \frac{S_n(A)(x)}{n} - \nu(A) - \varepsilon\omega(A) \right| \le \varepsilon\delta.$$

Proof. This follows immediately from Proposition 1. □

Exercise 17. *Let $f : M \to M$ be an Anosov diffeomorphism. Show that there is a neighbourhood $\mathcal{U}(f)$ such that the following holds. Let $\{f_j\}$ be a sequance with $f_j \in \mathcal{U}$ and let*

$$F_{k,n} = f_n \circ \dots f_{k+1} \circ f_k.$$

Prove that

$$(a) \quad \exists \mu_n(A) = \lim_{k \to -\infty} \int A(F_{k,n}(x))dx.$$

(b) $\forall A, n$ the map $\{f_j\} \to \mu_n(\{f_j\}$ is C^1.

Exercise 18. * *Prove Theorem 6.*

(a) ([11]) Let φ_ε be the conjugation $f_\varepsilon \varphi_\varepsilon = \varphi_\varepsilon f$. Show that $\forall x$ the map $\varepsilon \to \varphi_\varepsilon(x)$ is smooth.

(b) Use (a) and the fact that the SRB measure is the unique measure satisfying

$$h(\mu_\varepsilon) = \int \ln\det(df_\varepsilon | E_u(\varepsilon))(x) d\mu_\varepsilon$$

to prove Theorem 6.

Exercise 19. [17] *Let f_ε be a one-parameter family such that f_0 is a time one map of a geodesic flow on a surface of negative curvature.*

(a) $W^c(f_0)$ and $W^c(f_\varepsilon)$ are conjugated. Show that this conjugation φ_ε can be chosen so that $\forall x$ the map $\varepsilon \to \varphi_\varepsilon(x)$ is smooth.

(b) Use (a) to show that $\forall x$ the map $\varepsilon \to E^c(x, \varepsilon)$ is differentiable at 0.

(c) Use (b) and the fact that $E^u \oplus E^s(f_0)$ is c^∞ to show that there is a quadratic form $c(X)$ such that if μ_ε is a u-Gibbs state for f_ε then

$$\lambda_c(\mu_\varepsilon) \sim c(\frac{df_\varepsilon}{d\varepsilon})\varepsilon^2.$$

(d) Show that c is not identically equal to zero.

Exercise 20. *([5]) Let $M = \mathbb{H}^2/\Gamma$ where Γ is a cocomapct lattice. Consider a particle moving in a constant electric field subject to a Gaussian thermostat. The equation of motion of the particle lifted to $\mathbb{H}^2$ is given in the upper halfplane model by*

$$x' = y^2 p_x \quad y' = y^2 p_y$$

$$p_x' = E_x - \frac{p_x E_x + p_y E_y}{p_x^2 + p_y^2} p_x \quad p_y' = -y^2(p_x^2 + p_y^2) + E_y - \frac{p_x E_x + p_y E_y}{p_x^2 + p_y^2} p_y$$

where $E_x + iE_y = \varepsilon\psi(z)$ where $\psi(z)$ is homomorphic in $\mathbb{H}^2$ and

$$\forall \gamma(z) = \frac{az+b}{cz+d} \in \Gamma \quad \psi(\gamma(z)) = (cz+d)^2\psi(z).$$

Let $\tilde{M}$ be a covering such that $M = \tilde{M}/\mathbb{Z}$.

(a) In the notation of Exercise 16 show that for Lebesgue almost all x

$$\exists d(\varepsilon) = \lim_{t\to\infty} \frac{x(q(\varepsilon,t))}{t} \quad \text{and} \quad \exists d = \lim_{\varepsilon\to 0} \frac{d(\varepsilon)}{\varepsilon}.$$

(b) Let initial positions of particles be distributed as in Exercise 16(b) with $\rho(s) \equiv 1$, $N \equiv 1$. Let $J(\varepsilon, n, T)$ denote the (algebraic) number of particle which have crossed M_n up to time T that is $J(\varepsilon, n, T) = \mathrm{Card}($ particles such that $q(\varepsilon, 0) < n$, $q(\varepsilon, T) > n+1) - \mathrm{Card}($ particles such that $q(\varepsilon, 0) > n+1$, $q(\varepsilon, T) < n)$. Deduce from (a)

$$\exists j(\varepsilon) = \lim_{T\to\infty} \frac{J(\varepsilon, n, T)}{T} \quad \text{and} \quad \exists j = \lim_{\varepsilon\to 0} \frac{j(\varepsilon)}{\varepsilon}.$$

2.2. **Conclusion.** u-Gibbs states play an important role in the study of statistical properties of partially hyperbolic systems, there are several situations where they can be computed explicitly and they are stable with respect to changes of parameters. Thus if we get some information in the model which involves partially hyperbolic systems when it persists under the vagueness coming from the model construction. However there are still many open questions about u-Gibbs states of general partially hyperbolic systems especially in higher dimensions so it is an interesting area of research.

References to Section 2. Results about the smooth dependence of Gibbs states for Anosov systems and some applications are discussed in [24]. The expression for the first derivatives we derive here is taken from [36]. Applications of differentiability to statistical mechanics can be found in [37].

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