# DOUBLE THETA POLYNOMIALS AND EQUIVARIANT GIAMBELLI FORMULAS

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Abstract. We use Young's raising operators to introduce and study *double theta polynomials*, which specialize to both the theta polynomials of Buch, Kresch, and Tamvakis, and to double (or factorial) Schur $S$-polynomials and $Q$-polynomials. These double theta polynomials give Giambelli formulas which represent the equivariant Schubert classes in the torus-equivariant cohomology ring of symplectic Grassmannians, and we employ them to obtain a new presentation of this ring in terms of intrinsic generators and relations.

## 0. Introduction

Fix the nonnegative integer $k$ and let $\mathrm{IG} = \mathrm{IG}(n-k, 2n)$ be the Grassmannian which parametrizes isotropic subspaces of dimension $n-k$ in the vector space $\mathbb{C}^{2n}$, equipped with a symplectic form. In 2008, Buch, Kresch, and Tamvakis [BKT2] introduced the theta polynomials $\Theta_\lambda(c)$, a family of polynomials in independent variables $c_p$, defined using Young's raising operators [Y]. When the $c_p$ are mapped to the special Schubert classes, which are the Chern classes of the tautological quotient vector bundle over IG, then the $\Theta_\lambda(c)$ represent the Schubert classes in $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$. These theta polynomials give a combinatorially explicit, intrinsic solution to the *Giambelli problem* for the cohomology ring of IG.

This paper is concerned with Giambelli formulas for the *equivariant Schubert classes* in the equivariant cohomology ring $\mathrm{H}_T^*(\mathrm{IG})$, where $T$ denotes a maximal torus of the complex symplectic group. As explained in [Gr] and [T3], such formulas are equivalent to corresponding ones within the framework of degeneracy loci of vector bundles [F]. In 2009, the theta polynomials $\Theta_\lambda(c)$ were extended to obtain representing polynomials in the torus-equivariant cohomology ring of IG, and more generally, of any isotropic partial flag variety, by the first author [T2]. The *equivariant Giambelli problem* was thus solved in a uniform manner for any classical $G/P$ space, in terms of positive combinatorial formulas which are *native to* $G/P$. See [T3] for an exposition of this work.

It has been known for some time (cf. [KL, Ka, I, Mi2]) that the equivariant Schubert classes of the usual (type A) Grassmannian and of the Lagrangian Grassmannian $\mathrm{LG}(n, 2n)$ may be represented by Jacobi-Trudi type determinants and Schur Pfaffians which generalize the classical results of [G, P]. The polynomials which appear in these formulas are called double (or factorial) Schur $S$-polynomials and $Q$-polynomials, respectively (cf. [BL, Iv]). Our aim here is to use raising operators to define *double theta polynomials* $\Theta_\lambda(c\,|\,t)$, which advance the theory of the single

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theta polynomials $\Theta_\lambda(c)$ from [BKT2] to the equivariant setting, and specialize to both of the aforementioned versions of double Schur polynomials. The $\Theta_\lambda(c\,|\,t)$ represent the equivariant Schubert classes on IG, but differ from the equivariant Giambelli polynomials given in [T2]. It follows that the two theories must agree up to the ideal $I^{(k)}$ of *classical relations* (3) among the variables $c_p$. We discuss this in detail in §4.3 and Corollary 2.

For the rest of this paper, $c = (c_1, c_2, \ldots)$ and $t = (t_1, t_2, \ldots)$ will denote two families of commuting variables. We set $c_0 = 1$ and $c_p = 0$ for any $p < 0$. For any integers $j \geq 0$ and $r \geq 1$, define the elementary and complete symmetric polynomials $e_j(t_1, \ldots, t_r)$ and $h_j(t_1, \ldots, t_r)$ by the generating series

$$\prod_{i=1}^{r}(1+t_iz) = \sum_{j=0}^{\infty} e_j(t_1,\ldots,t_r)z^j \quad \text{and} \quad \prod_{i=1}^{r}(1-t_iz)^{-1} = \sum_{j=0}^{\infty} h_j(t_1,\ldots,t_r)z^j,$$

respectively. Let $e_j^r(t) := e_j(t_1,\ldots,t_r)$, $h_j^r(t) := h_j(t_1,\ldots,t_r)$, and $e_j^0(t) = h_j^0(t) := \delta_{0j}$, where $\delta_{0j}$ denotes the Kronecker delta. Furthermore, if $r < 0$ then define $h_j^r(t) := e_j^{-r}(t)$.

We will work throughout with integer sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ which are assumed to have finite support, when they appear as subscripts. For any positive integers $i < j$ and integer sequence $\alpha$, define the operator $R_{ij}$ by

$$R_{ij}(\alpha) := (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

A *raising operator* $R$ is any monomial in the basic operators $R_{ij}$. The sequence $\alpha$ is a *composition* if $\alpha_i \geq 0$ for all $i$, and a *partition* if $\alpha_i \geq \alpha_{i+1} \geq 0$ for all $i \geq 1$. As is customary, we will identify partitions with their Young diagram of boxes.

Fix the nonnegative integer $k$. For any integers $p$ and $r$, define

$$c_p^r := \sum_{j=0}^{p} c_{p-j}\, h_j^r(-t)$$

and for any integer sequences $\alpha$ and $\beta$, let

$$c_\alpha^\beta := \prod_{i\geq 1} c_{\alpha_i}^{\beta_i} \quad \text{and} \quad c_\alpha := c_\alpha^0 = \prod_{i\geq 1} c_{\alpha_i}.$$

Given any raising operator $R$, define $R\, c_\alpha^\beta := c_{R\alpha}^\beta$. It is important that the variables $c_{\alpha_i}^{\beta_i}$ in the monomials $c_\alpha^\beta$ are regarded as *noncommuting* for the purposes of this action.

We say that a partition $\lambda$ is *$k$-strict* if all parts $\lambda_i$ which are strictly greater than $k$ are distinct. To any such $\lambda$, we attach a finite set of pairs

$$\mathcal{C}(\lambda) := \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid 1 \leq i < j \ \text{ and } \ \lambda_i + \lambda_j > 2k + j - i\}$$

and a sequence $\beta(\lambda) = \{\beta_j(\lambda)\}_{j\geq 1}$ defined by

$$\beta_j(\lambda) := k + 1 - \lambda_j + \#\{i < j \mid (i,j) \notin \mathcal{C}(\lambda)\}, \quad \text{for all} \ \ j \geq 1.$$

Following [BKT2], consider the raising operator expression $R^\lambda$ given by

$$R^\lambda := \prod_{i<j}(1 - R_{ij}) \prod_{(i,j)\in\mathcal{C}(\lambda)} (1 + R_{ij})^{-1}.$$

**Definition 1.** For any $k$-strict partition $\lambda$, the *double theta polynomial* $\Theta_\lambda(c\,|\,t)$ is defined by

$$\Theta_\lambda(c\,|\,t) := R^\lambda\, c_\lambda^{\beta(\lambda)}.$$

The single theta polynomial $\Theta_\lambda(c)$ of [BKT2] is given by

$$\Theta_\lambda(c) = \Theta_\lambda(c\,|\,0) = R^\lambda\, c_\lambda.$$

We next relate the double theta polynomials $\Theta_\lambda(c\,|\,t)$ to the torus-equivariant Schubert classes on IG. Let $\{e_1,\ldots,e_{2n}\}$ denote the standard symplectic basis of $\mathbb{C}^{2n}$ and let $F_i$ be the subspace spanned by the first $i$ vectors of this basis, so that $F_{n-i}^\perp = F_{n+i}$ for $0 \leq i \leq n$. Let $B_n$ denote the stabilizer of the flag $F_\bullet$ in the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$, and let $T_n$ be the associated maximal torus in the Borel subgroup $B_n$. The $T_n$-equivariant cohomology ring $\mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n),\mathbb{Z})$ is defined as the cohomology ring of the Borel mixing space $ET_n \times^{T_n} \mathrm{IG}$. The Schubert varieties in IG are the closures of the $B_n$-orbits, and are indexed by the $k$-strict partitions $\lambda$ whose Young diagram fits in an $(n-k)\times(n+k)$ rectangle. Any such $\lambda$ defines a Schubert variety $X_\lambda = X_\lambda(F_\bullet)$ of codimension $|\lambda| := \sum_i \lambda_i$ by

$$X_\lambda := \{\Sigma \in \mathrm{IG} \mid \dim(\Sigma \cap F_{n+\beta_j(\lambda)}) \geq j \quad \forall\, 1 \leq j \leq n-k\}\,. \tag{1}$$

Since $X_\lambda$ is stable under the action of $T_n$, we obtain an *equivariant Schubert class* $[X_\lambda]^{T_n} := [ET_n \times^{T_n} X_\lambda]$ in $\mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$.

Following [BKT2, §5.2] and [IMN1, §5], we consider the *stable* equivariant cohomology ring of $\mathrm{IG}(n-k,2n)$, denoted by $\mathbb{H}_T(\mathrm{IG}_k)$. The latter is defined by

$$\mathbb{H}_T(\mathrm{IG}_k) := \varprojlim \mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n)),$$

where the inverse limit of the system

$$\cdots \to \mathrm{H}^*_{T_{n+1}}(\mathrm{IG}(n+1-k,2n+2)) \to \mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n)) \to \cdots \tag{2}$$

is taken in the category of graded algebras. The surjections (2) are induced from the natural inclusions $W_n \hookrightarrow W_{n+1}$ of the Weyl groups of type C, as in [BH, §2], [IMN1, §10], and §3 of the present work. Moreover, the variables $t_i$ are identified with the characters of the maximal tori $T_n$ in a compatible fashion, as in loc. cit. One then has that $\mathbb{H}_T(\mathrm{IG}_k)$ is a free $\mathbb{Z}[t]$-algebra with a basis of stable equivariant Schubert classes

$$\sigma_\lambda := \varprojlim [X_\lambda]^{T_n},$$

one for every $k$-strict partition $\lambda$. We view $\mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$ as a $\mathbb{Z}[t]$-module via the natural projection map $\mathbb{Z}[t] \to \mathbb{Z}[t_1,\ldots,t_n]$.

Consider the graded polynomial ring $\mathbb{Z}[c] := \mathbb{Z}[c_1,c_2,\ldots]$ where the element $c_p$ has degree $p$ for each $p \geq 1$. Let $I^{(k)} \subset \mathbb{Z}[c]$ be the homogeneous ideal generated by the relations

$$\frac{1-R_{12}}{1+R_{12}}\, c_{p,p} = c_pc_p + 2\sum_{i=1}^{p}(-1)^i c_{p+i}c_{p-i} = 0 \quad \text{for } p > k, \tag{3}$$

and define the quotient ring $C^{(k)} := \mathbb{Z}[c]/I^{(k)}$. We call the graded polynomial ring $C^{(k)}[t]$ the *ring of double theta polynomials.*

**Theorem 1.** *The polynomials $\Theta_\lambda(c\,|\,t)$, as $\lambda$ runs over all $k$-strict partitions, form a free $\mathbb{Z}[t]$-basis of $C^{(k)}[t]$. There is an isomorphism of graded $\mathbb{Z}[t]$-algebras*

$$\pi : C^{(k)}[t] \to \mathbb{H}_T(\mathrm{IG}_k)$$

*such that $\Theta_\lambda(c\,|\,t)$ is mapped to $\sigma_\lambda$, for every $k$-strict partition $\lambda$. For every $n \geq 1$, the morphism $\pi$ induces a surjective homomorphism of graded $\mathbb{Z}[t]$-algebras*

$$\pi_n : C^{(k)}[t] \to \mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$$

*which maps $\Theta_\lambda(c\,|\,t)$ to $[X_\lambda]^{T_n}$, if $\lambda$ fits inside an $(n-k)\times(n+k)$ rectangle, and to zero, otherwise. The equivariant cohomology ring $\mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$ is presented as a quotient of $C^{(k)}[t]$ modulo the relations*

$$\Theta_p(c\,|\,t) = 0, \quad p \geq n+k+1 \tag{4}$$

*and*

$$\Theta_{(1^p)}(c\,|\,t) = 0, \quad n-k+1 \leq p \leq n+k, \tag{5}$$

*where $(1^p)$ denotes the partition $(1,\ldots,1)$ with $p$ parts.*

Theorem 1 has a direct analogue for the torus-equivariant cohomology ring of the odd orthogonal Grassmannian $\mathrm{OG}(n-k,2n+1)$; this follows from known results (cf. [T3, §2.2 and §5.1]). In this case, the polynomials $2^{-\ell_k(\lambda)}\Theta_\lambda(c\,|\,t)$ represent the equivariant Schubert classes, where $\ell_k(\lambda)$ denotes the number of parts $\lambda_i$ of $\lambda$ which are strictly bigger than $k$. Theorem 1 therefore generalizes the results of Pragacz [P, §6] and Ikeda and Naruse [I, IN] to the equivariant cohomology of all symplectic and odd orthogonal Grassmannians.

Let

$$0 \to E' \to E \to E'' \to 0 \tag{6}$$

denote the universal exact sequence of vector bundles over $\mathrm{IG}(n-k,2n)$, with $E$ the trivial bundle of rank $2n$ and $E'$ the tautological subbundle of rank $n-k$. The $T_n$-equivariant vector bundles $E'$, $E$, and $F_j$ have equivariant Chern classes, denoted by $c^T_p(E')$, $c^T_p(E)$, and $c^T_p(F_j)$, respectively. Define $c^T_p(E-E'-F_j)$ by the equation of total Chern classes $c^T(E-E'-F_j) := c^T(E)c^T(E')^{-1}c^T(F_j)^{-1}$. We now have the following Chern class formula for the Schubert classes $[X_\lambda]^T$.

**Corollary 1.** *Let $\lambda$ be a $k$-strict partition that fits inside an $(n-k)\times(n+k)$ rectangle. Then we have*

$$[X_\lambda]^T = \Theta_\lambda(E-E'-F_{n+\beta(\lambda)}) = R^\lambda\, c^T_\lambda(E-E'-F_{n+\beta(\lambda)}) \tag{7}$$

*in the equivariant cohomology ring $\mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$.*

The equivariant Chern polynomial in (7) is interpreted as the image of $\Theta_\lambda(c) = R^\lambda\, c_\lambda$ under the $\mathbb{Z}$-linear map which sends the noncommutative monomial $c_\alpha$ to $\prod_j c^T_{\alpha_j}(E-E'-F_{n+\beta_j(\lambda)})$, for every integer sequence $\alpha$. Formula (7) is a natural generalization of Kazarian's multi-Pfaffian formula [Ka, Thm. 1.1] to arbitrary equivariant Schubert classes on symplectic Grassmannians.

The double theta polynomials $\Theta_\lambda(c\,|\,t)$ were defined in Wilson's 2010 University of Maryland Ph.D. thesis [W], whose aim was to apply the raising operator approach of [BKT2] to the theory of factorial Schur polynomials and equivariant Giambelli formulas.[1] In her dissertation, Wilson gave a direct proof that the $\Theta_\lambda(c\,|\,t)$ satisfy the *equivariant Chevalley rule* in $\mathbb{H}_T(\mathrm{IG}_k)$, i.e., the combinatorial formula for the expansion of the products $\sigma_1\cdot\sigma_\lambda$ in the basis of stable equivariant Schubert classes. This fits into a program for proving Theorem 1, and its extension to equivariant

[1] Wilson actually worked with the principal specialization $\Theta_\lambda(x,z\,|\,t)$ of $\Theta_\lambda(c\,|\,t)$ in the ring of type C double Schubert polynomials of [IMN1].

quantum cohomology, by applying Mihalcea's characterization theorem [Mi1, Cor. 8.2] (see §3). The proof of the Chevalley rule for the $\Theta_\lambda(c\,|\,t)$ which we present in §1 and §2 uses the technical tools for working with raising operators constructed in [BKT2], but the argument is much simpler, since it avoids the general Substitution Rule employed in op. cit.

It was conjectured by Wilson in [W] that the $\Theta_\lambda(c\,|\,t)$ represent the stable equivariant Schubert classes on IG. Ikeda and Matsumura [IM, Thm. 1.2] established this result, which is the main ingredient behind the new presentation of the equivariant cohomology ring $\mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$ displayed in Theorem 1. A key technical achievement in their proof was to show that the $\Theta_\lambda(c\,|\,t)$ are compatible with the action of the (left) divided difference operators on $C^{(k)}[t]$. In Section 6, we explain how this fact may be combined with a formula for the equivariant Schubert class of a point on $\mathrm{IG}(n-k,2n)$, to obtain a proof of Wilson's conjecture; the argument found in [IM] is different, and uses localization in equivariant cohomology.

It is easy to see that the polynomial $\Theta_\lambda(c\,|\,t)$ may be written formally as a sum of Schur Pfaffians (cf. Proposition 2). This identity was used in [IM] in the proof of the compatibility of the $\Theta_\lambda(c\,|\,t)$ with divided differences, allowing them to avoid the language of raising operators entirely. In Section 5, we eliminate this feature of their argument, and work directly with the raising operator expressions $R^\lambda$. This makes the proof more transparent, and leads naturally to a companion theory of *double eta polynomials* $H_\lambda(c\,|\,t)$, which represent the equivariant Schubert classes on even orthogonal Grassmannians; see [T4] for further details.

The role of the first author of this article has been mostly expository, extending the point of view found in [T2, T3] to the present setting, and comparing the results with various earlier formulas in the literature. We emphasize that the double theta polynomials $\Theta_\lambda(c\,|\,t)$ defined here are new, and are not a special case of the type C double Schubert polynomials of [IMN1], which represent the equivariant Schubert classes on complete symplectic flag varieties. Recently, Hudson, Ikeda, Matsumura, and Naruse [HIMN] proved an analogue of our cohomological formula in connective $K$-theory, and Anderson and Fulton [AF] obtained related Chern class formulas for more general degeneracy loci. The latter work features a powerful geometric approach to the theory, which is uniform for all four classical Lie types.

This paper is organized as follows. Section 1 establishes some preliminary lemmas which are required in our proof that the $\Theta_\lambda(c\,|\,t)$ obey the equivariant Chevalley rule (Theorem 2); the latter is completed in Sections 2 and 3. Although Theorem 2 is a consequence of Theorem 1, it is a much earlier result, and the independent proof given here exhibits alternating properties of the $\Theta_\lambda(c\,|\,t)$ which are useful in other contexts. Section 4 is concerned with how the $\Theta_\lambda(c\,|\,t)$ behave in special cases, and relates them to some earlier formulas. In Section 5, we observe that the action of the divided differences on $C^{(k)}[t]$ lifts to $\mathbb{Z}[c,t]$, and prove the key result about them and the $\Theta_\lambda(c\,|\,t)$ (Proposition 5) using our raising operator approach. The proof of Theorem 1 and its corollaries is contained in Section 6.

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## 1. Preliminary results

We begin with a purely formal lemma regarding the elements $c_p^r$ of $\mathbb{Z}[c,t]$. For $r<0$ we let $t_r := t_{-r}$.

**Lemma 1.** *Suppose that $p,r\in\mathbb{Z}$.*

(a) *Assume that $r>0$. Then we have*

$$c_p^r = c_p^{r-1} - t_r\, c_{p-1}^r.$$

(b) *Assume that $r\leq 0$. Then we have*

$$c_p^r = c_p^{r-1} + t_{r-1}\, c_{p-1}^r.$$

*Proof.* For any positive integer $m$, we have the basic equations

$$e_j^m(-t) = e_j^{m-1}(-t) - t_m\, e_{j-1}^{m-1}(-t) \quad\text{and}\quad h_j^m(-t) = h_j^{m-1}(-t) - t_m\, h_{j-1}^m(-t).$$

If $r>0$, it follows that

$$c_p^r = \sum_{j=0}^p c_{p-j}\, h_j^r(-t) = \sum_{j=0}^p c_{p-j}(h_j^{r-1}(-t) - t_r h_{j-1}^r(-t))$$

and the result of part (a) is proved. For (b), let $m := |r|+1$ and compute

$$\begin{aligned} c_p^r = \sum_{j=0}^{m-1} c_{p-j}\, e_j^{m-1}(-t) &= \sum_{j=0}^{m-1} c_{p-j}(e_j^m(-t) + t_m\, e_{j-1}^{m-1}(-t)) \\ &= \sum_{j=0}^{m-1} c_{p-j}\, e_j^m(-t) + t_m \sum_{j=1}^{m-1} c_{p-j}\, e_{j-1}^{m-1}(-t) \\ &= \sum_{j=0}^{m} c_{p-j}\, e_j^m(-t) + t_m \sum_{s=0}^{m-1} c_{p-1-s}\, e_s^{m-1}(-t). \end{aligned}$$

(In the last equality, notice that we have added the term

$$A := c_{p-m}\,(-t_1)\cdots(-t_m)$$

to the first sum, and added $-A$ to the second sum.) The result follows. □

Let $\Delta^\circ := \{(i,j)\in\mathbb{N}\times\mathbb{N} \mid 1\leq i<j\}$, equipped with the partial order $\leq$ defined by $(i',j')\leq(i,j)$ if and only if $i'\leq i$ and $j'\leq j$. A finite subset $D$ of $\Delta^\circ$ is a *valid set of pairs* if it is an order ideal in $\Delta^\circ$. An *outer corner* of a valid set of pairs $D$ is a pair $(i,j)\in\Delta^\circ\smallsetminus D$ such that $D\cup\{(i,j)\}$ is also a valid set of pairs.

**Definition 2.** Given a valid set of pairs $D$ and an integer sequence $\mu$, we define sequences $a=a(D)$ and $\gamma=\gamma(D,\mu)$ by

$$a_j(D) := \#\{i<j \mid (i,j)\notin D\} \quad\text{and}\quad \gamma_j(D,\mu) := k+1-\mu_j+a_j(D)$$

and the raising operator

$$R^D := \prod_{i<j}(1-R_{ij}) \prod_{i<j\,:\,(i,j)\in D}(1+R_{ij})^{-1}.$$

Finally, let

$$T(D,\mu) := R^D\, c_\mu^{\gamma(D,\mu)}.$$

Let $\epsilon_j$ denote the $j$-th standard basis vector in $\mathbb{Z}^\ell$. The next result is the analogue of [BKT2, Lemma 1.4] that we require here.

**Lemma 2.** *Let $(i,j)$ be an outer corner of the valid set of pairs $D$, and suppose that the integer sequence $\mu$ satisfies $\mu_i > k \geq \mu_j$. Let $\rho := r\epsilon_j$ for some integer $r \leq 1$ and set $\gamma := \gamma(D, \mu+\rho)$. Then we have*

$$T(D,\mu+\rho) = T(D\cup(i,j),\mu+\rho) + T(D\cup(i,j),\mu+R_{ij}\rho) + (t_{\gamma_i-1}-t_{\gamma_j})\,R^{D\cup(i,j)}\,c^{\gamma}_{\mu+\rho-\epsilon_j}$$

*in $\mathbb{Z}[c,t]$. In particular, if $\mu_i+\mu_j+\rho_j = 2k+1+a_j(D)$, then*

$$T(D,\mu+\rho) = T(D\cup(i,j),\mu+\rho) + T(D\cup(i,j),\mu+R_{ij}\rho). \tag{8}$$

*Proof.* Observe that since $(i,j)$ is an outer corner of $D$, we have

$$a_i(D) = a_i(D\cup(i,j)) = 0,$$

while $a_j(D\cup(i,j)) = a_j(D)-1$. It follows that

$$\gamma(D\cup(i,j),\mu+\rho) = \gamma(D,\mu+\rho)-\epsilon_j.$$

For any integer $p$, Lemma 1(a) gives

$$c_p^{\gamma_j} = c_p^{\gamma_j-1} - t_{\gamma_j}\,c_{p-1}^{\gamma_j}$$

and hence

$$R^{D\cup(i,j)}\,c^{\gamma}_{\mu+\rho} = T(D\cup(i,j),\mu+\rho) - t_{\gamma_j}R^{D\cup(i,j)}\,c^{\gamma}_{\mu+\rho-\epsilon_j}. \tag{9}$$

Notice that $\gamma_i = k+1-\mu_i \leq 0$. Therefore Lemma 1(b) gives

$$c_{p+1}^{\gamma_i} = c_{p+1}^{\gamma_i-1} + t_{\gamma_i-1}\,c_p^{\gamma_i}$$

and we deduce that

$$R^{D\cup(i,j)}\,c^{\gamma}_{\mu+R_{ij}\rho} = T(D\cup(i,j),\mu+R_{ij}\rho) + t_{\gamma_i-1}\,R^{D\cup(i,j)}\,c^{\gamma}_{\mu+\rho-\epsilon_j}. \tag{10}$$

Since we have

$$T(D,\mu+\rho) = R^D\,c^{\gamma}_{\mu+\rho} = R^{D\cup(i,j)}\,c^{\gamma}_{\mu+\rho} + R^{D\cup(i,j)}\,c^{\gamma}_{\mu+R_{ij}\rho},$$

the first equality follows by combining (9) with (10). To prove (8), note that $|\gamma_i-1| = \mu_i-k$ and $\gamma_j = k+1+a_j(D)-\mu_j-\rho_j$, therefore $t_{\gamma_i-1}-t_{\gamma_j} = 0$. □

**Lemma 3.** *Let $\lambda = (\lambda_1,\ldots,\lambda_{j-1})$ and $\mu = (\mu_{j+2},\ldots,\mu_\ell)$ be integer vectors. Let $D$ be a valid set of pairs such that $(j,j+1)\notin D$ and for each $h<j$, $(h,j)\in D$ if and only if $(h,j+1)\in D$. Then for any integers $r$ and $s$, we have*

$$T(D,(\lambda,r,s,\mu)) = -T(D,(\lambda,s-1,r+1,\mu))$$

*in $\mathbb{Z}[c,t]$. In particular, if $\lambda_{j+1} = \lambda_j+1$, then $T(D,\lambda)=0$.*

*Proof.* If $\alpha = (\alpha_1,\ldots,\alpha_\ell)$ is any integer sequence and $1\leq j<\ell$, define

$$s_j(\alpha) := (\alpha_1,\ldots,\alpha_{j-1},\alpha_{j+1},\alpha_j,\alpha_{j+2},\ldots,\alpha_\ell).$$

The hypotheses on $D$ and $j$ imply that

$$\gamma(D,(\lambda,s-1,r+1,\mu)) = s_j(\gamma(D,(\lambda,r,s,\mu))).$$

The argument is now the same as in the proof of [BKT2, Lemma 1.2]; the details are spelled out in [W, Lemma 10]. □

Throughout this paper, we will often write equalities that hold only in the rings $C^{(k)}$ and $C^{(k)}[t]$, where we have imposed the relations (3) on the generators $c_p$. Whenever these relations are needed, we will emphasize this by noting that the equalities are true in $C^{(k)}[t]$ rather than in $\mathbb{Z}[c,t]$.

**Lemma 4.** *Given $p>k$ and $q>k$, we have*

$$\frac{1-R_{12}}{1+R_{12}}\, c_{(p,q)}^{(k+1-p,k+1-q)} = -\frac{1-R_{12}}{1+R_{12}}\, c_{(q,p)}^{(k+1-q,k+1-p)}$$

*in $C^{(k)}[t]$.*

*Proof.* The relations (3) readily imply that

$$\frac{1-R_{12}}{1+R_{12}}\, c_{(r,s)} = -\frac{1-R_{12}}{1+R_{12}}\, c_{(s,r)}$$

whenever $r+s>2k$ (see [BKT2, Eqn. (9)]). For any raising operator $R$, we have

$$R\, c_{(p,q)}^{(k+1-p,k+1-q)} = \sum_{i=0}^{p}\sum_{j=0}^{q} R\, c_{(p-i,q-j)}\, e_i^{p-k-1}(-t) e_j^{q-k-1}(-t).$$

We deduce that

$$\begin{aligned}\frac{1-R_{12}}{1+R_{12}}\, c_{(p,q)}^{(k+1-p,k+1-q)} &= \sum_{i=0}^{p}\sum_{j=0}^{q} \frac{1-R_{12}}{1+R_{12}}\, c_{(p-i,q-j)}\, e_i^{p-k-1}(-t) e_j^{q-k-1}(-t) \\ &= -\sum_{j=0}^{q}\sum_{i=0}^{p} \frac{1-R_{12}}{1+R_{12}}\, c_{(q-j,p-i)}\, e_j^{q-k-1}(-t) e_i^{p-k-1}(-t) \\ &= -\frac{1-R_{12}}{1+R_{12}}\, c_{(q,p)}^{(k+1-q,k+1-p)}.\end{aligned}$$

□

Lemma 4 admits the following generalization.

**Lemma 5.** *Let $\lambda=(\lambda_1,\ldots,\lambda_{j-1})$ and $\mu=(\mu_{j+2},\ldots,\mu_\ell)$ be integer vectors. Let $D$ be a valid set of pairs such that $(j,j+1)\in D$ and for each $h>j+1$, $(j,h)\in D$ if and only if $(j+1,h)\in D$. If $r>k$ and $s>k$, then we have*

$$T(D,(\lambda,r,s,\mu)) = -T(D,(\lambda,s,r,\mu))$$

*in $C^{(k)}[t]$. In particular, if $\lambda_{j+1}=\lambda_j$, then $T(D,\lambda)=0$.*

*Proof.* The hypotheses on $D$ and $j$ imply that

$$\gamma(D,(\lambda,s,r,\mu)) = s_j(\gamma(D,(\lambda,r,s,\mu))).$$

The rest of the proof follows [BKT2, Lemma 1.3], using Lemma 4 in place of [BKT2, Eqn. (9)]; see [W, Lemma 11] for the details. □

## 2. The Chevalley formula for double theta polynomials

We begin this section with a basis theorem for the $\mathbb{Z}[t]$-algebra $C^{(k)}[t]$.

**Proposition 1.** *The monomials $c_\lambda$, the single theta polynomials $\Theta_\lambda(c)$, and the double theta polynomials $\Theta_\lambda(c\,|\,t)$ form three $\mathbb{Z}[t]$-bases of $C^{(k)}[t]$, as $\lambda$ runs over all $k$-strict partitions.*

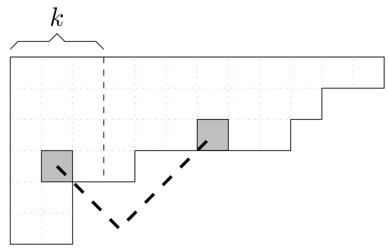


FIGURE 1. Two $k$-related boxes in a $k$-strict Young diagram.

*Proof.* It follows from [BKT2, Prop. 5.2] that the monomials $c_\lambda$ and the single theta polynomials $\Theta_\lambda(c)$ for $\lambda$ a $k$-strict partition form two $\mathbb{Z}$-bases of $C^{(k)}$. We deduce that these two families are also $\mathbb{Z}[t]$-bases of $C^{(k)}[t]$. By expanding the raising operator definition of $\Theta_\lambda(c\,|\,t)$, we obtain that

$$\Theta_\lambda(c\,|\,t) = c_\lambda + \sum_\mu a_{\lambda\mu}\, c_\mu$$

where $a_{\lambda\mu} \in \mathbb{Z}[t]$ and the sum is over $k$-strict partitions $\mu$ with $\mu \succ \lambda$ in dominance order or $|\mu| < |\lambda|$. Therefore, the $\Theta_\lambda(c\,|\,t)$ for $\lambda$ $k$-strict form another $\mathbb{Z}[t]$-basis of $C^{(k)}[t]$. □

For the remainder of this section, $\Theta_\lambda$ will be used to denote $\Theta_\lambda(c\,|\,t)$. To state the Chevalley formula for the double theta polynomials $\Theta_\lambda$, we need to recall certain definitions from [BKT1, §1.2] for the Pieri products with the divisor class.

We say that the box $[r,c]$ in row $r$ and column $c$ of a Young diagram is *$k$-related* to the box $[r',c']$ if $|c-k-1|+r = |c'-k-1|+r'$. The two grey boxes in the Young diagram of Figure 1 are $k$-related.

For any two $k$-strict partitions $\lambda$ and $\mu$, we write $\lambda \to \mu$ if $\mu$ may be obtained by (i) adding a box to $\lambda$ or (ii) removing $r$ boxes from one of the first $k$ columns of $\lambda$ and adding $r+1$ boxes to a single row of the result, so that the removed boxes and the bottom box of $\mu$ in that column are each $k$-related to one of the added boxes. If $\lambda \to \mu$, let $e_{\lambda\mu} = 2$ if $\mu \supset \lambda$ and the added box in $\mu$ is not in column $k+1$ and is not $k$-related to a bottom box in one of the first $k$ columns of $\lambda$, and otherwise set $e_{\lambda\mu} = 1$.

Fix a $k$-strict partition $\lambda$ and let $\ell = \ell(\lambda)$ denote the *length* of $\lambda$, that is, the number of non-zero parts $\lambda_i$. Recall that the *$k$-length* of $\lambda$ is

$$\ell_k = \ell_k(\lambda) := \#\{\lambda_i \mid \lambda_i > k\}.$$

**Theorem 2** ([W]). *The equation*

$$\Theta_1 \cdot \Theta_\lambda = \left(\sum_{j=k+1}^{k+\ell} t_j + \sum_{j=1}^{\ell_k} t_{\lambda_j-k} - \sum_{j=\ell_k+1}^{\ell} t_{\beta_j(\lambda)}\right)\Theta_\lambda + \sum_{\lambda\to\mu} e_{\lambda\mu}\,\Theta_\mu \tag{11}$$

*holds in $C^{(k)}[t]$, where the sum is over all $k$-strict partitions $\mu$ with $\lambda \to \mu$.*

*Proof.* We begin by studying the partitions $\mu$ which appear on the right hand side of the product rule (11). It is an easy exercise to show that $\mathcal{C}(\mu) \supset \mathcal{C}(\lambda)$ for all such $\mu$ (a more general result is proved in [BKT2, Lemma 2.1]).

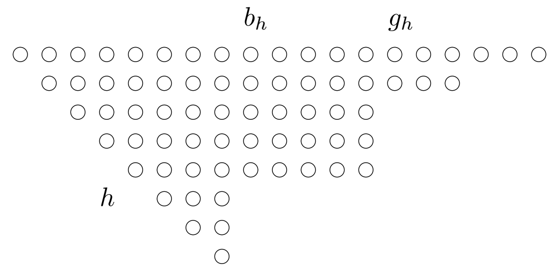


FIGURE 2. A set of pairs $\mathcal{C}$ with $h = 6$, $b_h = 10$, and $g_h = 15$.

**Definition 3.** Suppose that $\mu$ is a $k$-strict partition such that $\lambda \to \mu$. We say that $\mu$ and the Chevalley term $\Theta_\mu$ are *of type I* if $\mathcal{C}(\mu) = \mathcal{C}(\lambda)$; otherwise, $\mu$ and $\Theta_\mu$ are said to be *of type II*.

Suppose that $\lambda \to \mu$. It is then straightforward to check the following assertions: First, if $\mu$ is of type I, then $e_{\lambda\mu} = 1$. Second, if $\mu$ is of type II, then exactly one of the following three possibilities holds: (a) $e_{\lambda\mu} = 2$, in which case $\mu \smallsetminus \lambda$ is a single box in some row $h$, and $\mathcal{C}(\mu) \smallsetminus \mathcal{C}(\lambda)$ consists of a single pair in row $h$; (b) $\mu \smallsetminus \lambda$ is a single box in some row $h$ with $\mu_h \leq k$, and the pairs in $\mathcal{C}(\mu) \smallsetminus \mathcal{C}(\lambda)$ are all contained in column $h$; or (c) $\mu \smallsetminus \lambda$ has $r+1$ boxes in some row $h$ and $\mathcal{C}(\mu) \smallsetminus \mathcal{C}(\lambda)$ consists of $r$ pairs in row $h$. We say that $\mu$ and the Chevalley term $\Theta_\mu$ are of type I, II(a), II(b), or II(c), respectively.

Define $\mathcal{C} := \{(i,j) \in \mathcal{C}(\lambda) \mid j \leq \ell\}$. We represent the valid set of pairs $\mathcal{C}$ as positions above the main diagonal of a matrix, denoted by dots in Figure 2. For $1 \leq h \leq \ell_k + 1$, we let $b_h := \min\{j > \ell_k \mid (h,j) \notin \mathcal{C}\}$ and $g_h := b_{h-1}$ (by convention, we set $g_1 = \ell + 1$). Figure 2 illustrates these invariants.

**Lemma 6.** *If $2 \leq h \leq \ell_k + 1$ then we have $\lambda_{h-1} - \lambda_h \geq g_h - b_h + 1$.*

*Proof.* The result is the same as [BKT2, Lemma 3.5], but we repeat the argument here for completeness. The inequality is clear if $b_h = g_h$, as $\lambda$ is $k$-strict and $h \leq \ell_k + 1$. If $b_h < g_h$, then since $(h-1, g_h - 1) \in \mathcal{C}$ and $(h, b_h) \notin \mathcal{C}$, we obtain

$$\lambda_{h-1} - \lambda_h > (2k + g_h - h - \lambda_{g_h - 1}) + (\lambda_{b_h} + h - b_h - 2k) \geq g_h - b_h.$$

□

For any $h \geq 1$, let $\lambda^h := \lambda + \epsilon_h$. Set $\gamma' := (\beta_1(\lambda), \ldots, \beta_\ell(\lambda), 0)$ and

$$\gamma := \gamma(\mathcal{C}, \lambda^{\ell+1}) = (\beta_1(\lambda), \ldots, \beta_\ell(\lambda), k + \ell).$$

For any $d \geq 1$ define the raising operator $R^\lambda_d$ by

$$R^\lambda_d := \prod_{1 \leq i < j \leq d} (1 - R_{ij}) \prod_{i<j \,:\, (i,j) \in \mathcal{C}} (1 + R_{ij})^{-1}.$$

We compute that

$$c_1 \cdot \Theta_\lambda = c_1 \cdot R^\lambda_\ell \, c_\lambda^{\beta(\lambda)} = R^\lambda_{\ell+1} \cdot \prod_{i=1}^{\ell} (1 - R_{i,\ell+1})^{-1} \, c^{\gamma'}_{\lambda^{\ell+1}}$$

$$= R^\lambda_{\ell+1} \cdot \prod_{i=1}^{\ell} (1 + R_{i,\ell+1}) \, c^{\gamma'}_{\lambda^{\ell+1}} = R^\lambda_{\ell+1} \cdot (1 + \sum_{i=1}^{\ell} R_{i,\ell+1}) \, c^{\gamma'}_{\lambda^{\ell+1}}$$

and therefore

$$c_1\cdot\Theta_\lambda=\sum_{h=1}^{\ell+1}R^{\mathcal{C}}\,c^{\gamma'}_{\lambda^h}. \tag{12}$$

We wish to replace $\gamma'$ by $\gamma$ in (12); note that this substitution only affects the term $R^{\mathcal{C}}\,c^{\gamma'}_{\lambda^{\ell+1}}$. Since $c^0_1=c_1=c^{k+\ell}_1+\sum_{j=1}^{k+\ell}t_j$, we have

$$R^{\mathcal{C}}\,c^{\gamma'}_{\lambda^{\ell+1}}=R^{\mathcal{C}}\,c^{\gamma}_{\lambda^{\ell+1}}+\left(\sum_{j=1}^{k+\ell}t_j\right)\Theta_\lambda$$

and hence

$$c_1\cdot\Theta_\lambda=\left(\sum_{j=1}^{k+\ell}t_j\right)\Theta_\lambda+\sum_{h=1}^{\ell+1}R^{\mathcal{C}}\,c^{\gamma}_{\lambda^h}. \tag{13}$$

We examine the terms $R^{\mathcal{C}}\,c^{\gamma}_{\lambda^h}$ for $h\in[1,\ell+1]$, which appear on the right hand side of equation (13). If $h\leq\ell_k$, then $\lambda_h>k$ and $\gamma_h=k+1-\lambda_h\leq 0$. For any integer $p$, Lemma 1(b) gives

$$c^{\gamma_h}_{p+1}=t_{\lambda_h-k}\,c^{k+1-\lambda_h}_p+c^{k-\lambda_h}_{p+1}.$$

It follows that

$$R^{\mathcal{C}}\,c^{\gamma}_{\lambda^h}=t_{\lambda_h-k}\,\Theta_\lambda+T(\mathcal{C},\lambda^h).$$

If $\ell_k<h\leq\ell$, then $\lambda_h\leq k$ and $\gamma_h=k+1+a_h-\lambda_h>0$. Lemma 1(a) gives

$$c^{\gamma_h}_{p+1}=-t_{\gamma_h}c^{\gamma_h}_p+c^{\gamma_h-1}_{p+1}$$

and hence

$$R^{\mathcal{C}}\,c^{\gamma}_{\lambda^h}=-t_{\gamma_h}\Theta_\lambda+T(\mathcal{C},\lambda^h).$$

Finally, we have $R^{\mathcal{C}}\,c^{\gamma}_{\lambda^{\ell+1}}=T(\mathcal{C},\lambda^{\ell+1})$. We deduce that

$$c_1\cdot\Theta_\lambda=\left(\sum_{j=1}^{k+\ell}t_j+\sum_{j=1}^{\ell_k}t_{\lambda_j-k}-\sum_{j=\ell_k+1}^{\ell}t_{\beta_j(\lambda)}\right)\Theta_\lambda+\sum_{h=1}^{\ell+1}T(\mathcal{C},\lambda^h). \tag{14}$$

We claim that

$$\sum_{h=1}^{\ell+1}T(\mathcal{C},\lambda^h)=\sum_{\lambda\to\mu}e_{\lambda\mu}\,\Theta_\mu. \tag{15}$$

The proof of (15) proceeds by showing that for every $h\in[1,\ell+1]$, the term $T(\mathcal{C},\lambda^h)$ is equal to (i) zero, or (ii) a Chevalley term $\Theta_\mu$, or (iii) a sum $\Theta_\mu+\Theta_{\mu'}$ of two distinct Chevalley terms, exactly one of which is of type II(a). The terms $T(\mathcal{C},\lambda^h)$ are decomposed exactly as in the non-equivariant setting of [BKT2], using Lemmas 2, 3, and 5 as the main tools. However the analysis here is much easier, so we provide a short self-contained argument below.

Given any integer sequence $\mu$, define a weight condition $\mathrm{W}(i,j)$ on $\mu$ by

$$\mathrm{W}(i,j)\ :\ \mu_i+\mu_j>2k+j-i\,.$$

**Case 1.** Terms $T(\mathcal{C},\lambda^h)$ with $(h-1,h)\notin\mathcal{C}$ (this condition always holds if $h>\ell_k+1$). Notice that we must have $\lambda_h\leq k$.

**a)** Suppose that there is no outer corner $(r,h)$ of $\mathcal{C}$ in column $h$ such that $\mathrm{W}(r,h)$ holds for $\lambda^h$. If $\lambda_h = \lambda_{h-1}$, we have $T(\mathcal{C},\lambda^h)=0$ by Lemma 3 with $j=h-1$ (note that $(r,h-1)\in\mathcal{C}$ implies that $(r,h)\in\mathcal{C}$, so the conditions of the lemma hold). Otherwise, $T(\mathcal{C},\lambda^h)=\Theta_{\lambda^h}$ is a Chevalley term of type I (in the special case that $h=\ell_k+1$ we must have $\lambda_h<k$, so the result is still true).

**b)** If there exists an outer corner $(r,h)$ of $\mathcal{C}$ in column $h$ such that $\mathrm{W}(r,h)$ holds for $\lambda^h$, then let $s<h$ be maximal such that $\mathrm{W}(s,h)$ holds for $\lambda^h$. Observe that in this case,

$$\lambda_s+\lambda_{h-1}\geq\lambda_s+\lambda_h\geq 2k+h-s>2k+(h-1)-s,$$

and hence $(s,h-1)\in\mathcal{C}$ or $s=h-1$. Notice also that we must have $\lambda_{i-1}=\lambda_i+1$ for $i=r+1,\ldots,s$.

By definition, we have $\gamma_h=k+1+a_h-\lambda_h$ and $\gamma_r=k+1-\lambda_r$. We claim that $\lambda_r+\lambda_h=2k+a_h$. Indeed, we know that $(r,h)\notin\mathcal{C}$ and $(\lambda_h+1)+\lambda_r>2k+h-r$, therefore $\lambda_r+\lambda_h=2k+h-r$. The claim now follows, since $a_h=h-r$.

Let

$$E_j:=\{(r,h),\ldots,(j,h)\},\quad \text{for } j\in[r,s].$$

Using Lemma 2 and the claim gives

$$T(\mathcal{C},\lambda^h)=T(\mathcal{C}\cup E_r,\lambda^h)+T(\mathcal{C}\cup E_r,\lambda^r).$$

Continuing in the same manner, we obtain

$$T(\mathcal{C},\lambda^h)=T(\mathcal{C}\cup E_s,\lambda^h)+\sum_{j=r}^{s}T(\mathcal{C}\cup E_j,\lambda^j).$$

Since $\lambda_{i-1}=\lambda_i+1$ for $i=r+1,\ldots,s$, Lemma 5 implies that all terms in the sum over $j$ vanish, except for the initial one. Therefore, we have

$$T(\mathcal{C},\lambda^h)=T(\mathcal{C}\cup E_r,\lambda^r)+T(\mathcal{C}\cup E_s,\lambda^h).$$

If $\lambda_h=\lambda_{h-1}$, then Lemma 3 implies that $T(\mathcal{C}\cup E_s,\lambda^h)=0$. This must be the case since $\lambda_h+1>\lambda_{h-1}$ implies that $(s+1,h)$ is not an outer corner of $\mathcal{C}\cup E_s$. If $h=\ell_k+1$ and $\lambda_h=k$, then $\lambda_{h-1}=k+1$, $s=h-1$, and $T(\mathcal{C}\cup E_s,\lambda^h)=T(\mathcal{C}\cup E_{h-1},\lambda^h)=0$, by Lemma 5. The terms that remain after these cancellations are Chevalley terms; note that $T(\mathcal{C}\cup E_r,\lambda^r)=\Theta_{\lambda^r}$ is of type II(a), and $T(\mathcal{C}\cup E_s,\lambda^h)=\Theta_{\lambda^h}$ is of type II(b).

**Case 2.** Terms $T(\mathcal{C},\lambda^h)$ with $(h-1,h)\in\mathcal{C}$ (and hence $h\leq\ell_k+1$). Notice that $a_h=0$ for all such $h$, so $\gamma_h=k+1-\lambda_h$.

**a)** Suppose that there is no outer corner $(h,c)$ of $\mathcal{C}$ such that $\mathrm{W}(h,c)$ holds for $\lambda^h$. If $\lambda_h>k$ and $\lambda_{h-1}=\lambda_h+1$, then we have $T(\mathcal{C},\lambda^h)=0$ by Lemma 5. In all other cases, $T(\mathcal{C},\lambda^h)=\Theta_{\lambda^h}$ is a Chevalley term of type I.

**b)** Suppose that there exists an outer corner $(h,c)$ of $\mathcal{C}$ such that $\mathrm{W}(h,c)$ holds for $\lambda^h$. It follows that $b_h=c$ and $\lambda_h+\lambda_c=2k+c-h$. Setting $g=g_h$, we have $(h-1,g)\notin\mathcal{C}$, and hence $\lambda_{h-1}+\lambda_g\leq 2k+g-h+1$. Lemma 6 implies that

$$\begin{aligned} g-c+1 &\leq \lambda_{h-1}-\lambda_h=\lambda_{h-1}+\lambda_c-2k-c+h \\ &\leq (2k+g-h+1-\lambda_g)+\lambda_c-2k-c+h=g-c+1+(\lambda_c-\lambda_g). \end{aligned}$$

We deduce that if $\lambda_c=\lambda_g$, then $\lambda_{h-1}-\lambda_h=g-c+1$.

Note that we have $\lambda_h>k$. Let $d\in[c,g)$ be maximal such that $\lambda_c=\lambda_d$. Define

$$E'_j:=\{(h,c),\ldots,(h,j)\}\ \text{ and }\ R_j:=R_{hc}\cdots R_{hj}.$$

Applying Lemma 2 repeatedly, and using $\lambda_h + \lambda_c = 2k + c - h$, we obtain

$$T(\mathcal{C},\lambda^h) = T(\mathcal{C}\cup E'_c,\lambda^h) + T(\mathcal{C}\cup E'_d, R_d\lambda^h) + \sum_{j=c+1}^{d} T(\mathcal{C}\cup E'_j, R_{j-1}\lambda^h). \tag{16}$$

Lemma 3 implies that all the terms in the sum in (16) are zero. Therefore

$$T(\mathcal{C},\lambda^h) = T(\mathcal{C}\cup E'_c,\lambda^h) + T(\mathcal{C}\cup E'_d, R_d\lambda^h).$$

If $\lambda_c > \lambda_g$, then we obtain two Chevalley terms; the first summand $\Theta_{\lambda^h}$ is of type II(a), and the second $\Theta_{R_d\lambda^h}$ is of type II(c), with $r = d - c + 1$. If $\lambda_c = \lambda_g$, then since $\lambda_{h-1} - \lambda_h = g - c + 1$, we conclude from Lemma 5 that the last term $T(\mathcal{C}\cup E'_d, R_d\lambda^h)$ vanishes.

The claim (15) now follows easily, by combining the above two cases. Observe in particular that each of the Chevalley terms $\Theta_\mu$ of type II(a) appears twice in the decomposition of the left hand side of (15): once from a term $T(\mathcal{C},\lambda^h)$ with $\mu \neq \lambda^h$, in Case 1(b), and once from a term $T(\mathcal{C},\lambda^h)$ with $\mu = \lambda^h$, in Case 2(b). Finally, equations (14) and (15) imply (11), since $\Theta_1 = c_1^k = c_1 - (t_1 + \cdots + t_k)$. □

**Example 1.** Let $k = 2$ and consider the 2-strict partition $\lambda = (7,4,3,2,1,1)$. The element of the Weyl group $W_8$ associated to $\lambda$ is $w_\lambda = (4,8,\overline{5},\overline{2},\overline{1},3,6,7)$ ($w_\lambda$ is defined in §3). We have $\mathcal{C} = \{(1,2),(1,3),(1,4),(2,3)\}$, $\gamma = \gamma(\mathcal{C},\lambda^7) = (-4,-1,0,3,6,7,8)$, $\ell_k = 3$, and $\ell = 6$. The Chevalley formula in this case is

$$\begin{aligned}\Theta_1\cdot\Theta_\lambda = (t_1+t_2+t_4+2t_5+t_8)\,\Theta_\lambda + \Theta_{(7,4,3,2,1,1,1)} + \Theta_{(7,4,3,2,2,1)} \\ + 2\,\Theta_{(7,5,3,2,1,1)} + \Theta_{(7,6,3,1,1,1)} + 2\,\Theta_{(8,4,3,2,1,1)} + \Theta_{(10,4,3,2)}.\end{aligned}$$

In the above rule, the partition $(7,4,3,2,1,1,1)$ is of type I, $(7,5,3,2,1,1)$ and $(8,4,3,2,1,1)$ are of type II(a), $(7,4,3,2,2,1)$ is of type II(b), and $(7,6,3,1,1,1)$ and $(10,4,3,2)$ are of type II(c). The terms $T(\mathcal{C},\lambda^h)$ in the sum (15) are expanded as follows:

$$T(\mathcal{C},\lambda^1) = T(\mathcal{C}\cup E'_5,\lambda^1) + T(\mathcal{C}\cup E'_6, R_6\lambda^1) = \Theta_{(8,4,3,2,1,1)} + \Theta_{(10,4,3,2)}\,;$$
$$T(\mathcal{C},\lambda^2) = T(\mathcal{C}\cup E'_4,\lambda^2) + T(\mathcal{C}\cup E'_4, R_4\lambda^2) = \Theta_{(7,5,3,2,1,1)} + \Theta_{(7,6,3,1,1,1)}\,;$$
$$T(\mathcal{C},\lambda^3) = T(\mathcal{C},\lambda^6) = 0\,; \qquad T(\mathcal{C},\lambda^7) = \Theta_{(7,4,3,2,1,1,1)}\,;$$
$$T(\mathcal{C},\lambda^4) = T(\mathcal{C}\cup E_2,\lambda^2) = \Theta_{(7,5,3,2,1,1)}\,;$$
$$T(\mathcal{C},\lambda^5) = T(\mathcal{C}\cup E_1,\lambda^5) + T(\mathcal{C}\cup E_1,\lambda^1) = \Theta_{(7,4,3,2,2,1)} + \Theta_{(8,4,3,2,1,1)}\,.$$

## 3. Transition to the hyperoctahedral group

The Weyl group for the root system of type $\mathrm{C}_n$ is the *hyperoctahedral group* $W_n$ of signed permutations on the set $\{1,\ldots,n\}$. We adopt the notation where a bar over an integer denotes a negative sign. The group $W_n$ is generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$ and the sign change $s_0(1) = \overline{1}$. The natural inclusion $W_n \hookrightarrow W_{n+1}$ is defined by adding the fixed point $n+1$, and we set $W_\infty = \cup_n W_n$. The length of an element $w$ in $W_\infty$ is denoted by $\ell(w)$.

An element $w \in W_\infty$ is called *k-Grassmannian* if we have $\ell(ws_i) = \ell(w) + 1$ for all $i \neq k$; the set of all $k$-Grassmannian elements is denoted by $W^{(k)}$. The elements of $W^{(k)}$ are the minimal length coset representatives in $W_\infty/W_{(k)}$, where $W_{(k)}$ is the subgroup of $W_\infty$ generated by the $s_i$ for $i \neq k$. There is a canonical bijection between $k$-strict partitions and $W^{(k)}$ (see [BKT2, §6]). The element of

$W^{(k)}$ corresponding to $\lambda$ is denoted by $w_\lambda$, and lies in $W_n$ if and only if $\lambda$ fits inside an $(n-k)\times(n+k)$ rectangle; the set of all such $k$-strict partitions is denoted by $\mathcal{P}(k,n)$. Suppose that $w_\lambda = (w_1,\ldots,w_n) \in W_n$, so that $0 < w_1 < \cdots < w_k$ and $w_{k+1} < \cdots < w_n$. Then we deduce from [BKT1, §4.1 and §4.4] that the bijection is given by the equations

$$\lambda_i = \begin{cases} k + |w_{k+i}| & \text{if } w_{k+i} < 0, \\ \#\{p \leq k \,:\, w_p > w_{k+i}\} & \text{if } w_{k+i} > 0. \end{cases} \tag{17}$$

Moreover, we have

$$\mathcal{C}(\lambda) = \{(i,j) \in \Delta^\circ \mid w_{k+i} + w_{k+j} < 0\} \tag{18}$$

and

$$\beta_j(\lambda) = \begin{cases} w_{k+j} + 1 & \text{if } w_{k+j} < 0, \\ w_{k+j} & \text{if } w_{k+j} > 0. \end{cases} \tag{19}$$

We are now ready to justify that the formula displayed in Theorem 2 coincides with the equivariant Chevalley formula for the products $\sigma_1 \cdot \sigma_\lambda$ in the stable ring $\mathbb{H}_T(\mathrm{IG}_k)$. It follows from [BKT1, Thm. 1.1] and [IMN1, Lemma 6.8] that

$$\sigma_1 \cdot \sigma_\lambda = (\sigma_1|_{w_\lambda}) \cdot \sigma_\lambda + \sum_{\lambda \to \mu} e_{\lambda\mu}\, \sigma_\mu,$$

where $\sigma_1|_{w_\lambda} \in \mathbb{Z}[t]$ denotes the localization of the stable equivariant Schubert class $\sigma_1$ at the torus-fixed point indexed by $w_\lambda$. The polynomial $\sigma_1|_{w_\lambda}$ may be computed e.g. as the image of the type C double Schubert polynomial representing $\sigma_1$ under the homomorphism $\Phi_{w_\lambda}$ of [IMN1, §6.1]; this gives

$$\sigma_1|_{w_\lambda} = 2\sum_{j=1}^{\ell_k(\lambda)} t_{\lambda_j - k} + \sum_{j=1}^{k}(t_{w_j} - t_j). \tag{20}$$

The fact that the term (20) coincides with the $t$-linear coefficient in equation (11) amounts to the identity

$$\sum_{j=1}^{k} t_{w_j} + \sum_{j=1}^{\ell_k} t_{\lambda_j - k} + \sum_{j=\ell_k+1}^{\ell} t_{\beta_j(\lambda)} = \sum_{j=1}^{k+\ell} t_j. \tag{21}$$

The equations (19) imply that

$$\sum_{j=1}^{\ell_k} t_{\lambda_j - k} + \sum_{j=\ell_k+1}^{\ell} t_{\beta_j(\lambda)} = \sum_{j=1}^{\ell_k} t_{|w_{k+j}|} + \sum_{j=\ell_k+1}^{\ell} t_{w_{k+j}}.$$

Since $w$ is a signed *permutation*, the equality (21) follows.

A theorem of Mihalcea [Mi1, Cor. 8.2] proves that the equivariant Chevalley rule characterizes the $\mathbb{Z}[t]$-algebra $\mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$, when it can be verified for a given $\mathbb{Z}[t]$-basis of the latter ring. However, an analogous characterization is not available for the stable equivariant cohomology ring $\mathbb{H}_T(\mathrm{IG}_k)$. To complete the proof that the polynomials $\Theta_\lambda(c\,|\,t)$ represent equivariant Schubert classes, Theorem 2 could be combined with a *vanishing theorem* that determines their behavior under the natural projection $\pi_n$ of §6.1 to the finite ring $\mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$. The vanishing is easy to show using Lemma 1 in the Lagrangian case when $k=0$, and similarly for maximal orthogonal and type A Grassmannians. This method gives new, intrinsic proofs that double Schur $S$-, $Q$-, and $P$-polynomials represent equivariant Schubert classes, which apply to equivariant *quantum* cohomology as well, along the lines of [Mi2, IMN2]. It would be desirable to extend these arguments to arbitrary isotropic Grassmannians; for the non-equivariant case, see [BKT3, BKT4].

## 4. Examples and relations with other polynomials

In this section, we will prove some general facts about the double theta polynomials $\Theta_\lambda(c\,|\,t)$ and study how they specialize and relate to earlier formulas.

**4.1.** Let $\alpha = \{\alpha_j\}_{1\leq j\leq \ell}$ and $\rho = \{\rho_j\}_{1\leq j\leq \ell}$ be two integer sequences. Consider the following Schur type determinant:

$$S_\alpha^\rho(c\,|\,t) := \prod_{i<j}(1-R_{ij})\,c_\alpha^\rho = \det\left(c_{\alpha_i+j-i}^{\rho_i}\right)_{1\leq i,j\leq \ell}$$

which specializes when $t=0$ to the *Schur $S$-polynomial*

$$S_\alpha(c) := \det(c_{\alpha_i+j-i})_{1\leq i,j\leq \ell}.$$

Furthermore, define the Schur type Pfaffian:

$$Q_\alpha^\rho(c\,|\,t) := \prod_{i<j}\frac{1-R_{ij}}{1+R_{ij}}\,c_\alpha^\rho = \text{Pfaffian}\left(\frac{1-R_{12}}{1+R_{12}}\,c_{\alpha_i,\alpha_j}^{\rho_i,\rho_j}\right)_{i<j}.$$

(The matrix in the Pfaffian has size $r\times r$, where $r$ is the least even integer such that $r\geq \ell$.) This specializes when $t=0$ to the *Schur $Q$-polynomial*

$$Q_\alpha(c) := \text{Pfaffian}\left(\frac{1-R_{12}}{1+R_{12}}c_{\alpha_i,\alpha_j}\right)_{i<j},$$

where the entries in the Pfaffian satisfy

$$\frac{1-R_{12}}{1+R_{12}}\,c_{\alpha_i,\alpha_j} = c_{\alpha_i}c_{\alpha_j} - 2c_{\alpha_i+1}c_{\alpha_j-1} + 2c_{\alpha_i+2}c_{\alpha_j-2} - \cdots$$

Let $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ be a $k$-strict partition of length $\ell$, and define

$$\mathcal{C}_\ell(\lambda) := \{(i,j)\in\Delta^\circ\,|\,j\leq \ell \ \text{ and } \ \lambda_i+\lambda_j > 2k+j-i\}.$$

and

$$\mathcal{A}_\ell(\lambda) := \{(i,j)\in\Delta^\circ\,|\,j\leq \ell \ \text{ and } \ \lambda_i+\lambda_j \leq 2k+j-i\}.$$

Recall that a *multiset* is a set in which members are allowed to appear more than once, with finite multiplicity. The next result generalizes formulas found in [BKT2, Prop. 5.9] and [BKT3, Prop. 2] to the case of double theta polynomials.

**Proposition 2.** *For any $k$-strict partition $\lambda$ of length $\ell$, we have*

$$\Theta_\lambda(c\,|\,t) = \prod_{(i,j)\in\mathcal{C}_\ell(\lambda)}(1-R_{ij}+R_{ij}^2-\cdots)\,S_\lambda^{\beta(\lambda)}(c\,|\,t)$$

*and*

$$\Theta_\lambda(c\,|\,t) = \prod_{(i,j)\in\mathcal{A}_\ell(\lambda)}(1+R_{ij})\,Q_\lambda^{\beta(\lambda)}(c\,|\,t) = \sum_{\nu\in\mathcal{N}} Q_\nu^{\beta(\lambda)}(c\,|\,t)$$

*in $\mathbb{Z}[c,t]$, where $\mathcal{N}$ is the multiset of integer vectors defined by*

$$\mathcal{N} := \left\{\prod_{(i,j)\in S} R_{ij}\,\lambda \mid S\subset \mathcal{A}_\ell(\lambda)\right\}.$$

*Proof.* The equalities are an immediate consequence of the formal raising operator identities

$$R^\lambda_\ell = \prod_{(i,j)\in\mathcal{C}_\ell(\lambda)} (1+R_{ij})^{-1} \prod_{1\le i<j\le \ell}(1-R_{ij})$$

and

$$R^\lambda_\ell = \prod_{(i,j)\in\mathcal{A}_\ell(\lambda)} (1+R_{ij}) \prod_{1\le i<j\le \ell}\frac{1-R_{ij}}{1+R_{ij}}.$$

□

**4.2.** We next examine the extreme cases when the parts of the partition $\lambda$ are all small or all large, respectively, with respect to $k$.

If $\lambda_i+\lambda_j\le 2k+j-i$ for all $i<j$, then we have

$$\Theta_\lambda(c\,|\,t) = S^{\beta(\lambda)}_\lambda(c\,|\,t) = \det\left(c^{k+i-\lambda_i}_{\lambda_i+j-i}\right)_{1\le i,j\le\ell}. \tag{22}$$

If $\lambda_i\le k$ for all $i\ge 1$, the determinant in (22) is a *double Schur $S$-polynomial*. Suppose that $x=(x_1,x_2,\ldots)$ is another list of commuting independent variables. The specialization of $\Theta_\lambda(c\,|\,t)$ which maps each $c_r$ to $e_r(x_1,\ldots,x_k)$ is called a factorial Schur function, and was studied in [BL] and [M, I.3, Ex. 20].

**Proposition 3.** *Suppose that $\lambda_i+\lambda_j\le 2k+j-i$ for all $i<j$. Then we have*

$$\Theta_\lambda(c\,|\,t) = \sum_{\mu\subset\lambda} S_\mu(c)\det\left(h^{k+i-\lambda_i}_{\lambda_i-\mu_j+j-i}(-t)\right)_{1\le i,j\le\ell} \tag{23}$$

*in $\mathbb{Z}[c,t]$, where the sum is over all partitions $\mu$ contained in $\lambda$.*

*Proof.* Set $\delta_\ell=(\ell-1,\ldots,1,0)$. For any composition $\alpha$, we have $S_{\lambda-\alpha}(c)=0$ in $\mathbb{Z}[c]$, unless $\lambda-\alpha+\delta_\ell = w(\mu+\delta_\ell)$ for some partition $\mu\subset\lambda$, in which case $S_{\lambda-\alpha}(c)=(-1)^wS_\mu(c)$. For any raising operator $R$, we have

$$R\,c^{\beta(\lambda)}_\lambda = \sum_{\alpha\ge 0} c_{R\lambda-\alpha}\,h^{\beta(\lambda)}_\alpha(-t) = \sum_{\alpha\ge 0}(R\,c_{\lambda-\alpha})\,h^{\beta(\lambda)}_\alpha(-t)$$

where the sums are over all compositions $\alpha$. We therefore obtain

$$\Theta_\lambda(c\,|\,t) = \prod_{i<j}(1-R_{ij})\,c^{\beta(\lambda)}_\lambda = \sum_{\alpha\ge 0} S_{\lambda-\alpha}(c)\,h^{\beta(\lambda)}_\alpha(-t)$$

$$= \sum_{\mu\subset\lambda} S_\mu(c)\sum_{w\in S_\ell}(-1)^w h^{\beta(\lambda)}_{\lambda+\delta_\ell-w(\mu+\delta_\ell)}(-t) = \sum_{\mu\subset\lambda}S_\mu(c)\det\left(h^{k+i-\lambda_i}_{\lambda_i-\mu_j+j-i}(-t)\right)_{1\le i,j\le\ell}.$$

□

Suppose now that $\lambda_i>k$ for all nonzero parts $\lambda_i$. Then we have

$$\Theta_\lambda(c\,|\,t) = Q^{\beta(\lambda)}_\lambda(c\,|\,t) = \text{Pfaffian}\left(\frac{1-R_{12}}{1+R_{12}}\,c^{k+1-\lambda_i,k+1-\lambda_j}_{(\lambda_i,\lambda_j)}\right)_{i<j}. \tag{24}$$

In 2000, Kazarian [Ka] proved a multi-Pfaffian formula for those equivariant Schubert classes $[X_\lambda]^T$ on isotropic Grassmannians where all the subbundles $F_j$ involved in the defining conditions (1) are isotropic. It is easily seen that the Pfaffian (24) maps to Kazarian's formula under the geometrization map $\pi_n$ of §6.1.

When $k=0$, the Pfaffian in (24) becomes the *double Schur $Q$-polynomial*

$$Q_\lambda(c\,|\,t) = \text{Pfaffian}\left(\frac{1-R_{12}}{1+R_{12}}\,c^{1-\lambda_i,1-\lambda_j}_{(\lambda_i,\lambda_j)}\right)_{i<j}. \tag{25}$$

Define the Schur $Q$-functions $q_j(x)$ by the generating series

$$\prod_{i=1}^{\infty}\frac{1+x_iz}{1-x_iz}=\sum_{j=0}^{\infty}q_j(x)z^j.$$

Then the specialization of $Q_\lambda(c\,|\,t)$ which maps $c_r$ to $q_r(x)$ for every $r\geq 1$ may be identified with the factorial Schur $Q$-functions introduced by Ivanov [Iv]. The connection between Ivanov's theory and the equivariant cohomology of the Lagrangian Grassmanian $\mathrm{LG}(n,2n)$ was pointed out by Ikeda [I].

**Proposition 4.** *Suppose that $\lambda_i>k$ for $1\leq i\leq \ell$. Then we have*

$$\Theta_\lambda(c\,|\,t)=\sum_{\mu\subset\lambda}Q_\mu(c)\det\left(h^{k+1-\lambda_i}_{\lambda_i-\mu_j}(-t)\right)_{1\leq i,j\leq \ell} \tag{26}$$

*in $C^{(k)}[t]$, where the sum is over all strict partitions $\mu\subset\lambda$ with $\mu_i>k$ for $1\leq i\leq\ell$.*

*Proof.* Arguing as in the proof of Proposition 3, we obtain

$$\Theta_\lambda(c\,|\,t)=\sum_{\alpha\geq 0}Q_{\lambda-\alpha}(c)\,h_\alpha^{\beta(\lambda)}(-t)$$

where the sum is over all compositions $\alpha$. Observe that the polynomial $h_\alpha^{\beta(\lambda)}(-t)=e_\alpha^{-\beta(\lambda)}(-t)$ vanishes unless $\alpha_j\leq -\beta_j(\lambda)=\lambda_j-k-1$ for each $j$, which implies that $\lambda_j-\alpha_j>k$ for all $j$. Therefore, according to [BKT2, Lemma 1.3], the $Q$-polynomial $Q_{\lambda-\alpha}(c)$ is alternating in the indices $\lambda_j-\alpha_j$. It follows that

$$\Theta_\lambda(c\,|\,t)=\sum_{\mu\subset\lambda}Q_\mu(c)\sum_{w\in S_\ell}(-1)^w e^{-\beta(\lambda)}_{\lambda-w(\mu)}(-t)=\sum_{\mu\subset\lambda}Q_\mu(c)\det\left(e^{\lambda_i-k-1}_{\lambda_i-\mu_j}(-t)\right)_{1\leq i,j\leq\ell}$$

where the sums over $\mu\subset\lambda$ are as in the statement of the proposition. □

**4.3.** The general degeneracy locus formulas of [T2, Eqn. (26) and Thm. 3] produce many different answers for each indexing Weyl group element $w$, depending on a choice of two compatible sequences $\mathfrak{a}$ and $\mathfrak{b}$, which specify the symmetries of the resulting expression. However, once this latter choice is fixed, these formulas are *uniquely determined* (cf. [T3, §5.3]). We recall here the formula given in [T3, Ex. 12(b)] for the equivariant Schubert classes on symplectic Grassmannians.

The symmetric group $S_n$ is the subgroup of $W_n$ generated by the transpositions $s_i$ for $1\leq i\leq n-1$; we let $S_\infty:=\cup_n S_n$ be the corresponding subgroup of $W_\infty$. For every permutation $u\in S_\infty$, let $\mathfrak{S}_u(t)$ denote the type A Schubert polynomial of Lascoux and Schützenberger [LS] indexed by $u$ (our notation follows [T3, §5]). We say that a factorization $w_\lambda=uv$ in $W_\infty$ is reduced if $\ell(w_\lambda)=\ell(u)+\ell(v)$. In any such factorization, the right factor $v=w_\mu$ is also $k$-Grassmannian for some $k$-strict partition $\mu$.

Consider the polynomial

$$\Omega_\lambda(c\,|\,t):=\sum_{uw_\mu=w_\lambda}\Theta_\mu(c)\mathfrak{S}_{u^{-1}}(-t) \tag{27}$$

where the sum is over all reduced factorizations $uw_\mu=w_\lambda$ with $u\in S_\infty$. One can show that the partitions $\mu$ which appear in (27) are all contained in $\lambda$ and satisfy $\ell_k(\mu)=\ell_k(\lambda)=\ell_k$. It is furthermore proved in [T2] that $\Omega_\lambda(c\,|\,t)$ represents the stable equivariant Schubert class $\sigma_\lambda$ indexed by $\lambda$. Theorem 1 implies that $\Omega_\lambda(c\,|\,t)$ must agree with $\Theta_\lambda(c\,|\,t)$ up to the relations (3) among the variables $c_r$ in $C^{(k)}$

(see Corollary 2). We give a direct proof of this below, in the two extreme cases considered in §4.2.

If $\lambda_i \leq k$ for all $i$, then $w_\lambda$ is a Grassmannian element of $S_\infty$, and hence is *fully commutative* in the sense of [St]. There is a one-to-one correspondence between reduced factorizations $w_\lambda = uw_\mu$ and partitions $\mu \subset \lambda$. For each such $\mu$, we have $\Theta_\mu(c) = S_\mu(c)$, and the Schubert polynomial $\mathfrak{S}_{u^{-1}}(-t)$ is a flagged skew Schur polynomial. The latter may be computed using [BJS, Thm. 2.2], which proves that $\mathfrak{S}_{u^{-1}}(-t) = \det\left(h^{k+i-\lambda_i}_{\lambda_i-\mu_j+j-i}(-t)\right)_{1\leq i,j\leq \ell}$. Formula (23) shows that in this case, $\Omega_\lambda(c\,|\,t) = \Theta_\lambda(c\,|\,t)$ as polynomials in $\mathbb{Z}[c,t]$.

If $\lambda_i > k$ for $1 \leq i \leq \ell$, then $w_\lambda$ will not be a fully commutative element of $W_\infty$, in general. However there is a one-to-one correspondence between reduced factorizations $w_\lambda = uw_\mu$ with $u \in S_\infty$ and $k$-strict partitions $\mu \subset \lambda$ with $\ell_k(\mu) = \ell_k(\lambda) = \ell$. For each such $\mu$, we have $\Theta_\mu(c) = Q_\mu(c)$, and can use e.g. the reduced words for $w_\lambda$ and $w_\mu$ given in [T1, Ex. 9] to deduce that $u$ is fully commutative. According to [BJS, Thm. 2.2] again, the Schubert polynomial $\mathfrak{S}_{u^{-1}}(-t)$ is equal to $\det\left(h^{k+1-\lambda_i}_{\lambda_i-\mu_j}(-t)\right)_{1\leq i,j\leq \ell}$. We thus see that $\Omega_\lambda(c\,|\,t)$ agrees with the right hand side of formula (26), and hence $\Omega_\lambda(c\,|\,t) = \Theta_\lambda(c\,|\,t)$ in $C^{(k)}[t]$.

**Example 2.** Let $k = 0$, so that the partition $\lambda$ is strict, and $\Theta_\lambda(c)$ and $\Theta_\lambda(c\,|\,t)$ specialize to the single and double $Q$-polynomials $Q_\lambda(c)$ and $Q_\lambda(c\,|\,t)$, respectively. Following [T3, Ex. 12(b)], we have

$$\Omega_\lambda(c\,|\,t) = \sum_{\mu\subset\lambda} Q_\mu(c) \det\left(e^{\lambda_i-1}_{\lambda_i-\mu_j}(-t)\right)_{1\leq i,j\leq \ell} \tag{28}$$

summed over all strict partitions $\mu \subset \lambda$ with $\ell(\mu) = \ell(\lambda) = \ell$. Taking $\lambda = (3,1)$ in (28) gives

$$\Omega_{3,1}(c\,|\,t) = Q_{3,1}(c) - Q_{2,1}(c)(t_1+t_2) = (c_3c_1 - 2c_4) - (c_2c_1 - 2c_3)(t_1+t_2).$$

On the other hand, equation (25) above gives

$$\begin{aligned} Q_{3,1}(c\,|\,t) &= \frac{1-R_{12}}{1+R_{12}}\, c^{-2,0}_{(3,1)} = c_3^{-2}c_1^0 - 2\,c_4^{-2}c_0^0 \\ &= (c_3 - c_2(t_1+t_2) + c_1t_1t_2)c_1 - 2\,(c_4 - c_3(t_1+t_2) + c_2t_1t_2). \end{aligned}$$

Observe that $Q_{3,1}(c\,|\,t) - \Omega_{3,1}(c\,|\,t) = (c_1^2 - 2c_2)t_1t_2$, and thus the two polynomials are not equal in $\mathbb{Z}[c,t]$. However, since the relation $c_1^2 = 2c_2$ holds in $C^{(0)}$, we have agreement in $C^{(0)}[t]$, as expected.

## 5. Divided difference operators on $\mathbb{Z}[c,t]$

There is an action of $W_\infty$ on $\mathbb{Z}[c,t]$ by ring automorphisms, defined as follows. The simple reflections $s_i$ for $i > 0$ act by interchanging $t_i$ and $t_{i+1}$ and leaving all the remaining variables fixed. The reflection $s_0$ maps $t_1$ to $-t_1$, fixes the $t_j$ for $j \geq 2$, and satisfies, for each $p \geq 0$,

$$s_0(c_p) = c_p + 2\sum_{j=1}^{p}(-t_1)^j c_{p-j}.$$

The latter equation can be written in the form

$$s_0\left(\sum_{p=0}^{\infty} c_p u^p\right) = \frac{1-t_1u}{1+t_1u}\cdot\sum_{p=0}^{\infty} c_p u^p \tag{29}$$

where $u$ is a formal variable with $s_i(u) = u$ for each $i$. It follows that

$$s_0s_1s_0s_1\left(\sum_{p=0}^{\infty} c_p u^p\right) = \frac{(1-t_1u)(1-t_2u)}{(1+t_1u)(1+t_2u)}\cdot\sum_{p=0}^{\infty} c_p u^p = s_1s_0s_1s_0\left(\sum_{p=0}^{\infty} c_p u^p\right),$$

and therefore that the braid relations for $W_\infty$ are satisfied on $\mathbb{Z}[c,t]$.

For every $i \geq 0$, we define the *divided difference operator* $\partial_i$ on $\mathbb{Z}[c,t]$ by

$$\partial_0 f := \frac{f - s_0 f}{2t_1}, \qquad \partial_i f := \frac{f - s_i f}{t_{i+1} - t_i}, \quad \text{if } i \geq 1.$$

The operators $\partial_i$ correspond to the *left* divided differences $\delta_i$ studied in [IMN1] and [IM]. However, the definition given here is more general than that found in op. cit., since the $\partial_i$ act on the polynomial ring $\mathbb{Z}[c,t]$ rather than on its quotient $C^{(k)}[t]$. Note that for each $i \geq 0$, the $\partial_i$ satisfy the Leibnitz rule

$$\partial_i(fg) = (\partial_i f)g + (s_i f)\partial_i g.$$

We next recall from [IM, §5.1] some important identities satisfied by these operators, with statements and proofs in our setting.

**Lemma 7.** *We have*

$$s_i(c_p^r) = \begin{cases} c_p^r & \text{if } r \neq \pm i, \\ c_p^{i+1} + t_i c_{p-1}^{i+1} & \text{if } r = i > 0, \\ c_p^{-i+1} - t_{i+1} c_{p-1}^{-i+1} & \text{if } r = -i \leq 0. \end{cases}$$

*Proof.* Since $c_p^r$ is symmetric in $(t_1, \ldots, t_{|r|})$ when $r \neq 0$, the identity $s_i(c_p^r) = c_p^r$ for $r \neq \pm i$ is clear if $i > 0$. For $r > 0$, we have

$$\sum_{p=0}^{\infty} c_p^r u^p = \left(\sum_{i=0}^{\infty} c_i u^i\right)\prod_{j=1}^{r}\frac{1}{1+t_ju} \tag{30}$$

and we apply $s_0$ to both sides and use (29) to prove that $s_0(c_p^r) = c_p^r$ for all $p$; the argument when $r < 0$ is similar. The rest of the proof is straightforward. ☐

**Lemma 8.** *Suppose that $p, r \in \mathbb{Z}$.*

a) *For all $i \geq 0$, we have*

$$\partial_i c_p^r = \begin{cases} c_{p-1}^{r+1} & \text{if } r = \pm i, \\ 0 & \text{otherwise.} \end{cases}$$

b) *For all $i \geq 1$, we have*

$$\partial_i(c_p^{-i} c_q^i) = c_{p-1}^{-i+1} c_q^{i+1} + c_p^{-i+1} c_{q-1}^{i+1}.$$

*Proof.* For part (a), observe that if $r \neq \pm i$, then $\partial_i c_p^r = 0$ by Lemma 7. If $r = i > 0$, then we compute easily using (30) that

$$\partial_i\left(\sum_{p=0}^{\infty} c_p^r u^p\right) = \left(\sum_{p=0}^{\infty} c_p^r u^{p+1}\right)\prod_{j=1}^{r+1}\frac{1}{1+t_ju}$$

from which the desired result follows. Work similarly when $r = i = 0$ or $r = -i < 0$.

For part (b), we use the Leibniz rule and Lemmas 1 and 7 to compute

$$\begin{aligned}\partial_i(c_p^{-i}c_q^i) &= \partial_i(c_p^{-i})c_q^i + s_i(c_p^{-i})\partial_i(c_q^i) \\ &= c_{p-1}^{-i+1}c_q^i + (c_p^{-i+1} - t_{i+1}c_{p-1}^{-i+1})c_{q-1}^{i+1} \\ &= c_{p-1}^{-i+1}(c_q^i - t_{i+1}c_{q-1}^{i+1}) + c_p^{-i+1}c_{q-1}^{i+1} \\ &= c_{p-1}^{-i+1}c_q^{i+1} + c_p^{-i+1}c_{q-1}^{i+1}.\end{aligned}$$

□

The following result was proved differently in [IM], working in a ring $\mathcal{R}_\infty^{(k)}$ which is isomorphic to $C^{(k)}[t]$, and not in the polynomial ring $\mathbb{Z}[c,t]$.

**Proposition 5.** *Let $\lambda$ and $\mu$ be $k$-strict partitions such that $|\lambda| = |\mu| + 1$ and $w_\lambda = s_i w_\mu$ for some simple reflection $s_i \in W_\infty$. Then we have*

$$\partial_i \Theta_\lambda(c\,|\,t) = \Theta_\mu(c\,|\,t)$$

*in $\mathbb{Z}[c,t]$.*

*Proof.* Let $w = w_\lambda$, $\beta = \beta(\lambda)$, and $\beta' = \beta(\mu)$. Following [IM, Lemmas 3.4, 3.5], there are 4 possible cases for $w$, discussed below. In each case, we have $\mu \subset \lambda$, so that $\mu_p = \lambda_p - 1$ for some $p \geq 1$ and $\mu_j = \lambda_j$ for all $j \neq p$, and the properties listed are an easy consequence of equations (17), (18), and (19).

(a) $w = (\cdots \overline{1} \cdots)$. In this case we have $i = 0$, $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$, $\beta_p = i$, $\beta'_p = i+1$, while $\beta_j = \beta'_j$ for all $j \neq p$.

(b) $w = (\cdots i+1 \cdots i \cdots)$. In this case $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$, $\beta_p = i$, $\beta'_p = i+1$, and $\beta_j = \beta'_j$ for all $j \neq p$.

(c) $w = (\cdots i \cdots \overline{i+1} \cdots)$. In this case $\mathcal{C}(\lambda) = \mathcal{C}(\mu)$, $\beta_p = -i$ and $\beta'_p = -i+1$, and $\beta_j = \beta'_j$ for all $j \neq p$.

(d) $w = (\cdots \overline{i+1} \cdots i \cdots)$. In this case $\mathcal{C}(\lambda) = \mathcal{C}(\mu) \cup \{(p,q)\}$, where $w_{k+p} = -i-1$ and $w_{k+q} = i$. It follows that $\beta_p = -i$ and $\beta_q = i$. We see similarly that $\beta'_p = -i+1$ and $\beta'_q = i+1$, while $\beta_j = \beta'_j$ for all $j \notin \{p,q\}$.

In cases (a), (b), or (c), it follows using the Leibniz rule and Proposition 8(a) that for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, we have

$$\begin{aligned}\partial_i c_\alpha^{\beta(\lambda)} &= c_{(\alpha_1,\ldots,\alpha_{p-1})}^{(\beta_1,\ldots,\beta_{p-1})}\left(\partial_i(c_{\alpha_p}^{\beta_p})c_{(\alpha_{p+1},\ldots,\alpha_\ell)}^{(\beta_{p+1},\ldots,\beta_\ell)} + s_i(c_{\alpha_p}^{\beta_p})\partial_i(c_{(\alpha_{p+1},\ldots,\alpha_\ell)}^{(\beta_{p+1},\ldots,\beta_\ell)})\right) \\ &= c_{(\alpha_1,\ldots,\alpha_{p-1})}^{(\beta_1,\ldots,\beta_{p-1})}\left(c_{\alpha_p-1}^{\beta_p+1}c_{(\alpha_{p+1},\ldots,\alpha_\ell)}^{(\beta_{p+1},\ldots,\beta_\ell)} + s_i(c_{\alpha_p}^{\beta_p})\cdot 0\right) = c_{(\alpha_1,\ldots,\alpha_p-1,\ldots,\alpha_\ell)}^{(\beta_1,\ldots,\beta_p+1,\ldots,\beta_\ell)} = c_{\alpha-\epsilon_p}^{\beta(\mu)}.\end{aligned}$$

Since $\lambda - \epsilon_p = \mu$, we deduce that if $R$ is any raising operator, then

$$\partial_i R\, c_\lambda^{\beta(\lambda)} = \partial_i c_{R\lambda}^{\beta(\lambda)} = c_{R\lambda-\epsilon_p}^{\beta(\mu)} = R\, c_\mu^{\beta(\mu)}.$$

As $R^\lambda = R^\mu$, we conclude that

$$\partial_i \Theta_\lambda(c\,|\,t) = \partial_i R^\lambda c_\lambda^{\beta(\lambda)} = R^\mu c_\mu^{\beta(\mu)} = \Theta_\mu(c\,|\,t).$$

In case (d), it follows from the Leibniz rule as in the proof of Proposition 8(b) that for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, we have

$$\begin{aligned}\partial_i c_\alpha^{\beta(\lambda)} &= \partial_i c_{(\alpha_1,\ldots,\alpha_p,\ldots,\alpha_q,\ldots,\alpha_\ell)}^{(\beta_1,\ldots,-i,\ldots,i,\ldots,\beta_\ell)} \\ &= c_{(\alpha_1,\ldots,\alpha_p-1,\ldots,\alpha_q,\ldots,\alpha_\ell)}^{(\beta_1,\ldots,-i+1,\ldots,i+1,\ldots,\beta_\ell)} + c_{(\alpha_1,\ldots,\alpha_p,\ldots,\alpha_q-1,\ldots,\alpha_\ell)}^{(\beta_1,\ldots,-i+1,\ldots,i+1,\ldots,\beta_\ell)} = c_{\alpha-\epsilon_p}^{\beta(\mu)} + c_{\alpha-\epsilon_q}^{\beta(\mu)}.\end{aligned}$$

Since $\lambda - \epsilon_p = \mu$, we deduce that if $R$ is any raising operator, then

$$\partial_i R\, c_\lambda^{\beta(\lambda)} = \partial_i c_{R\lambda}^{\beta(\lambda)} = c_{R\lambda-\epsilon_p}^{\beta(\mu)} + c_{R\lambda-\epsilon_q}^{\beta(\mu)} = R\, c_\mu^{\beta(\mu)} + RR_{pq}\, c_\mu^{\beta(\mu)}.$$

As $R^\lambda + R^\lambda R_{pq} = R^\mu$, we conclude that

$$\partial_i \Theta_\lambda(c\,|\,t) = \partial_i R^\lambda c_\lambda^{\beta(\lambda)} = R^\lambda c_\mu^{\beta(\mu)} + R^\lambda R_{pq} c_\mu^{\beta(\mu)} = R^\mu c_\mu^{\beta(\mu)} = \Theta_\mu(c\,|\,t).$$

□

## 6. The proof of Theorem 1

**6.1. The geometrization map.** Let

$$0 \to E' \to E \to E'' \to 0$$

be the tautological sequence (6) of vector bundles over $\mathrm{IG}(n-k, 2n)$, and define the subbundles $F_j$ of $E$ for $0 \leq j \leq 2n$ as in the introduction. Let $IM_n := ET_n \times^{T_n} \mathrm{IG}$ denote the Borel mixing space for the action of the torus $T_n$ on IG. The $T_n$-equivariant vector bundles $E', E, E'', F_j$ over IG induce vector bundles over $IM_n$. Their Chern classes in $\mathrm{H}^*(IM_n, \mathbb{Z}) = \mathrm{H}^*_{T_n}(\mathrm{IG}(n-k, 2n))$ are called *equivariant Chern classes* and denoted by $c_p^T(E')$, $c_p^T(E)$, etc.

Recall that $c_p^T(E - E' - F_j)$ for $p \geq 0$ is defined by the total Chern class equation

$$c^T(E - E' - F_j) := c^T(E) c^T(E')^{-1} c^T(F_j)^{-1}.$$

Let $\mathrm{t}_i := -c_1^T(F_{n+1-i}/F_{n-i})$ for $1 \leq i \leq n$. Following [IMN1, §10] and [T2, Thm. 3], we define the *geometrization map* $\pi_n$ as the $\mathbb{Z}[t]$-algebra homomorphism

$$\pi_n : C^{(k)}[t] \to \mathrm{H}^*_{T_n}(\mathrm{IG}(n-k, 2n))$$

determined by setting

$$\pi_n(c_p) := c_p^T(E - E' - F_n) \ \text{ for all } p,$$

$$\pi_n(t_i) := \begin{cases} \mathrm{t}_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n. \end{cases}$$

Since $\mathrm{t}_1 \ldots, \mathrm{t}_r$ are the (equivariant) Chern roots of $F_{n+r}/F_n$ for $1 \leq r \leq n$, it follows that

$$\pi_n(c_p^r) = \sum_{j=0}^{p} c_{p-j}^T(E - E' - F_n) h_j^r(-\mathrm{t}) = c_p^T(E - E' - F_{n+r}) \tag{31}$$

for $-n \leq r \leq n$. This can be extended to $r \in \mathbb{Z}$ if we set $F_j = F_{2n} = E$ for $j > 2n$ and $F_j = 0$ for $j < 0$. For example, if $\lambda$ is any partition in $\mathcal{P}(k, n)$, then we have

$$\pi_n(\Theta_\lambda(c\,|\,t)) = \pi_n(R^\lambda\, c_\lambda^{\beta(\lambda)}) = R^\lambda\, c_\lambda^T(E - E' - F_{n+\beta(\lambda)}), \tag{32}$$

in agreement with formula (7).

The embedding of $W_n$ into $W_{n+1}$ defined in §3 induces maps of equivariant cohomology rings $\mathrm{H}^*_{T_{n+1}}(\mathrm{IG}(n+1-k,2n+2)) \to \mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$ which are compatible with the morphisms $\pi_n$. We thus obtain an induced $\mathbb{Z}[t]$-algebra homomorphism

$$\pi : C^{(k)}[t] \to \mathbb{H}_T(\mathrm{IG}_k),$$

which we will show has the properties listed in Theorem 1.

**6.2. The equivariant Schubert class of a point.** Fix a rank $n$ and let

$$\lambda_0 := (n+k, n+k-1, \ldots, 2k+1)$$

be the $k$-strict partition associated to the $k$-Grassmannian element of maximal length in $W_n$. In this short section we discuss three different proofs that

$$\pi_n(\Theta_{\lambda_0}(c\,|\,t)) = [X_{\lambda_0}]^{T_n}. \tag{33}$$

Equation (24) gives

$$\Theta_{\lambda_0}(c\,|\,t) = Q_{\lambda_0}^{(1-n,2-n,\ldots,-k)}(c\,|\,t)$$

and by applying (31) we obtain that

$$\pi_n(\Theta_{\lambda_0}(c\,|\,t)) = Q_{\lambda_0}(E - E' - F_{(1,2,\ldots,n-k)}). \tag{34}$$

Observe now that the right hand side of (34) coincides with Kazarian's multi-Pfaffian formula [Ka, Thm. 1.1] for the cohomology class of the degeneracy locus which correponds to $[X_{\lambda_0}]^{T_n}$ (compare with Corollary 1). Therefore (33) follows directly from a known result, which is proved geometrically in op. cit.

A second proof of (33) is obtained by using the equality

$$\Theta_{\lambda_0}(c\,|\,t) = \sum_{uw_\mu = w_{\lambda_0}} Q_\mu(c)\mathfrak{S}_{u^{-1}}(-t) = \Omega_{\lambda_0}(c\,|\,t) \tag{35}$$

in $C^{(k)}[t]$, which is shown in §4.3. We then appeal to [T2, Eqn. (26) and Thm. 3], which, in this special situation, give the equality $\pi_n(\Omega_{\lambda_0}(c\,|\,t)) = [X_{\lambda_0}]^{T_n}$. Note that the proof of this in op. cit. uses the fact that the single $Q$-polynomials $Q_\mu(c)$ in (35) represent the corresponding cohomological Schubert classes on IG. This last result is a special case of the main theorem of [BKT2].

Finally, a third proof of (33) is provided by Ikeda and Matsumura in [IM, §8.2], starting from the Pfaffian formula [IMN1, Thm. 1.2] for the equivariant Schubert class of a point on the complete symplectic flag variety $\mathrm{Sp}_{2n}/B$. From this, using the left divided differences, they derive a Pfaffian formula for the top equivariant Schubert class on any symplectic partial flag variety. In particular, one recovers the Pfaffian $\Theta_{\lambda_0}(c\,|\,t)$ which represents the class $[X_{\lambda_0}]^{T_n}$.

**6.3. Proof of Theorem 1.** We have shown in Proposition 1 that the $\Theta_\lambda(c\,|\,t)$ for $\lambda$ $k$-strict form a $\mathbb{Z}[t]$-basis of $C^{(k)}[t]$. Following [IM, §3.4], for any $k$-strict partition $\lambda \in \mathcal{P}(k,n)$, write $w_\lambda w_{\lambda_0} = s_{a_1}\cdots s_{a_r}$ as a product of simple reflections $s_{a_j}$ in $W_n$, with $r = |\lambda_0| - |\lambda|$. Since $w_{\lambda_0}^2 = 1$, we deduce from Proposition 5 that

$$\Theta_\lambda(c\,|\,t) = \partial_{a_1} \circ \cdots \circ \partial_{a_r}(\Theta_{\lambda_0}(c\,|\,t)) \tag{36}$$

holds in $\mathbb{Z}[c,t]$.

The action of the operators $\partial_i$ on $\mathbb{Z}[c,t]$ induces an action on the quotient ring $C^{(k)}[t]$. Moreover, the corresponding left divided differences $\delta_i$ on $H_n := \mathrm{H}^*_{T_n}(\mathrm{IG}(n-k,2n))$ from [IMN1, §2.5] are compatible with the geometrization map $\pi_n : C^{(k)}[t] \to H_n$. According to [IMN1, Prop. 2.3] (see also [T3, Eqn. (62)]), we have $\delta_i([X_\lambda]^{T_n}) = [X_\mu]^{T_n}$ whenever $|\lambda| = |\mu|+1$ and $w_\lambda = s_i w_\mu$ for some simple reflection $s_i$. It follows from this and equations (33) and (36) that

$$\pi_n(\Theta_\lambda(c\,|\,t)) = [X_\lambda]^{T_n}. \tag{37}$$

The fact that $\pi_n(\Theta_\lambda(c\,|\,t)) = 0$ whenever $\lambda \notin \mathcal{P}(k,n)$, or equivalently $w_\lambda \notin W_n$, is now a consequence of the vanishing property for equivariant Schubert classes (see e.g. [IMN1, Prop. 7.7]). The induced map $\pi : C^{(k)}[t] \to \mathbb{H}_T(\mathrm{IG}_k)$ satisfies $\pi(\Theta_\lambda(c\,|\,t)) = \sigma_\lambda$ for all $k$-strict $\lambda$, and is a $\mathbb{Z}[t]$-algebra isomorphism because the $\Theta_\lambda(c\,|\,t)$ and $\sigma_\lambda$ for $\lambda$ $k$-strict form $\mathbb{Z}[t]$-bases of the respective algebras.

We are left with proving the last assertion in Theorem 1, about the presentation of the $\mathbb{Z}[t]$-algebra $H_n$. Let $I_n$ be the ideal of $C^{(k)}[t]$ generated by the $\Theta_\lambda(c\,|\,t)$ for $\lambda \notin \mathcal{P}(k,n)$, and $J_n$ be the ideal of $C^{(k)}[t]$ generated by the relations (4) and (5). Since $J_n \subset I_n$, there is a surjection of $\mathbb{Z}[t]$-algebras

$$\psi : C^{(k)}[t]/J_n \to C^{(k)}[t]/I_n\,.$$

We have established that $H_n \cong C^{(k)}[t]/I_n$, so it suffices to show that $\mathrm{Ker}(\psi) = 0$.

Consider the ideal $I$ of $\mathbb{Z}[t]$ generated by the $t_i$ for all $i \geq 1$, and let $M := C^{(k)}[t]/J_n$. When $k \geq 1$, we have the formal identity

$$\Theta_{(1^p)}(c) = \det(c_{1+j-i})_{1\leq i,j\leq p} \tag{38}$$

which is a specialization of (22). We deduce from (38) and the presentation of $\mathrm{H}^*(\mathrm{IG}(n-k,2n))$ given in [BKT1, Thm. 1.2] that the polynomials $\Theta_\lambda(c)$ for $\lambda \in \mathcal{P}(k,n)$ generate $M/IM$ as a $\mathbb{Z}$-module. It follows from [Mi2, Lemma 4.1] that the $\Theta_\lambda(c\,|\,t)$ for $\lambda \in \mathcal{P}(k,n)$ generate $M$ as a $\mathbb{Z}[t]$-module. Now suppose that $\psi$ maps a general element $\sum_\lambda a_\lambda(t)\Theta_\lambda(c\,|\,t) + J_n$ of $M$ to zero, where $a_\lambda(t) \in \mathbb{Z}[t]$ and the sum is over $\lambda \in \mathcal{P}(k,n)$. Then $\sum_\lambda a_\lambda(t)\Theta_\lambda(c\,|\,t) \in I_n$. Since $I_n$ is equal to the $\mathbb{Z}[t]$-submodule of $C^{(k)}[t]$ with basis $\Theta_\lambda(c\,|\,t)$ for $\lambda \notin \mathcal{P}(k,n)$, we conclude that $a_\lambda(t) = 0$, for all $\lambda$. This completes the proof of Theorem 1, and Corollary 1 follows from equations (32) and (37).

According to [T2], the polynomials $\Omega_\lambda(c\,|\,t)$ of §4.3 represent the stable equivariant Schubert classes $\sigma_\lambda$ in $\mathbb{H}_T(\mathrm{IG}_k)$ under the geometrization map $\pi$. The next result is therefore an immediate consequence of Theorem 1.

**Corollary 2.** *Let $\lambda$ be any $k$-strict partition. Then we have*

$$\Theta_\lambda(c\,|\,t) = \sum_{uw_\mu = w_\lambda} \Theta_\mu(c)\mathfrak{S}_{u^{-1}}(-t) \tag{39}$$

*in the ring $C^{(k)}[t]$, where the sum is over all reduced factorizations $uw_\mu = w_\lambda$ with $u \in S_\infty$.*

We remark that the right hand side of (39) is the unique expansion of $\Theta_\lambda(c\,|\,t)$ as a $\mathbb{Z}$-linear combination in the product basis $\{\Theta_\mu(c)\mathfrak{S}_u(-t)\}$ of $C^{(k)}[t]$, where $\mu$ ranges over all $k$-strict partitions and $u$ lies in $S_\infty$. It is instructive to write equation (39) in the following way:

$$R^\lambda\, c_\lambda^{\beta(\lambda)} = \sum_{uw_\mu = w_\lambda} (R^\mu\, c_\mu)\mathfrak{S}_{u^{-1}}(-t). \tag{40}$$

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