THE CONNECTION BETWEEN REPRESENTATION THEORY AND SCHUBERT CALCULUS

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1. Introduction

The irreducible polynomial representations of the general linear group $GL_n(\mathbb{C})$ are parametrized by integer partitions λ with at most n parts. Given any two such representations V^λ and V^μ, one has a decomposition of the tensor product

$$V^\lambda \otimes V^\mu = \sum_\nu c_{\lambda\mu}^\nu \, V^\nu \tag{1}$$

into irreducible representations V^ν of GL_n.

Let $G(m,n)$ denote the Grassmannian of complex m-dimensional linear subspaces of $\mathbb{C}^{m+n}$. The cohomology ring of $G(m,n)$ has a natural geometric basis of Schubert classes σ_λ, and there is a cup product decomposition

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \, \sigma_\nu. \tag{2}$$

The structure constants $c_{\lambda\mu}^\nu$ determine the classical Schubert calculus on $G(m,n)$.

It has been known for some time that the integers $c_{\lambda\mu}^\nu$ in formulas (1) and (2) coincide, as long as the meaningless Schubert classes in (2) are interpreted as zero. Following the work of Giambelli [G1] [G2], this is proved formally by relating both products to the multiplication of Schur S-polynomials; a precise argument along these lines was given by Lesieur [Les]. It is natural to ask for a more direct, conceptual explanation of this fact. This question has appeared every so often in print; some recent examples are [F2, §6.2] and [Len, §1].

Our aim here is to describe a direct and natural connection between the representation theory of GL_n and the Schubert calculus, which goes via the Chern-Weil theory of characteristic classes. Indeed, since the Grassmannian is a universal carrier for the Chern classes of principal GL_n-bundles, it is not so surprising that the cohomology ring of $G(m,n)$ is related to the representation ring of GL_n. From this point of view, we can also understand why a result of this sort fails to hold for other types of Lie groups: what makes GL_n special is the fact that it sits naturally as a dense open subset of its own Lie algebra (see Sec. 2). The key observation is that the Chern-Weil homomorphism extends to a ring homomorphism from the (polynomial) representation ring $\mathrm{R}_+(GL_n)$ to $H^*(G(m,n))$, which sends the natural basis elements of the first ring to the Schubert classes.

The relation between Schubert calculus and the multiplication of Schur polynomials has been investigated before by Horrocks [Ho] and Carrell [C]. Although

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the approach in [Ho] is closest to the one here, the main ideas go back to the fundamental works of Chern [Ch1], Weil [W], and H. Cartan [Car]. We provide an exposition where the various ingredients from representation theory, differential geometry, topology of fiber bundles, and Schubert calculus are each presented in turn. In Sec. 6, we apply Grothendieck's construction of the Chern classes of Lie group representations to look for an analogue of these results in the other Lie types.

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2. Representations and Schur functors

We are concerned here with the polynomial representations of the general linear group $GL_n(\mathbb{C})$. A matrix representation $\pi : GL_n \to GL_N$ of GL_n is *polynomial* if the entries of $\pi(g)$ are polynomials in the entries of $g \in GL_n$. The character of π is the function $\chi : GL_n \to \mathbb{C}$ defined by $\chi(g) = \mathrm{Tr}(\pi(g))$. The polynomial representation ring $\mathrm{R} = \mathrm{R}_+(GL_n)$ is the $\mathbb{R}$-algebra generated by the polynomial characters of GL_n. The ring R may be identified with the real vector space spanned by the irreducible polynomial GL_n-representations (up to isomorphism), with the ring structure given by the tensor product. We use real instead of integer coefficients here because the Chern-Weil construction in the sequel will involve de Rham cohomology groups.

The group GL_n acts by conjugation on its Lie algebra $\mathfrak{gl}_n$ (the space of all $n \times n$ matrices). This induces a GL_n action on the ring $\mathrm{Pol}(\mathfrak{gl}_n)$ of polynomial functions on $\mathfrak{gl}_n$ with real coefficients; we denote by $\mathrm{Pol}(\mathfrak{gl}_n)^{GL_n}$ the corresponding ring of invariants. Since GL_n is a dense open subset of $\mathfrak{gl}_n$, any character $\chi \in \mathrm{R}$ extends to a unique element of $\mathrm{Pol}(\mathfrak{gl}_n)^{GL_n}$. Conversely, for any invariant polynomial $f \in \mathrm{Pol}(\mathfrak{gl}_n)^{GL_n}$, the restriction of f to GL_n is clearly a polynomial class function, and hence an element of R. Thus we obtain a canonical ring isomorphism

$$\phi : \mathrm{R}_+(GL_n) \to \mathrm{Pol}(\mathfrak{gl}_n)^{GL_n}. \tag{3}$$

In contrast, there is no satisfactory analogue of the morphism ϕ for the other types of Lie groups (see Sec. 6). Note that there is already the problem of defining 'polynomial representations' for a general complex Lie group.[1]

Following Schur [S1] [S2], the irreducible polynomial representations of GL_n are parametrized by integer partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$ of length (i.e. number of nonzero parts λ_i) at most n. For each partition λ there is a *Schur functor* $\mathfrak{s}_\lambda : \mathfrak{V} \to \mathfrak{V}$, where $\mathfrak{V}$ denotes the category of finite dimensional $\mathbb{C}$-vector spaces and $\mathbb{C}$-linear maps. If $V = \mathbb{C}^n$ is the standard representation of GL_n, then the irreducible representation corresponding to λ is the Schur module $V^\lambda = \mathfrak{s}_\lambda(V)$. In the language of Lie theory, V^λ is the GL_n-representation with highest weight λ. The character of V^λ is a *Schur polynomial* in the eigenvalues of $g \in GL_n$.

For completeness, we briefly describe Weyl's construction of $\mathfrak{s}_\lambda(V)$, for any complex vector space V. First, we identify the partition λ with its Young diagram of boxes; this is an array of $p = \sum \lambda_i$ boxes arranged in left-justified rows, with λ_i boxes in the ith row. Number the boxes in λ with the integers $1, \ldots, p$ in order

[1]For a connected reductive complex Lie group G with Lie algebra $\mathfrak{g}$, Knutson suggests to define a ring of polynomial representations of G as the image of the injective map $\mathrm{Pol}(\mathfrak{g})^G \to \mathrm{Pol}(G)^G$ which is induced by pullback along a generalized Cayley transform $G \to \mathfrak{g}$ (see [KM]).

going from left to right and top to bottom. The resulting standard tableaux T is illustrated below on the Young diagram of $\lambda = (4, 3, 2)$.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & 9 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

Let R (respectively C) denote the subgroup of the symmetric group S_p consisting of elements which permute the entries of each row (respectively column) of T among themselves. Consider the elements

$$a_\lambda = \sum_{u \in C} \operatorname{sgn}(u)\, u, \qquad b_\lambda = \sum_{v \in R} v$$

of the group algebra $\mathbb{C}[S_p]$, and define the *Young symmetrizer* $c_\lambda = a_\lambda b_\lambda$.

For any vector space V, S_p acts on the right on the p-fold tensor product $M = V^{\otimes p}$ by permuting the factors, and this action commutes with the left action of $GL(V)$ on M. The *Schur module* $\mathfrak{s}_\lambda(V)$ is defined as the image of the map $M \to M$ that is right multiplication by c_λ. This construction is functorial, in the sense that a linear map $f : V \to W$ of vector spaces determines a linear map $\mathfrak{s}_\lambda(f) : \mathfrak{s}_\lambda(V) \to \mathfrak{s}_\lambda(W)$, with $\mathfrak{s}_\lambda(f_1 \circ f_2) = \mathfrak{s}_\lambda(f_1) \circ \mathfrak{s}_\lambda(f_2)$ and $\mathfrak{s}_\lambda(\mathrm{id}_V) = \mathrm{id}_{\mathfrak{s}_\lambda(V)}$. In passing, we note that the *Specht modules* $\mathbb{C}[S_p]c_\lambda$ form a complete set of irreducible representations of the symmetric group S_p, as λ varies over all partitions of p. For more information on Schur modules the reader may consult e.g. [FH], [F1, §8], [Gr], and [Ma].

Example 1. Let $p = 2$ and write $S_2 = \{1, \sigma\}$. The two relevant partitions λ are (2) and $(1, 1)$, with respective Young symmetrizers $c_2 = 1 + \sigma$ and $c_{1,1} = 1 - \sigma$. For any vector space V, there is a decomposition $V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V$, and we see that $\mathfrak{s}_2(V) = \mathrm{Sym}^2 V$ and $\mathfrak{s}_{1,1}(V) = \wedge^2 V$.

Given an $n \times n$ matrix $A = \{a_{ij}\}$ of indeterminates, we define the *Schur matrix* to be $\mathfrak{s}_\lambda(A)$. This is a square matrix of size $\dim(V^\lambda)$ whose entries are polynomials in the a_{ij} with integer coefficients. Moreover, the map ϕ in (3) sends V^λ to the invariant polynomial $A \mapsto \mathrm{Tr}(\mathfrak{s}_\lambda(A))$. It is surprising that no explicit formula for the entries of $\mathfrak{s}_\lambda(A)$ is known, except in special cases. In general, there are several algorithms available for this computation, some rather classical (see [CLL], [Cl], [DKR], and [GK] for a sample).

Example 2. The fundamental representations of GL_n are the exterior powers $\wedge^k V$, for $V = \mathbb{C}^n$ and $k = 1, \ldots, n$, which correspond to the partitions $\lambda = (1^k)$. The Schur matrix $\wedge^k A$ has order $\binom{n}{k}$, and its rows (resp. columns) are indexed by the k-element subsets of the n rows (resp. columns) of A. The entries of $\wedge^k A$ are the determinants of the $k \times k$ minors in A. The corresponding invariant polynomials for $k = 1$ and $k = n$ are given by $\mathrm{Tr}(A)$ and $\det(A)$, respectively.

If A is a diagonal matrix with eigenvalues $x_1, \ldots, x_n$, then $\mathrm{Tr}(\mathfrak{s}_\lambda(A))$ is the Schur polynomial $s_\lambda(x_1, \ldots, x_n)$. There is much to say about these important polynomials (see e.g. [M, §I]), but we shall refrain from doing so because they are not used in the sequel. We simply note here that since the Schur polynomials form a basis of the ring of symmetric polynomials in $x_1, \dots, x_n$, it follows that the rings in (3) are both isomorphic to $\mathbb{R}[x_1, \dots, x_n]^{S_n}$.

3. The Chern-Weil homomorphism

The homomorphism which is the subject of this section is a fundamental tool in the theory of characteristic classes. Our main references are the monographs [D], [GHV], and [GuS], all of which contain detailed expositions of Chern-Weil theory and the related work of H. Cartan.

Consider a principal GL_n-bundle $P \to M$ over a differentiable manifold M, and choose an open cover $\{U_\alpha\}$ of M which trivializes P. The transition functions for P with respect to this cover are a system of morphisms $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n$ satisfying the cocycle condition $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ on the intersections $U_\alpha \cap U_\beta \cap U_\gamma$. We may identify $P \to M$ with the rank n complex vector bundle with the same transition functions $g_{\alpha\beta}$; in other words, with the associated vector bundle for the standard representation of GL_n on $\mathbb{C}^n$.

For each $g \in GL_n$, let $R_g : P \to P$ denote the right action of GL_n on P, and for each $x \in P$, let $v_x : \mathfrak{gl}_n \to T_x P$ be the differential of the map $g \mapsto xg$. A *connection* on P is a smooth $\mathfrak{gl}_n$-valued 1-form $\theta \in \mathcal{A}^1(P, \mathfrak{gl}_n)$ satisfying (i) $\theta_x v_x = \mathrm{id}$ for all $x \in P$, and (ii) $R_g^* \theta = \mathrm{Ad}(g^{-1})\theta$, for all $g \in GL_n$. Note that (ii) asserts that θ is equivariant with respect to the adjoint representation of GL_n. Connections always exist, provided the base manifold M admits partitions of unity.

The *curvature* $\Theta \in \mathcal{A}^2(P, \mathfrak{gl}_n)$ of the connection θ is defined by the Maurer-Cartan structure equation $\Theta = d\theta + \frac{1}{2}[\theta, \theta]$ (this is Élie Cartan, the father of Henri Cartan). A basic theorem of Weil states that for any homogeneous polynomial $f \in \mathrm{Pol}(\mathfrak{gl}_n)^{GL_n}$ of degree k, (i) the differential form $f(\frac{i}{2\pi}\Theta)$ is closed in $\mathcal{A}^{2k}(M, \mathbb{R})$, and (ii) its de Rham cohomology class does not depend on the connection θ. The resulting map

$$\mathrm{Pol}(\mathfrak{gl}_n)^{GL_n} \to H^*(M, \mathbb{R}); \qquad f \mapsto f(\frac{i}{2\pi}\Theta)$$

is an algebra homomorphism called the *Chern-Weil homomorphism*. We will continue to use cohomology with real coefficients unless otherwise indicated.

In topology, one constructs a contractible space EGL_n on which GL_n acts freely, and with quotient equal to the classifying space BGL_n. Every principal GL_n-bundle over M is a pullback of the universal bundle $EGL_n \to BGL_n$; in this way, the isomorphism classes of principal GL_n-bundles over M are in one-to-one correspondence with the homotopy classes of maps from M to BGL_n. Moreover, the previous definitions of connection and curvature have universal analogues, and there is a homomorphism $\mathrm{Pol}(\mathfrak{gl}_n)^{GL_n} \to H^*(BGL_n)$ (see [NR] or [D, §5,6] for a detailed construction). Since the quotient $GL_n/U(n)$ of GL_n by the unitary group $U(n)$ is diffeomorphic to a Euclidean space, the inclusion $U(n) \hookrightarrow GL_n$ induces an isomorphism $H^*(BGL_n) \to H^*(BU(n))$. We deduce that there is a universal Chern-Weil homomorphism

$$\psi : \mathrm{Pol}(\mathfrak{gl}_n)^{GL_n} \to H^*(BU(n)). \tag{4}$$

H. Cartan [Car] proved that the map ψ is an isomorphism of polynomial rings. The cohomology of $BU(n)$ is thus identified with the ring of characteristic classes of principal GL_n-bundles (or complex vector bundles). If the polynomial $f \in \mathrm{Pol}(\mathfrak{gl}_n)^{GL_n}$ has integer coefficients, then its image under the Chern-Weil map (4) lies in $H^*(BU(n), \mathbb{Z})$ (which we identify with its image in $H^*(BU(n), \mathbb{R})$).

Example 3. Let $E \to M$ be a rank n complex vector bundle over M. The Chern-Weil homomorphism sends the invariant polynomial $A \mapsto \mathrm{Tr}(\wedge^k A)$ of the previous section to the k-th Chern class $c_k(E) \in H^{2k}(M, \mathbb{Z})$.

4. Schubert varieties and Schubert forms

The cohomology ring of the Grassmannian $X = G(m,n)$ has a basis of Schubert classes σ_λ, one for each partition λ whose Young diagram is contained in an $n \times m$ rectangle (equivalently, λ is a partition of length at most n with $\lambda_1 \leqslant m$). The Schubert class σ_λ may be defined using the Poincaré duality isomorphism between cohomology and homology, as follows: σ_λ is the element of $H^*(X)$ whose cap product with the fundamental class of X is the homology class of a *Schubert variety* X_λ, described below. If the diagram of λ does not fit in the above rectangle, then we set $\sigma_\lambda = 0$.

To define X_λ, consider the fixed complete flag of subspaces

$$0 = F_0 \subset F_1 \subset \cdots \subset F_N = \mathbb{C}^N$$

where $N = m + n$ and $F_j = \mathbb{C}^j \times \{0\} \subset \mathbb{C}^N$, for $0 \leqslant j \leqslant N$. Let λ' denote the conjugate partition, whose diagram is the transpose of the diagram of λ. The variety X_λ consists of those m-dimensional subspaces U such that $\dim(U \cap F_{n+i-\lambda'_i}) \geqslant i$, for $1 \leqslant i \leqslant m$. The complex codimension of X_λ in X is given by the weight $|\lambda| = \sum \lambda_i$ of λ. Note that our indexing convention for Schubert varieties is the transpose of the usual one, as found for example in [F2, §4].

There is a tautological short exact sequence of vector bundles

$$0 \to S \to E \to Q \to 0 \tag{5}$$

over X, where $E = X \times \mathbb{C}^N$ is the trivial bundle of rank N over X, S denotes the tautological rank m subbundle of E, and Q the quotient bundle. The total space of S is the submanifold of the product $X \times \mathbb{C}^N$ consisting of those pairs $([U], v)$ with $v \in U$, and the projection $X \times \mathbb{C}^N \to X$ restricts to the map from S to X. Observe that the fiber of S (resp. Q) over a point $[U]$ in X which corresponds to the subspace $U \subset \mathbb{C}^N$ is given by U itself (resp. $\mathbb{C}^N/U$). From now on it will be convenient to work with the quotient bundle Q and to think of $X = G(m,n)$ as parametrizing rank n quotients of $\mathbb{C}^N$.

According to [BT, §23] (see also [MS, §14] and [Hu, §18]), the *infinite Grassmannian* $G(\infty, n) = \lim_{m\to\infty} G(m,n)$ provides a model for the classifying space $BU(n)$ of the unitary group. There is a natural sequence of inclusions

$$\cdots \hookrightarrow G(m-1,n) \hookrightarrow G(m,n) \hookrightarrow G(m+1,n) \hookrightarrow \cdots$$

and one defines the space $G(\infty,n)$ as the union of all the finite Grassmannians $G(m,n)$ over $m \geqslant 1$, with the inductive topology. The inclusion $G(m,n) \hookrightarrow G(\infty,n) = BU(n)$ induces a surjection

$$\zeta : H^*(BU(n)) \to H^*(G(m,n)).$$

Moreover, there is a universal rank n quotient bundle $Q \to G(\infty,n)$ which corresponds to the universal bundle $EU(n) \to BU(n)$.

We let $\rho = \zeta \circ \psi \circ \phi$ denote the composite of the three maps

$$\mathrm{R}_+(GL_n) \xrightarrow{\phi} \mathrm{Pol}(\mathfrak{gl}_n)^{GL_n} \xrightarrow{\psi} H^*(BU(n)) \xrightarrow{\zeta} H^*(G(m,n)). \tag{6}$$

The connection between the representation ring of GL_n and the cohomology of $G(m,n)$ is exhibited in the following result.[2]

Theorem 1. *For every λ, the ring homomorphism $\rho\colon \mathrm{R}_+(GL_n) \to H^*(G(m,n))$ maps the class of the irreducible representation V^λ to the Schubert class σ_λ.*

As a prelude to the proof of Theorem 1, we will describe an equivalent method of constructing the morphism ρ. This involves putting a $U(N)$-invariant hermitian metric on the vector bundle $Q \to X$ and using the canonical induced linear connection to define *Schubert forms* Ω_λ which represent the Schubert classes in the de Rham cohomology of X. We will denote by $\mathcal{A}^k(X)$ (respectively $\mathcal{A}^k(X,Q)$) the real vector space of C^∞ k-forms on X (respectively, Q-valued k-forms on X). A connection on Q is a $\mathbb{C}$-linear map $D : \mathcal{A}^0(X,Q) \to \mathcal{A}^1(X,Q)$ such that

$$D(f \cdot s) = df \otimes s + f \cdot ds \tag{7}$$

for all functions $f \in \mathcal{A}^0(X)$ and sections $s \in \mathcal{A}^0(X,Q)$. The type decomposition $\mathcal{A}^1(X,Q) = \mathcal{A}^{1,0}(X,Q) \bigoplus \mathcal{A}^{0,1}(X,Q)$ of differential forms induces a decomposition $D = D^{1,0} + D^{0,1}$ of each connection D on Q.

The standard hermitian metric on $\mathbb{C}^N$ gives a metric on the trivial vector bundle in (5) and induces a metric on the subbundle S. One obtains a hermitian metric h on the quotient bundle Q by identifying it with the orthogonal complement of S. The metric h induces a unique connection $D = D(Q,h)$ such that $D^{0,1} = \overline{\partial}_Q$ and D is *unitary*, i.e.

$$d\,h(s,t) = h(Ds,t) + h(s,Dt), \text{ for all } s,t \in \mathcal{A}^0(X,Q).$$

The connection D is called the *hermitian holomorphic connection* of (Q,h). We extend D to Q-valued forms by using the Leibniz rule

$$D(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \otimes Ds.$$

The curvature of D is the composite

$$\widetilde{\Omega} = D^2 : \mathcal{A}^0(X,Q) \to \mathcal{A}^2(X,Q).$$

By applying (7) twice we compute that $\widetilde{\Omega}(f \cdot s) = f \cdot \widetilde{\Omega}(s)$, hence the map $\widetilde{\Omega}$ is $\mathcal{A}^0(X)$-linear. We deduce that $\widetilde{\Omega} \in \mathcal{A}^2(X, \mathrm{End}(Q))$. In fact, $\widetilde{\Omega} = D^{1,1} \in \mathcal{A}^{1,1}(X, \mathrm{End}(Q))$, because $D^{0,2} = \overline{\partial}_Q^2 = 0$, so $D^{2,0}$ also vanishes by unitarity. It follows that locally, we can identify $\widetilde{\Omega}$ with an $n \times n$ matrix of $(1,1)$-forms on X.

Let $\Omega = \frac{i}{2\pi}\widetilde{\Omega}$. For each partition λ, define the Schubert form $\Omega_\lambda = \mathrm{Tr}(\mathfrak{s}_\lambda(\Omega))$; this is a closed form of type $(|\lambda|,|\lambda|)$ on X. Since the hermitian vector bundle (Q,h) is $U(N)$-equivariant (for the natural $U(N)$ action on X), the Schubert forms Ω_λ are $U(N)$-invariant. The class of Ω_λ in the de Rham cohomology group $H^{2|\lambda|}(X)$ coincides with the image $\rho(V^\lambda)$ in Theorem 1. One has an equation

$$\mathrm{Tr}(\mathfrak{s}_\lambda(\Omega)) \wedge \mathrm{Tr}(\mathfrak{s}_\mu(\Omega)) = \mathrm{Tr}(\mathfrak{s}_\lambda(\Omega) \otimes \mathfrak{s}_\mu(\Omega)) = \sum_\nu c_{\lambda\mu}^\nu \mathrm{Tr}(\mathfrak{s}_\nu(\Omega))$$

of differential forms on X, and hence a formula

$$[\Omega_\lambda] \cdot [\Omega_\mu] = [\Omega_\lambda \wedge \Omega_\mu] = \sum_\nu c_{\lambda\mu}^\nu [\Omega_\nu]$$

in $H^*(X)$, with the constants $c_{\lambda\mu}^\nu$ defined as in (1).

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[2]The pun here and in the title of this paper is intended.

To prove the theorem, we must show that $[\Omega_\lambda]$ is equal to the Schubert class σ_λ. The link between Schubert classes and GL_n-modules is then evident: each Schubert class σ_λ is represented in de Rham cohomology by a unique[3] $U(N)$-invariant form, given as the trace of the Schur functor $\mathfrak{s}_\lambda$ on the curvature matrix Ω. This was first demonstrated by Horrocks [Ho]; we give a different proof in the next section.

5. Proof of Theorem 1

For each integer k with $1 \leqslant k \leqslant n$, define the *special Schubert class* $\sigma^k = \sigma_{(1^k)}$ to be the class corresponding to the partition $(1^k) = (1, 1, \ldots, 1)$ of weight k. Over a century ago, Pieri [P] proved the following multiplication rule in $H^*(X)$:

$$\sigma_\lambda \cdot \sigma^k = \sum \sigma_\mu, \tag{8}$$

summed over all partitions μ whose diagram is obtained by adding k boxes to the diagram of λ, no two in the same row (a more recent proof is given in [HP]).

A straightforward consequence of (8) is the formula

$$\sigma_\lambda = \det(\sigma^{\lambda'_i+j-i})_{1\leqslant i,j\leqslant m},$$

due to Giambelli [G1], which shows that the special classes generate the cohomology ring of $G(m, n)$ (the corresponding determinantal formula for Schur polynomials was discovered by Jacobi [J]). Moreover, one knows that the same Pieri rule governs the tensor product decomposition

$$V^\lambda \otimes \wedge^k V = \sum V^\mu$$

of GL_n-modules (several different proofs of this are found in [FH, §I.6] and [Z, §79]). Therefore, Theorem 1 will follow from the equality $\sigma^k = [\mathrm{Tr}(\wedge^k\Omega)]$, where $\mathrm{Tr}(\wedge^k\Omega) = c_k(Q, h)$ is the *k-th Chern form* of (Q, h). Equivalently, we must show that for all partitions λ and all $k = 1, \ldots, n$,

$$\int_{X_\lambda} \mathrm{Tr}(\wedge^k\Omega) = \delta(\lambda;\, (1^k)), \tag{9}$$

where $\delta(\lambda;\, \mu)$ is Kronecker's delta.

The integrals (9) were computed by Chern in [Ch1, §2], who wrote the integrand using the Maurer-Cartan forms of the unitary group $U(N)$. The point is that any invariant form on $X = U(N)/(U(m) \times U(n))$ pulls back to a differential form on $U(N)$, which can be expressed in terms of the basic invariant forms on $U(N)$.

We will use the differential forms ω_{ij} and $\overline{\omega}_{ij}$ defined in [T, §5], of type $(1, 0)$ and $(0, 1)$, respectively, which are a scalar multiple of the Maurer-Cartan forms. To describe them, let $\mathfrak{h} = \{\mathrm{diag}(t_1, \ldots, t_N) \,|\, t_i \in \mathbb{C}\}$ be the Cartan subalgebra of diagonal matrices in $\mathfrak{gl}_N$. Consider the set of roots

$$\Delta = \{t_i - t_j \,|\, 1 \leqslant i \neq j \leqslant N\} \subset \mathfrak{h}^*$$

and denote the root $t_i - t_j$ by the pair ij. The adjoint representation of $\mathfrak{h}$ on $\mathfrak{gl}_N$ determines a decomposition

$$\mathfrak{gl}_N = \mathfrak{h} \oplus \sum_{ij\in\Delta} \mathbb{C}\, e_{ij},$$

where e_{ij} is the matrix with 1 as the ij-th entry and zeroes elsewhere.

[3]Since the Grassmannian X is a hermitian symmetric space, the $U(N)$-invariant forms are harmonic for the natural invariant metric on X coming from the Kähler form Ω_1.

Let $\overline{e}_{ij} = -e_{ji}$ and consider the linearly independent set

$$B_X = \{e_{ij}, \overline{e}_{ij} \mid i \leqslant m < j\}.$$

Extend B_X to a basis B of $\mathfrak{gl}_N$ and let B^* be the dual basis of $\mathfrak{gl}_N^*$. For every e_{ij} and $\overline{e}_{ij}$ in B_X, let ω^{ij} and $\overline{\omega}^{ij}$ be the corresponding dual basis vectors in B^*, which we shall regard as left invariant complex one-forms on $U(N)$. Let $\omega_{ij} = \gamma\,\omega^{ij}$ and $\overline{\omega}_{ij} = \gamma\,\overline{\omega}^{ij}$, where γ is a constant such that $\gamma^2 = \frac{i}{2\pi}$.

If $\pi : U(N) \to X$ denotes the quotient map, then any smooth form η on X pulls back to

$$\pi^*\eta = \sum f_{a_1\ldots a_r b_1\ldots b_s}\,\omega_{a_1} \wedge \ldots \wedge \omega_{a_r} \wedge \overline{\omega}_{b_1} \wedge \ldots \wedge \overline{\omega}_{b_s} \tag{10}$$

on $U(N)$, with coefficients $f_{a_1\ldots a_r b_1\ldots b_s} \in C^\infty(U(N))$. The differential forms on $U(N)$ which arise in this way are exactly those which are invariant under the action of the group $H = U(m) \times U(n)$ (H acts on $C^\infty(U(N))$ by right translation and on $\mathrm{Span}\{\omega_{ij}, \overline{\omega}_{ij} \mid i \leqslant m < j\}$ by the dual of the adjoint representation). A smooth form η on X is of (p, q) type precisely when each summand on the right hand side of (10) involves p unbarred and q barred terms.

The curvature matrix Ω can now be written explicitly (see [GS, §4] and [T, Prop. 2]); one has $\Omega = \{\Gamma_{\alpha\beta}\}_{m+1\leqslant\alpha,\beta\leqslant N}$ with

$$\Gamma_{\alpha\beta} = \sum_{i=1}^{m} \omega_{i\alpha} \wedge \overline{\omega}_{i\beta}. \tag{11}$$

Applying the defining expansion of a determinant in terms of its entries, we obtain

$$\mathrm{Tr}(\wedge^k(\Omega)) = \frac{1}{k!}\sum \mathrm{sgn}(\alpha_1, \ldots, \alpha_k;\, \beta_1, \ldots, \beta_k)\,\Gamma_{\alpha_1\beta_1} \cdots \Gamma_{\alpha_k\beta_k}, \tag{12}$$

where the sum is over all indices $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ from the set $\{m+1, \ldots, N\}$, and $\mathrm{sgn}(\alpha_1, \ldots, \alpha_k;\, \beta_1, \ldots, \beta_k)$ is zero except when $(\beta_1, \ldots, \beta_k)$ is a permutation of $(\alpha_1, \ldots, \alpha_k)$, in which case it equals $+1$ or -1 according to the sign of the permutation. Equations (11) and (12) correspond exactly to [Ch1, (12) and (13)], and (9) is proved in loc. cit., Theorem 5, by directly integrating the forms (12) over the Schubert varieties in X (see also [Ch3, IV.2]).

Remarks. 1) Although the differential forms on the right hand side of (12) appear in [Ch1], their interpretation as invariant polynomials in the entries of the curvature matrix $\Omega = \Omega(Q, h)$ does not. This connection is explained clearly later, in Chern's University of Chicago notes [Ch4, §12] and in [Ch5, §8].

2) For the purposes of the proof, it is not necessary to know that the ring homomorphisms in (3) and (4) are isomorphisms. Furthermore, each of the three morphisms in (6) maps integral classes to integral classes. If $\mathcal{R}_+(GL_n)$ denotes the $\mathbb{Z}$-submodule of $\mathrm{R}_+(GL_n)$ spanned by the polynomial characters of GL_n, we obtain a natural induced homomorphism

$$\rho_{\mathbb{Z}} : \mathcal{R}_+(GL_n) \longrightarrow H^*(G(m,n), \mathbb{Z})$$

which sends $[V^\lambda]$ to the Schubert class σ_λ for every partition λ.

6. The problem in other Lie types

In this section, we look for an analogue of Theorem 1 with GL_n replaced by a group of a different Lie type. Let G be a complex connected reductive Lie group, T a maximal torus in G and let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and T, respectively. The adjoint action of G on $\mathfrak{g}$ induces an action on the ring $\mathrm{Pol}(\mathfrak{g})$ of polynomial functions on $\mathfrak{g}$ with real coefficients. If K is a maximal compact subgroup of G, then Cartan's theorem [Car] gives a ring isomorphism

$$\psi_G : \mathrm{Pol}(\mathfrak{g})^G \to H^*(BK) \tag{13}$$

as before.

If G is not of type A, then extending (13) – on either side – to a sequence analogous to (6) is problematic. We will go one step further by using the natural λ-ring structure on the representation ring of G. Our main reference for λ-rings is [FL], while we learned much of the material that follows from [Fa], [O], and [Be].

A *λ-ring* is a commutative ring A with a sequence of operations $\lambda^i : A \to A$ such that $\lambda^0 = 1$, $\lambda^1 = \mathrm{id}_A$ and

$$\lambda^k(x+y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y)$$

for all $k \geqslant 1$ and for all $x, y \in A$. In addition, we require that there are formulas

$$\lambda^k(xy) = P_k(\lambda^1(x), \ldots, \lambda^k(x), \lambda^1(y), \ldots, \lambda^k(y))$$

and

$$\lambda^k(\lambda^\ell(x)) = P_{k,\ell}(\lambda^1(x), \ldots, \lambda^{k\ell}(x))$$

where P_k and $P_{k,\ell}$ are universal polynomials with integer coefficients, independent of the ring A. Setting $\lambda_t(x) = \sum_i \lambda^i(x)\, t^i$, we have $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$.

Let $\mathcal{R}(G)$ denote the integral representation ring of G. Then $\mathcal{R}(G)$ is a λ-ring, with the λ-operations $\lambda^i : \mathcal{R}(G) \to \mathcal{R}(G)$ induced by the exterior powers of representations. There is a unique λ-ring homomorphism $\epsilon : \mathcal{R}(G) \to \mathbb{Z}$, called the augmentation, which associates to each G-representation V its dimension $\epsilon(V)$.

Grothendieck [SGA6] introduced λ-rings in his work extending the Hirzebruch-Riemann-Roch theorem to a relative, functorial setting. He showed that the purely algebraic data of a λ-ring A together with an augmentation map $A \to \mathbb{Z}$ suffice to define characteristic classes for the elements of A. The Chern classes constructed in this way take values in the graded ring $\mathrm{gr}\, A$ associated to a certain filtration on A coming from the λ-structure, known as the *γ-filtration*. Grothendieck applied this theory for A equal to the K-theory group of vector bundles on an algebraic variety; we will take $A = \mathcal{R}(G)$ in the sequel.

The γ-operations γ^i on $\mathcal{R}(G)$ are defined by the formula

$$\gamma_t(x) = \lambda_{t/(1-t)}(x) = \sum_i \gamma^i(x)\, t^i, \quad \forall x \in \mathcal{R}(G).$$

The γ-filtration is the decreasing sequence $\{F^k\}_{k\geqslant 0}$ where $F^0 = \mathcal{R}(G)$, $F^1 = \mathrm{Ker}(\epsilon)$ and F^k is spanned by the elements $\gamma^{i_1}(x_1)\cdots\gamma^{i_r}(x_r)$ with $x_1, \ldots, x_r \in F^1$ and $\sum_{p=1}^r i_p \geqslant k$. Let $\mathrm{gr}\,\mathcal{R}(G) = \bigoplus_{k\geqslant 0} F^k/F^{k+1}$ be the associated graded ring. For each element $x \in \mathcal{R}(G)$, there are Chern classes $c_k(x)$ with values in $\mathrm{gr}^k\mathcal{R}(G)$. By definition, $c_k(x) = \gamma^k(x - \epsilon(x))$.

Example 4. Suppose that the torus T has rank n, and let Λ be its character group. We can identify Λ with the multiplicative group of monomials $\alpha_1^{m_1}\cdots\alpha_n^{m_n}$ with $m_i\in\mathbb{Z}$, where $\alpha_i^{\pm 1}$ corresponds to the map $(t_1,\ldots,t_n)\mapsto t_i^{\pm 1}$, for $1\leqslant i\leqslant n$. We then have

$$\mathcal{R}(T)=\mathbb{Z}[\Lambda]=\mathbb{Z}[\alpha_1,\alpha_1^{-1},\ldots,\alpha_n,\alpha_n^{-1}].$$

The relations $\lambda_t(\xi)=1+\xi t$ and $\epsilon(\xi)=1$ for $\xi\in\Lambda$ determine the λ-structure and augmentation on $\mathcal{R}(T)$. It is straightforward to check that the γ-filtration of $\mathcal{R}(T)$ coincides with its F^1-adic filtration, that is, $F^k=(F^1)^k$ for all $k\geqslant 1$. The map $\xi\mapsto\xi-1$ gives a canonical isomorphism from Λ to the additive group $\mathrm{gr}^1\mathcal{R}(T)=F^1/F^2$. If $u_i\in\mathrm{gr}^1\mathcal{R}(T)$ denotes the image of α_i under this map, then the elements $u_1,\ldots,u_n$ are algebraically independent over $\mathbb{Z}$, and we have $\mathrm{gr}\,\mathcal{R}(T)=\mathbb{Z}[u_1,\ldots,u_n]\cong\mathrm{Sym}(\Lambda)$ (see e.g. [O, Prop. 3.2]). We may thus identify the k-th Chern class of $\xi_1+\ldots+\xi_r$ (where $\xi_i\in\Lambda$) with the element $e_k(\xi_1,\ldots,\xi_r)\in\mathrm{Sym}(\Lambda)$.

The Weyl group W acts on T, hence on the ring $\mathcal{R}(T)$, and the restriction map $\eta:\mathcal{R}(G)\to\mathcal{R}(T)$ induces an isomorphism of λ-rings $\mathcal{R}(G)\to\mathcal{R}(T)^W$. Let us now pass to real coefficients: define $\mathrm{R}(G)=\mathcal{R}(G)\otimes\mathbb{R}$, $\mathrm{gr}\,\mathrm{R}(G)=\mathrm{gr}\,\mathcal{R}(G)\otimes\mathbb{R}$ and similarly for the torus T. One can then show that the map η respects the γ-filtrations of both $\mathrm{R}(G)$ and $\mathrm{R}(T)$, and thus induces a graded ring homomorphism $\mathrm{gr}(\eta):\mathrm{gr}\,\mathrm{R}(G)\to\mathrm{gr}\,\mathrm{R}(T)$, which maps $\mathrm{gr}\,\mathrm{R}(G)$ isomorphically onto the invariant subring $(\mathrm{gr}\,\mathrm{R}(T))^W$ (see [Fa], [O], and [Be] for further discussion and proofs).

Assume that the Lie groups G, T and their Lie algebras are defined over the real numbers, and view $\mathfrak{h}$ as a vector space over $\mathbb{R}$. Applying the map which takes a character in Λ to its derivative in $\mathfrak{h}^*$, we can identify $(\mathrm{gr}\,\mathrm{R}(T))^W\cong(\mathrm{Sym}(\Lambda))^W\otimes\mathbb{R}$ with the algebra $\mathrm{Pol}(\mathfrak{h})^W$ of W-invariant polynomial functions on $\mathfrak{h}$ (again with real coefficients). Chevalley proved that the restriction homomorphism $\mathrm{Pol}(\mathfrak{g})\to\mathrm{Pol}(\mathfrak{h})$ maps $\mathrm{Pol}(\mathfrak{g})^G$ isomorphically onto $\mathrm{Pol}(\mathfrak{h})^W$. Combining this with the preceeding ingredients shows that (13) extends to a sequence of isomorphisms

$$\mathrm{gr}\,\mathrm{R}(G)\longrightarrow\mathrm{Pol}(\mathfrak{g})^G\xrightarrow{\psi_G}H^*(BK)$$

which respects the natural grading in all three rings.

Example 5. Let x be the element of $\mathrm{R}(GL_n)$ which corresponds to the standard representation of $GL_n(\mathbb{C})$ on $\mathbb{C}^n$. It is easy to see that the k-th Chern class $c_k(x)$ (in the above sense) is the invariant polynomial $A\mapsto\mathrm{Tr}(\wedge^k A)$ on $\mathfrak{gl}_n$. Theorem 1 shows that the Chern classes of the standard representation of GL_n map naturally to the special Schubert classes in $G(m,n)$. The fact that the target ring $\mathrm{gr}\,\mathrm{R}(GL_n)$ for the $\mathrm{R}(GL_n)$ Chern classes is naturally graded isomorphic to $\mathrm{R}_+(GL_n)$ is a type A phenomenon.

Example 6. Let $G=Sp_{2n}(\mathbb{C})$ be the symplectic group of rank n and $K=Sp(2n)$ the homonymous compact subgroup. Let $y\in\mathrm{R}(Sp_{2n})$ be the class of the standard representation of G on $\mathbb{C}^{2n}$. In this case the $2k$-th Chern class $c_{2k}(y)$ is the invariant polynomial $A\mapsto\mathrm{Tr}(\wedge^{2k}A)$ on $\mathfrak{g}=\mathfrak{sp}_{2n}$, and these classes for $1\leqslant k\leqslant n$ generate the algebra $\mathrm{Pol}(\mathfrak{g})^G$ (the odd Chern classes of y vanish). Observe that $\mathrm{Pol}(\mathfrak{g})^G$ is isomorphic to $\mathrm{Pol}(\mathfrak{gl}_n)^{GL_n}$, up to a doubling of degrees.

The classifying space $BSp(2n)$ may be identified with the infinite quaternionic Grassmannian $G_{\mathbb{H}}(\infty,n)$, which is the inductive limit of the finite Grassmannians

$Y = G_{\mathbb{H}}(m,n)$ over all $m > 0$. Note that $Y = Sp(2N)/(Sp(2m) \times Sp(2n))$ is a compact oriented manifold of real dimension $4mn$, which parametrizes m-dimensional left $\mathbb{H}$-linear subspaces of $\mathbb{H}^N$. There is a rank n quotient bundle $Q \to Y$ which is a pullback of the universal quotient bundle over $G_{\mathbb{H}}(\infty, n)$. The cell decomposition and cohomology ring of Y are identical to those of the complex Grassmannian $G(m,n)$, again up to a doubling of degrees (see [B1], [GHV, Vol. III, Chp. XI], and [PR, Appendix A]). In particular, for each partition λ as in Sec. 4, there is a 'Schubert variety' Y_λ in Y, defined by the same inequalities that define X_λ in $G(m,n)$, and a 'Schubert class' $\tau_\lambda = [Y_\lambda] \in H^{4|\lambda|}(Y)$. By arguing as in Sec. 5, one can show that the induced ring homomorphism

$$\operatorname{gr} \mathrm{R}(Sp_{2n}) \longrightarrow \mathrm{Pol}(\mathfrak{sp}_{2n})^{Sp_{2n}} \xrightarrow{\psi_{Sp}} H^*(BSp(2n)) \longrightarrow H^*(G_{\mathbb{H}}(m,n))$$

maps the generators $c_{2k}(y)$ to the 'special Schubert classes' $\tau_{(1^k)}$.

We remark that the irreducible characters χ^λ of Sp_{2n} are parametrized by partitions of length at most n, as is the case for the polynomial characters of GL_n. Moreover, in the product decomposition

$$\chi^\lambda \chi^\mu = \sum_\nu c_{\lambda\mu}^\nu \, \chi^\nu,$$

the structure constants $c_{\lambda\mu}^\nu$ agree with those in (1), whenever $|\nu| = |\lambda| + |\mu|$ (see e.g. [KT, Prop. 2.5.2]).

In the case of the orthogonal groups $G = O_n(\mathbb{C})$, the classifying space $BO(n)$ is an infinite real Grassmannian, which is a limit of finite dimensional real Grassmannians $G_{\mathbb{R}}(m,n)$. However there is no clear analogue of $GL_n(\mathbb{C})$ Schubert calculus on these manifolds, some of which are not even orientable.

7. Concluding remarks

In this final section, we briefly discuss some of the early works where the representation theory of Lie groups was applied to study the cohomology of homogeneous spaces, and which relate somehow to the present paper. For more details about the early investigations of the Chern-Weil circle of ideas and its applications, we recommend Chern's address at the 1950 International Congress [Ch2], Weil's letters, written in 1949 and first published in [W], the survey articles by Samelson [Sa] and Borel [B2], and the historical notes in [GHV, Vol. III].

In the middle of the last century, there emerged two effectively different approaches to the study of the cohomology of principal bundles and homogeneous spaces of Lie groups. The techniques used in this paper derive from É. Cartan's method of invariant differential forms [Ca] and its later extension by H. Cartan [Car]. An alternative approach, espoused by Borel [B1] among others, used the methods of classical algebraic topology. The theorems proved by these two schools were frequently identical, although the results obtained by topological methods were usually with $\mathbb{Z}$ or $\mathbb{Z}_p$ coefficients. These latter techniques were applied by Borel and Hirzebruch [BH] to connect the theory of characteristic classes to the cohomology of homogeneous spaces G/U, by interpreting the former as elementary symmetric functions in certain roots of G (or their squares).

Ehresmann [E] investigated the topology of complex Grassmann manifolds (and other hermitian symmetric spaces) by studying the algebra of K-invariant differential forms on them ($K = U(N)$ for $X = G(m,n)$). This relies on the fact that the invariant forms are harmonic for the natural hermitian structure on X, which implies that the ring of all such forms is isomorphic to $H^*(X)$. Kostant [K1] [K2] later found analogues of these results for arbitrary (generalized) flag manifolds. The representation theory used to determine the K-invariant forms in this program does not directly relate the multiplicities $c_{\lambda\mu}^{\nu}$ in equations (1) and (2). Note however that the cited works of É. Cartan and Ehresmann were used by Chern in his fundamental paper on the characteristic classes of complex manifolds [Ch1]. More recently, Stoll [St] used fiber integration to study the algebra of invariant forms on the Grassmannian, but his work does not address the question posed in the Introduction.

Following [SGA6], [V] and [Be], the isomorphism between $\mathrm{grR}(G)$ and $\mathrm{Pol}(\mathfrak{g})^G$ in Sec. 6 may be used to construct the Chern-Weil (or characteristic) homomorphism in algebraic geometry. Let $P \to X$ be a principal G-bundle over a smooth algebraic variety X and let $CH^*(X)$ denote the Chow group of algebraic cycles on X modulo rational equivalence. The Grothendieck group $K(X)$ of vector bundles on X is a λ-ring, with the λ-operations induced by exterior powers. According to [SGA6, Exp. XIV], the graded ring $\mathrm{gr}K(X) \otimes \mathbb{R}$ is canonically isomorphic to the real Chow ring $CH^*_{\mathbb{R}}(X) = CH^*(X) \otimes \mathbb{R}$. There is a natural λ-ring homomorphism $\pi : \mathcal{R}(G) \to K(X)$, defined by sending a representation $G \to GL(E)$ to the associated vector bundle $P \times_G E$ over X. The characteristic homomorphism is the induced map

$$\mathrm{gr}(\pi)_{\mathbb{R}} : \mathrm{Pol}(\mathfrak{g})^G \longrightarrow CH^*_{\mathbb{R}}(X).$$

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