Figure 1: Paddle wheel dipped into fluid flowing in the xy plane.

NOTES ON CURL AND ALL THAT JAZZ

These notes are an attempt to give a more geometric/physical meaning to the notions of curl, divergence, and thereby illuminate the theorems of Green, Gauss, and Stokes.

1. An intuitive description of the curl in two dimensions

The curl of a vector field $\mathbf{F} = (u(x,y,z), v(x,y,z), w(x,y,z))$ is defined to be

$$\text{curl}(\mathbf{F}) = (w_y - v_z, u_z - w_x, v_x - u_y). \tag{1}$$

To gain an intuitive understanding of the curl, we shall think of the vector field as the velocity field of a fluid. The function u is the x component of the velocity, v is the y component, and w is the z component. We begin with a simpler situation, two-dimensional flow, where $\mathbf{F} = (u(x,y),\ v(x,y))$. Then

$$\text{curl}(\mathbf{F}) = (v_x - u_y)\mathbf{k}. \tag{2}$$

We imagine dipping a paddle wheel into the fluid with the paddle wheel oriented so that the axis of rotation is in the $\mathbf{k}$ direction as in Figure 1. Now let us consider a simple flow, where u and v are linear in x and y:
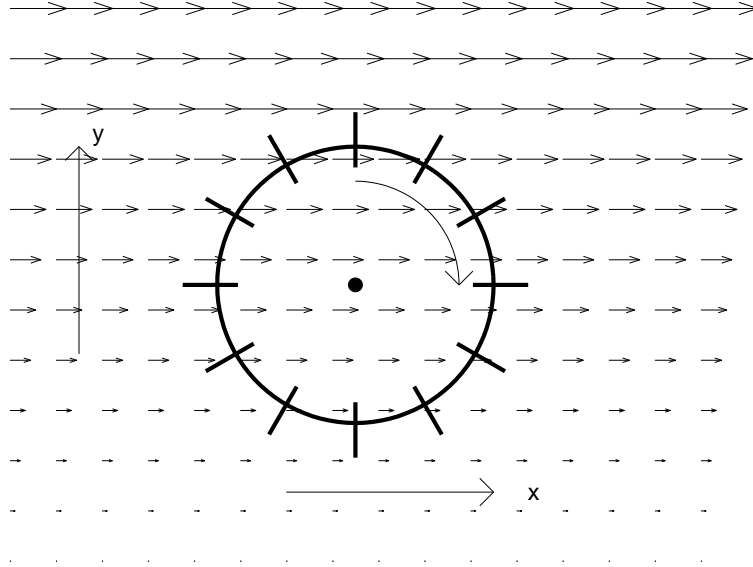
Figure 2: Vector field $\mathbf{F}_1 = (y/2 + 1, 0)$ producing clockwise rotation of paddle wheel.

$$
\begin{aligned}
u(x, y) &= A + u_1 x + u_2 y \\
v(x, y) &= B + v_1 x + v_2 y.
\end{aligned}
\tag{3}
$$

First we consider the flow where the fluid moves only in the x direction, $\mathbf{F}_1 = (u(x, y), 0) = (A + u_1 x + u_2 y, 0)$. Note that the velocity component $(A + u_1 x, 0)$ is the same along any vertical line, and thus will not cause any rotation. However, if $u_2 > 0$, the component $(u_2 y, 0)$ increases along a vertical line, making the velocity at the top of wheel greater than at the bottom. This will cause the wheel to rotate in a *clockwise* direction. If $u_2 < 0$, the wheel will rotate in the *counterclockwise* direction. Therefore $\mathbf{F}_1$ causes rotation only when $u_2 \neq 0$. See Figure 2.

Next we consider a simple flow $\mathbf{F}_2 = (0, B + v_1 x + v_2 y)$. This flow moves the fluid particles in the y direction. Now the component $(0, B + v_2 y)$ is constant along any horizontal line and will not cause the wheel to rotate. However, the component $(0, v_2 x)$ increases from left to right if $v_2 > 0$, causing the wheel to rotate in the *counterclockwise* direction. If $v_2 < 0$, the wheel will rotate in the *clockwise* direction. Hence $\mathbf{F}_2$ will cause rotation if $v_2 \neq 0$. See Figure 3. The combined flow $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ will cause rotation if $v_1 - u_2 \neq 0$. If $v_1 - u_2 > 0$, the rotation will be *counterclockwise* and if $v_1 - u_2 < 0$, the rotation will be *clockwise*. The quantity $|v_1 - u_2|$ will be a measure of the speed of rotation. In this special case when the vector field $\mathbf{F}$ is given by (3), $v_1 - u_2 = v_x - u_y$.
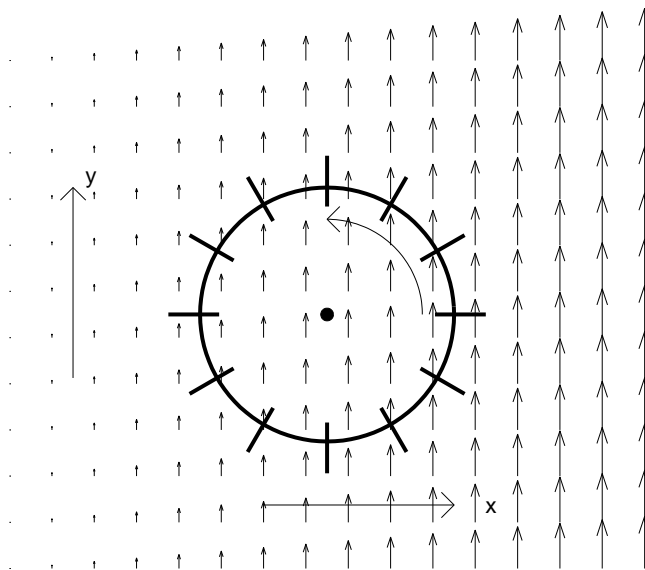
Figure 3: Vector field $\mathbf{F}_2 = (0, x/2 + 1)$ causing paddle wheel to rotate in counterclockwise direction.

Now let $\mathbf{F} = (u(x, y), v(x, y))$ be a general two-dimensional vector field and let $p_0 = (x_0, y_0)$. We make a linear approximation to each component of $\mathbf{F}$ at p_0.

$$
\begin{aligned}
u(x, y) &\approx u(p_0) + u_x(p_0)(x - x_0) + u_y(p_0)(y - y_0) \quad\quad (4) \\
&\approx A + u_x(p_0)x + u_y(p_0)y \\
v(x, y) &\approx v(p_0) + v_x(p_0)(x - x_0) + v_y(p_0)(y - y_0) \quad\quad (5) \\
&\approx B + v_x(p_0)x + v_y(p_0)y.
\end{aligned}
$$

For (x, y) close to (x_0, y_0), the linear approximations (4) and (5) provide a good description of the vector field. If we apply the previous discussion to these linear approximations, we deduce that the quantity $v_x(p_0) - u_y(p_0)$ is a measure of the ability of the fluid to rotate a small paddle wheel centered at p_0, with the quantity $v_x(p_0) - u_y(p_0)$ giving the direction and speed of rotation. In fact, we will see in the Example (iii) below that $|v_x(p_0) - u_y(p_0)|$ is twice the angular velocity at which the paddle wheel rotates.

Examples

(i) Let $\mathbf{F} = (1 - y^2, 0)$ on the strip $\{-\infty < x < \infty, \ -1 \leq y \leq 1\}$. $\mathbf{F}$ is the velocity field of water flowing in the channel occupied by the strip. Note the velocity is zero on the top and bottom edges of the strip, and is greatest in the middle of the strip ($y = 0$). It is easy to see that $\operatorname{curl}(\mathbf{F}) = 2y\mathbf{k}$. This means that

a paddle wheel placed in the strip in $y > 0$ will rotate in the counterclockwise direction, and in the clockwise direction in $y < 0$. On the line $y = 0$, it will not rotate.

(ii) Let

$$\mathbf{F}(x, y) = \frac{(-y, x)}{r^\alpha}, \quad \alpha > 0.$$

A fluid particle carried by this flow will circle the origin in a counterclockwise fashion. We might expect the curl of this vector field to always be positive. However, we find that

$$\text{curl}(\mathbf{F}) = \left(\frac{2 - \alpha}{r^\alpha}\right)\mathbf{k}, \quad (x, y) \neq (0, 0).$$

Hence the paddle wheel dipped into the fluid will rotate counterclockwise when $\alpha < 2$, clockwise when $\alpha > 2$, and will not rotate at all when $\alpha = 2$.

(iii) The case $\alpha = 0$ can also be interpreted as the velocity field of the points on a rigid disk rotating in the counter clockwise direction. Let $\omega > 0$ be a real number and consider the vector field

$$\mathbf{F}(x, y) = (-\omega y, \ \omega x).$$

The points on the circle of radius r travel with speed $\sqrt{(\omega x)^2 + (\omega y)^2} = \omega r$. In other words, the angular velocity of the disk is ω. On the other hand, $v_x - u_y = 2\omega$. Thus the quantity $v_x - u_y$ is twice the angular velocity of the rotating disk. Consequently, we can interpret the quantity $|v_x(p_0) - u_y(p_0)|$ as twice the angular velocity of the paddle wheel dipped into the fluid at the point p_0.

Conclusion From these examples and from the discussion of the linear velocity fields described by (3), we see that the curl $v_x(x, y) - u_y(x, y)$ measures the amount of *shear* in the flow at the point (x, y). In other words, the curl at (x, y) is a measure of how much faster or slower nearby fluid particles are moving. Example (iii) of the rotating disk allows us to interpret the curl as twice the angular velocity of a paddle wheel dipped into the fluid at the point (x, y).

2. Circulation

Usually the line integral of a vector field $\mathbf{F}$ along an oriented curve C is motivated physically by assuming $\mathbf{F}$ represents a force defined at each point $\mathbf{p} = (x, y, z)$ on or near the curve C. Then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the work done by the force when a body is moved along the curve in the direction of the positive orientation of the curve. We shall give a different meaning to this same line integral by assuming that $\mathbf{F} = (u, v, w)$ is the velocity field of a fluid.
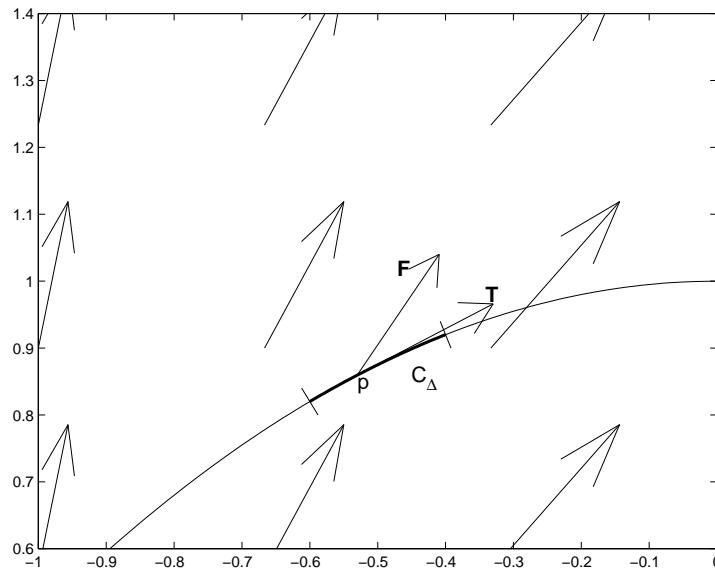
Figure 4: Fluid velocity and tangential displacement of fluid particles

At each point $\mathbf{p} = (x, y, z)$ where the vector field $\mathbf{F}$ is defined, the velocity vector $\mathbf{F} = (u, v, w)$ is tangent to the path of the fluid particle through $\mathbf{p}$. These fluid paths are called streamlines. Now let C be an oriented curve and let $\mathbf{p} \in C$. Let $\mathbf{T}$ be the unit tangent vector to C at $\mathbf{p}$, pointing in the direction of the positive orientation of C. Then $F_T = \mathbf{F}(\mathbf{p}) \cdot \mathbf{T}$ is the tangential component of the fluid velocity (see Figure 4). Over a short time interval Δt, the fluid particle at $\mathbf{p}$ is displaced a distance $\Delta t F_T$ in the tangential direction when $F_T > 0$. Let C_Δ be a short piece of the curve with arc length Δs that contains $\mathbf{p}$. We assume that C_Δ is so short that $\mathbf{F}$ is practically constant on C_Δ. Then when $F_T > 0$,

$$\Delta t F_T \Delta s$$

is an area that is a measure of the tangential displacement of the fluid particles in C_Δ over the time interval Δt. If we divide out Δt, we see that $F_T \Delta s$ is the *rate of tangential displacement* of fluid particles in C_Δ. If $F_T < 0$, the tangential displacement of fluid particles is in the direction opposite to the direction of $\mathbf{T}$. Now we sum over the short pieces C_Δ that make up C, and take the limit as $\Delta s \to 0$. The resulting limit is the line integral

$$\int_C F_T \, ds = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Thus when $\mathbf{F}$ is a fluid velocity, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the rate of net tangential displacement of the fluid along the curve in the direction specified by

the orientation of C. This line integral may be negative. When C is an oriented, closed curve,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C u\,dx + v\,dy + w\,dz \tag{6}$$

is called the *circulation* of the fluid in the direction of the positive orientation of C. It is the rate of net tangential displacement of fluid particles along the curve C in the direction specified by the orientation of C. The circulation has the units of area/time.

Example

When we speak of the circulation around a closed curve C, we do not mean that the fluid particles actually go around the curve like cars on a race track. Consider the vector field $\mathbf{F}_2 = (0, x/2 + 1)$ of Figure 3. Let C be the boundary of the unit square $0 \leq x, y, \leq 1$, oriented in the counterclockwise direction. We label the edges as in Figure 5. The rate of fluid displacement along the right edge C_2, where the unit tangent vector is $\mathbf{T} = (0, 1)$, is

$$\int_{C_2} \mathbf{F}_2 \cdot d\mathbf{r} = \int_0^1 (3/2)dy = 3/2.$$

The rate of fluid displacement along the left edge C_4, where $\mathbf{T} = (0, -1)$ is

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (-1)dy = -1.$$

There is no displacement of the fluid along the top and bottom edges C_1 and C_3 because $\mathbf{F} \cdot \mathbf{T} = 0$ there. Hence the circulation around C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = 3/2 - 1 = 1/2.$$

The circulation around C is positive even though all fluid particles are moving vertically.

3. Green's theorem

Next we shall relate the pointwise quantity $v_x(x, y) - u_y(x, y)$, which measures the shear in the flow at the point at (x, y), to the macroscopic quantity which is the circulation around the boundary of some set. First we consider a rectangle $R = \{a \leq x \leq b \ c \leq y \leq d\}$. The boundary of R is oriented in the counterclockwise direction, and broken down into four parts, one for each edge (see Figure 5).
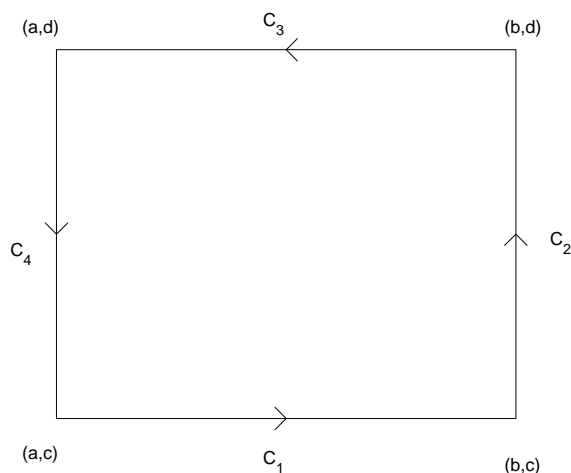
Figure 5: Rectangle R with boundary $C = C_1 \cup C_2 \cup C_3 \cup C_4$.

Recall that in one dimension, we relate the pointwise behavior of the derivative $f'(x)$ to the values of f at the endpoints of an interval via the fundamental theorem of calculus:

$$f(b) - f(a) = \int_a^b f'(x)dx.$$

Let the vector field $\mathbf{F} = (u(x,y), v(x,y))$ be the velocity field of a fluid. For a point $(x,y) \in R$, $v_x(x,y) - u_y(x,y)$ is the shear in the flow at that point. We use the fundamental theorem of calculus to relate this pointwise quantity to the values of the velocity on the edges of the rectangle. Let $\mathbf{T}_2 = (0,1)$ and $\mathbf{T}_4 = (0,-1)$ be the unit tangent vectors on the sides C_2 and C_4. Now for a fixed y,

$$\int_a^b v_x(x,y)dx = v(b,y) - v(a,y) \ .$$

The right side of this equation is just the difference in the velocities on the left and right sides of the rectangle. In terms of the unit tangent vectors $\mathbf{T}_2$ and $\mathbf{T}_4$, we write this equation as

$$\int_a^b u_x(x,y)dx = \mathbf{F} \cdot \mathbf{T}_2(b,y) + \mathbf{F} \cdot \mathbf{T}_4(a,y).$$

Now the right side of the equation is the sum of the tangential components of the velocity on the left and right edges of the rectangle at height y. We integrate this

expression in y to find:

$$\int_c^d \int_a^b v_x(x,y) \; dxdy \;=\; \int_c^d \mathbf{F} \cdot \mathbf{T}_2(b,y) \; dy + \int_c^d \mathbf{F} \cdot \mathbf{T}_2(a,y) \; dy$$
$$=\; \int_{C_4} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

The integral over C_2 is the rate at which fluid moves up the right edge, and the integral over C_4 is the rate at which fluid moves down the left edge. Of course these rates may be negative. Next we fix x, and integrate vertically,

$$\int_c^d -u_y(x,y)dy \;=\; u(x,c) - u(x,d)$$
$$=\; \mathbf{F} \cdot \mathbf{T}_1(x,c) + \mathbf{F} \cdot \mathbf{T}_3(x,d).$$

This is the sum of the tangential components of velocity at the top and bottom edges of the rectangle. We integrate this expression in x,

$$\int_a^b \int_c^d -u_y(x,y)dydx \;=\; \int_a^b \mathbf{F} \cdot \mathbf{T}_1(x,c) \; dx + \int_a^b \mathbf{F} \cdot \mathbf{T}_3(x,d)$$
$$=\; \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}.$$

The result is the rate at which fluid moves along the top and bottom edges of the rectangle. The rate at which fluid moves along all four edges of the rectangle is

$$\int_a^b \int_c^d [v_x(x,y) - u_y(x,y)] \; dxdy \;=\; \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \right) \mathbf{F} \cdot d\mathbf{r}$$
$$=\; \int_C \mathbf{F} \cdot d\mathbf{r}. \tag{7}$$

Equation (7) can be interpreted as follows. The left side is an integral of of the curl over R. It is the total amount of the shear in the flow in the rectangle R. The right side is the rate of tangential fluid displacement along the edges of R; it is the circulation.

Equation (7) is a preliminary version of Green's theorem. To extend this result to more general sets than rectangles, it will suffice to know that (7) also holds on triangles. In fact, now that we have seen the motivation for Green's theorem, we can give a quick proof for the class of sets which are both vertically and horizontally simple.
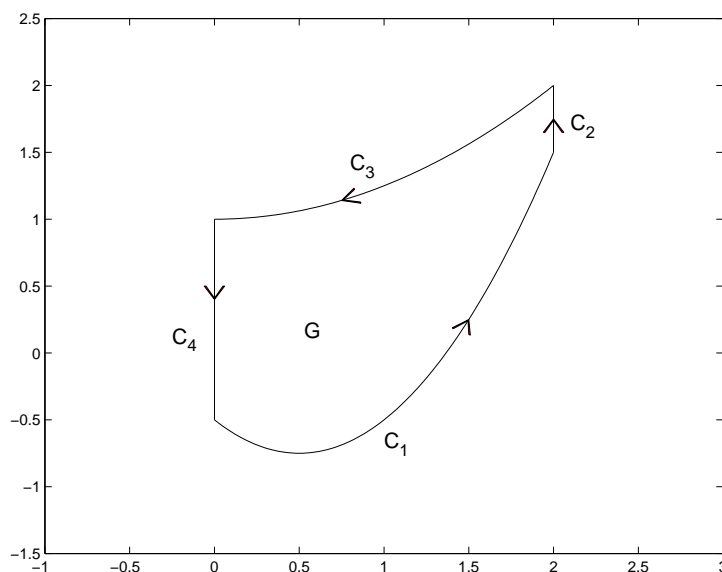
Figure 6: Vertically simple set G with oriented boundary C.

Recall that a set G is vertically simple if there are functions $f(x)$ and $g(x)$, $x \in [a, b]$, such that

$$G = \{(x, y) : f(x) \leq y \leq g(x), \ a \leq x \leq b\}.$$

We assume that $C = C_1 \cup C_2 \cup C_3 \cup C_4$ is oriented in the counterclockwise direction (see Figure 6). Now C_1 is parameterized by $x \to (x, f(x))$ and C_3^- is parametrized by $x \to (x, g(x))$. First we consider a vector field of the form $\mathbf{F}_1 = (u(x, y), 0)$. We see that

$$\int_{C_1} \mathbf{F}_1 \cdot d\mathbf{r} = \int_{C_1} u \ dx = \int_a^b u(x, f(x)) \ dx,$$

and

$$\int_{C_3} \mathbf{F}_1 \cdot d\mathbf{r} = \int_{C_3} u \ dx = -\int_{C_3^-} u \ dx = -\int_a^b u(x, g(x)) \ dx.$$

Then

$$
\begin{aligned}
\int\int_G u_y(x, y) \ dA(x, y) &= \int_a^b \int_{f(x)}^{g(x)} u_y(x, y) \ dxdy \\
&= \int_a^b [u(x, g(x)) - u(x, f(x))] \ dx \\
&= \int_{C_3^-} u \ dx - \int_{C_1} u \ dx
\end{aligned}
$$

$$= -\int_C u \; dx = -\int_C \mathbf{F}_1 \cdot d\mathbf{r} \tag{8}$$

because $\int_{C_3^-} u \; dx = -\int_{C_3} u \; dx$ and $\int_{C_2} u \; dx = \int_{C_4} u \; dx = 0$.

Next assume that G is horizontally simple and that the vector field is $\mathbf{F}_2 = (0, v(x, y))$. The same kind of argument shows that

$$\int\int_G v_x(x, y) \; dA(x, y) = \int_C v \; dy = \int_C \mathbf{F}_2 \cdot d\mathbf{r}. \tag{9}$$

If we assume that G is both vertically and horizontally simple, we can add together (8) and (9) to deduce that

$$\int\int_G [v_x - u_y] \; dA(x, y) = \int_C u \; dx + v \; dy \tag{10}$$

or

$$\int\int_G \mathrm{curl}(\mathbf{F}) \cdot \mathbf{k} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = (u, v) = \mathbf{F}_1 + \mathbf{F}_2$. Equation (10) is Green's theorem which we now see holds for sets G which are both vertically and horizontally simple.

Example

(i) Consider the velocity field of the rotating disk, given by $\mathbf{F}(x, y) = (-\omega y, \omega x)$. We let G be the disk of radius $a > 0$, and we take C as the circle of radius a, oriented in the counterclockwise direction. We parametrize C with $\mathbf{r}(t) = (a \cos t, a \sin t)$, $0 \le t \le 2\pi$. Then the circulation

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_0^{2\pi} a^2 \omega(\cos^2(t) + \sin^2(t)) dt = 2\pi a^2 \omega.$$

We have already seen that $v_x - u_y = 2\omega$ for this velocity field. The circulation is just twice the angular velocity of the disk multiplied by the area of the disk, in agreement with equation (10).

(ii) Let $\mathbf{F} = (1, x)$ and let G be the unit disk, $x^2 + y^2 < 1$, with boundary C being the unit circle oriented in the counterclockwise direction. The curl is $v_x - u_y \equiv 1$, so according to equation (10),

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int\int_G dA = \pi.$$

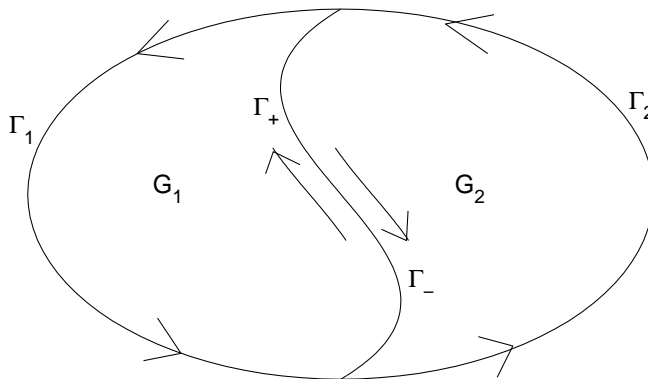The circulation is positive although none of the fluid particles go around the circle; they pass through it.

Figure 7: Circulation around sets G_1 and G_2.

Now we want to extend Green's theorem to more general sets. First we make an important observation about the circulation. Let G_1 and G_2 be sets as indicated in Figure 7. Let $C_1 \equiv \Gamma_1 \cup \Gamma_+$ be the boundary of G_1 and $C_2 \equiv \Gamma_2 \cup \Gamma_-$ be the boundary of G_2, both oriented in the counterclockwise direction. Let $C = \Gamma_1 \cup \Gamma_2$ be the boundary of $G = G_1 \cup G_2$. Let $\mathbf{F}$ be any two-dimensional vector field defined on G. We note that

$$\int_{\Gamma_-} \mathbf{F} \cdot d\mathbf{r} = -\int_{\Gamma_+} \mathbf{F} \cdot d\mathbf{r}$$

because Γ_- and Γ_+ are oriented in opposite directions. Hence

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{\Gamma_1} + \int_{\Gamma_2} \\
&= \int_{\Gamma_1} + \int_{\Gamma_+} + \int_{\Gamma_2} + \int_{\Gamma_-} \\
&= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.
\end{aligned}
$$

We can repeat this argument over and over again as we subdivide G into more disjoint pieces G_j. The result can be stated as follows:
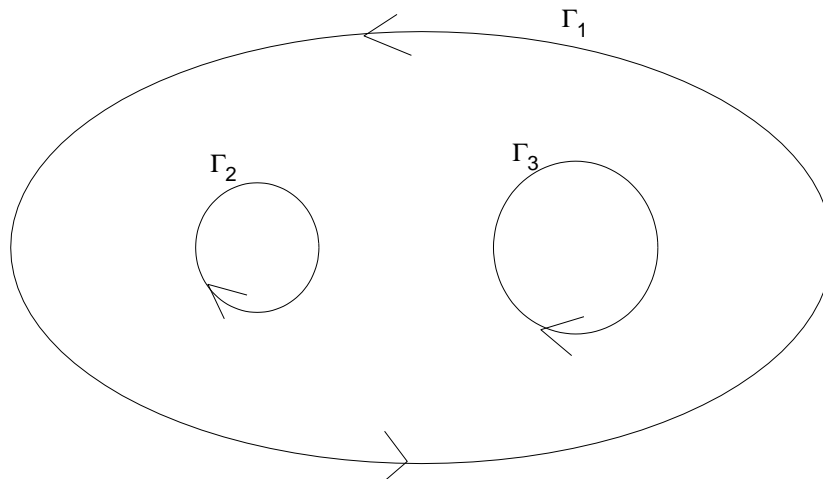
Additivity of the circulation

12

Figure 8: Boundaries Γ_1, Γ_2 and Γ_3 of set G with holes, and their orientations.

Let G be a set in the xy plane and let $\mathbf{F}$ be a continously differentiable vector field on G. Suppose that G is subdivided into a finite number of subsets G_j which do not overlap, except on their boundaries. Let C be the boundary of G and let C_j be the boundary of the subset G_j. We assume that C and the C_j are oriented in the counterclockwise direction. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \sum_j \int_{C_j} \mathbf{F} \cdot d\mathbf{r}.$$

We remark that if G has holes in it, the boundary of G may have several components. The positive orientation around the interior holes is in the clockwise direction. In Figure 8, we see that boundary C of G consists of $C = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. $\int_C \mathbf{F} \cdot d\mathbf{r} = (\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3})\mathbf{F} \cdot d\mathbf{r}$ with the orientations shown in Figure 8.

Now we are ready to prove the general form of

Green's Theorem Let G be a bounded set in $\mathbb{R}^2$ with boundary C, perhaps consisting of several components. Let C have the positive orientation. Let $\mathbf{F} = (u(x, y), v(x, y))$ be continuously differentiable on G. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int \int_G [v_x(x, y) - u_y(x, y)] dA(x, y). \tag{11}$$

We know that Green's Theorem holds over triangles. We shall reduce the general case to that one by introducing a triangulation of G. A *triangulation* of
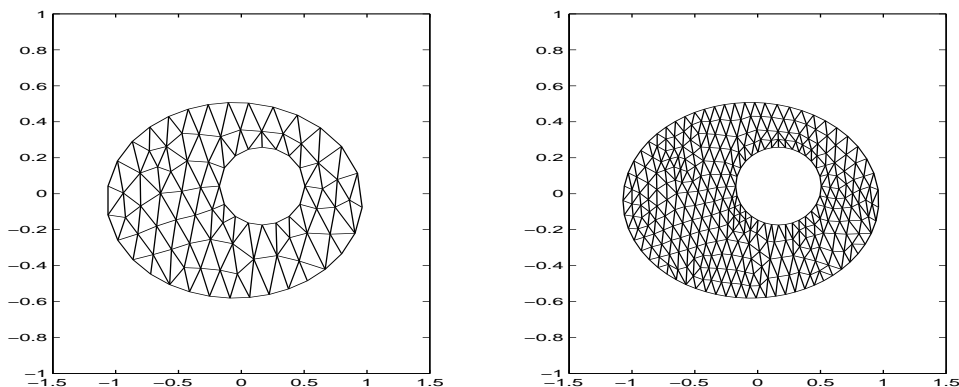
Figure 9: Coarse mesh triangulation on left, with refinement on right.

G is a mesh on G with triangular cells (see Figure 9). We can refine the mesh by subdividing the triangles, as shown in the right side of Figure 9.

Let $\hat{G}$ be the union of the triangles T_j in a triangulation of G. $\hat{G}$ will be a good approximation to G when the mesh is very fine. Let $\hat{C}$ be the boundary of $\hat{G}$. We use the additive property of the circulation to deduce

$$\int_{\hat{C}} \mathbf{F} \cdot d\mathbf{r} = \sum_j \int_{C_j} \mathbf{F} \cdot d\mathbf{r}$$

where C_j is the boundary of the triangle T_j with the counterclockwise orientation. Now for each j, we apply Green's theorem to the triangle T_j:

$$\int_{C_j} \mathbf{F} \cdot d\mathbf{r} = \int\int_{T_j} [v_x(x,y) - u_y(x,y)]dA(x,y).$$

Then summing over the triangles that make up $\hat{G}$, we see that

$$\int_{\hat{C}} \mathbf{F} \cdot d\mathbf{r} = \int\int_{\hat{G}} [v_x(x,y) - u_y(x,y)]dA(x,y).$$

Finally we take the limit on each side of this equation as the mesh becomes finer and finer, to deduce

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_G [v_x(x,y) - u_y(x,y)]dA(x,y).$$

4. Stokes' Theorem

Stokes' theorem is an extension of Green's theorem to a piece of oriented surface Σ in three-dimensional space. We will see that the circulation around the boundary C of Σ is equal to the integral over Σ of a pointwise quantity related to the curl of the vector field.

We shall make a simple extension of Green's theorem that yields Stokes' theorem for a flat piece of surface. We suppose that Σ is a flat surface lying in the plane $ax + by + cz = d$, with closed boundary curve C parameterized by $\mathbf{r}(t) = (x(t), y(t), z(t)), t_0 \leq t \leq t_1$. Without loss of generality, we can suppose $c \neq 0$, solve for z and relabel the coefficients. Thus we can suppose that the plane is given by $z = ax + by + h$. In this case the bounding curve C is parameterized by $\mathbf{r}(t) = (x(t), y(t), ax(t) + by(t) + h)$. We let $\tilde{\mathbf{r}}(t) = (x(t), y(t))$ be the projection of $\mathbf{r}(t)$ onto the plane $z = 0$.

Let the vector field $\mathbf{F}$ have three components: $\mathbf{F} = (u, v, w)$ where each function depends on x, y, z. Since the tangent vector to $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = (x'(t), y'(t), ax'(t) + by'(t))$$

the integral around the curve C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} [(u + aw)x' + (v + bw)y'(t)]dt$$

where u, v, w are evaluated at $(x(t), y(t), ax(t) + y(t) + h)$. We want to write this integral as $\int \tilde{\mathbf{F}} \cdot d\tilde{\mathbf{r}}$ for an appropriate two dimensional vector field. To this end we let

$$\tilde{\mathbf{F}}(x, y) = (u + aw, v + bw)$$

where u, v and w are evaluated at $(x, y, ax + by + h)$. Now we can write the integral around C as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t_0}^{t_1} \tilde{\mathbf{F}}(x(t), y(t)) \cdot \tilde{\mathbf{r}}'(t)$$
$$= \int_{\tilde{C}} \tilde{\mathbf{F}} \cdot d\tilde{\mathbf{r}}.$$

The curve $\tilde{C}$, which is parameterized by $\tilde{\mathbf{r}}(t)$, lies in the plane $z = 0$ and encloses a region G that is the projection of the piece of surface Σ onto the plane $z = 0$. Then using Green's theorem we can write

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\tilde{C}} \tilde{\mathbf{F}} \cdot d\tilde{\mathbf{r}}$$
$$= \int_G \text{curl}(\tilde{\mathbf{F}}) \cdot \mathbf{k} \, dA(x, y)$$

It remains to convert the integral over G into a surface integral on Σ. We see that

$$
\begin{aligned}
\mathrm{curl}(\tilde{\mathbf{F}}) \cdot \mathbf{k} &= (\tilde{F}_2)_x - (\tilde{F}_1)_y \\
&= v_x + av_z + b(w_x + aw_z) - [u_y + bu_z + a(w_y + bw_z)] \\
&= -a(w_y - v_z) - b(u_z - w_x) + v_x - u_y \\
&= \mathrm{curl}(\mathbf{F}) \cdot (-a, -b, 1)
\end{aligned}
$$

However the unit normal to the plane $z = ax + by + h$ is

$$
\mathbf{n} = \frac{(-a, -b, 1)}{\sqrt{a^2 + b^2 + 1}}.
$$

and the element of surface area on the plane is $dS = \sqrt{a^2 + b^2 + 1}\, dA(x, y)$. Hence

$$
\begin{aligned}
\mathrm{curl}(\tilde{\mathbf{F}}) \cdot \mathbf{k}\, dA(x, y) &= \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\sqrt{a^2 + b^2 + 1}\, dA(x, y) \\
&= \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\, dS.
\end{aligned}
$$

We conclude that

$$
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_\Sigma \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\, dS. \tag{12}
$$

Equation (12) is Stoke's theorem in the special case of a flat piece of surface.

Now we shall derive the general form of Stokes' theorem in the same way that we derived Green's theorem from the special case for triangles. Let Σ be an orientable, bounded piece of surface. We can approximate the surface Σ with a collection of flat triangular patches. In general, only the vertices of the triangular patches will touch the surface. The union of these patches,

$$
\hat{\Sigma} = \cup T_j,
$$

will be a good approximation to Σ when the patches are small enough. The boundary curve $\hat{C}$ of $\hat{\Sigma}$ will also be a good approximation to the boundary curve C of Σ. An example is shown in Figure 10. We choose the normal directions to the patches so that they vary continuously from patch to patch, and that as the patches get smaller and smaller, the normals on the triangular patches converge to the normals on the surface Σ.

The additivity of the circulation also holds in three dimensions. In Figure 11 we see a surface $\Sigma = \Sigma_1 \cup \Sigma_2$. Let C be the boundary curve of Σ, C_1 the boundary curve of Σ_1, and C_2, the boundary curve of Σ_2 with the indicated orientations. Then

$$
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}
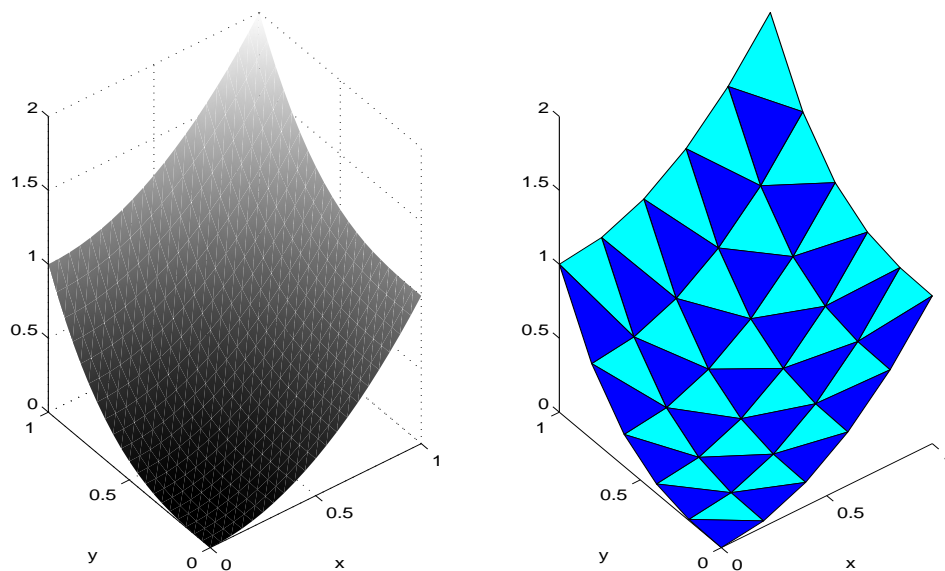$$

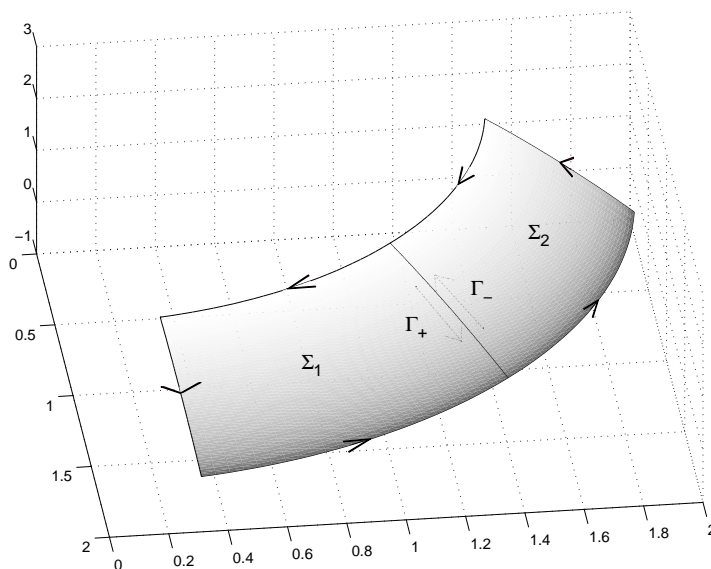Figure 10: Surface $z = x^2 + y^3$ on left, triangulation right.

Figure 11: Surface Σ consisting of two pieces, Σ_1 and Σ_2, with common boundary oriented in opposite directions, Γ_+ and Γ_-.

because the integrals on Γ_+ and Γ_- cancel out.

We apply this additivity to the boundaries of the triangular patches of the approximating surface $\hat{\Sigma}$. For any continuously differentiable vector field $\mathbf{F} = (u(x, y, z), v(x, y, z), w(x, y, z))$, we have

$$\int_{\hat{C}} \mathbf{F} \cdot d\mathbf{r} = \sum_j \int_{C_j} \mathbf{F} \cdot d\mathbf{r} \tag{13}$$

where C_j is the boundary of the triangle T_j. Now we apply the special case of Stokes' theorem, equation (12) to each triangle T_j:

$$\int_{\hat{C}} \mathbf{F} \cdot d\mathbf{r} = \sum_j \int_{T_j} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n} \ dS$$

$$= \int_{\hat{\Sigma}} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n} \ dS$$

Now take the limit on both sides of the last expressions as the triangulation becomes finer and finer. It follows that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\Sigma} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n} \ dS.$$

We have derived Stokes' theorem in the general case.

Stokes' theorem can be used to further our understanding of the notion of curl. Let Σ_r be a flat disk of radius $r > 0$ centered at the point $\mathbf{p}_0 = (x_0, y_0, z_0)$. Let the normal to the disk be $\mathbf{n}$, and let C_r be the boundary of Σ_r oriented so that it turns counterclockwise with respect to the direction of $\mathbf{n}$ (see Figure 12). Then by Stokes' theorem,

$$\int_{C_r} \mathbf{F} \cdot d\mathbf{r} = \int_{\Sigma_r} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n} \ dS.$$

Divide both sides of this equation by the area of the disk, πr^2, and take the limit as $r \to 0$. Assuming $\mathbf{F}$ is continuously differentiable, we find,

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{C_r} \mathbf{F} \cdot d\mathbf{r} = \mathrm{curl}(\mathbf{F})(\mathbf{p}_0) \cdot \mathbf{n}.$$

Thus the limiting value of the circulation in the plane perpendicular to $\mathbf{n}$ is $\mathbf{n} \cdot \mathrm{curl}(\mathbf{F})(\mathbf{p}_0)$.

We use this observation to interpret each component of the curl. If we choose $\mathbf{n} = \mathbf{k}$, then $\mathrm{curl}(\mathbf{F}) \cdot \mathbf{k} = v_x - u_y$ is twice the angular velocity of the rotation of the
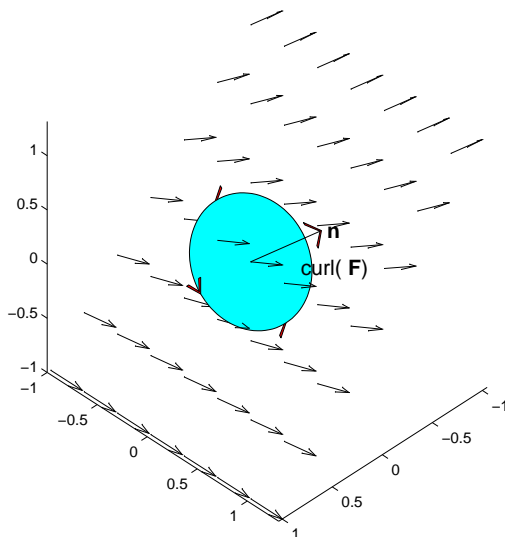
Figure 12: Curl of vector field **F** and circulation around small disk of radius r with normal vector **n**.

paddle wheel when it is placed in the plane $z = z_0$ i.e., parallel to the xy plane. When $\mathbf{n} = \mathbf{j}$, $\mathrm{curl}(\mathbf{F}) \cdot \mathbf{j} = u_z - w_x$ is twice the angular velocity of the paddle wheel when it is placed in the plane $y = y_0$ i.e., parallel to the xz plane, and similarly for the first component of the curl $w_y - v_z$. This fact is essentially what was proved in our first version of Stokes' theorem for a piece of planar surface. The angular velocity of the paddle wheel will be greatest in that plane with **n** pointing in the same direction as $\mathrm{curl}(\mathbf{F})(\mathbf{p}_0)$.

Continuing in this vein, we can interpret Stokes' theorem in the general case as follows. At each point $\mathbf{p} \in \Sigma$, $\mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}$ is twice the angular velocity of the paddle wheel when placed in the tangent plane to Σ at the point $\mathbf{p}$. It a measure of the shear of the projection of the vector field **F** onto the tangent plane. It is this quantity that is integrated over Σ in the statement of Stokes' theorem.

5. The divergence

The divergence of a vector field $\mathbf{F} = (u(x, y, z), v(x, y, z), w(x, y, z))$ is defined to be the *scalar* quantity

$$\mathrm{div}(\mathbf{F}) = u_x(x, y, z) + v_y(x, y, z) + w_z(x, y, z).$$

The choice of the derivatives u_x, v_y, w_z which appear in the definition of the divergence has a very natural intuitive meaning when the vector field is **F** is thought of

as the velocity field of a fluid. To see this, we consider the two-dimensional case.

$$\mathbf{F} = (u(x,y), v(x,y)). \tag{14}$$

Now let R be a rectangle consisting of fluid particles. Let $R(\Delta t)$ be the image set of all the fluid particles in R moved by the flow for a time interval Δt. How does the area of R change when it is moved by the flow ? In three dimensions, we would ask how does the volume of a solid rectangle change when displaced by the flow.

In the time interval $[0, \Delta t]$, the fluid particle at the point (x, y) when $t = 0$ is moved by the flow to a point $(x(\Delta t), y(\Delta t))$. If Δt is quite small,

$$x(\Delta t) \approx f(x, y) = x + \Delta t u(x, y) \tag{15}$$
$$y(\Delta t) \approx g(x, y) = y + \Delta t v(x, y)) \tag{16}$$

The mapping

$$(x, y) \rightarrow (f(x, y), g(x, y))$$

takes the rectangle R onto a set $\tilde{R}(\Delta t)$ that approximates the exact image $R(\Delta t)$. See Figure 13. We know that such a mapping changes area by a factor of the Jacobian matrix. In fact

$$\text{area}(R(\Delta t)) \approx \text{area}(\tilde{R}(\Delta t)) = \int\int_{R(\tilde{\Delta t})} dA = \int\int_R |J(\Delta t)| dA$$

where $J(\Delta t)$ is the Jacobian of the mapping (16). Now

$$
\begin{aligned}
J(\Delta t) &= \det \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \\
&= \det \begin{bmatrix} 1 + \Delta t u_x & \Delta t u_y \\ \Delta t v_x & 1 + \Delta t v_y \end{bmatrix} \\
&= 1 + \Delta t(u_x + v_y) + (\Delta t)^2(u_x v_y - u_y v_x)
\end{aligned}
$$

Because $J(0) = 1$, $|J(\Delta t)| = J(\Delta t)$ for Δt small enough. Hence

$$\text{area}(\tilde{R}(\Delta t)) - \text{area}(R) = \Delta t \int\int_R (u_x + v_y) dA + \Delta t^2 \int\int_R (u_x v_y - u_y v_x) dA.$$

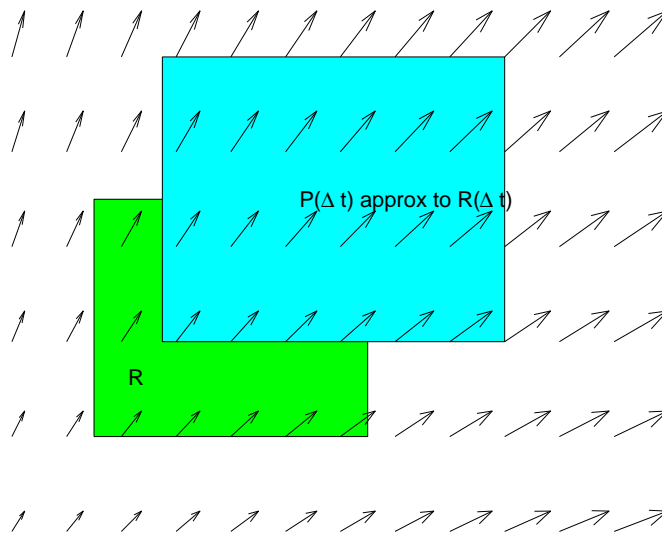The *rate of change* of $t \rightarrow \text{area}(R(t))$ at $t = 0$ is

Figure 13: Rectangle R and approximate image $\tilde{R}(\Delta t)$ under the mapping $(x, y) \to (x + \Delta t u(x, y), y + \Delta t v(x, y))$.

$$
\begin{aligned}
\lim_{\Delta t \to 0} \frac{\text{area}(R(\Delta t)) - \text{area}(R)}{\Delta t} &= \lim_{\Delta t \to 0} \frac{\text{area}(\tilde{R}(\Delta t)) - \text{area}(R)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \int\int_R (u_x + v_y) dA + \Delta t \int\int (u_x v_y - u_y v_x) dA \\
&= \int\int_R (u_x + v_y) dA
\end{aligned}
$$

If $u_x > 0$ and $v_y > 0$, the rectangle is expanded in both directions. If $u_x > 0$ and $v_y < 0$, the rectangle is lengthened in the x direction and shortened in the y direction. The area is increased if $u_x + v_y > 0$, and reduced if $u_x + v_y < 0$. If $u_x + v_y = 0$, the shape of the rectangle may change but the area remains the same.

We see that the quantities u_y and v_x, which were important in the effect of curl($\mathbf{F}$), do not effect the rate of change of the area caused by the flow. In fact, the formula for a general flow $\mathbf{F} = (u(x, y), v(x, y))$ is

$$
\frac{d}{dt}\text{area}(G(t)) = \int\int_G \text{div}(\mathbf{F}) \; dA(x, y). \tag{17}
$$

Examples

(i) Consider the vector field $\mathbf{F}(x,y) = (x^2/2, 0)$. Clearly $\mathrm{div}(\mathbf{F}) = x$. The flow is horizontal, from left to right. A set of fluid particles starting out in the half plane $x < 0$ will be compressed in the x direction, until it flows past the line $x = 0$. At this time it will be lengthened in the x direction because the right (leading) edge of the set will be moving faster than the left (trailing) edge.

(ii) Consider the vector field

$$\mathbf{F} = \frac{(x,y)}{r^\alpha}, \quad \alpha > 0.$$

This flow carries fluid particles away from the origin in a radial direction. What happens to a set of fluid particles that starts near the origin? We calculate

$$\mathrm{div}(\mathbf{F}) = \frac{2 - \alpha}{r^\alpha}, \quad (x,y) \neq (0,0).$$

Thus the set of fluid particles will expand in this flow if $\alpha < 2$, but will be compressed if $\alpha > 2$. If $\alpha = 2$, the shape of the set will change but the area will remain the same.

6. The divergence theorem

The divergence theorem in two dimensions can be derived from Green's theorem. However, we shall give a direct derivation, based on the ideas of fluid flow. In section 4, we thought of the rectangle R as moving with the flow. Now let us think of it as fixed with the fluid particles moving through the sides of R. We shall establish a connection between the pointwise microscopic quantity $u_x(x,y) + v_y(x,y)$ and the flux of the fluid through the sides of the rectangle. Fix a value of y, $c \leq y \leq d$. Then

$$u(b,y) - u(a,y) = \int_a^b u_x(x,y)dx.$$

We integrate again in y,

$$\int_c^d u(b,y)dy - \int_c^d u(a,y)dy = \int\int_R u_x(x,y) \; dA(x,y).$$

Now

$$\int_c^d u(b,y)dy = \int_c^d \mathbf{F} \cdot \mathbf{n} \; dy$$

is the flux of the fluid through the right boundary where $\mathbf{n} = (1, 0)$. Similarly,

$$- \int_c^d u(a, y) dy = \int_c^d \mathbf{F} \cdot \mathbf{n} \, dy$$

is the flux of the fluid through the left boundary where $\mathbf{n} = (-1, 0)$. With the sides of the rectangle as labeled in Figure 5, we have

$$\int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds + \int_{C_4} \mathbf{F} \cdot \mathbf{n} \, ds = \int \int_R u_x(x, y) \, dA(x, y). \tag{18}$$

Note however, that we are not using the orientation of the boundary elements. The orientation is determined by the direction of the normal vector $\mathbf{n}$. This equation says that the flux through the left and right sides of the rectangle equals the integral of $u_x(x, y)$ over the rectangle. A similar integration of $v_y(x, y)$ yields the equation

$$\int_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds + \int_{C_3} \mathbf{F} \cdot \mathbf{n} \, ds = \int \int_R v_y(x, y) \, dA(x, y). \tag{19}$$

Adding together equations (18) and (19), we have

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int \int_R [u_x(x, y) + v_y(x, y)] \, dA(x, y). \tag{20}$$

Equation (20) is the divergence theorem for the special case of a rectangle. It is also easy to show this equation is valid for any set G which is both vertically and horizontally simple, in particular for triangles.

To extend this result to more general sets, we shall use the fact that the flux through the boundary of a set G is additive, just as the circulation is additive. The additivity of the flux can be see from Figure 14 where the set $G = G_1 \cup G_2$. The boundary of G is C, the boundary of G_1 is C_1 and the boundary of G_2 is C_2. The common part of the boundaries is Γ. The exterior normal $\mathbf{n}$ to G_1 on Γ points in the opposite direction to the exterior normal $\mathbf{n}$ to G_2 on Γ. This means that the rate at which fluid leaves G_1 through Γ is the same as the rate at which at which fluid enters G_2 through Γ. Hence the flux integrals over Γ cancel out. We see therefore that

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_{C_1} \mathbf{F} \cdot \mathbf{n} \, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{n} \, ds.$$
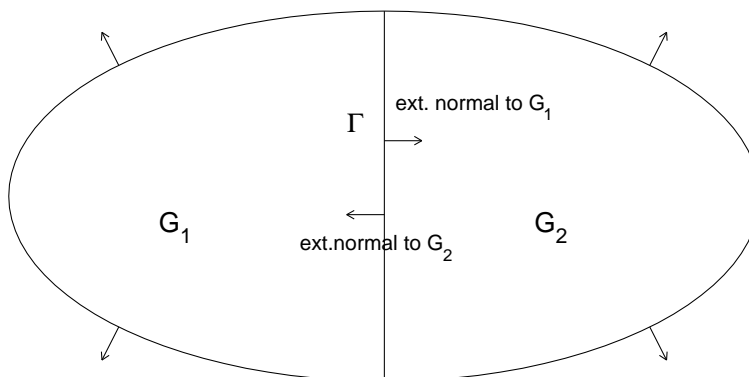
Figure 14: Set $G = G_1 \cup G_2$ with common boundary piece Γ and exterior normals.

Now we are ready to derive

The divergence theorem

Let G be a bounded set with boundary C, perhaps consisting of several components. Let $\mathbf{n}$ be the exterior unit normal on C. Let $\mathbf{F} = (u(x,y), v(x,y))$ be a continously differentiable vector field on G. Then

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int\int_G [u_x(x,y) + v_y(x,y)] \, dA(x,y) = \int\int_G \mathrm{div}(\mathbf{F}) \, dA(x,y). \quad (21)$$

To derive this formula, we introduce a triangulation of G, as we did for Green's theorem. Let $\hat{G}$ be the union of the triangles T_j in the triangulation of G. Let $\hat{C}$ be the boundary of $\hat{G}$. We use the additive property of the flux to deduce

$$\int_{\hat{C}} \mathbf{F} \cdot \mathbf{n} \, ds = \sum_j \int_{C_j} \mathbf{F} \cdot \mathbf{n} \, ds$$

where C_j is the boundary of T_j and $\mathbf{n}$ is the exterior normal to T_j. Now for each j we apply the divergence theorem on the triangle T_j:

$$\int_{C_j} \mathbf{F} \cdot \mathbf{n} \, ds = \int\int_{T_j} \mathrm{div}(\mathbf{F}) \, dA(x,y).$$

Then summing over the triangles that make up $\hat{G}$, we see that

$$\int_{\hat{C}} \mathbf{F} \cdot \mathbf{n} \, ds = \int \int_{\hat{G}} \operatorname{div}(\mathbf{F}) \, dA(x, y).$$

Finally, we take the limit on each side of this equation as the mesh becomes finer and finer to deduce equation (21).

A similar proof works in three dimensions. Then for a bounded set $D \subset \mathbb{R}^3$, with bounding surface Σ, we have

$$\int \int \int_{D} \operatorname{div}(\mathbf{F}) \, dV(x, y, z) = \int \int_{\Sigma} \mathbf{F} \cdot \mathbf{n} \, dS.$$

The divergence theorem can be helpful in gaining further understanding of the divergence. Let B_r be the solid ball of radius $r > 0$ centered at $\mathbf{p}_0 = (x_0, y_0, z_0)$, let Σ_r be the sphere of radius $r > 0$ which bounds this ball. If we apply the divergence theorem over B_r and divide by the volume of the ball, $4\pi r^3 / 3$, we see that

$$\frac{1}{(4/3)\pi r^3} \int \int_{\Sigma_r} \mathbf{F} \cdot \mathbf{n} \, dS = \frac{1}{(4/3)\pi r^3} \int \int \int_{B_r} \operatorname{div}(\mathbf{F}) \, dV(x, y, z).$$

Assuming that $\mathbf{F}$ is continuously differentiable, we can take the limit as $r \to 0$ to deduce

$$\lim_{r \to 0} \frac{1}{(4/3)\pi r^3} \int \int_{\Sigma_r} \mathbf{F} \cdot \mathbf{n} \, dS = \operatorname{div}(\mathbf{F})(\mathbf{p}_0).$$

Thus $\operatorname{div}(\mathbf{F})(\mathbf{p})$ is the limiting value of the flux per unit volume at the point $\mathbf{p}_0$. In this view, the divergence is a measure of the rate at which the fluid is being absorbed or created at the point $\mathbf{p}_0$. This interpretation of the divergence is consistent with the interpretation we developed in Section 4 where we saw that the divergence of a fluid flow was the rate at which a volume of fluid was expanding or contracting. The divergence theorem allows us to make the connection. Let D be a bounded set of fluid particles in $\mathbb{R}^3$ and let $\mathbf{F} = (u, v, w)$ be the velocity field of a fluid. Let $D(t)$ be the location of the set of fluid particles which occupy D at time $t = 0$. The extension of equation (17) to three dimensions is

$$\frac{d}{dt} \operatorname{vol}(D(t)) = \int \int \int_{D(t)} \operatorname{div}(\mathbf{F}) \, dV. \tag{22}$$

On the other hand the divergence theorem says that for each t,

$$\int \int \int_{D(t)} \operatorname{div}(\mathbf{F}) \, dV = \int \int_{\Sigma(t)} \mathbf{F} \cdot \mathbf{n} \, dS \tag{23}$$

where $\Sigma(t)$ is the bounding surface of $D(t)$. Combining these two equations, we arrive at

$$\frac{d}{dt}\text{vol}(D(t)) = \int\int_{\Sigma(t)} \mathbf{F} \cdot \mathbf{n} \, dS. \tag{24}$$

This equation, called the Reynolds transport theorem, says that the rate of change of the volume of the fluid particles at time t is equal to the flux through the boundary of $D(t)$. Since the $D(t)$ is moving with the flow, it would seem that no fluid particles pass through the boundary. However, the flux integral of the right side of (33) should be interpreted to mean the flux though the boundary of $D(t)$ when it is frozen at time t.

6. Applications of the divergence theorem

The divergence theorem is one of the most important theorems used to derive the equations of mathematical physics. Our first example comes from electrostatics. Let $\mathbf{E} = (u, v, w)$ be the electric field produced by a charge density q. q has the units of coulombs/vol. $\mathbf{E}$ is the force experienced by a unit positive test charge at the point (x, y, z). Two things are known about the electric field:

(i) It is conservative, i.e., the line integral $\int_C \mathbf{E} \cdot d\mathbf{r} = 0$ for every closed curve C.

(ii) If Σ is any closed surface, bounding the set D,

$$\int_\Sigma \mathbf{E} \cdot \mathbf{n} \, dS = \int\int\int_D q \, dV.$$

If we apply the divergence theorem to the flux integral in (ii), we find that for each set D,

$$\int\int\int_D \text{div}(\mathbf{E}) \, dV = \int\int\int_D q \, dV.$$

On the other hand, (i) implies that there exists a scalar potential $\phi(x, y, z)$, the electrostatic potential, such that $-\nabla\phi = \mathbf{E}$. Thus we deduce from the last equation that

$$-\int\int\int_D \text{div}(\nabla\phi) \, dV = \int\int\int_D q \, dV$$

or

$$\int\int\int_D [q - \text{div}(\nabla\phi)] \, dV = 0$$

for all sets D. This implies that at each point (x, y, z), the electric potential ϕ satisfies the partial differential equation

$$-\text{div}(\nabla\phi) = q. \tag{25}$$

Now $\text{div}(\nabla\phi) = \phi_{xx} + \phi_{yy} + \phi_{zz} = \Delta\phi$. Equation (25) is usually written

$$-\Delta\phi = q. \tag{26}$$

Equation (26) is called the *Poisson* equation. It is one of the most important equations of physics.

For a second example, we turn to the diffusion of heat in a solid. Let $T(x, y, z)$ denote the temperature at point in a region of space. The *heat flux* is the vector field $\mathbf{F} = -\nabla T$. Heat flows from warm spots to cooler spots, in the opposite sense from the gradient of T. For any set D with bounding surface Σ, the heat flux through Σ must equal the rate at which heat is either produced or absorbed in the set D, which we state as

$$\int\int_{\Sigma} \mathbf{F} \cdot \mathbf{n} \, dV = \int\int\int_{D} q \, dV \tag{27}$$

where q is the heat source (or sink) in the set D. Again we apply the divergence theorem to the flux integral in (27) and deduce that

$$\int\int\int_{D} \text{div}(\mathbf{F}) \, dV = \int\int\int_{D} q \, dV$$

for all sets D. Since $\mathbf{F} = -\nabla T$, this implies that T also satisfies the Poisson equation, which in this setting we call the *steady state* heat equation,

$$-\Delta T = q. \tag{28}$$

It is often the case that for a given function q, one seeks a solution of the Poisson equation (26). It is clear that solutions of (26) are not unique. In fact, if ϕ is any solution of (26) for a given function q, then $\phi + f(x, y, z)$ is also a solution for the same q, provided f satisfies the *Laplace* equation:

$$\Delta f = 0.$$

Solution of the Laplace equation are called *harmonic* functions. An example is $f(x, y, z) = x^2 + y^2 - 2z^2$.

Physical restrictions usually tell us which one of the infinite number of solutions is the one that is appropriate in a particular physical setting. The physical

restrictions take the form of *boundary conditions* which we require a solution of (26) to satisfy. In the setting of electrostatics, we may require that the electrostatic potential be zero on the boundary of the region D where we seek a solution. Our problem then might be, for given charge density q in the set D, find that electrostatic potential ϕ such that

$$-\Delta\phi = q, \quad \text{in } D, \quad \text{and} \quad \phi = 0 \ \text{ on } \ \Sigma.$$

The combination of (26) and a boundary condition is called a *boundary value problem*.

Boundary value problems do not always have a solution. It may be necessary to place some additional restriction on the data (in this case, the given function q). Here is an example. Suppose that q is a heat source (or sink) in a set D, and that the boundary of D is insulated - no heat flows through the boundary. The boundary condition that expresses this situation is $\mathbf{F} \cdot \mathbf{n} = 0$, or in terms of the temperature T, $\nabla T \cdot \mathbf{n} = 0$ on the boundary. Here $\mathbf{n}$ is the unit exterior normal to the boundary surface Σ. The boundary value problem is thus

$$-\Delta T = q, \quad \text{in} \ \ D, \quad \text{and} \quad \nabla T \cdot \mathbf{n} = 0, \quad \text{on} \ \ \Sigma. \tag{29}$$

If no heat can escape the set, and the heat source q is positive (i.e., heat is only produced, not absorbed), it appears that the temperature T will rise. In this case the temperature function T would have to depend on time as well as position. However, the boundary value problem (29) describes *steady state* heat diffusion. To have a steady state solution, we must place some restriction on the heat source q. Equation (27), which gave rise to the steady state heat equation (28), says that for there to exist a steady state solution, we must have

$$\int\int\int q \, dV = \int\int_{\Sigma} (-\nabla T) \cdot \mathbf{n} \, dS = 0.$$

This equation says that the average value of the heat source q must be zero. If q is positive somewhere in D, heat is being produced there, and this must be balanced by some regions in D where q is negative, where heat is being absorbed.