Transverse geometry

The ‘space of leaves’ of a foliation $(V, \mathcal{F})$ can be described in terms of (M, Γ), with $M =$ *complete transversal* and $\Gamma =$ *holonomy pseudogroup*. The ‘natural’ ‘transverse coordinates’ form the crossed product algebra

$$\mathcal{A}_M^{\Gamma} := C_c^{\infty}(M) \rtimes \Gamma ,$$

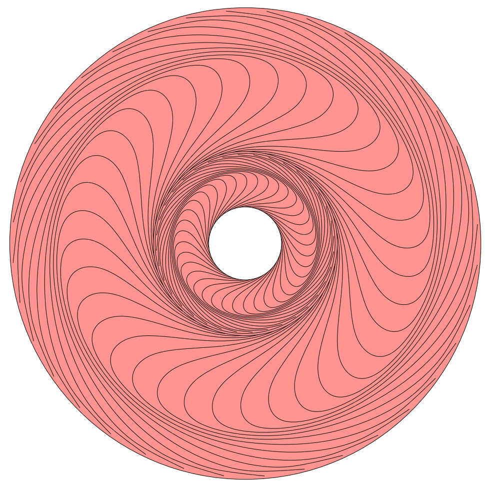
consisting of finite sums of monomials of the form

$$\sum f\, U_{\phi}^{*} , \qquad f \in C_c^{\infty}(FM) , \phi \in \Gamma ,$$

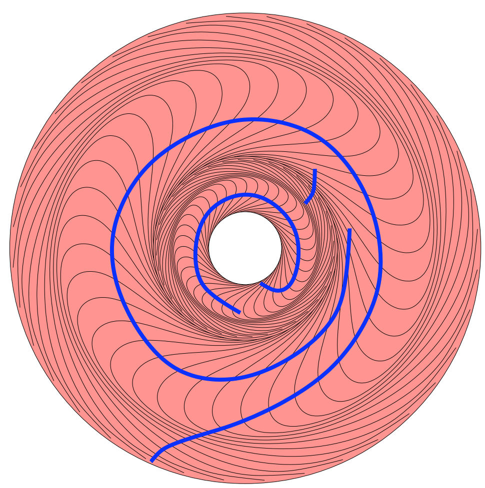
with the product

$$f\, U_{\phi}^{*} \cdot g\, U_{\psi}^{*} = (f \cdot g|\phi)\, U_{\psi\, \phi}^{*} .$$

How to find a geometric structure = spectral triple that is ‘invariant’ under the holonomy ? D cannot be taken elliptic, unless the foliation admits a transverse Riemannian structure.



Foliation



Transversals

A. Connes & H.M., *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5** (1995), Part I

Diff$^+(M)$-**invariant structure**

First, one replaces M by $PM = F^+M/SO(n)$, where $F^+M = J^1(M) =$ GL$^+(n, \mathbb{R})$–principal bundle of oriented frames on M. The sections of $\pi : PM \to M$ are precisely the Riemannian metrics on M.

Canonical structure on PM: the vertical subbundle $\mathcal{V} \subset T(PM)$, $\mathcal{V} = \mathrm{Ker}\, \pi_*$, has $GL^+(n, \mathbb{R})$-invariant Riemannian metric, since its fibers $\cong$ $GL^+(n, \mathbb{R})/SO(n)$. The bundle $\mathcal{N} = T(PM)/\mathcal{V}$ has tautological Riemannian structure: every point $q \in PM$ is an Euclidean structure on $T_{\pi(q)}(M) \cong \mathcal{N}_q$ via π_*.

Hypoelliptic signature operator

The *hypoelliptic signature operator* D on PM is uniquely determined by $Q = D|D|$,

$$Q = (d_V^* \, d_V - d_V \, d_V^*) \oplus \gamma_V \, (d_H + d_H^*) \, ,$$

acting on $\mathcal{H}_{PM} = L^2(\wedge^{\cdot}\mathcal{V}^* \otimes \wedge^{\cdot}\mathcal{N}^*, \text{vol}_{PM})$;
$d_V =$ vertical exterior derivative,
$\gamma_V =$ grading for the vertical signature,
$d_H =$ horizontal exterior differentiation with respect to a torsion-free connection,
$\text{vol}_{PM} =$ $\text{Diff}^+(M)$-invariant volume form.

*If $n \equiv 1 \text{ or } 2 \pmod 4$, one takes $PM \times S^1$ so that the dimension of the vertical fiber be even.

Theorem 1. *The operator Q is selfadjoint and so is D defined by $Q = D|D|$. Moreover, $(\mathcal{A}^{\Gamma}_{PM}, \mathcal{H}_{PM}, D)$ is a (nonunital) spectral triple with simple dimension spectrum*
$\Sigma_P = \{k \in \mathbb{Z}^+, \quad k \leqq p := \frac{n(n+1)}{2} + 2n\}$.

Proof – By means of adapted pseudodifferential calculus = a version of ΨDO for Heisenberg manifolds:

$$\lambda \cdot \xi = (\lambda\ \xi_v, \lambda^2\ \xi_n), \qquad \xi = (\xi_v, \xi_n)\ ,\ \lambda \in \mathbb{R}^*_+\ ,$$

$$\|\xi\|' = \left(\|\xi_v\|^4 + \|\xi_n\|^2\right)^{1/4}\ ,$$

$$\sigma'(x, \lambda\cdot\xi) = \lambda^q\ \sigma'(x,\xi), \quad \sigma' = q-\text{homogeneus.}$$

In particular, the residue density of $R \in \Psi' DO$

$$= \frac{1}{(2\pi)^{p-n}} \int_{\|\xi\|'=1} \sigma'_{-p}(R)(q,\xi)\, đ\xi\, dq\,.$$

Example (codimension 1): $S^1/\,\mathrm{Diff}(S^1)$

$$\mathcal{H} = L^2(FS^1 \times S^1, ds\, d\theta\, d\alpha) \otimes \mathbb{C}^2$$

$$Q = -2\partial_s\partial_\alpha\gamma_1 + \frac{1}{i}\, e^{-s}\partial_\theta\gamma_2 + \left(\partial_s^2 - \partial_\alpha^2 - \frac{1}{4}\right)\gamma_3\,,$$

where $\gamma_1, \gamma_2, \gamma_3$ are the Pauli matrices

$$\gamma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\ ,\ \gamma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\ \gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\ ;$$

the dimension spectrum is $\Sigma = \{0, 1, 2, 3, 4\}$. The components of the Chern character are $\{\varphi_1, \varphi_3\}$ and are given by:

$$\begin{aligned} \varphi_1(a^0, a^1) &= \Gamma\left(\frac{1}{2}\right) ⨍ (a^0[Q, a^1](Q^2)^{-1/2}) \\ &- \frac{1}{2!}\Gamma\left(\frac{3}{2}\right) ⨍ (a^0\nabla[Q, a^1](Q^2)^{-3/2}) \\ &+ \frac{1}{3!}\Gamma\left(\frac{5}{2}\right) ⨍ (a^0\nabla^2[Q, a^1](Q^2)^{-5/2}) \\ &- \frac{1}{4!}\Gamma\left(\frac{7}{2}\right) ⨍ (a^0\nabla^3[Q, a^1](Q^2)^{-7/2}) \end{aligned}$$

$$\begin{aligned}
&\varphi_3(a^0,a^1,a^2,a^3) = \\
&\quad \frac{1}{3i}\Gamma\left(\frac{3}{2}\right) ⨍ (a^0[Q,a^1][Q,a]\cdots[Q,a^3](Q^2)^{-3/2}) \\
&- \frac{1}{4!}\Gamma\left(\frac{5}{2}\right) ⨍ (a^0\nabla[Q,a^1][Q,a^2]\cdots[Q,a^3](Q^2)^{-5/2}) \\
&- \frac{1}{3\cdot 4}\Gamma\left(\frac{5}{2}\right) ⨍ (a^0[Q,a^1]\nabla([Q,a^2][Q,a^3](Q^2)^{-5/2} \\
&- \frac{1}{2\cdot 4}\Gamma\left(\frac{5}{2}\right) ⨍ (a^0[Q,a^1][Q,a^2]\nabla[Q,a^3](Q^2)^{-5/2}) \,.
\end{aligned}$$

The computation is purely symbolical, but requires the symbol σ'_{-4}, hence about 10^3 terms! It eventually yields the following result:

$$(\varphi_1)_{(1)}(a^1,a^1) = 0, \qquad \forall\, a^0,a^1 \in \mathcal{A};$$

in fact, each of the 4 terms turns out to be 0; on the other hand

$$(\varphi_3)_{(1)} = \frac{1}{12\,\pi^{3/2}}(\tilde{\mu} + b\psi),$$

where

$$\begin{aligned}
&\tilde{\mu}(f^0U_{\varphi_0}, f^1U_{\varphi_1}, \ldots, f^3U_{\varphi_3}) = 0, \quad \varphi_0\varphi_1\varphi_2\varphi_3 \neq 1 \\
&= \int f^0\varphi_0^*(df^1)\wedge(\varphi_0\varphi_1)^*(df^2)\wedge(\varphi_0\varphi_1\varphi_2)^*(df^3).
\end{aligned}$$

Underlying algebraic structure

W.l.o.g. can assume $M = \mathbb{R}^n$, with the *flat connection*; $\{X_k;\, 1 \le k \le n\}$, $\{Y_i^j;\, 1 \le i, j \le n\}$ horizontal, resp. vertical vector fields. The operator Q is built of these vector fields, and the cocycle involves iterated commutators of them acting on $\mathcal{A}^{\Gamma}_{FM}$.

E.g. in case $n = 1$,

$$Y = y \frac{\partial}{\partial y} \qquad \text{and} \qquad X = y \frac{\partial}{\partial x},$$

acting as

$$Y(f\,U_\varphi) = Y(f)\,U_\varphi\,, \qquad X(f\,U_\varphi) = X(f)\,U_\varphi\,.$$

However, while Y acts as derivation

$$Y(ab) = Y(a)\,b + a\,Y(b)\,, \qquad a, b \in \mathcal{A}^{\Gamma}\,.$$

X satisfies instead

$$X(ab) = X(a)\,b + a\,X(b) + \delta_1(a)\,Y(b)\,.$$

$$\delta_1(f\,U_{\varphi^{-1}}) \;=\; y\,\frac{d}{dx}\left(\log\frac{d\varphi}{dx}\right)\,fU_{\varphi^{-1}}\,.$$

δ_1 is a derivation,

$$\delta_1(ab) = \delta_1(a)\,b + a\,\delta_1(b)\,,$$

but its higher commutators with X

$$\delta_n(f\,U_{\varphi^{-1}}) \;=\; y^n\,\frac{d^n}{dx^n}\left(\log\frac{d\varphi}{dx}\right)\,fU_{\varphi^{-1}}\,, \qquad \forall\, n \geq 1\,,$$

satisfy more complicated Leibniz rules.

All this information can be encoded in a Hopf algebra $\mathcal{H}_1$. As algebra = universal enveloping algebra of the Lie algebra with presentation

$$[Y,X] = X\,, \qquad [Y,\delta_n] = n\,\delta_n\,,$$
$$[X,\delta_n] = \delta_{n+1}\,, \quad [\delta_k,\delta_\ell] = 0\,, \quad n,k,\ell \geq 1\,.$$

The coproduct is determined by

$$\begin{aligned}
\Delta\, Y &= Y \otimes 1 + 1 \otimes Y\,, \\
\Delta\, X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y \\
\Delta\, \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1, \\
\Delta(\delta_3) &= \delta_3 \otimes 1 + 1 \otimes \delta_3 + \\
&+ \ \delta_2 \otimes \delta_1 + 3\delta_1 \otimes \delta_2 + \delta_1^2 \otimes \delta_1;
\end{aligned}$$

the antipode is determined by

$$S(Y) = -Y\,,\; S(X) = -X + \delta_1 Y\,,\; S(\delta_1) = -\delta_1$$

and the counit is

$$\varepsilon(h) = \text{constant term of} \quad h \in \mathcal{H}_1\,.$$

The canonical trace τ_Γ on $\mathcal{A}^\Gamma$ satisfies

$$\tau_\Gamma(h(a)) = \delta(h)\,\tau_\Gamma(a)\,, \qquad \forall\, h \in \mathcal{H}_1\,, a \in \mathcal{A}.$$

where $\delta \in \mathcal{H}_1^*$ is the character

$$\delta(Y) = 1, \quad \delta(X) = 0, \quad \delta(\delta_n) = 0\,.$$

While $S^2 \neq \mathrm{Id}$, the δ-twisted antipode,

$$\widetilde{S}(h) = \delta(h_{(1)})\, S(h_{(2)})\,,$$

is involutive: $\widetilde{S}^2 = \mathrm{Id}$.

Finally, the cochains $\{\varphi_1, \varphi_3\}$ can be recognized as belonging to the range of a certain cohomological characteristic map.

More precisely, requiring the assignment

$$\chi_\Gamma(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) \\ = \tau_\Gamma(a^0\, h^1(a^1) \ldots h^n(a^n))\,,$$

to induce a characteristic homomorphism

$$\chi_\Gamma^* : HC^*_{\mathrm{Hopf}}(\mathcal{H}_1) \to HC^*(\mathcal{A}_\Gamma)\,,$$

practically dictates the definition of the Hopf cyclic cohomology.

[A. Connes & H.M., *Hopf algebras, cyclic Cohomology and the transverse index theorem*, Commun. Math. Phys. **198** (1998)]

$\mathcal{H} =$ Hopf algebra over a field k containing $\mathbb{Q}$, (δ, σ) =modular pair: $\delta \in \mathcal{H}^*$ character , and $\sigma \in \mathcal{H}$, $\Delta(\sigma) = \sigma \otimes \sigma$, $\varepsilon(\sigma) = 1$, with $\delta(\sigma) = 1$. One also requires $\widetilde{S}^2 = \mathrm{Id}$.

Then the following is a (co)cyclic structure:

$$
\begin{aligned}
\mathcal{H}^{\natural}_{(\delta,\sigma)} &= \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathcal{H}^{\otimes^n} : \\
\delta_0(h^1 \otimes \ldots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \ldots \otimes h^{n-1} \\
\delta_j(h^1 \otimes \ldots \otimes h^{n-1}) &= h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{n-1} \\
& \quad 1 \leq j \leq n-1 \\
\delta_n(h^1 \otimes \ldots \otimes h^{n-1}) &= h^1 \otimes \ldots \otimes h^{n-1} \otimes \sigma \\
\sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) &= h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1} \\
& \quad 0 \leq i \leq n \\
\tau_n(h^1 \otimes \ldots \otimes h^n) &= \widetilde{S}(h^1) \cdot (h^2 \otimes \ldots \otimes h^n \otimes \sigma).
\end{aligned}
$$

Equivalence of characteristic maps

[Gelfand-Fuchs-Bott-Haefliger] $\Longrightarrow$ Hopf

$J^\infty M := \{j_0^\infty(\psi);\, \psi : \mathbb{R}^n \to M\}$,
$\pi_1 : J^\infty M \to J^1 M = FM$ projection
with cross-section

$$\sigma_\nabla(u) = j_0^\infty(\mathrm{exp}_x^\nabla \circ u)\ , \qquad u \in F_x M$$

given by connection ∇; $\forall a \in \mathrm{GL}_n(\mathbb{R})$, $\forall\, \varphi \in \Gamma$

$$\sigma_\nabla \circ R_a = R_a \circ \sigma_\nabla \quad \text{and} \quad \sigma_{\nabla^\varphi} = \widetilde{\varphi}^{-1} \circ \sigma_\nabla \circ \widetilde{\varphi}.$$

Define $\quad \sigma_\nabla(\varphi_0, \ldots, \varphi_p) : \Delta^p \times FM \to J^\infty M$
by

$$\sigma_\nabla(\varphi_0, \ldots, \varphi_p)(t, u) = \sigma_{\nabla(\varphi_0, \ldots, \varphi_p; t)}(u)\,,$$

$$\text{where} \qquad \nabla(\varphi_0, \ldots, \varphi_p; t) = \sum_0^p\ t_i\, \nabla^{\varphi_i};$$

$$\sigma_\nabla(\varphi_0\,\varphi, \ldots, \varphi_p\varphi)(t, u) = \widetilde{\varphi}^{-1}\sigma_\nabla(\varphi_0, \ldots, \varphi_p)(t, \widetilde{\varphi}(u)).$$

$C^*(\mathfrak{a}_n) =$ *Gelfand-Fuchs* Lie algebra cohomology complex of $\mathfrak{a}_n$ = Lie algebra of formal vector fields on $\mathbb{R}^n$.

For $\varpi \in C^q(\mathfrak{a}_n)$, define $\forall \eta \in \Omega_c^m(FM)$,

$$\langle C_{p,m}(\varpi)(\varphi_0, \ldots, \varphi_p), \eta\rangle = (-1)^{\frac{m(m+1)}{2}} \int_{\Delta^p \times FM} \eta \wedge \sigma_\nabla(\varphi_0, \ldots, \varphi_p)^*(\widetilde{\varpi})$$

$$C_\nabla(\varpi) = \sum C_{p,m}(\varpi) : C^*(\mathfrak{a}_n) \to C^*\left(\Gamma; \Omega_c^*(FM)\right);$$

defines a map of (total) complexes,

$$C_\nabla(d\varpi) = (\delta + \partial) C_\nabla(\varpi).$$

For the relative (to SO_n) cohomology, one constructs similarly a homomorphism

$$H^*(\mathfrak{a}_n, SO_n) \to H^*\left(\Gamma; \Omega_c^*(PM)\right),$$

which can be followed by Connes' map $\Phi_*^\Gamma : H_\Gamma^*(PM) \to HC^*(\mathcal{A}_{PM}^\Gamma)$, yielding

$$\chi_{GF}^\Gamma : H^*(\mathfrak{a}_n, SO_n) \longrightarrow HC^*(\mathcal{A}_{PM}^\Gamma).$$

Composing χ^{Γ}_{GF} with the natural restriction

$$PHC^*(\mathcal{A}^{\Gamma}_{PM}) \to PHC^*(C^{\infty}_c(PM))$$

one recovers the Pontryagin classes of M as images of the universal Chern classes

$$c_{2i_1} \cdots c_{2i_k} \in H^*(\mathfrak{a}_n, SO_n), \quad 2i_1 + \ldots + 2i_k \leq n.$$

From Hopf cyclic to cyclic : $M = \mathbb{R}^n$

$$\chi_\tau(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau(a^0\, h^1(a^1) \ldots h^n(a^n)),$$

inducing characteristic homomorphism

$$\chi^{\Gamma}_{Hopf} : HC^*_{\mathrm{Hopf}}(\mathcal{H}_n, SO_n) \to HC^*(\mathcal{A}^{\Gamma}_{PM})_{(1)}.$$

Theorem 2. *There is a canonical isomorphism*

$$\kappa^*_n : H^*(\mathfrak{a}_n, SO_n) \xrightarrow{\simeq} PHC^*_{\mathrm{Hopf}}(\mathcal{H}_n, SO_n),$$

such that $\chi^{\Gamma}_{Hopf} \circ \kappa^*_n = \chi^{\Gamma}_{GF}$.

Summary: Transverse Index Theorem

Theorem 3. *There are canonical constructions for the following entities:*

- *a Hopf algebra* $\mathcal{H}_n$ *with modular character* δ, *and with* $(\delta, 1)$ *modular pair in involution;*
- *a co-cyclic structure for any Hopf algebra with a modular pair in involution* (δ, σ);
- *an isomorphism* κ_n^* *between the Gelfand-Fuks cohomology* $H^*_{\mathsf{GF}}(\mathfrak{a}_n)$, *resp.* $H^*_{\mathsf{GF}}(\mathfrak{a}_n, SO_n)$, *and* $HP^*(\mathcal{H}_n; \mathbb{C}_\delta)$, *resp.* $HP^*(\mathcal{H}_n, SO_n; \mathbb{C}_\delta)$;
- *an action of* $\mathcal{H}_n$ *on* $\mathcal{A}_\Gamma(F\mathbb{R}^n)$, *inducing a characteristic map* $\chi^*_\Gamma : HP^*(\mathcal{H}_n, SO_n; \mathbb{C}_\delta) \to HP^*_{(1)}(\mathcal{A}_\Gamma(P\mathbb{R}^n)) \cong H_*(P\mathbb{R}^n \times_\Gamma E\Gamma)$;
- *a class* $\mathcal{L}_n \in H^*_{\mathsf{GF}}(\mathfrak{a}_n, SO_n)$, *such that* $ch_*(\mathcal{A}_\Gamma(P\mathbb{R}^n), \mathcal{H}(P\mathbb{R}^n), D)_{(1)} = (\chi^*_\Gamma \circ \kappa^*_n)(\mathcal{L}_n)$.