Atiyah-Singer Revisited

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Minicourse of five lectures:

1. Dirac operator✓
2. Atiyah-Singer revisited
3. What is K-homology?
4. Beyond ellipticity
5. The Riemann-Roch theorem

The minicourse is based on joint work with Erik van Erp.

ATIYAH-SINGER REVISITED

Dedicated to the memory of Friedrich Hirzebruch.

This is an expository talk about the Atiyah-Singer index theorem.

1. Dirac operator of $\mathbb{R}^n$ will be defined.$\checkmark$
2. Some low dimensional examples of the theorem will be considered.$\checkmark$
3. A special case of the theorem will be proved, with the proof based on Bott periodicity.$\checkmark$
4. The proof will be outlined that the special case implies the full theorem.

Atiyah-Singer Index theorem

M compact C^∞ manifold without boundary

D an elliptic differential (or pseudo-differential) operator on M

$E^0, E^1, \quad C^\infty \quad \mathbb{C}$ vector bundles on M

$C^\infty(M, E^j)$ denotes the $\mathbb{C}$ vector space of all C^∞ sections of E^j.

$D\colon C^\infty(M, E^0) \longrightarrow C^\infty(M, E^1)$

D is a linear transformation of $\mathbb{C}$ vector spaces.

Atiyah-Singer Index theorem

M compact C^∞ manifold without boundary

D an elliptic differential (or pseudo-differential) operator on M

Index$(D) := \dim_{\mathbb{C}}$ (Kernel $D) - \dim_{\mathbb{C}}$ (Cokernel D)

Theorem (M.Atiyah and I.Singer)

Index (D) = (a topological formula)

Example

$M = S^1 = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1^2 + t_2^2 = 1\}$

$D_f \colon L^2(S^1) \longrightarrow L^2(S^1)$ is

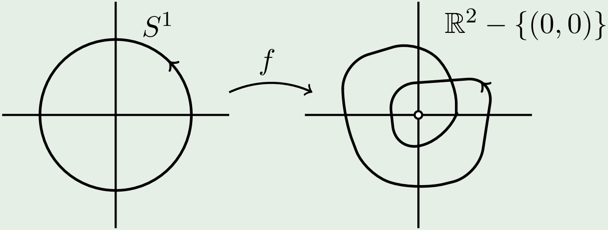
T_f	0
0	I

where $L^2(S^1) = L^2_+(S^1) \oplus L^2_-(S^1)$.

$L^2_+(S^1)$ has as orthonormal basis $e^{in\theta}$ with $n = 0, 1, 2, \ldots$
$L^2_-(S^1)$ has as orthonormal basis $e^{in\theta}$ with $n = -1, -2, -3, \ldots$.

Example

$f\colon S^1 \longrightarrow \mathbb{R}^2 - \{0\}$ is a C^∞ map.



$T_f\colon L^2_+(S^1) \longrightarrow L^2_+(S^1)$ is the composition

$L^2_+(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1) \longrightarrow L^2_+(S^1)$

$T_f\colon L^2_+(S^1) \longrightarrow L^2_+(S^1)$ is the Toeplitz operator associated to f

Example

Thus T_f is composition

$T_f\colon L^2_+(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1) \xrightarrow{P} L^2_+(S^1)$

where $L^2_+(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1)$ is $v \mapsto fv$

$fv(t_1, t_2) := f(t_1, t_2)v(t_1, t_2) \qquad \forall (t_1, t_2) \in S^1 \qquad \mathbb{R}^2 = \mathbb{C}$

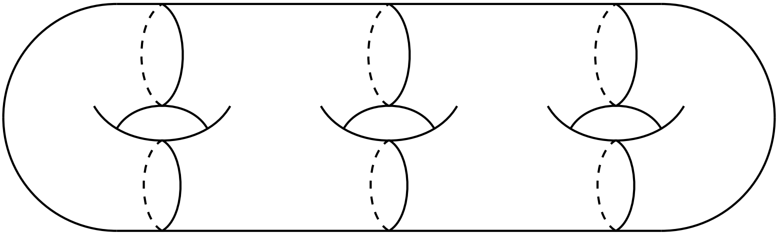
and $L^2(S^1) \xrightarrow{P} L^2_+(S^1)$ is the Hilbert space projection.

$D_f(v + w) := T_f(v) + w \qquad v \in L^2_+(S^1), \quad w \in L^2_-(S^1)$

Index$(D_f) =$ -winding number (f).

RIEMANN - ROCH

M compact connected Riemann surface



$$\text{genus of } M = \ \# \text{ of holes}$$
$$= \frac{1}{2}\,[\mathrm{rank} H_1(M;\mathbb{Z})]$$

D a divisor of M

D consists of a finite set of points of M $p_1, p_2, \ldots, p_l$ and an integer assigned to each point $n_1, n_2, \ldots, n_l$

Equivalently

D is a function $D\colon M \to \mathbb{Z}$ with finite support

Support$(D) = \{p \in M \mid D(p) \neq 0\}$

Support(D) is a finite subset of M

D a divisor on M

$\deg(D) := \sum_{p \in M} D(p)$

Remark

D_1, D_2 two divisors

$$D_1 \geqq D_2 \text{ iff } \forall p \in M, D_1(p) \geqq D_2(p)$$

Remark

D a divisor, $-D$ is

$$(-D)(p) = -D(p)$$

Example

Let $f\colon M \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function.

Define a divisor $\delta(f)$ by:

$$\delta(f)(p) = \begin{cases} 0 \text{ if } p \text{ is neither a zero nor a pole of } f \\ \text{order of the zero if } f(p) = 0 \\ -(\text{order of the pole}) \text{ if } p \text{ is a pole of } f \end{cases}$$

Example

Let w be a meromorphic 1-form on M. Locally w is $f(z)dz$ where f is a (locally defined) meromorphic function. Define a divisor $\delta(w)$ by:

$$\delta(w)(p) = \begin{cases} 0 \text{ if } p \text{ is neither a zero nor a pole of } w \\ \text{order of the zero if } w(p) = 0 \\ -(\text{order of the pole}) \text{ if } p \text{ is a pole of } w \end{cases}$$

D a divisor on M

$$H^0(M,D) := \left\{ \begin{array}{l} \text{meromorphic functions} \\ f\colon M \to \mathbb{C} \cup \{\infty\} \end{array} \middle| \, \delta(f) \geqq -D \right\}$$

$$H^1(M,D) := \left\{ \begin{array}{l} \quad\text{meromorphic 1-forms} \\ w \text{ on } M \end{array} \middle| \, \delta(w) \geqq D \right\}$$

Lemma

$H^0(M,D)$ and $H^1(M,D)$ are finite dimensional $\mathbb{C}$ vector spaces

$$\dim_{\mathbb{C}} H^0(M,D) < \infty$$
$$\dim_{\mathbb{C}} H^1(M,D) < \infty$$

Theorem (R. R.)

Let M be a compact connected Riemann surface and let D be a divisor on M. Then:

$$\dim_{\mathbb{C}} H^0(M, D) - \dim_{\mathbb{C}} H^1(M, D) = d - g + 1$$

$$d = \textit{degree } (D)$$
$$g = \textit{genus } (M)$$

HIRZEBRUCH-RIEMANN-ROCH

M non-singular projective algebraic variety / $\mathbb{C}$

E an algebraic vector bundle on M

$\underline{E}$ = sheaf of germs of algebraic sections of E

$H^j(M, \underline{E}) :=$ j-th cohomology of M using $\underline{E}$, $j = 0, 1, 2, 3, \ldots$

LEMMA

For all $j = 0, 1, 2, \ldots$ $\dim_{\mathbb{C}} H^j(M, \underline{E}) < \infty$.

For all $j > \dim_{\mathbb{C}}(M)$, $\quad H^j(M, \underline{E}) = 0$.

$$\chi(M, E) := \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(M, \underline{E})$$

$n = \dim_{\mathbb{C}}(M)$

THEOREM[HRR] Let M be a non-singular projective algebraic variety / $\mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M, E) = (ch(E) \cup Td(M))[M]$$

Hirzebruch-Riemann-Roch

Theorem (HRR)

Let M be a non-singular projective algebraic variety $/\,\mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M,E) = (ch(E) \cup Td(M))[M]$$

Various well-known structures on a C^∞ manifold M make M into a Spin^c manifold

$$
\begin{array}{ccc}
 & & \text{(complex-analytic)} \\
 & & \Downarrow \\
\text{(symplectic)} & \Rightarrow & \text{(almost complex)} \\
 & & \Downarrow \\
\text{(contact)} & \Rightarrow & \text{(stably almost complex)} \\
 & & \Downarrow \\
\text{Spin} & \Rightarrow & \text{Spin}^c \\
 & & \Downarrow \\
 & & \text{(oriented)}
\end{array}
$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds.

Two Out Of Three Lemma

Lemma

Let

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be a short exact sequence of $\mathbb{R}$*-vector bundles on* X*. If two out of three are Spin*^{c} *vector bundles, then so is the third.*

Definition

Let M be a C^∞ manifold (with or without boundary). M is a **Spin^c manifold** iff the tangent bundle TM of M is a Spin^c vector bundle on M.

The Two Out Of Three Lemma implies that the boundary ∂M of a Spin^c manifold M with boundary is again a Spin^c manifold.

A Spin^c manifold comes equipped with a first-order elliptic differential operator known as its Dirac operator.

If M is a Spin^c manifold, then $Td(M)$ is

$$Td(M) := \exp^{c_1(M)/2}\widehat{A}(M) \qquad\qquad Td(M) \in H^*(M;\mathbb{Q})$$

If M is a complex-analyic manifold, then M has Chern classes $c_1, c_2, \ldots, c_n$ and

$$\exp^{c_1(M)/2}\widehat{A}(M) = P_{Todd}(c_1, c_2, \ldots, c_n)$$

EXAMPLE. Let M be a compact complex-analytic manifold.
Set $\Omega^{p,q} = C^\infty(M, \Lambda^{p,q} T^* M)$
$\Omega^{p,q}$ is the $\mathbb{C}$ vector space of all C^∞ differential forms of type (p,q)
Dolbeault complex

$$0 \longrightarrow \Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \cdots \longrightarrow \Omega^{0,n} \longrightarrow 0$$

The Dirac operator (of the underlying Spin^c manifold) is the assembled Dolbeault complex

$$\bar\partial + \bar\partial^* \colon \bigoplus_j \Omega^{0,\,2j} \longrightarrow \bigoplus_j \Omega^{0,\,2j+1}$$

The index of this operator is the arithmetic genus of M — i.e. is the Euler number of the Dolbeault complex.

TWO POINTS OF VIEW ON SPIN^c MANIFOLDS

1. Spin^c is a slight strengthening of oriented. The oriented manifolds that occur in practice are Spin^c.

2. Spin^c is much weaker than complex-analytic. BUT the assempled Dolbeault complex survives (as the Dirac operator). AND the Todd class survives.

$$M \quad \mathrm{Spin}^c \Longrightarrow \quad \exists \quad Td(M) \in H^*(M;\mathbb{Q})$$

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M.

D_E denotes the Dirac operator of M tensored with E.

$$D_E \colon C^\infty(M, S^+ \otimes E) \longrightarrow C^\infty(M, S^- \otimes E)$$

S^+, S^- are the positive (negative) spinor bundles on M.

<u>THEOREM</u> Index$(D_E) = (ch(E) \cup Td(M))[M]$.

$\mathrm{K}_0(\cdot)$

Definition

Define an abelian group denoted $\mathrm{K}_0(\cdot)$ by considering pairs (M, E) such that:

1. M is a compact even-dimensional Spin^c manifold without boundary.
2. E is a $\mathbb{C}$ vector bundle on M.

Set $\mathrm{K}_0(\cdot) = \{(M, E)\}/ \sim$ where the the equivalence relation $\sim$ is generated by the three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

Addition in $\mathrm{K}_0(\cdot)$ is disjoint union.

$$(M, E) + (M', E') = (M \sqcup M', E \sqcup E')$$

In $\mathrm{K}_0(\cdot)$ the additive inverse of (M, E) is $(-M, E)$ where $-M$ denotes M with the Spin^c structure reversed.

$$-(M, E) = (-M, E)$$

Isomorphism (M, E) is isomorphic to (M', E') iff $\exists$ a diffeomorphism

$$\psi \colon M \to M'$$

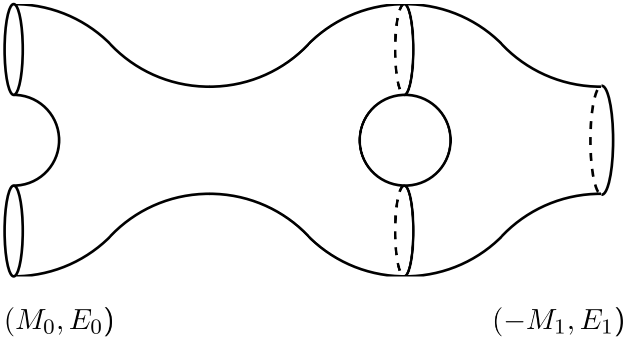
preserving the Spin^c-structures on M, M' and with

$$\psi^*(E') \cong E.$$

Bordism (M_0, E_0) is **bordant** to (M_1, E_1) iff $\exists$ (Ω, E) such that:

1. Ω is a compact odd-dimensional Spin^c manifold with boundary.
2. E is a $\mathbb{C}$ vector bundle on Ω.
3. $(\partial\Omega, E|_{\partial\Omega}) \cong (M_0, E_0) \sqcup (-M_1, E_1,)$

$-M_1$ is M_1 with the Spin^c structure reversed.


(M_0, E_0)
$(-M_1, E_1)$

Direct sum - disjoint union

Let E, E' be two $\mathbb{C}$ vector bundles on M

$$(M, E) \sqcup (M, E') \sim (M, E \oplus E')$$

Vector bundle modification

(M, E)

Let F be a Spin^c vector bundle on M

Assume that

$$\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M$$

for every fiber F_p of F

$$\mathbf{1}_{\mathbb{R}} = M \times \mathbb{R}$$

$$S(F \oplus \mathbf{1}_{\mathbb{R}}) := \text{unit sphere bundle of } F \oplus \mathbf{1}_{\mathbb{R}}$$

$$(M, E) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E)$$

$$\begin{array}{c} S(F \oplus \mathbf{1}_{\mathbb{R}}) \\ \Big\downarrow \pi \\ M \end{array}$$

This is a fibration with even-dimensional spheres as fibers.

$F \oplus \mathbf{1}_{\mathbb{R}}$ is a Spin^c vector bundle on M with odd-dimensional fibers.

The Spin^c structure for F causes there to appear on $S(F \oplus 1_{\mathbb{R}})$ a $\mathbb{C}$-vector bundle β whose restriction to each fiber of π is the Bott generator vector bundle of that even-dimensional sphere.

$$(M, E) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E)$$

Addition in $\mathrm{K}_0(\cdot)$ is disjoint union.

$$(M, E) + (M', E') = (M \sqcup M', E \sqcup E')$$

In $\mathrm{K}_0(\cdot)$ the additive inverse of (M, E) is $(-M, E)$ where $-M$ denotes M with the Spin^c structure reversed.

$$-(M, E) = (-M, E)$$

DEFINITION. (M, E) *bounds* $\Longleftrightarrow \exists\, (\Omega, \widetilde{E})$ with :

1. Ω is a compact odd-dimensional Spin^c manifold with boundary.
2. $\widetilde{E}$ is a $\mathbb{C}$ vector bundle on Ω.
3. $(\partial\Omega, \widetilde{E}|_{\partial\Omega}) \cong (M, E)$

REMARK. $(M, E) = 0$ in $K_0(\cdot) \Longleftrightarrow (M, E) \sim (M', E')$ where (M', E') bounds.

Consider the homomorphism of abelian groups

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

Notation

D_E is the Dirac operator of M tensored with E.

It is a corollary of Bott periodicity that this homomorphism of abelian groups is an isomorphism.

Equivalently, $\mathsf{Index}(D_E)$ is a complete invariant for the equivalence relation generated by the three elementary steps; i.e. $(M, E) \sim (M', E')$ if and only if $\mathrm{Index}(D_E) = \mathrm{Index}(D'_{E'})$.

BOTT PERIODICITY

$$\pi_j GL(n,\mathbb{C}) = \begin{cases} \mathbb{Z} & \text{j odd} \\ \\ 0 & \text{j even} \end{cases}$$

$$j = 0, 1, 2, \ldots, 2n-1$$

Why does Bott periodicity imply that

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

is an isomorphism?

To prove surjectivity must find an (M, E) with $\mathrm{Index}(D_E) = 1$.

e.g. Let $M = \mathbb{C}P^n$, and let E
be the trivial (complex) line bundle on $\mathbb{C}P^n$
$E{=}1_{\mathbb{C}} = \mathbb{C}P^n \times \mathbb{C}$
$\mathrm{Index}(\mathbb{C}P^n, 1_{\mathbb{C}}) = 1$

Thus Bott periodicity is not used in the proof of surjectivity.

Lemma used in the Proof of Injectivity

Given any (M, E) there exists an even-dimensional sphere S^{2n} and a $\mathbb{C}$-vector bundle F on S^{2n} with $(M, E) \sim (S^{2n}, F)$.

Bott periodicity is not used in the proof of this lemma.
The lemma is proved by a direct argument using the definition of the equivalence relation on the pairs (M, E).

Let r be a positive integer, and let $\mathrm{Vect}_{\mathbb{C}}(S^{2n}, r)$
be the set of isomorphism classes of $\mathbb{C}$ vector bundles on S^{2n}
of rank r, i.e. of fiber dimension r.

$$\mathrm{Vect}_{\mathbb{C}}(S^{2n}, r) \longleftrightarrow \pi_{2n-1} GL(r, \mathbb{C})$$

PROOF OF INJECTIVITY

Let (M, E) have $\mathrm{Index}(M, E) = 0$.

By the above lemma, we may assume that $(M, E) = (S^{2n}, F)$.

Using Bott periodicity plus the bijection

$$\mathrm{Vect}_{\mathbb{C}}(S^{2n}, r) \longleftrightarrow \pi_{2n-1} GL(r, \mathbb{C})$$

we may assume that F is of the form

$$F = \theta^p \oplus q\beta$$

$\theta^p = S^{2n} \times \mathbb{C}^p$ and β is the Bott generator vector bundle on S^{2n}.

Convention. If $q < 0$, then $q\beta = |q|\beta^*$.

$\mathrm{Index}(S^{2n}, \beta) = 1 \quad \mathrm{Index}(S^{2n}, \theta^p) = 0$
Therefore

$$\mathrm{Index}(S^{2n}, F) = 0 \Longrightarrow q = 0$$

Hence $(S^{2n}, F) = (S^{2n}, \theta^p)$. This bounds

$$(S^{2n}, \theta^p) = \partial(B^{2n+1}, B^{2n+1} \times \mathbb{C}^p)$$

and so is zero in $\mathrm{K}_0(\cdot)$.
QED

Define a homomorphism of abelian groups

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Q} \\ (M, E) &\longmapsto \big(ch(E) \cup \mathrm{Td}(M)\big)[M] \end{aligned}$$

where $ch(E)$ is the Chern character of E and $\mathrm{Td}(M)$ is the Todd class of M.

$ch(E) \in H^*(M, \mathbb{Q})$ and $\mathrm{Td}(M) \in H^*(M, \mathbb{Q})$.

$[M]$ is the orientation cycle of M. $[M] \in H_*(M, \mathbb{Z})$.

Granted that

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

is an isomorphism, to prove that these two homomorphisms are equal, it suffices to check one nonzero example.

Let X be a compact C^∞ manifold without boundary.
X is not required to be oriented.
X is not required to be even dimensional.
On X let

$$\delta : C^\infty(X, E_0) \longrightarrow C^\infty(X, E_1)$$

be an elliptic differential (or pseudo-differential) operator.

$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma) \in \mathrm{K}_0(\cdot)$, and

$$\mathrm{Index}(D_{E_\sigma}) = \mathrm{Index}(\delta).$$

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma)$$

$$\Big\Downarrow$$

$$\text{Index}(\delta) = (ch(E_\sigma) \cup \mathrm{Td}((S(TX \oplus 1_{\mathbb{R}})))[(S(TX \oplus 1_{\mathbb{R}})]$$

and this is the general Atiyah-Singer formula.

$S(TX \oplus 1_{\mathbb{R}})$ is the unit sphere bundle of $TX \oplus 1_{\mathbb{R}}$.
$S(TX \oplus 1_{\mathbb{R}})$ is even dimensional and is — in a natural way — a Spin^c manifold.

E_σ is the $\mathbb{C}$ vector bundle on $S(TX \oplus 1_{\mathbb{R}})$ obtained by doing a clutching construction using the symbol σ of δ.