G topological group, Hausdorff and paracompact

X topological space Hausdorff and paracompact

A principal G-bundle on X is a pair (P, π)

(1) P is a Hausdorff and paracompact topological space with a given continuous (right) action of G

$$\begin{aligned} P \times G &\to P \\ (p, g) &\mapsto pg \end{aligned}$$

(2) $\pi : P \to X$ is a continuous map, mapping P onto X

such that: Given any $x \in X$, there exists an open subset U of X with $x \in U$ and a homeomorphism

$$\varphi : U \times G \to \pi^{-1}(U)$$

with

$$\pi\,\varphi(u, g) = u \qquad \forall\,(u, g) \in U \times G$$

$$\varphi(u, g_1\,g_2) = \varphi(u, g_1)\,g_2 \qquad \forall\,(u, g_1, g_2) \in U \times G \times G$$

Remark. Such a $\varphi : U \times G \to \pi^{-1}(U)$ is referred to as a *local trivialization.*

Two principal G-bundles (P, π) and (Q, θ) are isomorphic if there exists a G-equivariant homeomorphism $f : P \longrightarrow Q$ with commutativity in the diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi"'] & Q \arrow[d, "\theta"] \\ X \arrow[r, "1_X"'] & X \end{tikzcd}$$

Let G, H be two topological groups and let (P, π) $\quad (Q, \theta)$ be a principal

G-bundle and a principal H-bundle on X.

A homomorphism of principal bundles from (P, π) to (Q, θ) is a pair (η, ρ) such that:

(i) ρ is a homomorphism of topological groups

$\rho : G \longrightarrow H$

(ii) $P \longrightarrow Q$ is a continuous map with commutativity in the diagrams

$$\begin{array}{ccc} P & \xrightarrow{\eta} & Q \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\theta} \\ X & \xrightarrow[1_X]{} & X \end{array} \qquad\qquad \begin{array}{ccc} P \times G & \xrightarrow{\eta \times \rho} & Q \times H \\ \downarrow & & \downarrow \\ P & \xrightarrow[\eta]{} & Q \end{array}$$

$$[\pi p = \theta(\eta p)] \qquad\qquad [\eta(pg) = (\eta p)(\rho g)]$$

Notation. A homomorphism of principal bundles on X will be denoted $\eta : P \longrightarrow Q$.

$\rho : G \longrightarrow H$ will be referred to as the homomorphism of topological groups underlying η.

Lemma. Let $\eta : P \to Q$ be a homomorphism of principal bundles on X with underlying homomorphism of topological groups $\rho : G \to H$. Then for any $x \in X$, there exists an open subset U of X with $x \in U$ and local trivializations

$$\varphi : U \times G \to \pi^{-1}(U)$$

$$\psi : U \times H \to \theta^{-1}(U)$$

such that the diagram

$$\begin{array}{ccc} U \times G & \xrightarrow{\varphi} & \pi^{-1}(U) \\ {\scriptstyle 1_U \times \rho}\downarrow & & \downarrow{\scriptstyle \eta} \\ U \times H & \xrightarrow[\psi]{} & \theta^{-1}(V) \end{array}$$

commutes.

<u>Example</u> E an $\mathbb{R}$ vector bundle on X.

$\dim_{\mathbb{R}}(E_p) = n \qquad\qquad p \in X$

$\Delta(E) =$

$$\left\{ (p, v_1, v_2, \ldots, v_n) \mid \begin{array}{r} p \in X \text{ and } v_1, v_2, \ldots, v_n \text{ is} \\ \text{a vector space basis for } Ep. \end{array} \right\}$$

$\Delta(E)$ topologized by $\Delta(E) \subset \overbrace{E \oplus E \oplus \cdots \oplus E}^{\longleftarrow\, n \,\longrightarrow}$

$\Delta(E) \times GL_n(\mathbb{R}) \longrightarrow \Delta(E)$

$(p, v_1, v_2, \ldots, v_n)\,[a_{ij}] = (p, w_1, w_2, \ldots, w_n)$

$$w_j = \sum_{i=1}^{n} a_{ij} v_i \qquad\qquad [a_{ij}] \in GL_n(\mathbb{R})$$

$\theta : \Delta(E) \longrightarrow X \qquad\qquad \theta(p, v_1, v_2, \ldots, v_n) = p$

$(\Delta(E), \theta)$ is a principal $GL_n(\mathbb{R})$ bundle on X.

$S^1 = \{\lambda \in \mathbb{C} \mid \lambda\overline{\lambda} = 1\}$

$\underline{n \geqq 3} \qquad \pi_1 SO(n) = \mathbb{Z}/2\mathbb{Z}$

$n \geqq 3 \qquad \pi_1 SO(n) = \mathbb{Z}/2\mathbb{Z}$

$Spin(n)$ IS THE NON-TRIVIAL 2-FOLD COVER OF $SO(n)$

$$Spin(n) \longrightarrow SO(n)$$

$$Spin^c(n) = S^1 \times_{(\mathbb{Z}/2\mathbb{Z})} Spin(n)$$

$$Spin(n) \longrightarrow Spin^c(n) \longrightarrow SO(n)$$

Spin(n) is the unique non-trivial 2-fold cover of $SO(n)$

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow Spin(n) \longrightarrow SO(n) \longrightarrow 1$$

$$Spin^c(n) = S^1 \underset{\mathbb{Z}/2\mathbb{Z}}{\times} Spin(n)$$

$$1 \longrightarrow S^1 \longrightarrow Spin^c(n) \xrightarrow{\rho} SO(n) \longrightarrow 1$$

Let $\in$ be the non-identity element in Kernel $(Spin(n) \longrightarrow SO(n))$.

$$Spin^c(n) = S^1 \times Spin(n)/\{(1, I)\ ,\ (-1, \in)\}$$

$$(\lambda, g) \sim (-\lambda, \in g)$$

$\underline{n = 2}$ $\quad Spin(2) = S^1 = SO(2)$

$$Spin(2) \longrightarrow SO(2)$$

$$\zeta \mapsto \zeta^2$$

$$Spin^c(2) = S^1 \underset{\mathbb{Z}/2\mathbb{Z}}{\times} Spin(2) \qquad \rho(\lambda, \zeta) = \zeta^2$$

$\underline{n = 1}$ $\quad Spin(1) = \mathbb{Z}/2\mathbb{Z} \qquad SO(1) = \bullet$

$$Spin^c(1) = S^1$$

$$\rho : S^1 \longrightarrow \bullet$$

Remark. Since $SO(n) \subset GL_n(\mathbb{R})$ can view the standard map $Spin^c(n) \longrightarrow SO(n)$ as $Spin^c(n) \longrightarrow GL_n(\mathbb{R})$.

datum

Definition. A $Spin^c$ datum for an $\mathbb{R}$ vector bundle E on X is a homomorphism of principal bundles $\eta : P \longrightarrow \Delta(E)$, where P is a principal $Spin^c(n)$-bundle on X $(n = \dim_{\mathbb{R}}(E_p))$ and the homomorphism of topological groups underlying η is the standard map $\rho : Spin^c(n) \longrightarrow GL_n(\mathbb{R})$.

Two $Spin^c$ data $\eta : P \longrightarrow \Delta(E)$, $\eta' : P' \longrightarrow \Delta(E)$ are isomorphic if there exists an isomorphism $f : P \longrightarrow P'$ of principal $Spin^c(n)$ bundles on X

with commutativity in the diagram

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "\eta"] & & P' \arrow[dl, "\eta'"] \\ & \Delta(E) & \end{tikzcd}$$

$[\eta = \eta' \circ f]$

Two $Spin^c$ data $\eta : P \longrightarrow \Delta(E)$, $\eta' : P' \longrightarrow \Delta(E)$ are homotopic if there exists a principal $Spin^c(n)$-bundle Q on X and a continuous map $\Phi : Q \times [0,1] \longrightarrow \Delta(E)$ such that

(i) For $t \in [0,1]$, each

$\Phi_t : Q \longrightarrow \Delta(E)$ is a $Spin^c$ data

$[\Phi_t(q) = \Phi(q,t)]$

$$\text{(ii)} \left\{ \begin{array}{l} \Phi_0 : Q \longrightarrow \Delta(E) \text{ is isomorphic to } \eta : P \longrightarrow \Delta(E) \\ \Phi_1 : Q \longrightarrow \Delta(E) \text{ is isomorphic to } \eta' : P' \longrightarrow \Delta(E). \end{array} \right\}$$

Definition A $Spin^c$ structure for E is an equivalence class of $Spin^c$ data, where the equivalence relation is homotopy.

A $Spin^c$ structure for E determines an orientation of E.

E an $\mathbb{R}$ vector bundle on X

$w_1(E), w_2(E), \ldots$ The Stiefel-Whitney classes of E

$w_j(E) \in H^j(X;\mathbb{Z}/2\mathbb{Z})$

Čech cohomology

E is orientable iff $w_1(E) = 0$.

If E is orientable, fix one orientation of E. The set of all possible orientations of E is then in $1-1$ correspondence with $H^\circ(X;\mathbb{Z}/2\mathbb{Z})$.

E is $Spin^c$-able iff $w_1(E) = 0$ and $w_2(E)$ is in the image of $H^2(X;\mathbb{Z}) \longrightarrow H^2(X;\mathbb{Z}/2\mathbb{Z})$.

If E is $Spin^c$-able, fix one $Spin^c$ structure for E. The set of all possible $Spin^c$ structures for E is then in $1-1$ correspondence with $H^\circ(X;\mathbb{Z}/2\mathbb{Z}) \times H^2(X;\mathbb{Z})$.

A <u>$Spin^c$ vector bundle</u> is an $\mathbb{R}$ vector bundle with a given $Spin^c$ structure.

A <u>*Spin*^{c} manifold</u> is a C^∞ manifold M (possibly with boundary) whose tangent bundle TM is a *Spin*^{c} vector bundle.

By forgetting some structure a complex vector bundle or a Spin vector bundle canonically becomes a Spin^c vector bundle

$$\begin{array}{ccc} & & \text{complex} \\ & & \Downarrow \\ \text{Spin} & \Rightarrow & \text{Spin}^c \\ & & \Downarrow \\ & & \text{oriented} \end{array}$$

A Spin^c structure for the $\mathbb{R}$ vector bundle E can be thought of as an orientation for E plus a slight extra bit of structure. Spin^c structures behave very much like orientations. For example, an orientation on two out of three $\mathbb{R}$ vector bundles in a short exact sequence determin an orientation on the third vector bundle. Analogous assertions are true for Spin^c structures.

Two out of three lemma.

Let $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ be an exact sequence of $\mathbb{R}$ vector bundles on X. If Spin^c structures are given for any two of E', E, E'' then a Spin^c structure is determined for the third.

Various well-known structures on a manifold M make M into a Spin^c manifold

$$\begin{array}{ccc}
 & & \text{(complex-analytic)} \\
 & & \Downarrow \\
\text{(symplectic)} & \Rightarrow & \text{(almost complex)} \\
 & & \Downarrow \\
\text{(contact)} & \Rightarrow & \text{(stably almost complex)} \\
 & & \Downarrow \\
\text{Spin} & \Rightarrow & \text{Spin}^c \\
 & & \Downarrow \\
 & & \text{(oriented)}
\end{array}$$

A Spin^c manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin^c manifolds.

Two out of three lemma.

Let $0 \to E' \to E \to E'' \to 0$ be an exact sequence of $\mathbb{R}$ vector bundles on X. If Spin^c structures are given for any two of E', E, E'' then a Spin^c structure is determined for the third.

Corollary. If M is a Spin^c manifold with boundary ∂M, then ∂M is (in a canonical way) a Spin^c manifold.

Proof.

set $\mathbf{1} = \partial M \times \mathbb{R}$

exact sequence

$0 \to T(\partial M) \to TM \mid \partial M \to \mathbf{1} \to 0$

$$\left(\begin{array}{c}\text{Spin}^c \text{ datum for } E\\[4pt] \eta : P \longrightarrow \Delta(E)\end{array}\right)$$

$$\rightsquigarrow$$

E IS AN $\mathbb{R}$ VECTOR BUNDLE ON X

$$\left(\begin{array}{c}\text{Spinor system for } E\\[4pt] (\epsilon, \langle , \rangle, F)\end{array}\right)$$

WHAT IS A "Spinor system for E" ?

The Clifford algebra

V finite dimensional $\mathbb{R}$ vector space

$<,>$ a positive definite, symmetric, bilinear

$\mathbb{R}$-valued inner product on V

tensor algebra:

$$TV = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \ldots$$

$$Cliff(V) = TV/(v \otimes v + < v, v > \cdot 1)$$

$(v \otimes v + < v, v > \cdot 1)$ denotes the two-sided ideal in TV generated by all elements of the form

$$v \otimes v + < v, v > \cdot 1 \qquad v \in V \qquad 1 \in \mathbb{R}$$

As a vector space over $\mathbb{R}$, $Cliff(V)$ is canonically isomorphic to the exterior algebra

$$\wedge^* V = \mathbb{R} \oplus V \oplus \wedge^2 V \oplus \cdots \oplus \wedge^n V \qquad n = \dim_{\mathbb{R}}(V)$$

Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of V.

The monomials

$$e_1^{\epsilon_1} e_2^{\epsilon_2} \cdots e_n^{\epsilon_n} \qquad \epsilon_j = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

form a vector space basis of $Cliff(V)$. The canonical isomorphism of $\mathbb{R}$ vector spaces

$$Cliff(V) \longleftrightarrow \wedge^* V$$

is: $e_1^{\epsilon_1} e_2^{\epsilon_2} \cdots e_n^{\epsilon_n} \longleftrightarrow e_1^{\epsilon_1} \wedge e_2^{\epsilon_2} \wedge \cdots \wedge e_n^{\epsilon_n}$

This isomorphism of $\mathbb{R}$ vector spaces does not depend on the choice of orthonormal basis of V.

$$\dim_{\mathbb{R}}(Cliff\ V) = 2^n \qquad n = \dim_{\mathbb{R}} V$$

In $Cliff\ V$: $\boxed{e_1, e_2, \ldots, e_n \text{ an orthonormal basis of } V}$

$$e_j^2 = -1 \qquad j = 1, 2, \ldots, n$$

$$e_i e_j + e_j e_i = 0 \qquad i \neq j$$

$V \subset Cliff\ V$

$$Cliff\ V = (Cliff\ V)_0 \oplus (Cliff\ V)_1$$

$Cliff(V)_0$	$Cliff(V)_1$
$\mathbb{R}$ vector space spanned by $e_1^{\epsilon_1} e_2^{\epsilon_2} \ldots e_n^{\epsilon_n}$ with $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ even	$\mathbb{R}$ vector space spanned by $e_1^{\epsilon_1} e_2^{\epsilon_2} \ldots e_n^{\epsilon_n}$ with $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n$ odd

This $\mathbb{Z}/2\mathbb{Z}$-grading of $Cliff\ V$ does not depend on the choice of orthonormal basis of V.

Take $\mathbb{R}^n$ with its usual inner product

$S^{n-1} \subset \mathbb{R}^n \subset Cliff\ (\mathbb{R}^n)$

The elements of S^{n-1} are invertible in $Cliff(\mathbb{R}^n)$.

<u>Definition</u>. $Pin(n)$ is the subgroup of the invertible elements of $Cliff(\mathbb{R}^n)$ generated by S^{n-1}.

$$Spin(n) = Pin(n) \cap Cliff(\mathbb{R}^n)_0$$

$$\rho : Spin(n) \longrightarrow SO(n)$$

$$(\rho g)(x) = g\, x\, g^{-1} \qquad \begin{array}{l} g \in S \\ x \in \mathbb{R}^n \end{array}$$

For $n \geqq 3$ this is the unique non-trivial 2-fold covering space of $SO(n)$

$$Cliff_{\mathbb{C}}\, V = \mathbb{C} \underset{\mathbb{R}}{\otimes} Cliff\ V$$

$Cliff_{\mathbb{C}}\, V$ is a C^* algebra

$$v \in V \subset Cliff\ V \subset Cliff_{\mathbb{C}}\, V$$

$$v^* = -v$$

$$\mathbb{C} \underset{\mathbb{R}}{\otimes} Cliff(\mathbb{R}^n)$$

$$Spin^c(n) = S^1 \underset{\mathbb{Z}/2\mathbb{Z}}{\times} Spin(n) \subset Cliff_{\mathbb{C}}(\mathbb{R}^n)$$

$Spin^c(n)$ is a subgroup of the group of unitary elements of the C^* algebra $Cliff_{\mathbb{C}}(\mathbb{R}^n)$.

$$\mathrm{Cliff}_{\mathbb{C}}(V) = \mathbb{C} \underset{\mathbb{R}}{\otimes} \mathrm{Cliff}(V)$$

$$V \subset \mathrm{Cliff}(V) \subset \mathrm{Cliff}_{\mathbb{C}}(V)$$

$\mathrm{Cliff}_{\mathbb{C}}(V)$ is a C^* algebra

$v \in V \qquad\qquad v^* = -v$

Choose an orthogonal basis $e_1, e_2, \ldots, e_n$ for V

$n = \dim_{\mathbb{R}}(V)$

<u>n even $\qquad n = 2r$</u>

$E_1, E_2, \ldots, E_n \qquad\qquad 2^r \times 2^r$ matrices

$$e_j \mapsto E_j \qquad\qquad \mathrm{Cliff}_{\mathbb{C}}(V) \xrightarrow[\cong]{} M(2^r, \mathbb{C})$$

Isomorphism of C^* algebras

n odd $\quad n = 2r+1 \qquad n = \dim_{\mathbb{R}}(V)$

$$\varphi_+ : \mathrm{Cliff}_{\mathbb{C}}(V) \to M(2^r, \mathbb{C})$$

$E_1, E_2, \ldots, E_n \qquad 2^r \times 2^r$ matrices

$$\varphi_+(e_j) = E_j \qquad j = 1, 2, \ldots, n$$

$$\varphi_- : \mathrm{Cliff}_{\mathbb{C}}(V) \to M(2^r, \mathbb{C})$$

$$\varphi_-(e_j) = -E_j \qquad j = 1, 2, \ldots, n$$

$$\varphi_+ \oplus \varphi_- : \mathrm{Cliff}_{\mathbb{C}}(V) \xrightarrow[\cong]{} M(2^r, \mathbb{C}) \oplus M(2^r, \mathbb{C})$$

Isomorphism of C^* algebras

Remark. These isomorphisms are non-canonical since they depend on the choice of an orthornormal basis for V.

Let E be an $\mathbb{R}$ vector bundle on X. Assume given an inner product $\langle\ ,\ \rangle$ for E. $\mathrm{Cliff}_{\mathbb{C}}(E)$ is the bundle of C^* algebras over X whose fibre at $p \in X$ is $\mathrm{Cliff}_{\mathbb{C}}(E_p)$.

Definition. An *Hermitian module* over $\mathrm{Cliff}_{\mathbb{C}}(E)$ is a complex vector bundle F on X with a $\mathbb{C}$-valued inner product $(\ ,\ )$ and a module structure

$$\mathrm{Cliff}_{\mathbb{C}}(E) \otimes F \to F$$

such that

(i) $(\ ,\ )$ makes each F_p into a finite dimensional Hilbert space

(ii) For each $p \in X$, the module map $\mathrm{Cliff}_{\mathbb{C}}(E_p) \to \mathcal{L}(F_p)$ is a unital homomorphism of C^* algebras.

Remark. Of course all structures here are assumed to be continuous. If X is a C^∞ manifold then we could take everything to be C^∞.

If E is oriented, define a section ω of $\mathrm{Cliff}_{\mathbb{C}}(E)$ as follows. Given $p \in X$, choose a positively oriented orthonormal basis $e_1, e_2, \ldots, e_n$ of E_p. For n even, $n = 2r$, set

$$\omega(p) = (i)^r \, e_1 \, e_2 \ldots e_{2r} \qquad n = 2r$$

For n odd, $n = 2r + 1$, set

$$\omega(p) = (i)^{r+1} \, e_1 \, e_2 \ldots e_{2r+1} \qquad n = 2r + 1$$

$\omega(p)$ does not depend on the choice of positively oriented orthonormal basis. In $\mathrm{Cliff}_{\mathbb{C}}(E_p)$ we have

$$(\omega(p))^2 = 1 \, .$$

If n is odd, then $\omega(p)$ is in the center of $\mathrm{Cliff}_{\mathbb{C}}(E_p)$. Note that to define ω, E must be *oriented.* Reversing the orientation will change ω to $-\omega$.

Definition. Let E be an $\mathbb{R}$ vector bundle on X. A Spinor system for E is a triple $(\epsilon, \langle , \rangle, F)$ such that:

(i) ϵ is an orientation of E

(ii) $\langle , \rangle$ is an inner product for E

(iii) F is an Hermitian module over $\mathrm{Cliff}_{\mathbb{C}}(E)$ with each F_p an irreducible module over $\mathrm{Cliff}_{\mathbb{C}}(E_p)$

(iv) If $n = \dim(E_p)$ is odd, then $\omega(p)$ acts as I on F_p

Remark. The irreducibility required in (iii) is equivalent to $\dim_{\mathbb{C}}(F_p) = 2^r$ where $n = 2r$ or $n = 2r + 1$. In (iv) note that $\omega(p)^2 = 1$. Since for (iv) n is odd, $\omega(p)$ is in the center of $\mathrm{Cliff}_{\mathbb{C}}(E_p)$. Hence the irreducibility of (iii) implies that $\omega(p)$ acts either by I or $-I$ on F_p. Thus (iv) normalizes the matter by requiring that $\omega(p)$ act as I. When $n = \dim_{\mathbb{R}}(E_p)$ is even no such normalization is made.

Terminology. If $(\epsilon, \langle , \rangle, F)$ is a Spinor system for E, then F is referred to as the Spinor bundle.

Suppose that $n = \dim_{\mathbb{R}}(E_p)$ is even. Let F_p^+ (F_p^-) be the $+1$ (-1) eigenspace of $\omega(p)$. We have a direct sum decomposition

$$F = F^+ \oplus F^-$$

where F^+, F^- are the 1/2-Spin bundles. F_p^+ (F_p^-) is the vector bundle of positive (negative) spinors.

$$\text{Assume}\begin{cases} X \times G \to X \\ G \text{ acts on } X \text{ by a right action} \\ \\ G \times Y \to Y \\ G \text{ acts on } Y \text{ by a left action} \end{cases}$$

Notation. $X \underset{G}{\times} Y = X \times Y/\sim \quad (xg, y) \sim (x, gy)$

Example. E $\mathbb{R}$ vector bundle on X

$$\Delta(E) \underset{\mathrm{GL}(n,\mathbb{R})}{\times} \mathbb{R}^n \cong E$$

$$((p, v_1, v_2, \ldots, v_n), (a_1, a_2, \ldots, a_n)) \mapsto a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

$E \quad \mathbb{R}$ vector bundle on X

$$\boxed{\begin{array}{l}\text{A Spin}^c \text{ datum } \eta : P \to \Delta(E) \text{ determines} \\ \text{a Spinor system } (\epsilon, \langle\,,\,\rangle, F) \text{ for } E.\end{array}}$$

$\underline{\epsilon \text{ and } \langle\,,\,\rangle}$ $\quad p \in X \quad$ An $\mathbb{R}$ basis $v_1, v_2, \ldots, v_n$ of E_p is positively oriented and orthonormal iff

$$(v_1, v_2, \ldots, v_n) \in \text{Image}\,(\eta)$$

Spinor bundle F

$n = 2r \qquad \text{or} \qquad n = 2r + 1$

$$F = P \underset{\text{Spin}^c(n)}{\times} \mathbb{C}^{2^r}$$

How does $\text{Spin}^c(n)$ act on $\mathbb{C}^{2^r}$?

n odd

$\mathrm{Spin}^c(n)$ has an irreducible representation known as its spin representation

$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^r, \mathbb{C})$

$n = 2r + 1$

n even

$\mathrm{Spin}^c(n)$ has two irreducible representations known as its 1/2-spin representations

$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C})$

$\mathrm{Spin}^c(n) \to \mathrm{GL}(2^{r-1}, \mathbb{C})$

The direct sum $\mathrm{Spin}^c(n) \to \mathrm{GL}(2^r, \mathbb{C})$ of these two representations is the spin representation of $\mathrm{Spin}^c(n)$

$n = 2r$

Consider $\mathbb{R}^n$ with its usual inner product and usual orthonormal basis $e_1, e_2, \ldots, e_n$

$\varphi : \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) \longrightarrow M(2^r, \mathbb{C})$

$\varphi(e_j) = E_j \qquad\qquad j = 1, 2, \ldots, n$

There is a canonical inclusion

$$\mathrm{Spin}^c(n) \subset \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n)$$

$\varphi : \mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^n) \longrightarrow M(2^r, \mathbb{C})$ restricted to $\mathrm{Spin}^c(n)$ maps $\mathrm{Spin}^c(n)$ to $2^r \times 2^r$ unitary matrices

$$\mathrm{Spin}^c(n) \longrightarrow U(2^r) \subset \mathrm{GL}(2^r, \mathbb{C})$$

This is the *Spin representation* of $\mathrm{Spin}^c(n)$

$\mathrm{Spin}^c(n)$ acts on $\mathbb{C}^{2^r}$ via this representation

M C^∞ manifold

∂M might be non-empty

$TM =$ the tangent bundle of M

$$\begin{pmatrix} \text{Spin}^c \text{ datum for } TM \\ \eta : P \to \Delta(TM) \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} \text{Spinor system for } TM \\ (\epsilon, \langle\ ,\ \rangle, F) \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} \text{Dirac operator} \\ D : C_c^\infty(M, F) \to C_c^\infty(M, F) \end{pmatrix}$$

$$\boxed{F \text{ is the Spinor bundle}}$$

$$C_c^\infty(M, F) = \{C^\infty \text{ sections with compact support of } F\}$$

$$D : C_c^\infty(M, F) \to C_c^\infty(M, F)$$

such that

(1) D is $\mathbb{C}$-linear

$$D(s_1 + s_2) = Ds_1 + Ds_2 \qquad s_j \in C_c^\infty(M, F)$$

$$D(\lambda x) = \lambda Ds \qquad \lambda \in \mathbb{C}$$

(2) If $f : M \to \mathbb{C}$ is a C^∞ function, then

$$D(fs) = (df)s + f(Ds)$$

(3) If $s_j \in C_c^\infty(M, F)$ then

$$\int_M (Ds_1 x, s_2 x) = \int_M (s_1 x, Ds_2 x) dx$$

(4) If dim M is even, then D is off-diagonal $F = F^+ \oplus F^-$

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

$D : C_c^\infty(M,F) \to C_c^\infty(M,F)$ is an elliptic first-order differential operator.

D can be viewed as an unbounded operator on the Hilbert space $L^2(M,F)$

$$(s_1, s_2) = \int_M (s_1 x, s_2 x) dx$$

$$D : C_c^\infty(M,F) \to C_c^\infty(M,F)$$

is a symmetric operator

Existence of D ?

YES – Construct D locally and patch together with a C^∞ partition of unity.

Uniqueness of D ?

YES – If D_0 and D_1 both satisfy (1)-(4) then $D_0 - D_1$ is a vector bundle map

$$D_0 - D_1 : F \to F$$

Hence D_0 and D_1 differ by lower order terms

Example.

$$n \text{ even}$$

$$S^n \subset \mathbb{R}^{n+1}$$

$D =$ Dirac operator of S^n

$F =$ Spinor bundle of S^n

$F = F^+ \oplus F^-$

$D : C^\infty(S^n, F) \longrightarrow C^\infty(S^n, F)$

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$$

$D^+ : C^\infty(S^n, F^+) \longrightarrow C^\infty(S^n, F^-)$

$\text{Index}\,(D^+) := \dim_{\mathbb{C}}(\text{Kernel}\, D^+) - \dim_{\mathbb{C}}(\text{Cokernel}\, D^+)$

Theorem. $\text{Index}\,(D^+) = 0$

Tensor D^+ with the Bott generator vector bundle β

$$D^+_\beta : C^\infty(S^n, F^+ \otimes \beta) \longrightarrow C^\infty(S^n, F^- \otimes \beta)$$

Theorem. Index $(D^+_\beta) = 1$