BEYOND ELLIPTICITY

Paul Baum
Penn State

Fields Institute
Toronto, Canada

June 20, 2013

Minicourse of five lectures:

1. Dirac operator✓
2. Atiyah-Singer revisited✓
3. What is K-homology?✓
4. Beyond ellipticity
5. The Riemann-Roch theorem

The minicourse is based on joint work with Erik van Erp.

BEYOND ELLIPTICITY

K-homology is the dual theory to K-theory. The BD (Baum-Douglas) isomorphism of Kasparov K-homology and K-cycle K-homology can be taken as providing a framework within which the Atiyah-Singer index theorem can be extended to certain non-elliptic operators. This talk will consider a class of non-elliptic differential operators on compact contact manifolds. These operators have been studied by a number of mathematicians. Working within the BD framework, the index problem will be solved for these operators. This is joint work with Erik van Erp.

FACT:
If M is a closed odd-dimensional C^∞ manifold and D is any elliptic differential operator on M, then $\text{Index}(D) = 0$.

EXAMPLE:
$M = S^3 = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1\}$
x_1, x_2, x_3, x_4 are the usual co-ordinate functions on $\mathbb{R}^4$.

$$x_j(a_1, a_2, a_3, a_4) = a_j \qquad j = 1, 2, 3, 4$$

$\partial/\partial x_j$ usual vector fields on $\mathbb{R}^4$ $\qquad j = 1, 2, 3, 4$

On S^3 consider the (tangent) vector fields V_1, V_2, V_3

$$V_1 = -x_2\partial/\partial x_1 + x_1\partial/\partial x_2 - x_4\partial/\partial x_3 + x_3\partial/\partial x_4$$

$$V_2 = -x_3\partial/\partial x_1 + x_4\partial/\partial x_2 + x_1\partial/\partial x_3 - x_2\partial/\partial x_4$$

$$V_3 = -x_4\partial/\partial x_1 - x_3\partial/\partial x_2 + x_2\partial/\partial x_3 + x_1\partial/\partial x_4$$

Let r be a positive integer and let $\gamma\colon S^3 \longrightarrow M(r,\mathbb{C})$ be a C^∞ map.
$M(r,\mathbb{C})$:= {r×r matrices of complex numbers}.
Form the operator $P_\gamma := 2i\gamma(V_1 \otimes I_r) - V_2^2 \otimes I_r - V_3^2 \otimes I_r$.
$I_r := r \times r$ identity matrix.

$$P_\gamma\colon C^\infty(S^3, S^3 \times \mathbb{C}^r) \longrightarrow C^\infty(S^3, S^3 \times \mathbb{C}^r)$$

$$P_\gamma := 2i\gamma(V_1 \otimes I_r) - V_2^2 \otimes I_r - V_3^2 \otimes I_r$$

$I_r := r \times r$ identity matrix. $\qquad i = \sqrt{-1}.$

$$P_\gamma \colon C^\infty(S^3, S^3 \times \mathbb{C}^r) \longrightarrow C^\infty(S^3, S^3 \times \mathbb{C}^r)$$

LEMMA.
Assume that for all $p \in S^3, \gamma(p)$ does not have any odd integers among its eigenvalues i.e.

$$\forall p \in S^3,\ \forall \lambda \in \{\ldots -3, -1, 1, 3, \ldots\} \Longrightarrow \lambda I_r - \gamma(p) \in GL(r, \mathbb{C})$$

then $\dim_{\mathbb{C}}$ (Kernel $P_\gamma) < \infty$ and $\dim_{\mathbb{C}}$ (Cokernel $P_\gamma) < \infty$.

With γ as in the above lemma, for each odd integer n , let

$$\gamma_n \colon S^3 \longrightarrow GL(r, \mathbb{C}) \quad \text{be}$$

$$p \longmapsto nI_r - \gamma(p)$$

By Bott periodicity if $r \geq 2$, then $\pi_3 GL(r, \mathbb{C}) = \mathbb{Z}$.
Hence for each odd integer n have the Bott number $\beta(\gamma_n)$.

PROPOSITION. With γ as above and $r \geq 2$

$$\mathrm{Index}(P_\gamma) = \sum_{n \text{ odd}} \beta(\gamma_n)$$

K-homology in topology

Problem

How can K-homology be taken from algebraic geometry to topology?

There are three ways in which this has been done:

Homotopy Theory K-homology is the homology theory determined by the Bott spectrum.

K-Cycles K-homology is the group of K-cycles.

C^*-algebras K-homology is the Kasparov group $KK^*(A, \mathbb{C})$.

M.F. Atiyah   Brown-Douglas-Fillmore

Let X be a finite CW complex.

$C(X) = \{\alpha : X \to \mathbb{C} \quad | \quad \alpha \text{ is continuous}\}$

$\mathcal{L}(\mathcal{H}) = \{\text{bounded operators } T : \mathcal{H} \to \mathcal{H}\}$

Any element in the Kasparov K-homology group $KK^0(C(X), \mathbb{C})$ is given by a 5-tuple $(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)$ such that :

- $\mathcal{H}_0$ and $\mathcal{H}_1$ are separable Hilbert spaces.
- $\psi_0 \colon C(X) \longrightarrow \mathcal{L}(\mathcal{H}_0)$ and $\psi_1 \colon C(X) \longrightarrow \mathcal{L}(\mathcal{H}_1)$ are unital $*$-homomorphisms.
- $T : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$ is a (bounded) Fredholm operator.
- For every $\alpha \in C(X)$ the commutator $T \circ \psi_0(\alpha) - \psi_1(\alpha) \circ T$ $\in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is compact.

$$KK^0(C(X), \mathbb{C}) := \{(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)\}/ \sim$$

$$KK^0(C(X),\mathbb{C}) := \{(\mathcal{H}_0,\psi_0,\mathcal{H}_1,\psi_1,T)\}/\sim$$

$$(\mathcal{H}_0,\psi_0,\mathcal{H}_1,\psi_1,T) + (\mathcal{H}_0',\psi_0',\mathcal{H}_1',\psi_1',T') = (\mathcal{H}_0\oplus\mathcal{H}_0',\psi_0\oplus\psi_0',\mathcal{H}_1\oplus\mathcal{H}_1',\psi_1\oplus\psi_1',T\oplus T')$$

$$-(\mathcal{H}_0,\psi_0,\mathcal{H}_1,\psi_1,T) = (\mathcal{H}_1,\psi_1,\mathcal{H}_0,\psi_0,T^*)$$

Let X be a finite CW complex.
Any element in the Kasparov K-homology group $KK^1(C(X), \mathbb{C})$ is given by a 3-tuple $(\mathcal{H}, \psi, T)$ such that :

- $\mathcal{H}$ is a separable Hilbert space.
- $\psi\colon C(X) \longrightarrow \mathcal{L}(\mathcal{H})$ is a unital $*$-homomorphism.
- $T : \mathcal{H} \longrightarrow \mathcal{H}$ is a (bounded) self-adjoint Fredholm operator.
- For every $\alpha \in C(X)$ the commutator $T \circ \psi(\alpha) - \psi(\alpha) \circ T \in \mathcal{L}(\mathcal{H})$ is compact.

$$KK^1(C(X), \mathbb{C}) := \{(\mathcal{H}, \psi, T)\}/\sim$$

$$(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T')$$

$$-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, -T)$$

Let X, Y be CW complexes and let $f\colon X \to Y$ be a continuous map.

Denote by $f^\natural\colon C(X) \leftarrow C(Y)$ the $*$-homomorphism

$$f^\natural(\alpha) := \alpha \circ f \qquad\qquad \alpha \in C(Y)$$

Then $f_*\colon KK^j(C(X), \mathbb{C}) \to KK^j(C(Y), \mathbb{C})$ is

$$f_*(\mathcal{H}, \psi, T) := (\mathcal{H}, \psi \circ f^\natural, T) \qquad\qquad j = 1$$

$$f_*(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) := (\mathcal{H}_0, \psi_0 \circ f^\natural, \mathcal{H}_1, \psi_1 \circ f^\natural, T) \qquad\qquad j = 0$$

Cycles for K-homology

Let X be a CW complex.

Definition

A K-cycle on X is a triple (M, E, φ) such that :

1. M is a compact Spin^c manifold without boundary.
2. E is a $\mathbb{C}$ vector bundle on M.
3. $\varphi \colon M \to X$ is a continuous map from M to X.

Chern character in K-theory

X finite CW complex

$$ch\colon K^j(X) \longrightarrow \bigoplus_l H^{j+2l}(X;\mathbb{Q})$$

$$j = 0, 1$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} K^j(X) \longrightarrow \bigoplus_l H^{j+2l}(X;\mathbb{Q})$$

is an isomorphism of $\mathbb{Q}$ vector spaces.

Chern character in K-homology

X finite CW complex $\qquad (M, E, \varphi) \mapsto \varphi_*(ch(E) \cup Td(M) \cap [M])$

$$ch\colon K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q})$$

$$j = 0, 1$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q})$$

is an isomorphism of $\mathbb{Q}$ vector spaces.

Theorem (B-Douglas-Taylor, B-Higson-Schick)

Let X be a finite CW complex.

Then for $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is an isomorphism.

For $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is $(M, E, \varphi) \mapsto \varphi_*[D_E]$

where

1. D_E is the Dirac operator of M tensored with E.
2. $[D_E] \in KK^j(C(M), \mathbb{C})$ is the element in the Kasparov K-homology of M determined by D_E.
3. $\varphi_* \colon KK^j(C(M), \mathbb{C}) \to KK^j(C(X), \mathbb{C})$ is the homomorphism of abelian groups determined by $\varphi \colon M \to X$.

Comparison of $K_*(X)$ and $KK^*(C(X), \mathbb{C})$

Given some analytic data on X (i.e. an index problem) it is usually easy to construct an element in $KK^*(C(X), \mathbb{C})$. This does not solve the given index problem. $KK^*(C(X), \mathbb{C})$ does not have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

$K_*(X)$ does have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

$$ch\colon K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q})$$

$$(M, E, \varphi) \mapsto \varphi_*(\mathrm{ch}(E) \cup Td(M) \cap [M])$$

With X a finite CW complex, suppose a datum (i.e. some analytical information) is given which then determines an element $\xi \in KK^j(C(X), \mathbb{C})$.

QUESTION : What does it mean to solve the index problem for ξ?

ANSWER : It means to explicitly construct the K-cycle (M, E, φ) such that

$$\mu(M, E, \varphi) = \xi$$

where $\mu \colon K_j(X) \to KK^j(C(X), \mathbb{C})$ is the natural map of abelian groups.

Suppose that $j = 0$ and that a K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been constructed. It then follows that for any $\mathbb{C}$ vector bundle F on X

$$\mathrm{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap ch(M, E, \varphi))$$

$\epsilon\colon X \longrightarrow \cdot$ $\qquad$ ϵ is the map of X to a point.

$$ch(M, E, \varphi) := \varphi_*(\mathrm{ch}(E) \cup Td(M) \cap [M])$$

EQUIVALENTLY Suppose that $j = 0$ and that a K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been constructed. It then follows that

$$\mathcal{I}(\xi) = \varphi_*(ch(E) \cup Td(M) \cap [M])$$

REMARK. If the construction of the K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been done correctly, then it will work in the equivariant case and in the case of families of operators.

Example

General case of the Atiyah-Singer index theorem

Let X be a compact C^∞ manifold without boundary.
X is not required to be oriented.
X is not required to be even dimensional.
On X let

$$\delta : C^\infty(X, E_0) \longrightarrow C^\infty(X, E_1)$$

be an elliptic differential (or pseudo-differential) operator.
Then δ determines an element

$$[\delta] \in KK^0(C(X), \mathbb{C})$$

The K-cycle on X – which solves the index problem for δ – is

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi).$$

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi)$$

$S(TX \oplus 1_{\mathbb{R}})$ is the unit sphere bundle of $TX \oplus 1_{\mathbb{R}}$.

$\pi\colon S(TX \oplus 1_{\mathbb{R}}) \longrightarrow X$ is the projection of $S(TX \oplus 1_{\mathbb{R}})$ onto X.

$S(TX \oplus 1_{\mathbb{R}})$ is even-dimensional and is a Spin^c manifold.

E_σ is the $\mathbb{C}$ vector bundle on $S(TX \oplus 1_{\mathbb{R}})$ obtained by doing a clutching construction using the symbol σ of δ.

$$\mu((S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi)) = [\delta]$$

$$\Downarrow$$

$$\text{Index}(\delta) = (ch(E_\sigma) \cup Td(S(TX \oplus 1_{\mathbb{R}})))[(S(TX \oplus 1_{\mathbb{R}})]$$

which is the general Atiyah-Singer formula.

Contact Manifolds

A contact manifold is an odd dimensional C^∞ manifold X
dimension$(X) = 2n+1$
with a given C^∞ 1-form θ such that

$$\theta(d\theta)^n \text{ is non zero at every } x \in X - i.e.\ \theta(d\theta)^n \text{ is a volume form for } X.$$

Let X be a compact connected contact manifold without boundary ($\partial X = \emptyset$).
Set dimension$(X) = 2n+1$.
Let r be a positive integer and let $\gamma\colon X \longrightarrow M(r,\mathbb{C})$ be a C^∞ map.
$M(r,\mathbb{C})$:= {r×r matrices of complex numbers}.

Assume: For each $x \in X$,
$\{\text{Eigenvalues of } \gamma(x)\} \cap \{\ldots, -n-4, -n-2, -n, n, n+2, n+4, \ldots\} = \emptyset$
i.e. $\forall x \in X$,
$\lambda \in \{\ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots\} \Longrightarrow \lambda I_r - \gamma(x) \in GL(r,\mathbb{C})$

$\gamma\colon X \longrightarrow M(r, \mathbb{C})$
Are assuming : $\forall x \in X$,
$\lambda \in \{\ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots\} \Longrightarrow \lambda I_r - \gamma(x) \in GL(r, \mathbb{C})$

Associated to γ is a differential operator P_γ which is hypoelliptic and Fredholm.

$$P_\gamma\colon C^\infty(X, X \times \mathbb{C}^r) \longrightarrow C^\infty(X, X \times \mathbb{C}^r)$$

P_γ is constructed as follows.

The sub-Laplacian Δ_H

Let H be the null-space of θ.

$$H = \{v \in TX \mid \theta(v) = 0\}$$

H is a C^∞ sub vector bundle of TX with

$$\text{For all } x \in X, \dim_{\mathbb{R}}(H_x) = 2n$$

The sub-Laplacian

$$\Delta_H \colon C^\infty(X) \to C^\infty(X)$$

is locally $-W_1^2 - W_2^2 - \cdots - W_{2n}^2$
where $W_1, W_2, \ldots, W_{2n}$ is a locally defined C^∞ orthonormal frame for H. These locally defined operators are then patched together using a C^∞ partition of unity to give the sub-Laplacian Δ_H.

The Reeb vector field

The Reeb vector field is the unique C^∞ vector field W on X with :

$$\theta(W) = 1 \text{ and } \forall v \in TX,\ d\theta(W, v) = 0$$

Let

$$\gamma\colon X \longrightarrow M(r, \mathbb{C})$$

be as above, $P_\gamma\colon C^\infty(X, X \times \mathbb{C}^r) \to C^\infty(X, X \times \mathbb{C}^r)$ is defined:

$$P_\gamma = i\gamma(W \otimes I_r) + (\Delta_H) \otimes I_r \qquad I_r = r \times r \text{ identity matrix} \quad i = \sqrt{-1}$$

P_γ is a differential operator (of order 2) and is hypoelliptic but not elliptic.

These operators P_γ have been studied by :

- R.Beals and P.Greiner *Calculus on Heisenberg Manifolds* Annals of Math. Studies 119 (1988).
- C.Epstein and R.Melrose.
- E. van Erp *The Atiyah-Singer index formula for subelliptic operators on contact manifolds. Part 1 and Part 2* Annals of Math. 171(2010).

A class of operators with somewhat similar analytic and topological properties has been studied by A. Connes and H. Moscovici.
M. Hilsum and G. Skandalis.

Set $T_\gamma = P_\gamma(I + P_\gamma^* P_\gamma)^{-1/2}$.

Let $\psi\colon C(X) \to \mathcal{L}(L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r)$ be

$$\psi(\alpha)(u_1, u_2, \ldots, u_r) = (\alpha u_1, \alpha u_2, \ldots, \alpha u_r)$$

where for $x \in X$ and $u \in L^2(X), (\alpha u)(x) = \alpha(x)u(x)$

$$\alpha \in C(X) \qquad\qquad u \in L^2(X)$$

Then

$$(L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r,\ \psi,\ \ L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r,\ \psi,\ \ T_\gamma) \in KK^0(C(X), \mathbb{C})$$

Denote this element of $KK^0(C(X), \mathbb{C})$ by $[P_\gamma]$.

$$[P_\gamma] \in KK^0(C(X), \mathbb{C})$$

$$[P_\gamma] \in KK^0(C(X), \mathbb{C})$$

QUESTION.What is the K-cycle that solves the index problem for $[P_\gamma]$?

ANSWER. To construct this K-cycle, first recall that the given 1-form θ which makes X a contact manifold also makes X a stably almost complex manifold :

$$(\text{contact}) \Longrightarrow (\text{stably almost complex})$$

(contact) $\Longrightarrow$ (stably almost complex)

Let θ, H, and W be as above. Then :

- $TX = H \oplus 1_{\mathbb{R}}$ where $1_{\mathbb{R}}$ is the (trivial) $\mathbb{R}$ line bundle spanned by W.
- A morphism of C^∞ $\mathbb{R}$ vector bundles $J : H \to H$ can be chosen with $J^2 = -I$ and $\forall x \in X$ and $u, v \in H_x$

$$d\theta(Ju, Jv) = d\theta(u, v) \qquad d\theta(Ju, u) \geq 0$$

- J is unique up to homotopy.

(contact) $\Longrightarrow$ (stably almost complex)

$J\colon H \to H$ is unique up to homotopy.
Once J has been chosen :

H is a C^∞ $\mathbb{C}$ vector bundle on X.

$$\Downarrow$$

$TX \oplus 1_{\mathbb{R}} = H \oplus 1_{\mathbb{R}} \oplus 1_{\mathbb{R}} = H \oplus 1_{\mathbb{C}}$ is a C^∞ $\mathbb{C}$ vector bundle on X.

$$\Downarrow$$

$$X \times S^1 \text{ is an almost complex manifold.}$$

REMARK. An almost complex manifold is a $\mathbb{C}^\infty$ manifold Ω with a given morphism $\zeta\colon T\Omega \to T\Omega$ of C^∞ $\mathbb{R}$ vector bundles on Ω such that

$$\zeta \circ \zeta = -I$$

The conjugate almost complex manifold is Ω with ζ replaced by $-\zeta$.

NOTATION. As above $X \times S^1$ is an almost complex manifold, $\overline{X \times S^1}$ denotes the conjugate almost complex manifold.

Since (almost complex)$\Longrightarrow$ (Spin^c), the disjoint union $X \times S^1 \sqcup \overline{X \times S^1}$ can be viewed as a Spin^c manifold.

Let

$$\pi\colon X \times S^1 \sqcup \overline{X \times S^1} \longrightarrow X$$

be the evident projection of $X \times S^1 \sqcup \overline{X \times S^1}$ onto X.
i.e.

$$\pi(x, \lambda) = x \qquad\qquad (x, \lambda) \in X \times S^1 \sqcup \overline{X \times S^1}$$

The solution K-cycle for $[P_\gamma]$ is $(X \times S^1 \sqcup \overline{X \times S^1},\ E_\gamma,\ \pi)$

$$E_\gamma = \left(\bigoplus_{j=0}^{j=N} L(\gamma, n+2j)\otimes\pi^*\mathrm{Sym}^j(H)\right) \bigsqcup \left(\bigoplus_{j=0}^{j=N} L(\gamma, -n-2j)\otimes\pi^*\mathrm{Sym}^j(H^*)\right)$$

1. "Sym^j" is " j-th symmetric power".
2. H^* is the dual vector bundle of H.
3. N is any positive integer such that : $n + 2N > \sup\{||\gamma(x)||, x \in X\}$.
4. $L(\gamma, n + 2j)$ is the $\mathbb{C}$ vector bundle on $X \times S^1$ obtained by doing a clutching construction using $(n + 2j)I_r - \gamma\colon X \to GL(r, \mathbb{C})$.
5. Similarly, $L(\gamma, -n - 2j)$ is obtained by doing a clutching construction using $(-n - 2j)I_r - \gamma\colon X \to GL(r, \mathbb{C})$.

Restriction of E_γ to $X \times S^1$

Let N be any positive integer such that :

$$n + 2N > \sup\{||\gamma(x)||, x \in X\}$$

The restriction of E_γ to $X \times S^1$ is:

$$E_\gamma \mid X \times S^1 = \bigoplus_{j=0}^{j=N} L(\gamma, n+2j) \otimes \pi^*\mathrm{Sym}^j(H)$$

Restriction of E_γ to $\overline{X \times S^1}$

The restriction of E_γ to $\overline{X \times S^1}$ is:

$$E_\gamma \mid \overline{X \times S^1} = \bigoplus_{j=0}^{j=N} L(\gamma, -n-2j) \otimes \pi^* \mathrm{Sym}^j(H^*)$$

Here H^* is the dual vector bundle of H:

$$H_x^* = \mathrm{Hom}_{\mathbb{C}}(H_x, \mathbb{C}) \qquad\qquad x \in X$$

$$E_\gamma = \Big(\bigoplus_{j=0}^{j=N} L(\gamma, n{+}2j)\otimes\pi^*\mathrm{Sym}^j(H)\Big)\bigsqcup\Big(\bigoplus_{j=0}^{j=N} L(\gamma, -n{-}2j)\otimes\pi^*\mathrm{Sym}^j(H^*)\Big)$$

Theorem (PB and Erik van Erp)

$$\mu(X \times S^1 \sqcup \overline{X \times S^1},\ E_\gamma\,,\ \pi) = [P_\gamma]$$