The Riemann-Roch Theorem

Paul Baum
Penn State

TIFR
Mumbai, India

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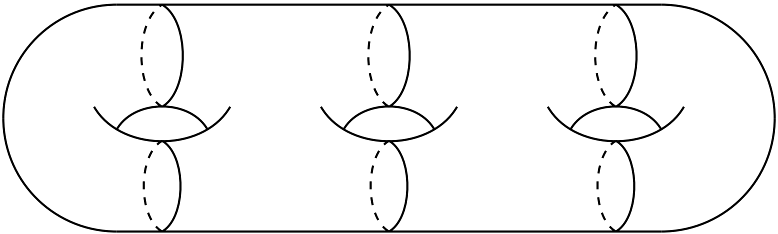
THE RIEMANN-ROCH THEOREM

Topics in this talk :

1. Classical Riemann-Roch
2. Hirzebruch-Riemann-Roch (HRR)
3. Grothendieck-Riemann-Roch (GRR)
4. RR for possibly singular varieties (Baum-Fulton-MacPherson)

CLASSSICAL RIEMANN - ROCH

M compact connected Riemann surface



$$\text{genus of } M = \ \# \text{ of holes}$$
$$= \frac{1}{2}\left[\mathrm{rank} H_1(M;\mathbb{Z})\right]$$

D a divisor of M

D consists of a finite set of points of M $p_1, p_2, \ldots, p_l$ and an integer assigned to each point $n_1, n_2, \ldots, n_l$

Equivalently

D is a function $D\colon M \to \mathbb{Z}$ with finite support

$\text{Support}(D) = \{p \in M \mid D(p) \neq 0\}$

$\text{Support}(D)$ is a finite subset of M

D a divisor on M

$\deg(D) := \sum_{p \in M} D(p)$

Remark

D_1, D_2 two divisors

$$D_1 \geqq D_2 \text{ iff } \forall p \in M, D_1(p) \geqq D_2(p)$$

Remark

D a divisor, $-D$ is

$$(-D)(p) = -D(p)$$

Example

Let $f\colon M \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function.

Define a divisor $\delta(f)$ by:

$$\delta(f)(p) = \begin{cases} 0 \text{ if } p \text{ is neither a zero nor a pole of } f \\ \text{order of the zero if } f(p) = 0 \\ -(\text{order of the pole}) \text{ if } p \text{ is a pole of } f \end{cases}$$

Example

Let ω be a meromorphic 1-form on M. Locally ω is $f(z)dz$ where f is a (locally defined) meromorphic function. Define a divisor $\delta(\omega)$ by:

$$\delta(\omega)(p) = \begin{cases} 0 \text{ if } p \text{ is neither a zero nor a pole of } \omega \\ \text{order of the zero if } \omega(p) = 0 \\ -(\text{order of the pole) if } p \text{ is a pole of } \omega \end{cases}$$

D a divisor on M

$$H^0(M,D) := \left\{ \begin{array}{l} \text{meromorphic functions} \\ f\colon M \to \mathbb{C} \cup \{\infty\} \end{array} \middle| \, \delta(f) \geqq -D \right\}$$

$$H^1(M,D) := \left\{ \begin{array}{l} \text{meromorphic 1-forms} \\ \omega \text{ on } M \end{array} \middle| \, \delta(\omega) \geqq D \right\}$$

Lemma

$H^0(M,D)$ and $H^1(M,D)$ are finite dimensional $\mathbb{C}$ vector spaces

$$\dim_{\mathbb{C}} H^0(M,D) < \infty$$
$$\dim_{\mathbb{C}} H^1(M,D) < \infty$$

Theorem (RR)

Let M be a compact connected Riemann surface and let D be a divisor on M. Then:

$$\dim_{\mathbb{C}} H^0(M, D) - \dim_{\mathbb{C}} H^1(M, D) = d - g + 1$$

$$d = \textit{degree}\ (D)$$
$$g = \textit{genus}\ (M)$$

HIRZEBRUCH-RIEMANN-ROCH

M non-singular projective algebraic variety / $\mathbb{C}$

E an algebraic vector bundle on M

$\underline{E}$ = sheaf of germs of algebraic sections of E

$H^j(M, \underline{E}) :=$ j-th cohomology of M using $\underline{E}$, $j = 0, 1, 2, 3, \ldots$

LEMMA

For all $j = 0, 1, 2, \ldots \dim_{\mathbb{C}} H^j(M, \underline{E}) < \infty$.

For all $j > \dim_{\mathbb{C}}(M), \quad H^j(M, \underline{E}) = 0$.

$$\chi(M, E) := \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(M, \underline{E})$$

$n = \dim_{\mathbb{C}}(M)$

THEOREM[HRR] Let M be a non-singular projective algebraic variety / $\mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M, E) = (ch(E) \cup Td(M))[M]$$

Hirzebruch-Riemann-Roch

Theorem (HRR)

Let M be a non-singular projective algebraic variety $/\,\mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M,E) = (ch(E) \cup Td(M))[M]$$

EXAMPLE. Let M be a compact complex-analytic manifold.
Set $\Omega^{p,q} = C^\infty(M, \Lambda^{p,q}T^*M)$
$\Omega^{p,q}$ is the $\mathbb{C}$ vector space of all C^∞ differential forms of type (p,q)
Dolbeault complex

$$0 \longrightarrow \Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \cdots \longrightarrow \Omega^{0,n} \longrightarrow 0$$

The Dirac operator (of the underlying Spin^c manifold) is the assembled Dolbeault complex

$$\bar\partial + \bar\partial^*\colon \bigoplus_j \Omega^{0,\,2j} \longrightarrow \bigoplus_j \Omega^{0,\,2j+1}$$

The index of this operator is the arithmetic genus of M — i.e. is the Euler number of the Dolbeault complex.

K-theory and K-homology in algebraic geometry

Let X be a (possibly singular) projective algebraic variety / $\mathbb{C}$.

Grothendieck defined two abelian groups:

$K^0_{alg}(X) =$ Grothendieck group of algebraic vector bundles on X.

$K_0^{alg}(X) =$ Grothendieck group of coherent algebraic sheaves on X.

$K^0_{alg}(X) =$ the algebraic geometry K-theory of X contravariant.

$K_0^{alg}(X) =$ the algebraic geometry K-homology of X covariant.

K-theory in algebraic geometry

$\mathsf{Vect}_{alg}X =$
set of isomorphism classes of algebraic vector bundles on X.

$\mathsf{A}(\mathsf{Vect}_{alg}X) =$ free abelian group
with one generator for each element $[E] \in \mathrm{Vect}_{alg}X$.

For each short exact sequence ξ

$$0 \to E' \to E \to E'' \to 0$$

of algebraic vector bundles on X, let $r(\xi) \in \mathrm{A}(\mathrm{Vect}_{alg}X)$ be

$$r(\xi) := [E'] + [E''] - [E]$$

K-theory in algebraic geometry

$\mathcal{R} \subset \mathrm{A}(\mathrm{Vect}_{alg}(X))$ is the subgroup of $\mathrm{A}(\mathrm{Vect}_{alg}X)$ generated by all $r(\xi) \in \mathrm{A}(\mathrm{Vect}_{alg}X)$.

DEFINITION. $K^0_{alg}(X) := \mathrm{A}(\mathrm{Vect}_{alg}X)/\mathcal{R}$

Let X, Y be (possibly singular) projective algebraic varieties $/\mathbb{C}$.
Let

$$f\colon X \longrightarrow Y$$

be a morphism of algebraic varieties.
Then have the map of abelian groups

$$f^*\colon K^0_{alg}(X) \longleftarrow K^0_{alg}(Y)$$

$$[f^*E] \leftarrow [E]$$

Vector bundles pull back. f^*E is the pull-back via f of E.

K-homology in algebraic geometry

$\mathcal{S}_{alg}X =$
set of isomorphism classes of coherent algebraic sheaves on X.

$\mathrm{A}(\mathcal{S}_{alg}X) =$ free abelian group
with one generator for each element $[\mathcal{E}] \in \mathcal{S}_{alg}X$.

For each short exact sequence ξ

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

of coherent algebraic sheaves on X, let $r(\xi) \in \mathrm{A}(\mathcal{S}_{alg}X)$ be

$$r(\xi) := [\mathcal{E}'] + [\mathcal{E}''] - [\mathcal{E}]$$

K-homology in algebraic geometry

$\mathfrak{R} \subset \mathrm{A}(\mathcal{S}_{alg}(X))$ is the subgroup of $\mathsf{A}(\mathcal{S}_{alg}X)$ generated by all $r(\xi) \in \mathrm{A}(\mathcal{S}_{alg}X)$.

DEFINITION. $K_0^{alg}(X) := \mathrm{A}(\mathcal{S}_{alg}X)/\mathfrak{R}$

Let X, Y be (possibly singular) projective algebraic varieties $/\mathbb{C}$. Let

$$f\colon X \longrightarrow Y$$

be a morphism of algebraic varieties.
Then have the map of abelian groups

$$f_*\colon K_0^{alg}(X) \longrightarrow K_0^{alg}(Y)$$

$$[\mathcal{E}] \mapsto \Sigma_j (-1)^j [(R^j f)\mathcal{E}]$$

$f\colon X \to Y$ morphism of algebraic varieties
$\mathcal{E}$ coherent algebraic sheaf on X
For $j \geq 0$, define a presheaf $(W^j f)\mathcal{E}$ on Y by

$$U \mapsto H^j(f^{-1}U;\, \mathcal{E}|f^{-1}U) \qquad\qquad U \text{ an open subset of } Y$$

Then

$$(R^j f)\mathcal{E} := \text{the sheafification of } (W^j f)\mathcal{E}$$

$f\colon X \to Y \quad$ morphism of algebraic varieties

$$f_*\colon K_0^{alg}(X) \longrightarrow K_0^{alg}(Y)$$
$$[\mathcal{E}] \mapsto \Sigma_j (-1)^j [(R^j f)\mathcal{E}]$$

SPECIAL CASE of $f_*\colon K_0^{alg}(X) \longrightarrow K_0^{alg}(Y)$

Y is a point. $Y = \cdot$

$\epsilon\colon X \to \cdot$ is the map of X to a point.

$K^0_{alg}(\cdot) = K_0^{alg}(\cdot) = \mathbb{Z}$

$\epsilon_*\colon K_0^{alg}(X) \to \ K_0^{alg}(\cdot) = \mathbb{Z}$

$\epsilon_*(\mathcal{E}) = \chi(X;\mathcal{E}) = \Sigma_j(-1)^j \dim_{\mathbb{C}} H^j(X;\mathcal{E})$

X non-singular $\Longrightarrow K^0_{alg}(X) \cong K_0^{alg}(X)$

Let X be non-singular.
Let E be an algebraic vector bundle on X.
$\underline{E}$ denotes the sheaf of germs of algebraic sections of E.
Then $E \mapsto \underline{E}$ is an isomorphism of abelian groups

$$K^0_{alg}(X) \longrightarrow K_0^{alg}(X)$$

This is Poincaré duality within the context of algebraic geometry K-theory&K-homology.

Grothendieck-Riemann-Roch

Theorem (GRR)

Let X, Y be non-singular projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} & K^0_{alg}(X) \longrightarrow K^0_{alg}(Y) & \\ ch(\ ) \cup Td(X) \quad \downarrow & & \downarrow \quad ch(\ ) \cup Td(Y) \\ & H^*(X;\mathbb{Q}) \longrightarrow H^*(Y;\mathbb{Q}) & \end{array}$$

WARNING!!!

The horizontal arrows in the GRR commutative diagram

$$\begin{array}{ccccc} & K^0_{alg}(X) & \longrightarrow & K^0_{alg}(Y) & \\ ch(\,)\cup Td(X) & \downarrow & & \downarrow & ch(\,)\cup Td(Y) \\ & H^*(X;\mathbb{Q}) & \longrightarrow & H^*(Y;\mathbb{Q}) & \end{array}$$

are wrong-way (i.e. Gysin) maps.

$$K^0_{alg}(X) \cong K_0^{alg}(X) \xrightarrow{f_*} K_0^{alg}(Y) \cong K^0_{alg}(Y)$$

$$H^*(X;\mathbb{Q}) \cong H_*(X;\mathbb{Q}) \xrightarrow{f_*} H_*(Y;\mathbb{Q}) \cong H^*(Y;\mathbb{Q})$$

Poincaré duality Poincaré duality

K-homology is the dual theory to K-theory.
How can K-homology be taken from algebraic geometry to topology?
There are three ways in which this has been done:

Homotopy Theory K-homology is the homology theory determined by the Bott spectrum.

Geometric Cycles K-homology is the group of K-cycles.

C* algebras K-homology is the Kasparov group $KK^*(A, \mathbb{C})$.

Riemann-Roch for possibly singular complex projective algebraic varieties

Let X be a (possibly singular) projective algebraic variety / $\mathbb{C}$

Then (Baum-Fulton-MacPherson) there are functorial maps

$\alpha_X : K^0_{alg}(X) \longrightarrow K^0_{top}(X)$ K-theory contravariant
natural transformation of contravariant functors

$\beta_X : K_0^{alg}(X) \longrightarrow K_0^{top}(X)$ K-homology covariant
natural transformation of covariant functors

Everything is natural. No wrong-way (i.e. Gysin) maps are used.

$\alpha_X \colon K^0_{alg}(X) \longrightarrow K^0_{top}(X)$
is the forgetful map which sends an algebraic vector bundle E to the underlying topological vector bundle of E.

$$\alpha_X(E) := E_{\text{topological}}$$

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} K^0_{alg}(X) & \longleftarrow & K^0_{alg}(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ K^0_{top}(X) & \longleftarrow & K^0_{top}(Y) \end{array}$$

i.e. natural transformation of contravariant functors

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} K^0_{alg}(X) & \longleftarrow & K^0_{alg}(Y) \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ K^0_{top}(X) & \longleftarrow & K^0_{top}(Y) \\ ch \downarrow & & \downarrow ch \\ H^*(X;\mathbb{Q}) & \longleftarrow & H^*(Y;\mathbb{Q}) \end{array}$$

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} K_0^{alg}(X) & \longrightarrow & K_0^{alg}(Y) \\ \beta_X \downarrow & & \downarrow \beta_Y \\ K_0^{top}(X) & \longrightarrow & K_0^{top}(Y) \end{array}$$

i.e. natural transformation of covariant functors

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$
\begin{array}{ccc}
K^0_{alg}(X) & \longleftarrow & K^0_{alg}(Y) \\
\alpha_X \downarrow & & \downarrow \alpha_Y \\
K^0_{top}(X) & \longleftarrow & K^0_{top}(Y) \\
ch \downarrow & & \downarrow ch \\
H^*(X;\mathbb{Q}) & \longleftarrow & H^*(Y;\mathbb{Q})
\end{array}
$$

Let X, Y be projective algebraic varieties $/\mathbb{C}$, and let $f : X \longrightarrow Y$ be a morphism of algebraic varieties. Then there is commutativity in the diagram :

$$\begin{array}{ccc} K_0^{alg}(X) & \longrightarrow & K_0^{alg}(Y) \\ \beta_X \downarrow & & \downarrow \beta_Y \\ K_0^{top}(X) & \longrightarrow & K_0^{top}(Y) \\ ch \downarrow & & \downarrow ch \\ H_*(X;\mathbb{Q}) & \longrightarrow & H_*(Y;\mathbb{Q}) \end{array}$$