Comparison Between Algebraic and Topological K-Theory for Banach Algebras and C^*-Algebras

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For a Banach algebra, one can define two kinds of K-theory: topological K-theory, which satisfies Bott periodicity, and algebraic K-theory, which usually does not. It was discovered, starting in the early 80's, that the "comparison map" from algebraic to topological K-theory is a surprisingly rich object. About the same time, it was also found that the *algebraic* (as opposed to topological) K-theory of operator algebras does have some direct applications in operator theory. This article will summarize what is known about these applications and the comparison map.

1 Some Problems in Operator Theory

1.1 Toeplitz operators and K-Theory

The connection between operator theory and K-theory has very old roots, although it took a long time for the connection to be understood. We begin with an example. Think of S^1 as the unit circle in the complex plane and let $\mathcal{H} \subset L^2(S^1)$ be the Hilbert space H^2 of functions all of whose negative Fourier coefficients vanish. In other words, if we identify functions with their formal Fourier expansions,

$$\mathcal{H} = \left\{ \sum_{n=0}^{\infty} c_n z^n \text{ with } \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\}.$$

Now let $f \in C(S^1)$ and let M_f be the operator of multiplication by f on $L^2(S^1)$. This operator does not necessarily map $\mathcal{H}$ into itself, so let $P\colon L^2(S^1) \to \mathcal{H}$ be the orthogonal projection and let $T_f = PM_f$, viewed as an operator from $\mathcal{H}$ to itself. This is called the *Toeplitz operator* with continuous symbol f. In terms of the orthonormal basis $e_0(z) = 1$, $e_1(z) =$

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z, $e_2(z) = z^2$, $\cdots$ of $\mathcal{H}$, T_f is given by the (one-sided) infinite matrix with entries $\langle T_f e_i, e_j\rangle = c_{j-i}$, where $f(z) = \sum c_n z^n$ is the formal Fourier expansion of f. This is precisely a *Toeplitz matrix*, i.e., a matrix with constant entries along any diagonal. The operator T_f may also be viewed as a singular integral operator, since by the Cauchy integral formula, one has

$$T_f \varphi(z) = \frac{1}{2\pi i} \oint_{S^1} \frac{f(\zeta)\varphi(\zeta)}{\zeta - z}\, d\zeta$$

for $|z| < 1$, and the same formula is "formally" valid for $|z| = 1$.

A natural question now arises: when is T_f invertible? And when this is the case, can one give a formula for the inverse? In other words, how does one solve the singular integral equation $T_f\varphi(z) = g(z)$? The following result is "classical" and was first proved by Krein back in the 1950's, though his formulation looked quite different.

Theorem 1.1. *Let T_f be the Toeplitz operator on H^2 defined as above, for $f \in C(S^1)$. Then T_f is invertible if and only if f is everywhere non-vanishing (so that f can be viewed as a map $S^1 \to \mathbb{C}^\times$) and if the winding number of f, i.e., the degree of the map $\frac{f}{|f|}\colon S^1 \to S^1$, is zero.*

Sketch of a modern proof. (For more details, see [17, Ch. 7, especially Theorem 7.23 and Proposition 7.24].) Let $\mathcal{T}$ be the C^*-algebra[2] generated by all the operators T_f, $f \in C(S^1)$, i.e., the norm closure of the algebra generated by these operators and their adjoints. $\mathcal{T}$ is called the *Toeplitz algebra*. The first thing to observe is that there is a surjective $*$-homomorphism $\sigma\colon \mathcal{T} \to C(S^1)$, the "symbol map," induced by $T_f \mapsto f$, fitting into a short exact sequence of C^*-algebras

$$0 \to \mathcal{K} \to \mathcal{T} \xrightarrow{\sigma} C(S^1) \to 0, \tag{1}$$

where $\mathcal{K}$ is the algebra of compact operators on H^2. In particular, $\mathcal{T}$ is commutative modulo compact operators.

To begin with, it is obvious that

$$T_f^* = (PM_fP)^*|_{H^2} = (PM_f^*P)\big|_{H^2} = (PM_{\bar f}P)\big|_{H^2} = T_{\bar f}$$

and that the map $f \mapsto T_f$ is linear, and

$$\|T_f\| = \|PM_f\| \le \|P\|\,\|M_f\| = \|f\|_\infty.$$

So since polynomials in z are dense in $C(S^1)$, for proving commutativity of $\mathcal{T}$ modulo compacts and multiplicativity of σ it is enough to check that $T_{z^j}T_{z^k} \equiv T_{z^{j+k}} \bmod \mathcal{K}$. This is immediate since

$$T_{z^j}T_{z^k}e_m = T_{z^{j+k}}e_m = e_{m+j+k}$$

[2] By definition, a C^*-algebra is a Banach algebra with involution $*$, isometrically $*$-isomorphic to a norm-closed self-adjoint algebra of operators on a Hilbert space.

for m sufficiently large ($m \geq |j| + |k|$). Thus $\mathcal{T}/(\mathcal{T} \cap \mathcal{K})$ is commutative, and σ by construction is surjective. Next, we show that $\mathcal{K} \subset \mathcal{T}$. For this it suffices to show that the action of $\mathcal{T}$ on $\mathcal{K}$ is irreducible, and since T_z is the unilateral shift (sending $e_j \mapsto e_{j+1}$), which is known to be irreducible, the result follows. (In fact, the rank-one operators $\xi \mapsto \langle \xi, e_j \rangle e_k$, which generate a dense subalgebra of $\mathcal{K}$, can all be written as polynomials in T_z and its adjoint $T_{z^{-1}}$. For example, $T_{z^{-1}}T_z - T_zT_{z^{-1}}$ is orthogonal projection onto the span of e_0.) Finally, we need to show that the kernel of σ is precisely $\mathcal{K}$; this can be checked by showing that the map $f \mapsto T_f \mod \mathcal{K}$ is an isometry — a detailed proof is in [17, proof of Theorem 7.11].

Now we get to the more interesting part of the proof, the part that involves K-theory. The idea is to use the long exact K-theory sequences

$$\begin{tikzcd} K_1(\mathcal{T}) \arrow[r, "\sigma_*"] \arrow[d] & K_1(C(S^1)) \arrow[r, "\partial"] \arrow[d] & K_0(\mathcal{K}) = \mathbb{Z} \arrow[d, equal] \\ 0 = K_1(\mathcal{L}) \arrow[r] & K_1(\mathcal{Q}) \arrow[r, "\partial"] & K_0(\mathcal{K}) = \mathbb{Z} \end{tikzcd} \tag{2}$$

associated to (1) and to the algebra $\mathcal{L}$ of all bounded linear operators on H^2 and its quotient $\mathcal{Q} = \mathcal{L}/\mathcal{K}$, the so-called *Calkin algebra*. The downward-pointing arrows here are induced by the inclusion $\mathcal{T} \hookrightarrow \mathcal{L}$. Note that we are using excision for K_0 to identify the relative groups $K_0(\mathcal{T}, \mathcal{K})$ and $K_0(\mathcal{L}, \mathcal{K})$ with $K_0(\mathcal{K}) = \mathbb{Z}$. Now one can show that $\partial([f])$ is (up to a sign depending on orientation conventions) the winding number of f. (To prove this, one can first show that $\partial([f])$ only depends on the homotopy class of f as a map $S^1 \to \mathbb{C}^\times$, and then compute for $f(z) = z$, which generates $\pi_1(S^1)$.) If T_f is invertible, then from (1), $\sigma(T_f) = f$ is invertible. And by exactness of (2), $\partial([f]) = 0$, so the winding number condition in the theorem is satisfied. In the other direction, suppose f is invertible in $C(S^1)$. Then f defines a class in $K_1(C(S^1))$ and $\partial([f])$ is an obstruction to lifting f to an invertible element of $\mathcal{T}$. So if the winding number condition in the theorem is satisfied, the obstruction vanishes. From the bottom part of the commuting diagram (2), together with the interpretation of the inverse image of $\mathcal{Q}^\times$ in $\mathcal{L}$ as the set of Fredholm operators and $\partial\colon K_1(\mathcal{Q}) \to K_0(\mathcal{K})$ as the Fredholm index, T_f is a Fredholm operator of index 0. Thus $\dim\ker T_f = \dim\ker T_f^* = \dim\ker T_{\bar{f}}$. But one can show that $\ker T_f$ and $\ker T_{\bar{f}}$ can't both be non-trivial [17, Proposition 7.24], so T_f is invertible. $\square$

1.2 *K*-Theory of Banach Algebras

The connection between Fredholm operators and K-theory, which appeared to some extent in the above proof, first appeared in [32]. This marked the beginning of formal connections between operator theory and K-theory. About the same time, Wood [69] noticed that topological K-theory can be defined for Banach algebras, in such a way that Bott periodicity holds, just as it does for topological K-theory of spaces. However, it took a while for specialists in Banach algebras to notice the possibilities that K-theory afforded for solving certain kinds of problems. Direct applications of K-theory to operator algebras did not surface until the early 70's, with publication of works like [58] and [10]. In the rest of this section, we will discuss a few of the other early connections between K-theory and problems in operator algebras, and in Section 2 which follows, we will discuss some of the motivation for studying the comparison map between algebraic and topological K-theory for Banach algebras.

In [58] and [59], Taylor began to consider direct applications of K-theory of Banach algebras to problems in harmonic analysis. Part of the motivation was to give new proofs of results like the Cohen idempotent theorem (which says that the idempotent finite measures on a locally compact abelian group are generated by those of the form $\chi(h)\,dh$, with H a compact subgroup, dh its Haar measure, and χ a character on H). One of the things he found was:

Theorem 1.2 (Taylor). *If A is a unital commutative Banach algebra and if X is its maximal ideal space, then the Gelfand transform $A \to C(X)$ induces an isomorphism on topological K-theory.*

An immediate corollary is that topological K-theory vanishes for the radical of A (the intersection of all the maximal ideals), and thus for purposes of studying topological K-theory, it is no loss of generality to assume that A is semisimple, or even that A is a C^*-algebra. The corresponding result for algebraic K_1 is easily seen to be false, however. (Just consider the algebra of dual numbers, $\mathbb{C}[x]/(x^2)$.)

1.3 Essentially Normal Operators

At about the same time, interest in K-theory for C^*-algebras began to explode, thanks to the work of Brown, Douglas, and Fillmore ("BDF" [6], [7]) on extensions of C^*-algebras, followed quickly by the work of Kasparov on "operator K-homology" ([39], [40]). The BDF work grew out of the study of a rather concrete problem in operator theory: classification of *essentially normal operators*, bounded operators T on an infinite-dimensional separable Hilbert space $\mathcal{H}$ for which $T^*T - TT^*$ is compact. Given such an operator, 1, T, T^*, and $\mathcal{K}$ (the algebra of compact operators) generate a C^*-algebra $E \subset \mathcal{L}$ containing $\mathcal{K}$ as an ideal and with $E/\mathcal{K} = A$ a unital commutative C^*-algebra, hence with $A \cong C(X)$, where X is the "essential spectrum" of T. Thus T defines an extension of C^*-algebras

$$0 \to \mathcal{K} \to E \xrightarrow{q} C(X) \to 0. \tag{3}$$

The similarity with (1) is not an accident; in fact, the Toeplitz extension is the special case where $\mathcal{H} = H^2$ and T is the Toeplitz operator T_z. The original problem was to determine when T can be written in the form $N + K$ with N normal (i.e., $N^*N = NN^*$) and K compact. (Clearly any operator T of the form $N + K$ satisfies the original condition $T^*T - TT^* \in \mathcal{K}$.) If we can write $T = N + K$ in this fashion, then the map $q(T) \mapsto N$ defines a splitting of the exact sequence (3) (assuming we choose N so that its spectrum is no larger than the essential spectrum of T). So classification of essentially normal operators comes down to classification of C^*-algebra extensions by $\mathcal{K}$, modulo split extensions. This was the motivation for the BDF project.

The important discovery in the BDF work was that extensions of the form (3) (modulo split extensions, in some sense) can be made into an abelian group $\mathrm{Ext}(X)$, and that Ext is part of a homology theory which is dual to (topological) K-theory. The addition operation on extensions makes use of the fact that $M_2(\mathcal{K}) \cong \mathcal{K}$. Given two such extensions E_1 and E_2, then

$$E_1 \oplus_A E_2 =_{\text{def}} \{(e_1, e_2) \in E_1 \oplus E_2 : e_1 \equiv e_2 \mod \mathcal{K}\}$$

is an extension of A by $\mathcal{K}\oplus\mathcal{K}$, and if we add to $E_1 \oplus_A E_2 \subset \mathcal{L}\oplus\mathcal{L} \subset M_2(\mathcal{L}) \cong \mathcal{L}$ the ideal $M_2(\mathcal{K}) \cong \mathcal{K}$, we get an extension of A by $\mathcal{K}$. In fact, Ext extends to a contravariant functor on a the category of separable nuclear C^*-algebras (where we replace $A = C(X)$ by more general C^*-algebras) — the duality with K-theory comes from the fact that the long exact K-theory sequence of (3) gives a homomorphism $\partial\colon\ K_1(A) \to K_0(\mathcal{K}) = \mathbb{Z}$ just as in the above proof of Theorem 1.1. And should this “primary obstruction” to splitting of (3) vanish, there is a secondary obstruction that comes from the exact sequence

$$0 \to K_0(\mathcal{K}) = \mathbb{Z} \to K_0(E) \xrightarrow{q_*} K_0(A) \to K_{-1}(\mathcal{K}) = 0\,,$$

which defines an element of $\mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z})$. In fact, Brown showed [10] that these invariants give rise to a “universal coefficient theorem” (UCT) exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}\big(K^0(X), \mathbb{Z}\big) \to \mathrm{Ext}(X) \to \mathrm{Hom}_{\mathbb{Z}}\big(K^{-1}(X), \mathbb{Z}\big) \to 0.$$

1.4 Smooth Extensions and K_2

A bounded operator T on a Hilbert space $\mathcal{H}$ is said to be of *determinant class* if $T - 1$ belongs to the ideal $\mathcal{L}^1 \subseteq \mathcal{L}(\mathcal{H})$ of trace-class operators. There is a well-defined notion of determinant for operators of determinant class. As expected, it is defined to be 0 if T is not invertible. If T is invertible, then one can show that $T = \exp(S)$ for some trace-class operator S, and we define $\det T = \det(\exp(S))$ to be $e^{\mathrm{Tr}(S)}$, according to the usual relationship between the trace and the determinant. (One needs to check that this is independent of the choice of S.) The determinant defined this way is multiplicative (on operators of determinant class); in fact it defines a homomorphism $\det\colon\ K_1(\mathcal{L}, \mathcal{L}^1) \to \mathbb{C}^\times$. Using this notion of determinant, Helton and Howe [27] defined an interesting invariant for a special subclass of the essentially normal operators. It was then shown by Brown ([9], [10]) that this invariant can be viewed as having something to do with algebraic K_2. The idea is this. Suppose one has an extension of the form (3), and suppose X is a smooth manifold (possibly with boundary). Inside E, which is an extension of $C(X)$ by $\mathcal{K}$, suppose one has a subalgebra $\mathfrak{A}$ which is an extension

$$0 \to \mathcal{L}^1 \to \mathfrak{A} \xrightarrow{q} C^\infty(X) \to 0. \tag{4}$$

of $C^\infty(X)$ by $\mathcal{L}^1$, the trace-class operators. Thus operators T in $\mathfrak{A}$ are not only essentially normal; they have trace-class self-commutators (i.e., $T^*T - TT^* \in \mathcal{L}^1$). Suppose T and S are two invertible operators in $\mathfrak{A}$. Then the images modulo $\mathcal{L}^1$ of T, T^*, S, and S^* commute, and so the multiplicative commutator $TST^{-1}S^{-1}$ is 1 modulo $\mathcal{L}^1$, and so is of determinant class. In particular, $\det(TST^{-1}S^{-1})$ is defined. Brown noticed that

$$\det(TST^{-1}S^{-1}) = \det \circ \partial\left(\{q(T),\, q(S)\}\right),$$

where $\partial\colon\ K_2(C^\infty(X)) \to K_1(\mathfrak{A}, \mathcal{L}^1)$ is the connecting map in the long exact K-theory sequence of (4), we view det as a function on $K_1(\mathfrak{A}, \mathcal{L}^1)$ via the natural map $K_1(\mathfrak{A}, \mathcal{L}^1) \to K_1(\mathcal{L}, \mathcal{L}^1)$, and $\{q(T), q(S)\} \in K_2(C^\infty(X))$ is the Steinberg symbol of the functions $q(T)$ and $q(S)$. In particular, one obtains the relation $\det(TST^{-1}S^{-1}) = 1$ when the symbols satisfy $q(T) + q(S) = 1$, which is not at all obvious from the operator-theoretic point of view.

1.5 Multiplicative Commutators

Algebraic K_1 and K_2 are also related to a number of other problems about multiplicative commutators in various operator algebras. For example, one has:

Theorem 1.3 (Brown and Schochet [8]). *$K_1(\mathcal{L}, \mathcal{K}) = 0$.*

This is proved by showing explicitly that every invertible operator $\equiv 1 \mod \mathcal{K}$ is a product of a finite number of (multiplicative) commutators of such operators. Thus there is a huge difference between the algebraic K-theory of $\mathcal{K}$ and that of $\mathcal{L}^1$. (Recall that we have the determinant map $\det\colon\ K_1(\mathfrak{A}, \mathcal{L}^1) \to \mathbb{C}^\times$, which is surjective.) Brown and Schochet also remark [8, Remark 3] that their methods also show that $K_1(\widetilde{\mathcal{K}}, \mathcal{K}) = 0$, with $\widetilde{\mathcal{K}} = \mathcal{K} + \mathbb{C} \cdot 1$ the algebra obtained by adjoining a unit to $\mathcal{K}$. (The two statements are not the same since K_1 does not in general satisfy the excision property.) A subsequent paper [11], using refinements of the same techniques, showed that the group of invertible operators in $\mathcal{L}$ which are $\equiv 1 \mod \mathcal{K}$ is perfect, with all even cohomology groups nontrivial. These groups are of course related by the Hurewicz homomorphism to the higher algebraic K-theory $K_*(\mathcal{L}, \mathcal{K})$ (about which we will say more later). A related later paper by de la Harpe and Skandalis [16] showed that if A is a *stable* C^*-algebra, i.e., if $A \cong A \otimes \mathcal{K}$,[3] then the connected component of the identity in the group of invertible operators of the form $1 + a$, $a \in \mathcal{K}$, is always perfect.

1.6 AF Algebras and Dimension Groups

One other important source for interest in K-theory of operator algebras comes from the study of so-called AF algebras, or C^*-algebra inductive limits of finite-dimensional semisimple algebras over $\mathbb{C}$. (The abbreviation AF stands for "approximately finite-dimensional.") Such algebras were first introduced by Bratteli [4], who showed how to classify them by means of equivalence classes of certain combinatorial constructs now called "Bratteli diagrams." However, this method of classification was almost uncomputable. A major breakthrough came a few years later when Elliott [19] showed that AF algebras are classified by their K_0 groups, together with the natural ordering on K_0 induced by the monoid of finitely generated projective modules, and in the unital case, the "order unit" corresponding to the rank-one free module. (The invariant consisting of K_0 and this extra order structure is often called the *dimension group*.) This classification theorem was made even more satisfying by a subsequent paper of Effros, Handelman, and Shen [18], which gave an abstract characterization of the possible dimension groups of AF algebras — they are exactly the unperforated ordered abelian groups satisfying the Riesz interpolation property. There has been much subsequent literature on classification of various classes of C^*-algebras via topological K-theory and the order structure on it, but we do not go into this here.

2 "Lie Groups Made Discrete" and Early Explorations

Topological K-theory, first introduced for compact spaces by Atiyah and Hirzebruch, was extended to Banach algebras as early as the work of Wood [69] in the mid-60's. As higher algebraic K-theory began to be developed in the 1970's, the question arose of trying to understand the similarities and

[3] Here $A \otimes \mathcal{K}$ is the C^*-algebra completion of the algebraic tensor product $A \odot \mathcal{K}$. For general C^*-algebras A and B, there can be more than one C^*-algebra completion of $A \odot B$, but there is always a maximal one $A \otimes_{\max} B$, defined by completing $A \odot B$ in the norm

$$\left\|\textstyle\sum_{i=1}^n a_i \otimes b_i\right\|_{\max} = \sup \left\{ \left\|\textstyle\sum_{i=1}^n \rho_1(a_i)\rho_2(b_i)\right\| : \rho_1 \text{ and } \rho_2 \text{ commuting representations of } A \text{ and } B \right\},$$

as well as a minimal one $A \otimes_{\min} B$, the completion of $A \odot B \subset \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ when A is represented on a Hilbert space $\mathcal{H}_1$ and B is represented on a Hilbert space $\mathcal{H}_2$. (One can show this is independent of the choices of faithful representations of A and B.) But if one of the two algebras is *nuclear*, and in particular if B is commutative or $B = \mathcal{K}$, all completions coincide.

differences between the two theories in the cases where both of them made sense. These explorations eventually went off in two different directions, with a certain overlap between them. The first of the directions had to do with relating purely algebraic and topological or "quasi-topological" K-theories for algebraic varieties, especially over $\mathbb{C}$. This subject is intimately connected with the Riemann-Roch problem (see [3] and [60], for example) and led to the development of semi-topological K-theory (see [22]). This line of development will *not* be the primary theme of this article, but the interested reader should consult the chapter by Friedlander and Walker for a treatment of at least some of this topic. Instead we will discuss another thread in the subject, of relating algebraic and topological K-theory for Banach algebras in general and for C^*-algebras in particular. This subject is also related to the use of algebraic K-theory as a language for discussing certain problems in operator theory.

2.1 Basic Concepts and Notations

In order to make it possible to give precise statements for all results, we begin by establishing some definitions and notation. The definitions here do not always coincide with those in use when the results were first established, but we have translated everything into terms consistent with these "modern" definitions.

First we need to make precise exactly what we mean by algebraic and topological K-theory for Banach algebras. Let A be a Banach algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. (The Banach norm $\|\cdot\|$ on A is implicit.) For the moment we assume A is unital, though it will be necessary from time to time to talk about non-unital Banach algebras as well. (Just as an example, stable C^*-algebras, which already appeared in Section 1.5 above, are necessarily non-unital.) By $K_n(A)$ we will mean the usual (Quillen) algebraic K-groups of A for $n \geq 0$. However, since the topological K-groups $K_n^{\mathrm{top}}(A)$ are periodic in n (with period 2 if $\mathbb{F} = \mathbb{C}$, period 8 if $\mathbb{F} = \mathbb{R}$), and since we want to compare $K_n(A)$ with $K_n^{\mathrm{top}}(A)$, it is also necessary to have a good definition of $K_n(A)$ for $n < 0$. Accordingly, we let $\mathbb{K}(A)$ be the non-connective delooping of the algebraic K-theory spectrum of A, as defined in [24] and [64], and let $K_n(A)$ denote the n-th homotopy group of $\mathbb{K}(A)$, whether or not n is positive. The groups $K_n(A)$ for $n < 0$ then agree with the "Bass negative K-groups" defined in [36] or [2], and in fact all the standard constructions of deloopings of the algebraic K-theory spectrum are known to be naturally equivalent [44, §§5–6].

By the same token, we let $\mathbb{K}^{\mathrm{top}}(A)$ be the topological K-theory spectrum of A. This is an Ω-spectrum in which every second (or eighth, depending on whether $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$) space is $GL(A)$, the infinite general linear group of A, with the Hausdorff group topology defined by the norm on A (*not* the discrete topology on $GL(A)$, which we'll denote by $GL(A)^\delta$, used to define $\mathbb{K}(A)$). More specifically, when $\mathbb{F} = \mathbb{C}$, $\mathbb{K}^{\mathrm{top}}(A)$ is given by the homotopy equivalences

$$K_0(A) \times BGL(A) \xrightarrow{\simeq} \Omega GL(A),$$
$$GL(A) \xrightarrow{\simeq} \Omega BGL(A) = \Omega\,(K_0(A) \times BGL(A))$$

of the Bott Periodicity Theorem [69], and by similar maps when $\mathbb{F} = \mathbb{R}$. The homotopy groups $K_n^{\mathrm{top}}(A)$ of $\mathbb{K}^{\mathrm{top}}(A)$ are thus periodic in n (with period 2 if $\mathbb{F} = \mathbb{C}$, period 8 if $\mathbb{F} = \mathbb{R}$).

Basic to what follows is [49, Theorem 1.1]:

Theorem 2.1. *Let A be a Banach algebra* (*over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$*). *There is a functorial "comparison map" of spectra $c\colon \mathbb{K}(A) \to \mathbb{K}^{\mathrm{top}}(A)$ induced by the "change of topology" map $GL(A)^\delta \to GL(A)$. The induced map $c_*\colon K_0(A) \to K_0^{\mathrm{top}}(A)$ is the identity, and the induced map $c_*\colon K_1(A) \to K_1^{\mathrm{top}}(A)$ is the quotient map $GL(A)/E(A) \to GL(A)/GL(A)_0$.* (*Here $E(A)$ is the group generated by the elementary matrices, and $GL(A)_0 \supseteq E(A)$ is the identity component of $GL(A)$.*)

Recall also that $\mathbb{K}(A)$ is a $\mathbb{K}(\mathbb{F})$-module spectrum and that $\mathbb{K}^{\mathrm{top}}(A)$ is a $\mathbb{K}^{\mathrm{top}}(\mathbb{F})$-module spectrum. The map c is compatible with the product structures, in that the diagram

$$\begin{tikzcd}
\mathbb{K}(\mathbb{F}) \times \mathbb{K}(A) \arrow[r, "\mu"] \arrow[d, "(c_{\mathbb{F}}, c_A)"] & \mathbb{K}(A) \arrow[d, "c"] \\
\mathbb{K}^{\mathrm{top}}(\mathbb{F}) \times \mathbb{K}^{\mathrm{top}}(A) \arrow[r, "\mu_{\mathrm{top}}"] & \mathbb{K}^{\mathrm{top}}(A),
\end{tikzcd}$$

μ denoting the multiplication maps, is homotopy commutative.

Proof (*Sketch*). The "change of topology" map of topological groups

$$GL(A)^\delta \to GL(A)$$

induces a map of classifying spaces $BGL(A)^\delta \to BGL(A)$. Apply the Quillen +-construction. Since $BGL(A)$ is already an H-space, this does nothing to $BGL(A)$, and we get a map $\left(BGL(A)^\delta\right)^+ \to BGL(A)$ and thus a map $K_0(A) \times \left(BGL(A)^\delta\right)^+ \to K_0(A) \times BGL(A)$. This is an infinite loop space map, and induces a map c of connective K-theory spectra $\mathbb{K}(A)\langle 0\rangle \to \mathbb{K}^{\mathrm{top}}(A)\langle 0\rangle$ with the desired properties. So it's only necessary to deloop it. This could be done using the Pedersen-Weibel construction in [44], or we can do it inductively, one step at a time, as follows. The single delooping of $K_0(A) \times \left(BGL(A)^\delta\right)^+$, which on the spectrum level we'll denote by $\Sigma\big(\mathbb{K}(A)\langle -1\rangle\big)$, is a direct summand in the K-theory space of the Laurent polynomial ring $A[t, t^{-1}]$, i.e., in $K_0(A[t, t^{-1}]) \times \left(BGL(A[t, t^{-1}])^\delta\right)^+$. Now by Stone-Weierstraß, $A[t, t^{-1}]$ is a dense subalgebra of the Banach algebra $C(S^1, A)$ (in the complex case), or of

$$\{f \in C(S^1, A_{\mathbb{C}}) : f(z^{-1}) = \overline{f(z)}\}$$

in the real case. (Note this is *not* the same as the algebra of real-valued continuous functions $S^1 \to A$, since the Laurent polynomial variable t should be identified with the complex variable z on the unit circle in the complex plane, and $z^{-1} = \bar{z}$.) Let us denote the completion of $A[t, t^{-1}]$ in both cases by ΣA, and call it the "suspension" of A. As before we have a map of spectra $\mathbb{K}(\Sigma A)\langle 0\rangle \to \mathbb{K}^{\mathrm{top}}(\Sigma A)\langle 0\rangle$. However, by the Fundamental Theorem of K-theory,

$$\mathbb{K}(A[t, t^{-1}]) \simeq \mathbb{K}(A) \oplus \Sigma\mathbb{K}(A) \oplus \text{Nil terms},$$

and similarly $\mathbb{K}^{\mathrm{top}}(\Sigma A) \simeq \mathbb{K}^{\mathrm{top}}(A) \oplus \Sigma\mathbb{K}^{\mathrm{top}}(A)$ by Bott periodicity (for KR in the real case). We thus obtain a commutative diagram of spectra

$$\begin{tikzcd}
\Sigma\big(\mathbb{K}(A)\langle -1\rangle\big) \arrow[rr, "\Sigma(c \text{ delooped})"] \arrow[d, hook, dashed] & & \Sigma\big(\mathbb{K}^{\mathrm{top}}(A)\langle -1\rangle\big) \arrow[d, hook, dashed] \\
\mathbb{K}(A[t, t^{-1}])\langle 0\rangle \arrow[r] & \mathbb{K}(\Sigma A)\langle 0\rangle \arrow[r, "c_{\Sigma A}"] & \mathbb{K}^{\mathrm{top}}(\Sigma A)\langle 0\rangle,
\end{tikzcd}$$

with the vertical dotted arrows split inclusions, which gives the inductive step.

The compatibility of the map c with products follows from the way the products are defined. The product in topological K-theory comes from a group homomorphism $\mu_{\mathrm{top}}\colon GL(\mathbb{F}) \times GL(A) \to GL(A)$ (see for example [48, Theorem 5.3.1, pp. 280–281], and the product in algebraic K-theory comes from a map $\mu\colon GL(\mathbb{F}) \times GL(A) \to GL(A)$ defined by exactly the same formula, so clearly the diagram

$$\begin{tikzcd}
GL(\mathbb{F})^{\delta} \times GL(A)^{\delta} \arrow[r, "\mu"] \arrow[d] & GL(A)^{\delta} \arrow[d] \\
GL(\mathbb{F}) \times GL(A) \arrow[r, "\mu_{\mathrm{top}}"] & GL(A),
\end{tikzcd}$$

commutes. So apply the classifying space functor, the plus construction, etc. □

Now we can formulate the basic problems to be studied in this article:

Problems 2.2.

1. How close is the map $c\colon \mathbb{K}(A) \to \mathbb{K}^{\mathrm{top}}(A)$ to being an equivalence?
2. When c is far from being an equivalence, can we still say anything intelligent about $\mathbb{K}(A)$?

We will sometimes consider K-theory with coefficients. With A as before, $\mathbb{K}(A; \mathbb{Z}/n)$, the algebraic K-theory spectrum with coefficients in $\mathbb{Z}/n$, is obtained by smashing $\mathbb{K}(A)$ with the mod n Moore spectrum (the cofiber of the map $\mathbb{S} \xrightarrow{n} \mathbb{S}$ of degree n, where $\mathbb{S}$ is the sphere spectrum). This definition agrees in positive degrees with, but is not precisely identical to, the (older) definition of mod n K-theory in [5].

2.2 Direct Calculation in the Abelian Case

In considering Problems 2.2(1–2), one must certainly begin with the case of the simplest Banach algebras, namely the archimedean local fields $\mathbb{R}$ and $\mathbb{C}$, and after that with commutative Banach algebras. Taylor's Theorem 1.2 shows that the study of the commutative case reduces to the study of the algebras of continuous functions, $C^{\mathbb{R}}(X)$ and $C^{\mathbb{C}}(X)$. Already in [43, §7], Milnor did a direct analysis of these cases in low dimensions, and found:

Theorem 2.3. *Let X be a compact Hausdorff space, let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $A = C^{\mathbb{F}}(X)$. Then the map $c_*\colon\ K_j(A) \to K_j^{\mathrm{top}}(A)$ is surjective for $j = 1$, with kernel $C(X, \mathbb{F}_0^\times)$, the continuous functions from X to the identity component of $\mathbb{F}^\times$. If $\mathbb{F} = \mathbb{R}$, since $\mathbb{R}_0^\times = \mathbb{R}_+^\times$ is contractible,*

$$\exp\colon\ C^{\mathbb{R}}(X) \xrightarrow{\cong} C(X, \mathbb{R}_0^\times),$$

while if $\mathbb{F} = \mathbb{C}$, since $\mathbb{C}^\times$ has the homotopy type of a circle,

$$\exp\colon\ C^{\mathbb{C}}(X) \twoheadrightarrow C(X, \mathbb{C}^\times)$$

with kernel $C(X,\mathbb{Z}) = C^0(X,\mathbb{Z})$ (Čech cohomology). Furthermore, c_* is surjective also for $j = 2$.

This shows in particular that c_* can have a huge kernel when $j = 1$, since $C^{\mathbb{R}}(X)$ is always a $\mathbb{Q}$-vector space of uncountable dimension. It is also true that c_* can have a huge kernel when $j = 2$, since for example by [43, Theorem 11.10], $K_2(\mathbb{R})$ and $K_2(\mathbb{C})$ must be uncountable, while on the other hand $K_2^{\mathrm{top}}(\mathbb{R}) = \mathbb{Z}/2$ and $K_2^{\mathrm{top}}(\mathbb{C}) = \mathbb{Z}$. So in general we cannot expect c_* to be close to an isomorphism, and we can already see that the presence of large uniquely divisible groups is part of the explanation. This suggests that examining c_* with finite coefficients might be more valuable.

2.3 "Lie Groups Made Discrete" and Suslin's Theorems on $K_*(\mathbb{R})$, $K_*(\mathbb{C})$

The algebraic K-theory of $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is more accessible than that of general Banach algebras, since it can be obtained from applying the Quillen $+$-construction to $BGL(\mathbb{F})^\delta$, and $GL(\mathbb{F})$ is an inductive limit of Lie groups. Thus understanding $\mathbb{K}(\mathbb{F};\, \mathbb{Z}/n)$ is related to understanding the group homology with finite coefficients of "Lie groups made discrete." This was studied by Friedlander (as early as the mid-1970's) and Friedlander-Mislin (see, e.g., [21]), using the machinery of étale homotopy theory, and by Milnor [42].

The most optimistic possible conjecture is that for any Lie group G, the natural map $BG^\delta \to BG$ is a homology isomorphism with finite coefficients. As Milnor shows in [42], this is indeed the case for solvable Lie groups. Milnor also proves that for G *any* Lie group with finitely many components, the map $H^*(BG; \mathbb{Z}/n) \to H^*(BG^\delta; \mathbb{Z}/n)$ is split injective.[4]

Around the same time as Milnor's work, Suslin began to investigate $\mathbb{K}(\mathbb{F}; \mathbb{Z}/n)$ (for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, as well as for more general local or algebraically closed fields) by using completely different techniques coming from algebraic geometry. We quickly summarize his remarkable results.

Theorem 2.4 (Suslin [52]). *If $F \hookrightarrow L$ is an extension of algebraically closed fields, then for any positive integer n, the induced map $\mathbb{K}(F; \mathbb{Z}/n) \to \mathbb{K}(L; \mathbb{Z}/n)$ is an equivalence.*

Comments on the proof. Suslin begins by observing that $L = \varinjlim A$, where A runs over the finitely generated F-subalgebras of L. Since F is algebraically closed, the Nullstellensatz implies that for any such A, the map $F \hookrightarrow A$ has an F-linear splitting, and in particular, $K_*(F; \mathbb{Z}/n) \to K_*(A; \mathbb{Z}/n)$ is split injective. Thus $K_*(F; \mathbb{Z}/n) \to K_*(L; \mathbb{Z}/n)$ is injective. However, this is the "trivial" part of the proof, as it would have applied just as well to the integral K-groups.

The finite coefficients are used (though the divisibility of $L^\times$ and of $\mathrm{Pic}^0(C)$, C a smooth curve over L) in the course of proving the rigidity theorem 2.5 below. This is then applied with A a smooth finitely generated F-subalgebra of L, $h_0\colon A \to L$ the inclusion, and $h_1\colon A \to L$ factoring through a an F-algebra homomorphism $A \to F$. Passage to the limit over all such A's gives the surjectivity of $K_*(F; \mathbb{Z}/n) \to K_*(A; \mathbb{Z}/n)$. ☐

The proof is completed with:

Theorem 2.5 (Suslin rigidity theorem [52]). *If $F \hookrightarrow L$ is an extension of algebraically closed fields, if A is a smooth affine F-algebra without zero-divisors, and if $h_0, h_1\colon A \to L$ are two F-homomorphisms, then for any positive integer n, $(h_0)_* \simeq (h_1)_*$ as maps $\mathbb{K}_*(A; \mathbb{Z}/n) \to \mathbb{K}_*(L; \mathbb{Z}/n)$.*

Theorem 2.4 implies:

Corollary 2.6. *If F is an algebraically closed field of characteristic 0, then $\mathbb{K}(F; \mathbb{Z}/n) \simeq \mathbb{K}(\mathbb{C}; \mathbb{Z}/n)$. And if F is an algebraically closed field of characteristic $p > 0$, then for $(n, p) = 1$, $K_i(F; \mathbb{Z}/n) \cong K_i^{\mathrm{top}}(\mathbb{C}; \mathbb{Z}/n)$.*

Proof. Theorem 2.4 implies that the homotopy type of $\mathbb{K}(F; \mathbb{Z}/n)$ is the same as for $F = \overline{\mathbb{Q}}$ (in the characteristic 0 case) or for $F = \overline{\mathbb{F}_p}$ (in the characteristic p case). The first statement follows from Theorem 2.4 applied to $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$; the second follows from Quillen's calculation [46] of the homotopy type of $\mathbb{K}(\mathbb{F}_q)$. ☐

More relevant for our purposes is:

[4] One might even hope that injectivity would be true for more general locally compact groups, but this cannot even be the case for general profinite groups, as demonstrated in [50].

Theorem 2.7 (Suslin [54]). *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then the comparison map c of Theorem 2.1 induces isomorphisms $c_*\colon K_j(\mathbb{F}; \mathbb{Z}/n) \xrightarrow{\cong} K_j^{\mathrm{top}}(\mathbb{F}; \mathbb{Z}/n)$ for all positive integers n and for all $j \geq 0$. We can rephrase this by saying that c induces an equivalence of spectra $\mathbb{K}(\mathbb{F}; \mathbb{Z}/n) \xrightarrow{\simeq} \mathbb{K}^{\mathrm{top}}(\mathbb{F}; \mathbb{Z}/n)\langle 0\rangle$, where the spectrum on the right is the connective topological K-theory spectrum, often denoted $\mathbf{bu}(\mathbb{Z}/n)$ or $\mathbf{bo}(\mathbb{Z}/n)$.*

Comparison of this result with Corollary 2.6 yields the remarkable conclusion that for algebraically closed fields F, the homotopy type of $\mathbb{K}(F; \mathbb{Z}/n)$ is almost independent of F. (The only variations show up when n is a multiple of the characteristic.) However, this is taking us somewhat far afield, as our interest here is in Banach algebras. The proof of Theorem 2.7 follows a surprising detour; it depends on:

Theorem 2.8 (Gabber [23], Gillet-Thomason [25]). *Let A be a commutative ring in which the integer $n > 0$ is invertible, and let $I \lhd A$ be an ideal contained in the radical of A, such that the pair (A, I) is Henselian. (This means that the conclusion of Hensel's Lemma holds for the map $A \twoheadrightarrow A/I$, i.e., that if $f \in A[t]$ and if the reduction $\bar{f} \in (A/I)[t]$ of f mod I has a root $\bar{\alpha} \in A/I$ such that $\bar{f}'(\bar{\alpha})$ is a unit in A/I, then $\bar{\alpha}$ can be lifted to a root α of f in A.) Then $K_*(A, A/I; \mathbb{Z}/n) = 0$.*

Comments on the proof of Theorem 2.7. Theorem 2.8 has a fairly obvious application to the computation of $K_*(\mathbb{Q}_p; \mathbb{Z}/n)$ or of mod n K-theory of other non-archimedean local fields F, since if $\mathcal{O}$ is the ring of integers in F and $\mathfrak{p}$ is its maximal ideal, then $(\mathcal{O}, \mathfrak{p})$ is Henselian, but the most ingenious part of [54] is the development of a trick for handling the case of the archimedean fields $\mathbb{R}$ and $\mathbb{C}$.

First there is a relatively straightforward reduction of the problem to proving that the identity map $BSL_k(\mathbb{F})^\delta \to BSL_k(\mathbb{F})$ induces an isomorphism on mod n homology in a range of dimensions (depending on k but increasing to infinity as $k \to \infty$). But since $G_k = SL_k(\mathbb{F})$ is a Lie group, it turns out that there is a good model for the fiber of the map $BG_k^\delta \to BG_k$, which Suslin denotes $(BG_k)_\varepsilon$, obtained by fixing a left-invariant Riemannian metric on G_k and choosing ε small enough so that if U_ε denotes the open ε-ball around the identity e of G_k, then there is a unique geodesic arc joining any two points in U_ε. This guarantees that any intersection of left translates of U_ε, if non-empty, is contractible. One then takes $(BG_k)_\varepsilon$ to be the geometric realization of the simplicial set whose m-simplices are m-tuples $[g_1, \dots, g_m]$ such that $U_\varepsilon \cap g_1 U_\varepsilon \cap \ldots \cap g_m U_\varepsilon \neq \emptyset$.

Now because of the Serre spectral sequence of the fibration

$$(BG_k)_\varepsilon \to BG_k^\delta \to BG_k$$

as well as Milnor's results, it turns out it suffices to prove that the natural map $(BG_k)_\varepsilon \to BG_k$ induces the zero map on mod n homology. To prove this, one similarly translates Theorem 2.8 into a statement about mod n homology, namely that the map $BGL_k(R,I) \to BGL(R,I)$ induces the zero map on mod n homology in the limit as $k \to \infty$. This is then used in a strange way — we take R to be the local ring of germs of $\mathbb{F}$-valued continuous functions on $\overbrace{G_k \times \cdots \times G_k}^{j}$ near $(e,\ldots,e)$, and I to be its maximal ideal of functions vanishing at $(e,\ldots,e)$. Disentangling everything turns out to give the result one needs in degree j, since j-chains on $(BG_k)_\varepsilon$ (where one can pass to the limit as $\varepsilon \to 0$) are basically elements of R.

One can also find an exposition of the proof in [51]. $\square$

2.4 Karoubi's Early Work on Algebraic *K*-Theory of Operator Algebras

The first substantial work on Problems 2.2 for infinite-dimensional Banach algebras, aside from the few special results already mentioned, was undertaken by Karoubi. In this subsection we summarize some of the results in two important papers of Karoubi, [37] and [38]. In all of this section, all Banach and C^*-algebras will be over $\mathbb{C}$, not $\mathbb{R}$.

In the category of C^*-algebras, it is rather artificial to restrict attention to unital algebras, so at this point it's necessary to say something about algebraic K-theory for *non-unital* algebras (over a field of characteristic zero). The problem is that algebraic K-theory does not in general satisfy excision , so that the algebraic K-theory of a non-unital algebra A should be interpreted as the *relative* K-theory of a pair (B,A), where B is an algebra containing A as an ideal. When A is a nonunital C^*-algebra, there are two canonical choices for B, both of which are C^*-algebras: $\widetilde{A} = A + 1\cdot\mathbb{C}$, the algebra obtained by adjoining a unit to A, and $\mathcal{M}(A)$, the *multiplier algebra* of A. The latter, first introduced in [33] and [12], is the *largest* unital C^*-algebra containing A as an *essential* ideal, just as $\widetilde{A}$ is the smallest such C^*-algebra. For example, if X is a locally compact Hausdorff space and if $A = C_0(X)$, $\widetilde{A} = C(X_+)$ and $\mathcal{M}(A) = C(\beta X)$, where X_+ is the one-point compactification of X and βX is the Stone-Čech compactification of X. It turns out that $\mathcal{M}(\mathcal{K}) = \mathcal{L}$, the algebra of bounded operators on the same Hilbert space where $\mathcal{K}$ is the algebra of compact operators. Below, when we talk about the algebraic K-theory of $\mathcal{K}$, we will implicitly mean the K-theory of $(\mathcal{L},\mathcal{K})$. (Later on, in section 3.2, it will turn out it doesn't matter, and the pair $(\widetilde{\mathcal{K}},\mathcal{K})$ would give the same results.)

Karoubi noticed that the periodicity of $K^{\mathrm{top}}(\mathbb{C})$ can be attributed to two special elements, the Bott element $\beta \in K_2^{\mathrm{top}}(\mathbb{C})$ and the inverse Bott element $\beta^{-1} \in K_{-2}^{\mathrm{top}}(\mathbb{C})$. The class β, once we use finite coefficients, does lie in the image of the comparison map $K_2(\mathbb{C};\mathbb{Z}/n) \to K_2^{\mathrm{top}}(\mathbb{C};\mathbb{Z}/n)$ of Theorem 2.1. (This follows immediately from Theorem 2.7, but it can also be proved directly — see [37, Proposition 5.5].) However, β^{-1} *cannot* lie in the image of the comparison map, even with finite coefficients, since $\mathbb{C}$ is a regular ring and thus its negative K-groups vanish. However, Karoubi noticed that topological K-theory is the same for $\mathbb{C}$ and for the algebra $\mathcal{K}$ of compact operators. (More precisely, the *non-unital* homomorphism $\mathbb{C} \hookrightarrow \mathcal{K}$ sending 1 to a rank-one projection induces an isomorphism on topological K-theory. The excision property of topological K-theory implies functoriality for *non-unital* homomorphisms.) And there *is* an algebraic inverse Bott element in $K_{-2}(\mathcal{K})$ which maps to $\beta^{-1} \in K_{-2}^{\mathrm{top}}(\mathbb{C})$ under the composite

$$K_{-2}(\mathcal{K}) \xrightarrow{c_{\mathcal{K}}} K_{-2}^{\mathrm{top}}(\mathcal{K}) \xrightarrow{\cong} K_{-2}^{\mathrm{top}}(\mathbb{C}) \cong \mathbb{Z}.$$

Karoubi proves this using two simple observations. The first is:

Theorem 2.9 ([37, Théorème 3.6]). *If A is a C^*-algebra (with or without unit), the map $c\colon K_{-1}(A) \to K_{-1}^{\mathrm{top}}(A)$ is surjective.*

Sketch of proof [37, §III]. It suffices to consider the case where A has a unit (since if A is non-unital, $K_{-1}(A) \cong K_{-1}(\widetilde{A})$, where $\widetilde{A}$ is the C^*-algebra obtained by adjoining a unit to A). Recall that the Bass definition of $K_{-1}(A)$ is in terms of a direct summand in $K_0(A[t, t^{-1}])$, and that the Laurent polynomial ring $A[t, t^{-1}]$ embeds densely in $C(S^1, A)$. But $K_0(C(S^1, A)) \cong K_0(A) \oplus K_1^{\mathrm{top}}(A)$, and $K_1^{\mathrm{top}}(A) \cong K_{-1}^{\mathrm{top}}(A)$ by Bott periodicity. So we just need to show that the summand $K_{-1}(A)$ in $K_0(A[t, t^{-1}])$ surjects onto $K_1^{\mathrm{top}}(A)$ under the map induced by the inclusion $A[t, t^{-1}] \hookrightarrow C(S^1, A)$. Since elements of $K_1^{\mathrm{top}}(A)$ are represented by unitary matrices over A and we can always replace A by $M_r(A)$ for some r, it suffices to show that if $u \in A$ is unitary (i.e., u is invertible and $u^{-1} = u^*$), the corresponding class in $K_1^{\mathrm{top}}(A)$ lies in the image of $K_0(A[t, t^{-1}])$. Since the C^*-algebra generated by u is a quotient of $C(S^1)$ (since u is normal and has spectrum in the unit circle), under a $*$-homomorphism sending the standard generator z of $C(S^1)$ (the indentity map $S^1 \to S^1 \subset \mathbb{C}$, when we think of S^1 as the unit circle in the complex plane) to u, it suffices to deal with the case where $A = C(S^1)$ and we are considering the class $[z]$. Then we just need to show that the Bott element in $K_1^{\mathrm{top}}(C(S^1)) \cong K_0(C(T^2))$ lies in the image of $K_0(C(S^1)[t, t^{-1}])$. However, one can write the Bott element out in terms of a very explicit 2×2 matrix with entries that are functions of z and t that are Laurent polynomials in the t-variable (see [37, pp. 269–270]), so that does it. $\square$

Now we obtain the desired result on the inverse Bott element as follows:

Theorem 2.10 (Karoubi). *The comparison map $c\colon K_{-2}(\mathcal{K}) \to K_{-2}^{\mathrm{top}}(\mathcal{K})$ is surjective.*

Proof. Consider the exact sequence of C^*-algebras

$$0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{Q} = \mathcal{L}/\mathcal{K} \to 0,$$

where $\mathcal{Q}$ is the Calkin algebra. Since $\mathcal{L}$, the algebra of all bounded operators on a separable Hilbert space, is "flasque" by the "Eilenberg swindle" (all finitely generated projective $\mathcal{L}$-modules are stably isomorphic to 0), all its K-groups, whether topological or algebraic, vanish. So now consider the commutative diagram of exact sequences:

$$\begin{tikzcd}
0 = K_{-1}(\mathcal{L}) \arrow[r] \arrow[d, equal] & K_{-1}(\mathcal{Q}) \arrow[r] \arrow[d] & K_{-2}(\mathcal{K}) \arrow[r] \arrow[d] & 0 = K_{-2}(\mathcal{L}) \arrow[d, equal] \\
0 = K_{-1}^{\mathrm{top}}(\mathcal{L}) \arrow[r] & K_{-1}^{\mathrm{top}}(\mathcal{Q}) \arrow[r] & K_{-2}^{\mathrm{top}}(\mathcal{K}) \arrow[r] & 0 = K_{-2}^{\mathrm{top}}(\mathcal{L}),
\end{tikzcd}$$

where the surjectivity of the arrow $K_{-1}(\mathcal{Q}) \to K_{-1}^{\mathrm{top}}(\mathcal{Q})$ follows from Theorem 2.9. The result follows by diagram chasing. $\square$

In fact because of the multiplicative structure on K-theory one can do much better than this, and Karoubi managed to prove:

Theorem 2.11 (Karoubi). *The comparison map $c\colon K_*(\mathcal{K};\, \mathbb{Z}/n) \to K_*^{\mathrm{top}}(\mathcal{K};\, \mathbb{Z}/n)$ is an isomorphism (in all degrees), and the map $c\colon K_j(\mathcal{K}) \to K_j^{\mathrm{top}}(\mathcal{K})$ is surjective for all j and an isomorphism for $j \le 0$.*

Proof. The first step is to prove the statement about K-theory with finite coefficients. Choose $\gamma \in K_{-2}(\mathcal{K})$ mapping to $\beta^{-1} \in K_{-2}^{\mathrm{top}}(\mathcal{K})$; this is possible by Theorem 2.10. Let β_n be the mod n Bott element in $K_2(\mathbb{C};\, \mathbb{Z}/n)$. (Recall Suslin's Theorem 2.7.) Then the cup-product $\beta_n \cdot \gamma \in K_0(\mathcal{K};\, \mathbb{Z}/n) \cong \mathbb{Z}/n$ maps to $\beta \cdot \beta^{-1} = 1 \in K_0^{\mathrm{top}}(\mathcal{K};\, \mathbb{Z}/n) \cong \mathbb{Z}/n$ (by the last part of Theorem 2.1, the compatibility with products), and so is 1. So the product with γ is inverse to the product with β_n on $K_*(\mathcal{K};\, \mathbb{Z}/n)$, and so $K_*(\mathcal{K};\, \mathbb{Z}/n)$ is Bott-periodic and canonically isomorphic to $K_*^{\mathrm{top}}(\mathcal{K};\, \mathbb{Z}/n) = \mathbb{Z}/n[\beta,\, \beta^{-1}]$.

Now we lift the mod n result to an integral result for K_2. Recall that by Theorem 1.3, $K_1(\mathcal{K}) = 0$. Because of this fact and the above result on mod n K-theory, we have the commuting diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & K_2(\mathcal{K}) \arrow[r, "n"] \arrow[d, "c"] & K_2(\mathcal{K}) \arrow[r] \arrow[d, "c"] & K_2(\mathcal{K};\, \mathbb{Z}/n) \cong \mathbb{Z}/n \arrow[r] \arrow[d, "c", "\cong"'] & K_1(\mathcal{K}) = 0 \arrow[d, equal] \\
0 \arrow[r] & K_2^{\mathrm{top}}(\mathcal{K}) \cong \mathbb{Z} \arrow[r, "n"] & K_2^{\mathrm{top}}(\mathcal{K}) \cong \mathbb{Z} \arrow[r] & K^{\mathrm{top}}(\mathcal{K};\, \mathbb{Z}/n) \cong \mathbb{Z}/n \arrow[r] & K_1^{\mathrm{top}}(\mathcal{K}) = 0.
\end{tikzcd}$$

From this it follows that the comparison map $c\colon\ K_2(\mathcal{K}) \to K_2^{\mathrm{top}}(\mathcal{K}) \cong \mathbb{Z}$ hits a generator mod n for each n, and thus this map is integrally surjective.

Hence we can choose an algebraic Bott element $\delta \in K_2(\mathcal{K})$ mapping to $\beta \in K_2^{\mathrm{top}}(\mathcal{K})$. We could then deduce that multiplication by γ is inverse to multiplication by δ, and thus that the algebraic K-theory of $\mathcal{K}$ is Bott-periodic and canonically isomorphic to the topological K-theory, provided we had a good cup-product structure on K-theory for non-commutative rings. Unfortunately there is a problem with this that comes from failure of excision in algebraic K-theory in positive degrees. This is exactly why Karoubi can only conclude that $c\colon K_j(\mathcal{K}) \to K_j^{\mathrm{top}}(\mathcal{K})$ is surjective for all j and an isomorphism for $j \leq 0$. $\square$

The above result on the K-theory of $\mathcal{K}$ (or rather, Karoubi's first partial results in this direction, since the paper [37] predated Theorem 2.11) motivated a rather audacious conjecture in [37] about the K-theory of stable C^*-algebras, which came to be known as the *Karoubi Conjecture*.

Conjecture 2.12 (Karoubi Conjecture [37]). *For any stable C^*-algebra A, the comparison map $c\colon \mathbb{K}(A) \to \mathbb{K}^{\mathrm{top}}(A)$ is an equivalence.*

The original formulation of this conjecture in Karoubi's paper seems a bit vague about what definition of algebraic K-theory should be used here for non-unital algebras. Fortunately we shall see later (section 3.2) that all possible definitions coincide. In fact it would appear that Karoubi wants to work with $K_*(A \otimes \mathcal{L}, A \otimes \mathcal{K})$, which presents a problem since the minimal C^*-algebra tensor product is not an exact functor in general. Fortunately all the difficulties resolve themselves *a posteriori*.

It is also worth mentioning that Karoubi's paper [38] deals not only with C^*-algebras, but also with Banach algebras, especially the Schatten ideals $\mathcal{L}^p(\mathcal{H})$ in $\mathcal{L}(\mathcal{H})$. (The ideal $\mathcal{L}^p(\mathcal{H})$, $1 \leq p < \infty$ is contained in $\mathcal{K}(\mathcal{H})$; a compact operator T lies in $\mathcal{L}^p(\mathcal{H})$ when the eigenvalues (counted with multiplicities) of the self-adjoint compact operator $(T^*T)^{\frac{1}{2}}$ form an l^p sequence. Thus $\mathcal{L}^1$ is the ideal of trace-class operators discussed previously.) All the ideals $\mathcal{L}^p$ have the same topological K-theory, but roughly speaking, the algebraic K-theory of $\mathcal{L}^p$ becomes more and more "stable" (resembling the K-theory of $\mathcal{K}$) as $p \to \infty$. This is reflected in:

Theorem 2.13 (Karoubi, [38, Propositions 3.5 and 3.9, Corollaire 4.2, and Théorème 4.13]). *For all $p \geq 1$, $K_{-1}(\mathcal{L}^p) = 0$ and $c\colon K_{-2}(\mathcal{L}^p) \to K_{-2}^{\mathrm{top}}(\mathcal{L}^p) \cong \mathbb{Z}$ is surjective. However, for integers $n \geq 1$, $c\colon K_{2n}(\mathcal{L}^p) \to K_{2n}^{\mathrm{top}}(\mathcal{L}^p) \cong \mathbb{Z}$ is the 0-map for $p \leq 2n-1$ and is surjective for $n = 1$, $p > 1$.*

The result for K_2 suggests that by using products one should obtain surjectivity of $c\colon K_{2n}(\mathcal{L}^p) \to K_{2n}^{\mathrm{top}}(\mathcal{L}^p) \cong \mathbb{Z}$ for p large enough compared with n, but failure of excision gets in the way of proving this in an elementary fashion. This issue is discussed in more detail in [68, §2], where additional results along these lines are obtained.

3 Recent Progress on Algebraic K-Theory of Operator Algebras

3.1 Algebraic K-Theory Invariants for Operator Algebras

For some purposes, it is useful to study the homotopy fiber $\mathbb{K}^{\mathrm{rel}}(A)$ of the comparison map $c\colon \mathbb{K}(A) \to \mathbb{K}^{\mathrm{top}}(A)$ of Theorem 2.1. We call this spectrum (or the set of its homotopy groups) the *relative K-theory*; it measures the difference between the algebraic and topological theories. Obviously we get a long exact sequence of K-groups

$$\cdots \to K_{j+1}^{\mathrm{top}}(A) \to K_j^{\mathrm{rel}}(A) \to K_j(A) \xrightarrow{c} K_j^{\mathrm{top}}(A) \to K_{j-1}^{\mathrm{rel}}(A) \to \cdots . \tag{5}$$

Since (for any unital Banach algebra A) $K_1(A)$ surjects onto $K_1^{\mathrm{top}}(A)$ and $K_0(A) \to K_0^{\mathrm{top}}(A)$ is an isomorphism, $K_0^{\mathrm{rel}}(A) = 0$. We have $K_j^{\mathrm{rel}}(\mathbb{C}) = \mathbb{Z}$ for $j = -3, -5, \cdots$ and $K_j^{\mathrm{rel}}(\mathbb{C}) = 0$ for other negative values of j. The Karoubi Conjecture (Conjecture 2.12) amounts to the assertion that $\mathbb{K}^{\mathrm{rel}}(A)$ is trivial for stable C^*-algebras.

A number of papers in the literature, such as [13], [14], [34], and [35], attempt to detect classes in relative K-theory through secondary index invariants or regulators. ("Primary" index invariants detect classes in topological K-theory.) For example, suppose τ is a p-summable Fredholm module over A. This consists of a representation of A on a Hilbert space $\mathcal{H}$, together with an operator $F \in \mathcal{L}(\mathcal{H})$ that satisfies $F^2 = 1$ and that commutes with A modulo the Schatten class $\mathcal{L}^p(\mathcal{H})$. When p is even, one additionally requires that $\mathcal{H}$ is $\mathbb{Z}/2$-graded, that the action of A on $\mathcal{H}$ preserves the grading, and that T is odd with respect to the grading. (The prototype for this situation is the case where $A = C^\infty(M)$, M a compact $(p-1)$-dimensional smooth manifold, and T is obtained by functional calculus from a first-order elliptic differential operator, such as the Dirac operator or signature operator.) In [13] and [14], Connes and Karoubi set up, for each $(p+1)$-summable Fredholm module τ, a commutative diagram with exact rows, where the top row comes from (5):

$$\begin{tikzcd}
K_{p+2}(A) \arrow[r, "c"] & K_{p+2}^{\mathrm{top}}(A) \arrow[r] \arrow[d, "\mathrm{Ind}_\tau"'] & K_{p+1}^{\mathrm{rel}}(A) \arrow[r] \arrow[d, "\mathrm{Ind}_\tau^{\mathrm{sec}}"'] & K_{p+1}(A) \arrow[r, "c"] \arrow[d, "\mathrm{Ind}_\tau^{\mathrm{sec}}"'] & K_{p+1}^{\mathrm{top}}(A) \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "2\pi i"] & \mathbb{C} \arrow[r, "\exp"] & \mathbb{C}^\times \arrow[r] & 0.
\end{tikzcd}$$

The downward arrow Ind_τ is the usual index and the downward arrows $\mathrm{Ind}_\tau^{\mathrm{sec}}$ are the secondary index invariants. When $A = C^\infty(S^1)$ (this is only a Fréchet algebra, but standard properties of topological K-theory for Banach algebras apply to it as well) and τ corresponds to the smooth Toeplitz extension (4), $\mathrm{Ind}_\tau^{\mathrm{sec}}$ recovers the determinant invariant discussed above in section 1.4. Other papers such as [34] and [35] relate other secondary invariants defined analytically (for example, via the eta invariant) to the Connes-Karoubi construction.

3.2 The Work of Suslin-Wodzicki on Excision

As we saw in section 2.4, the Karoubi Conjecture (Conjecture 2.12) and related conjectures about the K-theory of operator algebras are dependent on understanding to what extent the K-theory of nonunital Banach algebras satisfies excision. Work on this topic was begun by Wodzicki ([66], [67]) and completed in collaboration with Suslin [55]. Wodzicki started by studying excision in cyclic homology, then moved on to the study of rational K-theory, and finally Suslin and Wodzicki clarified the status of excision in integral algebraic K-theory. As the papers [67] and [55] are massive and deep, there is no room to discuss them here in detail, so we will be content with a short synopsis. For simplicity we specialize the results to algebras over a field F of characteristic 0, the only case of interest to us. Then (in [66]) Wodzicki calls an F-algebra A *homologically unital*, or *H-unital* for short, if the standard bar complex $B_\bullet(A)$ is acyclic, i.e., if $\mathrm{Tor}_\bullet^{\widetilde{A}}(F,F)=0$, where $\widetilde{A}=A+F\cdot 1$ is A with unit adjoined. In [66] and [67], Wodzicki shows that C^*-algebras, Banach algebras with bounded approximate unit [66, Proposition 5], and many familiar Fréchet algebras such as $\mathcal{S}(\mathbb{R}^n)$ [67, Corollary 6.3], are H-unital. Furthermore, any tensor product (over F) of an H-unital algebra with a unital F-algebra is H-unital [67, Corollary 9.7]. The main result of [66] is that an F-algebra satisfies excision in cyclic homology if and only if it is H-unital. It is also pointed out, as a consequence of Goodwillie's Theorem [26], that if an F-algebra satisfies excision in rational algebraic K-theory, then it must satisfy excision in cyclic homology and thus be H-unital.

In [55], Suslin and Wodzicki managed to prove the converse, that if A is an H-unital F-algebra, then A satisfies excision in rational algebraic K-theory, i.e., $K_\bullet(B,A)\otimes_{\mathbb{Z}}\mathbb{Q}$ is independent of B, for B an F-algebra containing A as an ideal. Since Weibel had already shown [65] that K-theory with $\mathbb{Z}/p$-coefficients satisfies excision for $\mathbb{Q}$-algebras, this implies:

Theorem 3.1 (Suslin-Wodzicki [55]). *Let A be an algebra over a field F of characteristic 0. Then A satisfies excision for algebraic K-theory if and only if A is H-unital. In particular, C^*-algebras satisfy excision for algebraic K-theory.*

The proof of the Suslin-Wodzicki Theorem is rather complicated, but ultimately, via the use of the Volodin approach to K-theory, it comes down to showing that the inclusion

$$A \hookrightarrow A_1 = \begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}$$

induces an isomorphism on Lie algebra homology $H_\bullet^{\mathrm{Lie}}(\mathfrak{gl}(A)) \cong H_\bullet^{\mathrm{Lie}}(\mathfrak{gl}(A_1))$. This in turn follows from showing that $HC_\bullet(A)\cong HC_\bullet(A_1)$, which can be deduced from the H-unitality of A. (By the way, if one is only interested in C^*-algebras A, then since they satisfy $A^2=A$, the proof in [55] can be shortened somewhat, as explained on page 89.)

3.3 Resolution of the Karoubi Conjecture

The Karoubi Conjecture is now known to be true, thanks to a combination of the work of Higson [28] and the Suslin-Wodzicki Theorem discussed above in Section 3.2. The method of Higson is somewhat indirect, and is based on the following intermediate result of independent interest:

Theorem 3.2 ([28, Theorem 3.2.2]). *Let k be a functor from the category of C^*-algebras and $*$-homomorphisms (or a suitable full subcategory, such as the category of separable C^*-algebras) to the category of abelian groups. Assume that k is **stable**, i.e., that the morphism $A \to A \otimes \mathcal{K}$ (C^*-algebra tensor product) given by $a \mapsto a \otimes e$, where e is a rank-one projection in $\mathcal{K}$, always induces an isomorphism $k(A) \to k(A \otimes \mathcal{K})$. Also assume that k is **split exact**, i.e., that it sends split short exact sequences of C^*-algebras to split short exact sequences of abelian groups. Then k is homotopy-invariant.*

A few ideas from the proof. The idea is to use the hypotheses to construct a pairing of k with Fredholm modules. More precisely, suppose $\varphi = (\varphi_+, \varphi_-)$ is a Fredholm pair; i.e., φ_+ and φ_- are $*$-representations of a C^*-algebra B on a Hilbert space $\mathcal{H}$, such that $\varphi_+(a) - \varphi_-(a) \in \mathcal{K}(\mathcal{H})$ for all $a \in B$. From this data, by a construction originally due to Cuntz, one gets a split short exact sequence (for any C^*-algebra A)

$$\begin{tikzcd} 0 \arrow[r] & A \otimes \mathcal{K} \arrow[r] & A \otimes B_\varphi \arrow[r, shift right, "p"'] & A \otimes B \arrow[l, dotted, shift right, "1\otimes\varphi"'] \arrow[r] & 0 \end{tikzcd},$$

where $B_\varphi = \{(b, x) \in B \oplus \mathcal{L}(\mathcal{H}) \mid \varphi(b) - x \in \mathcal{K}(\mathcal{H})\}$. (Note that this is independent of whether one uses φ_+ or φ_-.) Since k was assumed stable and split exact, we get a map

$$\varphi_*\colon\ k(A \otimes B) \to \ker(p_*) \xrightarrow{\cong} k(A \otimes \mathcal{K}) \xrightarrow{\cong} k(A)$$

with certain good functorial properties. The next step (which is not so difficult) is to show that this pairing can be expressed a pairing with Fredholm modules of the more conventional sort (where one has a $*$-representation φ of B on a Hilbert space $\mathcal{H}$ and a unitary operator F that commutes with the representation modulo compacts). One simply lets $\varphi_+ = \varphi$, $\varphi_- = \mathrm{Ad}(F) \circ \varphi$. Then one shows that this pairing is invariant under *operatorial homotopy*, i.e., norm-continuous deformation of the F, keeping φ fixed and with the "commutation modulo compacts" condition satisfied at all times. The final, and hardest, step is to construct an operatorial homotopy $\big(\varphi, \{F_t\}_{t\in[0,1]}\big)$ of Fredholm modules over $C([0,1])$, such that the pairing of k with (φ, F_0), $k(A \otimes C([0,1])) \to k(A)$, corresponds to evaluation of functions at 0, and the pairing of k with (φ, F_1) corresponds to evaluation of functions at 1. This step of the proof is highly reminiscent of the proof [40, §6, Theorem 1] that operatorial homotopy invariance of Kasparov's KK-functor implies homotopy invariance in the most general sense, and establishes the theorem. $\square$

From this and the Suslin-Wodzicki Theorem we immediately deduce

Theorem 3.3. *The Karoubi Conjecture is true. In other words, if* $A \cong A \otimes \mathcal{K}$ *is a stable* C^**-algebra, then the comparison map* $c\colon \mathbb{K}(A) \to \mathbb{K}^{\mathrm{top}}(A)$ *of Theorem* 2.1 *is an equivalence.*

Proof. For each integer j, let $k_j(A) = K_j(A \otimes \mathcal{K})$. Then k_j is a functor from C^*-algebras to abelian groups — note that since A is H-unital, we do not need to specify which unital algebra contains $A \otimes \mathcal{K}$ as an ideal, by Theorem 3.1. We claim this functor is split exact. Indeed, if

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] \arrow[l, dotted, bend right] & 0 \end{tikzcd}$$

is split exact, then so is

$$\begin{tikzcd} 0 \arrow[r] & A \otimes \mathcal{K} \arrow[r] & B \otimes \mathcal{K} \arrow[r] & C \otimes \mathcal{K} \arrow[r] \arrow[l, dotted, bend right] & 0 \end{tikzcd}$$

(because the C^*-algebra tensor product with $\mathcal{K}$ is an exact functor, since $\mathcal{K}$ is nuclear), and we can apply the long exact sequence in K-theory. Furthermore, k_j is stable, since if e is a rank-one projection in $\mathcal{K}$ and $\varphi\colon A \to A \otimes \mathcal{K}$ is given by $a \mapsto a \otimes e$, then $k_j(\varphi)\colon K_j(A \otimes \mathcal{K}) \to K_j(A \otimes \mathcal{K} \otimes \mathcal{K})$ is the morphism on K-theory induced by $a \otimes e \mapsto a \otimes e \otimes e$, and there is an isomorphism $\mathcal{K} \otimes \mathcal{K} \xrightarrow{\cong} \mathcal{K}$ sending $e \otimes e \mapsto e$. Hence by Theorem 3.2, k_j is homotopy-invariant.

Now we concude the proof by showing by induction that $c_*\colon k_j(A) \to K_j^{\mathrm{top}}(A)$ is an isomorphism for all C^*-algebras A and all j. Clearly this is true for $j = 0$. Next, we prove it for j positive. Assume by induction that $c_*\colon k_j(A) \to K_j^{\mathrm{top}}(A)$ is an isomorphism for all C^*-algebras A. We have a short exact sequence of C^*-algebras:

$$0 \to C_0((0,1)) \otimes A \to C_0([0,1)) \otimes A \to A \to 0.$$

The middle algebra is contractible, so by the homotopy invariance result just proved, $k_{j+1}(C_0([0,1)) \otimes A) = 0$ and $k_j(C_0([0,1)) \otimes A) = 0$. A similar result holds for topological K-theory. Thus the long exact sequences in K-theory give a commuting diagram

$$\begin{tikzcd} k_{j+1}(A) \arrow[r, "\partial", "\cong"'] \arrow[d, "c_*"'] & K_j(C_0((0,1)) \otimes A) \arrow[d, "c_*"', "\cong"] \\ K_{j+1}^{\mathrm{top}}(A) \arrow[r, "\partial", "\cong"'] & K_j^{\mathrm{top}}(C_0((0,1)) \otimes A). \end{tikzcd}$$

and thus $c_*\colon k_{j+1}(A) \to K_{j+1}^{\mathrm{top}}(A)$ is an isomorphism. This completes the inductive step.

The result for $j \le 0$ is already contained in [37, Théorème 5.18] and is essentially identical to the proof of Theorem 2.11, using the product structure on $K_*(\mathcal{K})$. □

Unfortunately this proof does not necessarily explain "why" the Karoubi Conjecture is true, since, unlike the proof of the Brown-Schochet Theorem (Theorem 1.3), it is not constructive.

A number of modifications or variants on Theorem 3.3 are now known. For example, one has the "unstable Karoubi Conjecture" in [53]: if A is a stable C^*-algebra, then the natural map $B\big(GL_n(A)^\delta\big) \to BGL_n(A)$ is an isomorphism on integral homology for all n. Here $GL_n(A)$ is to be interpreted as $GL_n(\widetilde{A}, A)$, i.e., the group of matrices in $GL_n(\widetilde{A})$ which are congruent to 1 modulo A. There is a Fréchet analogue of the Karoubi Conjecture in [57], with $\mathcal{K}$ replaced by the algebra of smoothing operators, or in other words by infinite matrices with rapidly decreasing entries, a version of the theorem for certain generalized stable algebras in [30], and a pro-C^*-algebra analogue in [31].

3.4 Other Miscellaneous Results

In this final section, we mention a number of other results and open problems related to algebraic K-theory of operator algebras. These involve K-regularity, negative K-theory, and K-theory with finite coefficients.

K-Regularity

We begin with a few results about K-regularity, or in other words, results that say that C^*-algebras behave somewhat like regular rings with respect to algebraic K-theory. As motivation for this subject, note that in [56], Swan defined a commutative ring R with unit, and with no nilpotent elements, to be *seminormal* if for any $b, c \in R$ with $b^3 = c^2$, there is an element $a \in R$ with $a^2 = b$ and $a^3 = c$. This condition guarantees that $\operatorname{Pic} R[X_1, \cdots, X_n] \cong \operatorname{Pic} R$ for all n, which we can call Pic-regularity. Swan's condition is clearly satisfied for commutative C^*-algebras, since if $R = C(X)$ for some compact Hausdorff space X, and if b and c are as indicated, one can take

$$a(x) = \begin{cases} c(x)/b(x), & b(x) \neq 0, \\ 0, & b(x) = c(x) = 0, \end{cases}$$

and check that a is continuous and thus lies in R. Hence commutative C^*-algebras are Pic-regular. This suggests that they might be K-regular as well, since Pic and K_0 are closely related.

In [28, §6], Higson proved the K-regularity of stable C^*-algebras as part of his work on the Karoubi Conjecture. In other words, we have

Theorem 3.4 (Higson; see also [30, Theorem 18]). *If A is a stable C^*-algebra, then for any n, the natural map $\mathbb{K}(A) \to \mathbb{K}(A[t_1, \cdots, t_n])$ (which is obviously split by the map induced by sending $t_j \mapsto 0$) is an equivalence. In other words, stable C^*-algebras are K-regular.*

Proof. For any j, the functor $k_j^n = A \mapsto K_j(A[t_1, \cdots, t_n])$ satisfies the conditions of Theorem 3.2. (Here we are using the fact that H-unitality of A implies H-unitality of the polynomial ring $A[t_1, \cdots, t_n]$.) Hence k_j^n is a homotopy functor. So we have an isomorphism $k_j^n(A \otimes C([0,1])) \cong k_j^n(A)$ induced in one direction by the inclusion of A in $A \otimes C([0,1]) \cong C([0,1], A)$ and in the other direction by evaluation at either 0 or 1. Now consider the homomorphism φ from $C([0,1], A)[t_1, \cdots, t_n]$ to itself defined by

$$\varphi(f)(s, t_1, \cdots, t_n) = f(s, st_1, \cdots, st_n), \qquad s \text{ the coordinate on } [0,1].$$

Then φ followed by evaluation at $s = 1$ is the identity on $A[t_1, \cdots, t_n]$, so it induces the identity on $K_j(A[t_1, \cdots, t_n])$, but on the other hand, φ followed by evaluation at $s = 0$ sends $A[t_1, \cdots, t_n]$ to A. Hence $K_j(A[t_1, \cdots, t_n])$ factors through $K_j(A)$. $\square$

Other results on K-regularity of C^*-algebras may be found in [49]. For example, there is some evidence there that all C^*-algebras should be K_0-regular (i.e., that one should have isomorphisms $K_0(A[t_1, \cdots, t_n]) \cong K_0(A)$ for all n, when A is a C^*-algebra). There are simple counterexamples there to show this cannot be true for Banach algebras. Commutative C^*-algebras are in some sense at the opposite extreme from stable C^*-algebras, and for these one has basically the same K-regularity result, though the method of proof is totally different.

Theorem 3.5 (Rosenberg [49, Theorem 3.1]). *If A is a commutative C^*-algebra, then for any n, the natural map $\mathbb{K}(A) \to \mathbb{K}(A[t_1, \cdots, t_n])$ (which is obviously split by the map induced by sending $t_j \mapsto 0$) is an equivalence. In other words, commutative C^*-algebras are K-regular.*

As observed in [49], to prove the general case, one may by excision (section 3.2) reduce to the case where A is unital, and one may by a transfer argument reduce to the case $\mathbb{F} = \mathbb{C}$. So we may take $A = C(Y)$. It was also observed in [49] that any finitely generated subalgebra $\mathbb{C}[f_1, \ldots, f_n]$ of A is reduced (contains no nilpotent elements), hence by the Nullstellensatz is isomorphic to the algebra $\mathbb{C}[X]$ of regular functions on some affine algebraic set $X \subseteq \mathbb{C}^N$, $N \leq n$, not necessarily irreducible. Then the inclusion $\mathbb{C}[f_1, \ldots, f_n] \hookrightarrow A$ is dual to a continuous map $Y \to X$. Thus it suffices to show:

Theorem 3.6. *Let $A = C(Y)$, where Y is a compact Hausdorff space, be a (complex) commutative C^*-algebra, and let $X \subseteq \mathbb{C}^N$ be an affine algebraic set. Suppose one is given a continuous map $\varphi\colon Y \to X$, and let $\varphi^*\colon \mathbb{C}[X] \to C(Y)$ be the dual map on functions. Then $(\varphi^*)_*$ vanishes identically on $N^j K_m(A)$ for any $j \geq 0$ and $m \geq 0$.*

Proof. The proof of this given in [49] was based on the (basically correct) idea of chopping up Y and factoring φ through smooth varieties, but the technical details were incorrect.[5] Indeed, as pointed out to me by Mark Walker, it was

[5] I thank Mark Walker for pointing this out to me.

claimed in [49] that one can find a closed covering of X such that a resolution of singularities $p\colon \widehat{X} \to X$ of X (in the sense of [29]) splits topologically over each member of the closed cover, and this simply isn't true. (It would be OK with a *locally* closed cover, however.) Walker [personal communication] has found another proof of Theorem 3.4; see also [22, Theorem 5.3]; to set the record straight, we give still another proof here.

Let $p\colon \widehat{X} \to X$ be a resolution of singularities of X (in the sense of [29]).[6] This has the following properties of interest to us:

1. $\widehat{X}$ is a smooth quasiprojective variety (not necessarily irreducible, since we aren't assuming this of X), and p is a proper surjective algebraic morphism.
2. There is a Zariski-closed subset X_1 of X, such that $X \setminus X_1$ is a smooth quasiprojective variety Zariski-dense in X, and such that if $\widehat{X}_1 = p^{-1}(X_1)$, then p gives an isomorphism from $\widehat{X} \setminus \widehat{X}_1$ to $X \setminus X_1$, and a proper surjective morphism from $\widehat{X}_1$ to X_1.

We now prove the theorem by induction on the dimension of X. To start the induction, if $\dim X = 0$, then X is necessarily smooth and the theorem is trivial. So assume we know the result when X has smaller dimension, and observe that the inductive hypothesis applies to the singular set X_1. Also note, as observed in [49], that there is no loss of generality in assuming $Y \subseteq X$. Let $Y_1 = Y \cap X_1$. From the diagram

$$\begin{tikzcd} Y_1 \arrow[r, hook] \arrow[d, "\varphi"] & Y \arrow[r] \arrow[d, "\varphi"] & (Y, Y_1) \arrow[d, "\varphi"] \\ X_1 \arrow[r, hook] & X \arrow[r] & (X, X_1), \end{tikzcd}$$

we get a commuting diagram of exact sequences of K-groups

$$\begin{tikzcd} N^jK_{m+1}(X_1) \arrow[r, "\partial"] \arrow[d, "(\varphi^*)_*"] & N^jK_m(X, X_1) \arrow[r] \arrow[d, "(\varphi^*)_*"] & N^jK_m(X) \arrow[r] \arrow[d, "(\varphi^*)_*"] & \cdots \\ N^jK_{m+1}(C(Y_1)) \arrow[r, "\partial"] & N^jK_m(C_0(Y \setminus Y_1)) \arrow[r] & N^jK_m(C(Y)) \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} \cdots \arrow[r] & N^jK_m(X) \arrow[r] \arrow[d, "(\varphi^*)_*"] & N^jK_m(X_1) \arrow[r, "\partial"] \arrow[d, "(\varphi^*)_*"] & N^jK_{m-1}(X, X_1) \arrow[d, "(\varphi^*)_*"] \\ \cdots \arrow[r] & N^jK_m(C(Y)) \arrow[r] & N^jK_m(C(Y_1)) \arrow[r, "\partial"] & N^jK_{m-1}(C_0(Y \setminus Y_1)). \end{tikzcd}$$

[6] We don't need the full force of the existence of a such a resolution, but it makes the argument a little easier. The interested reader can think of how to formulate everything without using $\widehat{X}$.

Here we have used excision (section 3.2) on the bottom rows and have identified K-theory of the coordinate ring of an affine variety with the K-theory of its category of vector bundles. The K-groups of (X, X_1) denote *relative* K-theory of vector bundles in the sense of [15], and NK-theory for varieties is defined by setting $NK_m(X) = \ker(K_m(X \times \mathbb{A}^1) \to K_m(X))$, etc. By inductive hypothesis, the maps $N^j K_m(X_1) \xrightarrow{(\varphi^*)_*} N^j K_{m+1}(C(Y_1))$ vanish, so by diagram chasing, it's enough to show that the maps $N^j K_m(X, X_1) \to N^j K_m(C_0(Y \setminus Y_1))$ vanish.

Since $X \setminus X_1$ is smooth, one might think this should be automatic, but that's not the case since algebraic K-theory doesn't satisfy excision. However, we are saved by the fact that we have excision in the *target* algebra. The map $p\colon \widehat{X} \to X$ is an isomorphism from $\widehat{X} \setminus \widehat{X}_1$ to $X \setminus X_1$, and induces maps $p^*\colon N^j K_m(X, X_1) \to N^j K_m(\widehat{X}, \widehat{X}_1)$. Since φ lifts over $Y \setminus Y_1$, the map $N^j K_m(X, X_1) \to N^j K_m(C_0(Y \setminus Y_1))$ factors through $N^j K_m(\widehat{X}, \widehat{X}_1)$. (Here the approach of [15] is essential since $\widehat{X}$ may not be affine, and so we can't work just with K-theory of rings.) But $N^j K_m(\widehat{X}, \widehat{X}_1)$ vanishes since $\widehat{X}$ and $\widehat{X}_1$ are smooth. □

Negative K-Theory

In [47] and [49], the author began a study of the *negative* algebraic K-theory of C^*-algebras. The most manageable case to study should be commutative C^*-algebras. By Theorem 3.5, such algebras are K-regular, so they satisfy the Fundamental Theorem in the simple form $K_j(A[t, t^{-1}]) \cong K_j(A) \oplus K_{j-1}(A)$. A conjecture from [47] and [49], complementary to the results of Higson in [28], is:

Conjecture 3.7 (Rosenberg). *Negative K-theory is a homotopy functor on the category of commutative C^*-algebras. Thus $X \mapsto K_j(C_0(X))$ is a homotopy functor on the category of locally compact Hausdorff spaces and proper maps when $j \le 0$.*

Corollary 3.8. *On the category of (second countable) locally compact Hausdorff spaces, $X \mapsto K_j(C_0(X))$ coincides with* ***connective*** *K-theory $bu^{-j}(X)$, for $j \le 0$.*

Proof (from [49]) that the Corollary follows from the Conjecture. Let

$$k^{-j}(X) = \begin{cases} K_j^{\mathrm{top}}(C_0(X)), & j > 0 \\ K_j(C_0(X)), & j \le 0. \end{cases}$$

Then Conjecture 3.8 implies that k^* is a homotopy functor, and it satisfies the excision and long exact sequence axioms, by Theorem 3.1 and the long exact sequences in algebraic and topological K-theory, pasted together at $j = 0$, where they coincide. It is also clear that k^* is additive on infinite disjoint unions, i.e., that $k^*(\coprod_i X_i) = \bigoplus_i k^*(X_i)$. Thus it is an additive cohomology theory (with compact supports). There is an obvious natural transformation of cohomology theories $k^* \to K^*$ (ordinary topological K-theory with compact supports), induced by $c_*\colon K_j(C_0(X)) \to K_j^{\text{top}}(C_0(X))$, which is an isomorphism on k^{-j}, $j \leq 0$. And k^* is a connective theory, since $\mathbb{C}$ is a regular ring and thus $k^{-j}(\text{pt}) = K_j(\mathbb{C}) = 0$ for $j < 0$. Thus by the universal property of the connective cover of a spectrum [1, p. 145], $k^* \to K^*$ factors through bu^*. Since $k^*(X) \to K^*(X)$ is an isomorphism for X a point, it is an isomorphism for any X with X_+ a finite CW-complex, and then by additivity, for X_+ any compact metric space (since any compact metric space is a countable inverse limit of finite complexes). □

While a proof of Conjecture 3.8 is outlined in [47], Mark Walker has kindly pointed out that the proof is faulty. The author still believes that the same method should work, and indeed it does in certain special cases, but it seems to be hard to get the technical details to work. In fact, it is even conceivable that negative K-theory is a homotopy functor for arbitrary C^*-algebras, but a proof of this would require a totally new technique.

K-Theory with Finite Coefficients

In this last section, we discuss results on K-theory with finite coefficients that generalize Theorem 2.7. These results can be viewed as analytic counterparts to the work of Friedlander-Mislin and Milnor discussed above in Section 2, and to the results of Thomason ([60], [61], [62], [63]) for algebraic varieties.

Theorem 3.9 (Fischer [20], Prasolov ([45], [41])). *Let A be a commutative C^*-algebra. Then the comparison map for A with finite coefficients,*

$$c\colon K_i(A;\mathbb{Z}/n) \to K_i^{\text{top}}(A;\mathbb{Z}/n)$$

is an isomorphism for $i \geq 0$.

The method of proof of this theorem is copied closely from the proof of Suslin's theorem, Theorem 2.7. Thus it relies on Theorem 2.8 on Henselian rings, and is quite special to the commutative case. However, it is conceivable that one has:

Conjecture 3.10 (Rosenberg [47, Conjecture 4.1]). *Let A be a C^*-algebra. Then the comparison map for A with finite coefficients,*

$$c\colon K_i(A;\mathbb{Z}/n) \to K_i^{\text{top}}(A;\mathbb{Z}/n)$$

is an isomorphism for $i \geq 0$.

In support of this, we have:

Theorem 3.11 (Rosenberg [47, Theorem 4.2]). *Let A be a type I C^*-algebra which has a finite composition series, each of whose composition factors has the form $A \otimes M_n(\mathbb{F})$ $(n \geq 0)$ or $A \otimes \mathcal{K}$, where A is commutative. Then the comparison map for A with finite coefficients,*

$$c\colon\ K_i(A;\mathbb{Z}/n) \to K_i^{\mathrm{top}}(A;\mathbb{Z}/n)$$

is an isomorphism for $i \geq 0$.

This is proved by piecing together Theorems 3.9 and 3.3, using excision (Theorem 3.1). The main obstruction to extending the proof to more general classes of C^*-algebras is the lack of a good result on (topological) inductive limits of C^*-algebras. Such a result would necessarily be delicate, because we know that *algebraic* K-theory behaves differently under *algebraic* inductive limits and *topological* inductive limits. For example, the *algebraic* inductive limit $\varinjlim M_n(\mathbb{C})$ has the same K-theory as $\mathbb{C}$, and thus its negative K-theory vanishes, whereas the C^*-algebra inductive limit $\varinjlim M_n(\mathbb{C})$ is $\mathcal{K}$, which has infinitely many non-zero negative K-groups.

References

1. John Frank Adams. *Infinite loop spaces*, volume 90 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J., 1978.
2. Hyman Bass. *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.
3. Paul Baum, William Fulton, and Robert MacPherson. Riemann-Roch and topological K theory for singular varieties. *Acta Math.*, 143(3-4):155–192, 1979.
4. Ola Bratteli. Inductive limits of finite dimensional C^*-algebras. *Trans. Amer. Math. Soc.*, 171:195–234, 1972.
5. William Browder. Algebraic K-theory with coefficients $\mathbb{Z}/p$. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), I*, volume 657 of *Lecture Notes in Math.*, pages 40–84. Springer, Berlin, 1978.
6. L. G. Brown, R. G. Douglas, and P. A. Fillmore. Unitary equivalence modulo the compact operators and extensions of C^*-algebras. In *Proceedings of a Conference on Operator Theory (Dalhousie Univ., Halifax, N.S., 1973)*, pages 58–128. Lecture Notes in Math., Vol. 345, Berlin, 1973. Springer.
7. L. G. Brown, R. G. Douglas, and P. A. Fillmore. Extensions of C^*-algebras and K-homology. *Ann. of Math.* (2), 105(2):265–324, 1977.
8. L. G. Brown and Claude Schochet. K_1 of the compact operators is zero. *Proc. Amer. Math. Soc.*, 59(1):119–122, 1976.
9. Lawrence G. Brown. The determinant invariant for operators with trace class self commutators. In *Proceedings of a Conference on Operator Theory (Dalhousie Univ., Halifax, N.S., 1973)*, pages 210–228. Lecture Notes in Math., Vol. 345, Berlin, 1973. Springer.
10. Lawrence G. Brown. Operator algebras and algebraic K-theory. *Bull. Amer. Math. Soc.*, 81(6):1119–1121, 1975.
11. Lawrence G. Brown, Pierre de la Harpe, and Claude Schochet. Perfection du groupe de Fredholm. *C. R. Acad. Sci. Paris Sér. A-B*, 290(3):A151–A154, 1980.

12. Robert C. Busby. Double centralizers and extensions of C^*-algebras. *Trans. Amer. Math. Soc.*, 132:79–99, 1968.
13. Alain Connes and Max Karoubi. Caractère multiplicatif d'un module de Fredholm. *C. R. Acad. Sci. Paris Sér. I Math.*, 299(19):963–968, 1984.
14. Alain Connes and Max Karoubi. Caractère multiplicatif d'un module de Fredholm. *K-Theory*, 2(3):431–463, 1988.
15. K. R. Coombes. Relative algebraic K-theory. *Invent. Math.*, 70(1):13–25, 1982/83. An appendix.
16. Pierre de la Harpe and Georges Skandalis. Sur la simplicité essentielle du groupe des inversibles et du groupe unitaire dans une C^*-algèbre simple. *J. Funct. Anal.*, 62(3):354–378, 1985.
17. Ronald G. Douglas. *Banach algebra techniques in operator theory.* Pure and Applied Mathematics, Vol. 49. Academic Press, New York, 1972. Reprinted by Springer-Verlag, New York, 1998.
18. Edward G. Effros, David E. Handelman, and Chao Liang Shen. Dimension groups and their affine representations. *Amer. J. Math.*, 102(2):385–407, 1980.
19. George A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *J. Algebra*, 38(1):29–44, 1976.
20. Thomas Fischer. K-theory of function rings. *J. Pure Appl. Algebra*, 69(1):33–50, 1990.
21. Eric M. Friedlander and Guido Mislin. Cohomology of classifying spaces of complex Lie groups and related discrete groups. *Comment. Math. Helv.*, 59(3):347–361, 1984.
22. Eric M. Friedlander and Mark E. Walker. Comparing K-theories for complex varieties. *Amer. J. Math.*, 123(5):779–810, 2001.
23. Ofer Gabber. K-theory of Henselian local rings and Henselian pairs. In *Algebraic K-theory, commutative algebra, and algebraic geometry* (*Santa Margherita Ligure, 1989*), volume 126 of *Contemp. Math.*, pages 59–70. Amer. Math. Soc., Providence, RI, 1992.
24. S. M. Gersten. On the spectrum of algebraic K-theory. *Bull. Amer. Math. Soc.*, 78:216–219, 1972.
25. Henri A. Gillet and Robert W. Thomason. The K-theory of strict Hensel local rings and a theorem of Suslin. *J. Pure Appl. Algebra*, 34(2-3):241–254, 1984.
26. Thomas G. Goodwillie. Relative algebraic K-theory and cyclic homology. *Ann. of Math.* (2), 124(2):347–402, 1986.
27. J. William Helton and Roger E. Howe. Integral operators: commutators, traces, index and homology. In *Proceedings of a Conference Operator Theory* (*Dalhousie Univ., Halifax, N.S., 1973*), pages 141–209. Lecture Notes in Math., Vol. 345, Berlin, 1973. Springer.
28. Nigel Higson. Algebraic K-theory of stable C^*-algebras. *Adv. in Math.*, 67(1):1–140, 1988.
29. Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2) 79 (1964), 109–203; ibid. (2)*, 79:205–326, 1964.
30. Hvedri Inassaridze. Algebraic K-theory of normed algebras. *K-Theory*, 21(1):25–56, 2000.
31. Hvedri Inassaridze and Tamaz Kandelaki. K-theory of stable generalized operator algebras. *K-Theory*, 27(2):103–110, 2002.
32. Klaus Jänich. Vektorraumbündel und der Raum der Fredholm-Operatoren. *Math. Ann.*, 161:129–142, 1965.

33. B. E. Johnson. An introduction to the theory of centralizers. *Proc. London Math. Soc. (3)*, 14:299–320, 1964.
34. Jerome Kaminker. Analytic invariants for algebraic K-theory. In *Current topics in operator algebras* (*Nara, 1990*), pages 246–254. World Sci. Publishing, River Edge, NJ, 1991.
35. Jerome Kaminker. Algebraic K-theory invariants for operator theory. In *Multivariable operator theory* (*Seattle, WA, 1993*), volume 185 of *Contemp. Math.*, pages 187–194. Amer. Math. Soc., Providence, RI, 1995.
36. Max Karoubi. Foncteurs dérivés et K-théorie. In *Séminaire Heidelberg-Saarbrücken-Strasbourg sur la K-théorie* (*1967/68*), Lecture Notes in Mathematics, Vol. 136, pages 107–186. Springer, Berlin, 1970.
37. Max Karoubi. K-théorie algébrique de certaines algèbres d'opérateurs. In *Algèbres d'opérateurs* (*Sém., Les Plans-sur-Bex, 1978*), volume 725 of *Lecture Notes in Math.*, pages 254–290. Springer, Berlin, 1979.
38. Max Karoubi. Homologie de groupes discrets associés à des algèbres d'opérateurs. *J. Operator Theory*, 15(1):109–161, 1986. With an appendix in English by Wilberd van der Kallen.
39. G. G. Kasparov. Topological invariants of elliptic operators. I. K-homology. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(4):796–838, 1975. translated in *Math. USSR-Izv*. 9:751–792, 1976.
40. G. G. Kasparov. The operator K-functor and extensions of C^*-algebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(3):571–636, 719, 1980. translated in *Math. USSR-Izv.* 16:513–572, 1981.
41. H. H. McFaden et al., editors. *Selected papers in K-theory*, volume 154 of *American Mathematical Society Translations, Series 2*. American Mathematical Society, Providence, RI, 1992. Translated from the Russian.
42. J. Milnor. On the homology of Lie groups made discrete. *Comment. Math. Helv.*, 58(1):72–85, 1983.
43. John Milnor. *Introduction to algebraic K-theory*. Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.
44. Erik K. Pedersen and Charles A. Weibel. A nonconnective delooping of algebraic K-theory. In *Algebraic and geometric topology* (*New Brunswick, N.J., 1983*), volume 1126 of *Lecture Notes in Math.*, pages 166–181. Springer, Berlin, 1985.
45. A. V. Prasolov. Algebraic K-theory of Banach algebras. *Dokl. Akad. Nauk BSSR*, 28(8):677–679, 1984. Reprinted in English translation in [41].
46. Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math.* (2), 96:552–586, 1972.
47. Jonathan Rosenberg. K and KK: topology and operator algebras. In *Operator theory: operator algebras and applications, Part 1* (*Durham, NH, 1988*), volume 51 of *Proc. Sympos. Pure Math.*, pages 445–480. Amer. Math. Soc., Providence, RI, 1990.
48. Jonathan Rosenberg. *Algebraic K-theory and its applications*, volume 147 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.
49. Jonathan Rosenberg. The algebraic K-theory of operator algebras. *K-Theory*, 12(1):75–99, 1997.
50. Victor Snaith. $H^i(BG^{\mathrm{top}};\mathbf{Z}/n)$ does not always inject into $H^i(BG^\delta;\mathbf{Z}/n)$. *C. R. Math. Rep. Acad. Sci. Canada*, 6(5):261–265, 1984.
51. Victor P. Snaith. Suslin's theorem and algebraic cobordism. Lecture notes, available at `www.maths.soton.ac.uk/pure/preprints.phtml`.

52. A. Suslin. On the K-theory of algebraically closed fields. *Invent. Math.*, 73(2):241–245, 1983.
53. A. A. Suslin. Homology stability for H-unital $\mathbf{Q}$-algebras. In *Mathematics in St. Petersburg*, volume 174 of *Amer. Math. Soc. Transl. Ser. 2*, pages 117–139. Amer. Math. Soc., Providence, RI, 1996.
54. Andrei A. Suslin. On the K-theory of local fields. *J. Pure Appl. Algebra*, 34(2-3):301–318, 1984.
55. Andrei A. Suslin and Mariusz Wodzicki. Excision in algebraic K-theory. *Ann. of Math.* (2), 136(1):51–122, 1992.
56. Richard G. Swan. On seminormality. *J. Algebra*, 67(1):210–229, 1980.
57. Joseph Tapia. K-théorie algébrique négative et K-théorie topologique de l'algèbre de Fréchet des opérateurs régularisants. *Ann. Sci. École Norm. Sup.* (4), 30(2):241–277, 1997.
58. J. L. Taylor. Banach algebras and topology. In *Algebras in analysis* (*Proc. Instructional Conf. and NATO Advanced Study Inst., Birmingham, 1973*), pages 118–186. Academic Press, London, 1975.
59. Joseph L. Taylor. Topological invariants of the maximal ideal space of a Banach algebra. *Advances in Math.*, 19(2):149–206, 1976.
60. R. W. Thomason. Riemann-Roch for algebraic versus topological K-theory. *J. Pure Appl. Algebra*, 27(1):87–109, 1983.
61. R. W. Thomason. Algebraic K-theory and étale cohomology. *Ann. Sci. École Norm. Sup.* (4), 18(3):437–552, 1985.
62. R. W. Thomason. Erratum to [61]. *Ann. Sci. École Norm. Sup.* (4), 22(4):675–677, 1989.
63. R. W. Thomason. Survey of algebraic vs. étale topological K-theory. In *Algebraic K-theory and algebraic number theory* (*Honolulu, HI, 1987*), volume 83 of *Contemp. Math.*, pages 393–443. Amer. Math. Soc., Providence, RI, 1989.
64. J. B. Wagoner. Delooping classifying spaces in algebraic K-theory. *Topology*, 11:349–370, 1972.
65. C. A. Weibel. Mayer-Vietoris sequences and mod p K-theory. In *Algebraic K-theory, Part I* (*Oberwolfach, 1980*), volume 966 of *Lecture Notes in Math.*, pages 390–407. Springer, Berlin, 1982.
66. Mariusz Wodzicki. The long exact sequence in cyclic homology associated with an extension of algebras. *C. R. Acad. Sci. Paris Sér. I Math.*, 306(9):399–403, 1988.
67. Mariusz Wodzicki. Excision in cyclic homology and in rational algebraic K-theory. *Ann. of Math.* (2), 129(3):591–639, 1989.
68. Mariusz Wodzicki. Algebraic K-theory and functional analysis. In *First European Congress of Mathematics, Vol. II* (*Paris, 1992*), volume 120 of *Progr. Math.*, pages 485–496. Birkhäuser, Basel, 1994.
69. R. Wood. Banach algebras and Bott periodicity. *Topology*, 4:371–389, 1965/1966.

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