Topology, C^*-Algebras, and String Duality

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Abstract. This book is the intertwined tale of three subjects: dualities in string theory, algebraic topology, and topics in C^*-algebras (especially algebras of continuous trace and the structure of crossed products). The objective is to show how these subjects are closely connected, and to explain recent work in topological T-duality. A few related more advanced topics are also included. This book is the outgrowth of an NSF/CBMS Regional Conference in the Mathematical Sciences, May 18–22, 2009, organized by Robert Doran and Greg Friedman at Texas Christian University.

Preface

This book is the outgrowth of an NSF/CBMS Regional Conference in the Mathematical Sciences, May 18–22, 2009, organized by Robert Doran and Greg Friedman at Texas Christian University. I am highly indebted to Bob and Greg for their tireless work in getting together funding for the conference, making all the logistical arrangements, recruiting participants and other speakers, and for keeping me on track during this entire process.

The subject of this book, identical to the subject of the conference, is interdisciplinary. Thus it involves a more-or-less equal mixture of topology, operator algebras, and physics. There is also a bit of algebraic geometry, especially in the last chapter. While I assume most readers of this book are probably somewhat familiar with at least one of these subjects, they may not necessarily be knowledgeable about all or even most of them. So I have tried to include some basics on each one. The expert reader can skip over these sections. Roughly speaking, each chapter of this book corresponds to a single lecture from the conference, but "fleshed out" a little more. I have also included some sections, marked with a star, that I didn't go into in detail in the lectures. These are more advanced, and someone just wanting an overview of the subject can skip these, though they might interest the more advanced reader. Since the book covers a lot of ground, it differs from most conventional mathematics books which follow the methodical "theorem-proof" style. There are of course plenty of theorems and proofs, but my main objective has been to show how several seemingly disparate subjects are closely linked with one another, and to give readers an overview of some areas of current research. In some places this happens at the expense of trying to cover everything systematically.

I would also like to the thank the Conference Board of the Mathematical Sciences and the National Science Foundation for their support. NSF Grant DMS-0735233 supported the conference, NSF Grant DMS-0602750 supported the entire Regional Conference program, and NSF Grants DMS-0504212 and DMS-0805003 supported the research that went into the preparation of the lectures and the writing of this book. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

Finally, I would also like to thank the Ruth M. Davis Professorship at the University of Maryland for partial support during the writing of this book, and would like thank the many participants at the CBMS conference for their lively participation and penetrating questions at the meeting, and for their valuable feedback on the first draft of this manuscript. They caught many misprints, errors, and omissions, and also made many useful suggestions for improvements, and for this I am very grateful.

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Contents

The symbol ✰ denotes either a more advanced section, or else a bit of a digression, which can be skipped without interrupting the flow of the rest of the book.

CHAPTER 1

Introduction and Motivation

1.1. Structure of Physical Theories

This section is intended to introduce certain key concepts from physics which we will use later in the book. The treatment will be rather superficial, and is intended for mathematicians without much background in physics. Any physicists reading this book can probably skip ahead to Section 1.2.

1.1.1. Fields and Lagrangians.

1.1.1.1. *Fields.* Since the time of Maxwell and Faraday in the 1860's, most physical theories have been expressed in terms of *fields*[1], e.g., the *gravitational field*, *electric field*, *magnetic field*, etc. Originally, "field" meant (to quote the *Oxford English Dictionary*) "a state or situation in which a force is exerted on any objects of a particular kind (e.g., electric charges) that are present," but came to be "frequently used as if it denoted an identifiable causal entity." Fields can be

- scalar-valued functions (physicists often call these *scalars*, which tends to confuse mathematicians because the word "function" is understood but not explicitly stated),
- sections of vector bundles (sometimes simply called *vectors*—again see the comment on the previous item),
- connections on principal bundles[2] (these are special cases of *gauge fields*), or
- sections of spinor bundles (often called *spinors*).

1.1.1.2. *Lagrangians and Least Action.* In classical physics, the fields satisfy a variational principle — they are extrema, or at least critical points, of the *action* S, which in turn is the integral of a local functional $\mathcal{L}$ called the *Lagrangian.* This is called the *principle of least action*. The Euler-Lagrange equations for critical points of the action are the *equations of motion*.

The idea that physics should be governed by variational principles is very old, and predates the notion of "field" itself. For example, Fermat around 1662 [**63**, "Synthesis ad Refractiones," pp. 173–179] proposed a theory of optics based on the

[1]In English, the word *field* has two mathematical meanings. One (a commutative ring in which every non-zero element is invertible) is in abstract algebra and is not relevant here. That term is translated in French as *corps* or in German as *Körper*. What we mean here is field as in "vector field," which is *champs* in French or *Feld* in German.

[2]A space E is called a principal G-bundle over another space X, for a topological group G, if there is a continuous free G-action on E with quotient space X and a continuous surjective open map $p\colon E \to X$ such that X has a covering by open sets U with each $p^{-1}(U)$ G-isomorphic to $U \times G$. A *connection* on a principal bundle, in the case where G is a Lie group and E and X are smooth manifolds, is a consistent way of choosing a "horizontal" subspace H_e of T_eE at each $e \in E$, so that $dp\colon H_e \to T_{p(e)}X$ is an isomorphism for all $e \in E$.

principle that light travels along curves which minimize its travel time, which is a functional of the path. This seemingly innocuous assumption turns out to imply Snell's law of refraction (when light passes from one medium to another, with different speeds in the two media), and also to imply that light does not necessarily travel along straight lines (if it is traveling through a medium with variable index of refraction, such as a fluid with varying density).

The principle of least action in mechanics is due to Lagrange (in his famous *Mécanique Analytique* of 1788) and to Hamilton, ca. 1835 [**74**]. But what we will need below is the least action principle applied to the theory of fields. The following two examples are the key ones for understanding 20th century physics. Yang-Mills theory (in a slightly more complicated form) is needed for understanding elementary particle physics; the Hilbert-Einstein action is basic to general relativity and thus to the understanding of gravity.

But before we get to the examples, we need to say something about Lorentz *vs.* Euclidean signatures in field theories. Indeed, a point which often confuses mathematicians trying to read the physics literature is a frequent shuttling back and forth between writing things in Lorentz and Euclidean signatures. The basic equations of physics do not treat space and time totally equally, in the sense that the natural metric on spacetime is a Lorentz metric, not a Riemannian one. However, in the Lorentz metric, most of the integrals one needs (such as the one computing the action) do not converge well, since one doesn't have positivity for the Lagrangian.

Physicists are therefore fond of what's called *Wick rotation*, replacing t by it and thus "analytically continuing" from Lorentz to Euclidean signature. This results in formulations which are better behaved mathematically but not as realistic physically. Still, one can often use this to some advantage, and we will sometimes do this without further ado.

Examples 1.1. Let M be a 4-manifold, say compact, representing spacetime.

(1) Yang-Mills Theory. The field for this theory is a connection A on a principal G-bundle, where G is some compact Lie group. The "field strength" F is the curvature, a $\mathfrak{g}$-valued 2-form. (Here $\mathfrak{g}$ is the Lie algebra of G.) The action is $S = \int_M \operatorname{Tr} F \wedge *F$. (Here $F \wedge *F$ is a 4-form with values in $\mathfrak{g}$; we take its pointwise trace and integrate.) If the bundle is non-trivial, then usually F cannot vanish, since Chern-Weil theory (Theorem 2.3 below) relates F to the characteristic classes of the bundle.

(2) General Relativity (in Euclidean signature). In this theory, the field is a Riemannian metric g on M. The action is $S = \int_M R\, d\mathrm{vol}$, where R is the scalar curvature of the metric. The associated field equation is Einstein's equation.

1.1.2. Classical to Quantum Physics.

1.1.2.1. *Quantum Mechanics.* Unlike classical mechanics, quantum mechanics is not deterministic, only probabilistic. The key property of quantum mechanics is the *Heisenberg uncertainty principle*, that observable quantities are represented by *noncommuting operators* A represented on a Hilbert space $\mathcal{H}$. In the quantum world, every particle has a wave-like aspect to it, and is represented by a wave function ψ, a unit vector in $\mathcal{H}$. The phase of ψ is not directly observable, only its amplitude, or more precisely, the *state*

$$\varphi_\psi(A) = A(\psi) = \langle A\psi, \psi\rangle$$

which computes the expected (or expectation) value of an observable A for a particle with the given wave function. But the phase is still important, since phenomena such as the Aharonov-Bohm effect (in which the phase of an electron beam is changed without changing the amplitude) and *interference* depend on it. (One of the important consequence of quantum mechanics is that there is no clear distinction between waves and particles. Every particle behaves somewhat like a wave, and two particles can "interfere" with each other, just as waves can. This is responsible for diffraction of electron beams, etc.)

1.1.2.2. *Quantum Fields.* The quantization of classical field theories, called *quantum field theory*, is based on *path integrals.* The idea (not always 100% rigorous in this formulation) is that *all fields contribute*, not just those that are critical points of the action (i.e., solutions of the classical field equations). Instead, one looks at the *partition function*

$$Z = \int e^{iS(\varphi)}\, d\varphi \text{ or } \int e^{-S(\varphi)}\, d\varphi\,, \tag{1.1}$$

depending on whether one is working in Lorentz or Euclidean signature. (Here we've taken $\hbar = 1$; if we didn't do this, since S has units of energy times time, we should divide S by $\hbar \approx 1.054 \times 10^{-34}\,\mathrm{J\,s}$ so as to get something dimensionless that we can exponentiate.) The problem is that the integration is over all possible fields φ, which live in an infinite dimensional space, so one needs to make sense of the integral through some sort of regularization procedure. By the *principle of stationary phase*, only fields close to the classical solutions should contribute much.

What one wants out of a quantum field theory is more than just the partition function; it is a prediction for the measured values of certain observable quantities. Consider a physical quantity A that takes a scalar value $A(\varphi)$ when the fields of the system are given by φ. Since all quantum theories are probabilistic in nature, the best that we can hope for is to compute the *expectation value* of A, which would be the average measured value of this quantity if we perform an experiment a large number of times. This expectation value $\langle A\rangle$ is also given by a path integral, namely

$$\langle A\rangle = \left(\int A(\varphi)\, e^{iS(\varphi)}\, d\varphi\right)/Z\,.$$

We see from this that the partition function is the universal denominator that comes into the calculation of $\langle A\rangle$ for any observable A.

1.2. Some Basics of String Theory

1.2.0.3. *Basic Ideas of String Theory.* The basic idea of string theory is to replace point particles (in conventional physics) by one-dimensional "strings." At ordinary (low) energies these strings are expected to be *extremely* short, on the order of the *Planck length*,

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \times 10^{-35}\,\mathrm{m}\,.$$

A string moving in time traces out a two-dimensional surface called a *worldsheet.* The most basic fields in string theory are thus maps $\varphi\colon \Sigma \to X$, where Σ is a 2-manifold (the worldsheet) and X is spacetime.

String theory offers [some] hope for combining gravity with the other forces of physics and quantum mechanics.

1.2.0.4. *Strings and Sigma-Models.* Let Σ be a string worldsheet and X the spacetime manifold. String theory is based on the *nonlinear sigma-model*, where the fundamental field is $\varphi\colon \Sigma \to X$ and the leading term in the action is

$$S(\varphi) = \frac{1}{4\pi\alpha'} \int_\Sigma \|\nabla\varphi\|^2 \, d\mathrm{vol}\,, \tag{1.2}$$

the energy of the map φ (in Euclidean signature). (In the physics literature, (1.2) is called the string sigma model action or Polyakov action.) The constant α', called the Regge slope parameter, represents (typical string length)^2, and $1/(2\pi\alpha')$ is the string tension. We have to add to this various gauge fields (giving rise to the fundamental particles) and a "gravity term" involving the scalar curvature of the metric on X. Usually we also require *supersymmetry*; this means the theory involves both bosons and fermions and there are symmetries interchanging the two.[3] (But this is a subject for a different course, such as the ones given by Freed [**64**] or Varadarajan [**166**].)

1.2.0.5. *The B-Field and H-Flux.* For various reasons, it's important to add to the action (1.2) another term (sometimes called the *Wess-Zumino term*) of the form

$$\frac{1}{4\pi\alpha'} \int_\Sigma \varphi^* B\,, \tag{1.3}$$

where B is a (locally defined) 2-form on spacetime, X. B is usually called the *B-field*. It need not be closed or even globally defined, just as long as it makes sense locally. (Recall the strings are really "small" in most cases.) But $H = dB$, a 3-form, should always be a well-defined closed 3-form on X, usually called the *H-flux*.

In fact, the 3-form H should correspond to an *integral* cohomology class, which we will also call (by abuse of language) the H-flux, even though this overlooks the torsion in $H^3(X)$. The reason for this is the following. Suppose $\varphi(\Sigma)$ is an embedded surface in X bounding two different 3-manifolds (think of solid handlebodies) M and M'. (See Figure 1.)

Since $\partial M = \partial M' = \varphi(\Sigma)$, Stokes' Theorem gives

$$\int_M H = \int_M dB = \int_{\partial M} B = \int_\Sigma \varphi^* B\,,$$

and similarly with M' in place of M. So

$$\int_M H = \int_{M'} H, \quad \text{and} \quad \int_N H = 0,$$

where N is the closed 3-dimensional submanifold of X obtained by gluing M and M' together along their common boundary. (To make N oriented, reverse the

[3] For those unfamiliar with the terminology here, a *boson* is a particle like a photon that obeys Bose-Einstein statistics; a *fermion* is a particle like the electron that obeys Fermi-Dirac statistics, or in other words, that satisfies the Pauli exclusion principle. The fact that the universe is as we see it depends very much on the fact that both kinds of particles are present. Fermions are always represented by spinor fields. One way of explaining the difference between bosons and fermions is that bosons transform under true representations of the rotation group $SO(n)$, whereas fermions transform under "genuine" representations of the double covering group $\mathrm{Spin}(n)$ that do not descend to $SO(n)$.

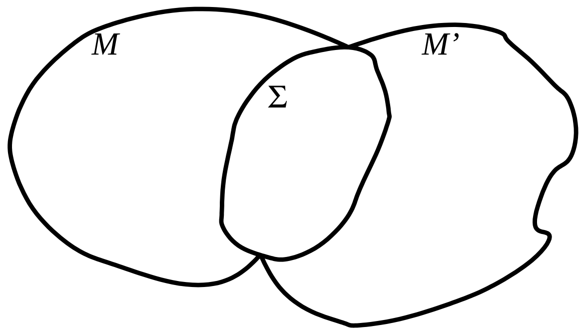


FIGURE 1. A Surface in Spacetime Bounding Two Handlebodies

orientation on M', so that $N = M \cup_{\varphi(\Sigma)} -M'$.) Thus, for the sake of consistency, we need H to pair to zero against every closed 3-dimensional submanifold N of X.

This seems to contradict the possibility of non-triviality of the de Rham class of H, but we've neglected one important thing. By formula (1.1), it is only e^{iS}, not S itself, that counts. So if we only have

$$\frac{1}{4\pi\alpha'}\int_M H \equiv \frac{1}{4\pi\alpha'}\int_{M'} H \mod 2\pi\mathbb{Z}\,,$$

or H integrating over each integral 3-homology class (like $[N]$) to something in $8\pi^2\alpha'\mathbb{Z}$, that's good enough for our purposes, and in physicists' language, the theory is *free of anomalies*.[4]

An interesting model for study, the *Wess-Zumino-Witten model* (WZW model) has $X = G$ a compact semisimple Lie group, say $SU(2)$, and H the canonical 3-form, coming from the trilinear pairing

$$(x, y, z) \mapsto \langle [x,\, y],\, z\rangle$$

on the Lie algebra $\mathfrak{g}$ (here $\langle\,,\,\rangle$ is an invariant inner product on $\mathfrak{g}$, such as the one coming from the Killing form). This H is closed but not exact, so B cannot be globally defined. (In fact, $\pi_3(X) \cong \mathbb{Z}$ for any compact simple Lie group G, and H, if normalized correctly, has an integral de Rham class dual to the image of the generator under the Hurewicz map.)

1.2.0.6. *D-Branes.* Physicists talk about both *closed* and *open* strings. The terminology doesn't quite match that of mathematicians. Both kinds of strings are given by compact manifolds, but in the "open" case there is a boundary. So to get a reasonable theory one has to impose boundary conditions. Usually, these are Dirichlet or Neumann conditions on some submanifold of X where the boundary of Σ must map. These submanifolds are traditionally called *D-branes*, "D" for *Dirichlet* and *brane* a back-formation[5] from *membrane*. Sometimes the name D-brane is retained even without Dirichlet boundary conditions. Physicists also talk

[4] In general, an *anomaly* is an inconsistency in a theory, usually due to nontriviality of some topological invariant. Thus physicists often look for topological conditions under which the anomalies will vanish.

[5] The *Oxford English Dictionary* defines this as "The formation of what looks like a root-word from an already existing word which might be (but is not) a derivative of the former." Here of course we are lopping off the *mem* in "membrane."

about Dp-branes or p-branes; that means branes with p space-like dimensions, or dimension $p+1$ (since usually one also has to allow for one time-like dimension). That leads to the seemingly paradoxical study of (-1)-branes, which just means points in X. Such one-point branes are also called *instantons*, as they are localized at a single "instant" in spacetime. A schematic picture of a D-brane is shown in Figure 2. Note that the open string is free to slide up and down the D-brane but not normal to it.

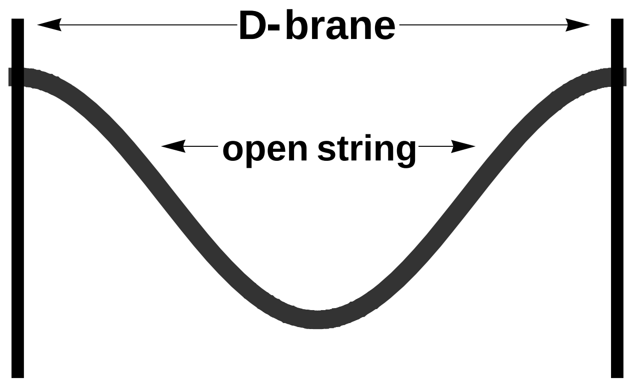


FIGURE 2. Schematic Picture of a D-Brane

In post-1995 string theory, the D-branes play a fundamental role, and are often viewed as fundamental objects in the theory. As we will see, they couple to the (Ramond-Ramond[6]) fields.

1.2.0.7. *The Five Different String Theories.* So far we have neglected to mention that there are really *five* different (supersymmetric) string theories, having slightly different fields and Lagrangians. The differences between them have to do with the gauge groups for some of the gauge fields, closed *vs.* open strings, orientation properties of the strings, and *chirality* (left- *vs.* right-handedness). We will not have time to go into the differences between them in detail, but most of the discussion in this book will focus implicitly on types IIA and IIB. In a nutshell, the five theories are:

- Type I. This is the one theory that involves *unoriented* strings, so that the string worldsheet Σ can be a non-orientable surface like a Klein bottle.
- Type IIA. A theory with oriented strings where left-moving and right-moving spinors have opposite handedness.
- Type IIB. A theory with oriented strings where left-moving and right-moving spinors have the same handedness.
- E_8 Heterotic. A theory where left-movers behave as in bosonic theory and right-movers behave as in supersymmetric theory, and the gauge group is the product of two copies of the exceptional Lie group E_8.

[6]The fields representing particles in string theory lie in different *sectors*, depending on what worldsheet boundary conditions they satisfy (see [**14**, pp. 122–124], [**133**, §10.2], or [**175**, §13.5]). These are called the Ramond (R) and Neveu-Schwarz (NS) sectors. Since one must put the left-moving and right-moving sectors together, that explains terms like "Ramond-Ramond" or "NS-NS."

- $SO(32)$ Heterotic. A theory where left-movers behave as in bosonic theory and right-movers behave as in supersymmetric theory, and the gauge group is a Lie group locally isomorphic to $SO(32)$.

1.2.0.8. *Textbook References on String Theory.* We have deliberately kept our treatment of string theory as short as possible, so the reader is urged to look at some of the standard books on the subject. In particular, this is where to read about the differences between the theories enumerated in Section 1.2.0.7 above. Here are my own personal favorites.

From the physics point of view:

- J. Polchinski, *String Theory*, 2 vols., Cambridge, 1998. [**132, 133**]
- B. Zwiebach, *A First Course in String Theory*, Cambridge, 2004. [**175**]
- K. and M. Becker and J. Schwarz, *String Theory and M-Theory*, Cambridge, 2007. [**14**]

From a more mathematical point of view:

- *Quantum Fields and Strings: A Course for Mathematicians*, 2 vols., Amer. Math. Soc. and IAS, 1999. [**51**]
- J. Jost, *Bosonic Strings: a Mathematical Treatment*, Amer. Math. Soc., 2001. [**89**]
- *Mirror Symmetry*, Amer. Math. Soc. and Clay Math. Inst., 2003. [**81**]

1.3. Dualities Related to String Theory

1.3.1. The Notion of Duality.

1.3.1.1. *What is a Duality?* A *duality* is a transformation between different-looking physical theories that, rather magically, have the same observable physics. Often, such dualities are part of a discrete group, such as $\mathbb{Z}/2$ or $\mathbb{Z}/4$ or $SL(2,\mathbb{Z})$.

EXAMPLE 1.2 (Electric-magnetic duality). There is a symmetry of Maxwell's equations in free space

$$\begin{aligned} \nabla\cdot E &= 0, & \nabla\cdot B &= 0, \\ \frac{\partial E}{\partial t} &= c\,\nabla\times B, & \frac{\partial B}{\partial t} &= -c\,\nabla\times E, \end{aligned} \tag{1.4}$$

given by $E\mapsto -B$, $B\mapsto E$. This is a duality of order 4.

A deep quantum extension of this duality was proposed by Dirac [**53**], and later generalized by Goddard, Nuyts, and Olive [**68**] and by Montonen and Olive [**115**]. Mathematicians who do not follow the physics literature might be interested to learn that when this duality is applied to gauge fields with nonabelian gauge group, such as those appearing in elementary particle theory, then the duality also switches a Lie group G with its Langlands dual $G^\vee$ [**93**], just as in the Langlands program in representation theory and automorphic forms [**32**]. But again, this is a subject for another book.

1.3.1.2. *Fourier Duality.*

EXAMPLE 1.3 (Configuration space-momentum space duality). Another example from standard quantum mechanics concerns the quantum harmonic oscillator (say in one dimension). For an object with mass m and a restoring force with "spring constant" k, the Hamiltonian is

$$H = \frac{k}{2}x^2 + \frac{1}{2m}p^2\,, \tag{1.5}$$

where p is the momentum. In classical mechanics, $p = m\dot{x}$. But in quantum mechanics (with $\hbar$ set to 1),

$$[x, p] = i\,. \tag{1.6}$$

We obtain a duality of (1.5) and (1.6) via $m \mapsto \frac{1}{k}$, $k \mapsto \frac{1}{m}$, $x \mapsto p$, $p \mapsto -x$. This is again a duality of order 4, and is closely related to the *Fourier transform*, since in the *Schrödinger representation* on $L^2(\mathbb{R})$, with x corresponding to the usual coordinate on $\mathbb{R}$, p is represented by the operator $-i\frac{d}{dx}$, whose Fourier transform is x. Recall, incidentally, that if Lebesgue measure is properly normalized, the Fourier transform is indeed a unitary operator on $L^2(\mathbb{R})$ of period 4, with the Hermite functions as eigenvectors.

1.3.2. T-Duality. One of the important dualities in string theory, called *T-duality* ("T" for "target space" or "torus"), will be the main subject of this book. This duality sets up an equivalence of string theories on two very different spacetime manifolds X and $X^\sharp$. The basic idea is that tori in X are replaced by their *dual tori* in $X^\sharp$. In the simplest case, that means that X has a circle factor of radius R and $X^\sharp$ has a circle factor of radius $\widetilde{R} = \frac{\alpha'}{R}$.[7] The duality also involves changes in the metric and the B-field, known as the *Buscher rules*, after Buscher, who derived them in 1987–88 [**37, 38**]. (A similar calculation was also done by Molera and Ovrut [**114**].)

1.3.2.1. *Derivation of T-Duality, Following Buscher.* Consider the simplest case. Take Σ a closed Riemannian 2-manifold and consider the action (1.2) for a map to a circle with radius R, gotten by integrating a 1-form ω on Σ with integral periods:

$$S(\omega) = \frac{1}{4\pi\alpha'}\int_\Sigma \frac{R^2}{\alpha'}\omega \wedge *\omega.$$

Add a new parameter μ, a kind of Lagrange multiplier, and consider instead

$$S(\omega, \mu) = \frac{1}{4\pi\alpha'}\int_\Sigma \left(\frac{R^2}{\alpha'}\omega \wedge *\omega + 2\mu\, d\omega\right).$$

For an extremum of S with respect to variations in μ, we need $d\omega = 0$, so we get back the original theory. But instead we can take the variation in ω.

$$\begin{aligned} \delta S &= \frac{R^2}{4\pi{\alpha'}^2}\int_\Sigma \left(\delta\omega \wedge *\omega + \omega \wedge *\delta\omega + \frac{2\alpha'}{R^2}\mu\, d\delta\omega\right) \\ &= \frac{R^2}{4\pi{\alpha'}^2}\int_\Sigma \delta\omega \wedge \left(2 * \omega + \frac{2\alpha'}{R^2} d\mu\right), \end{aligned}$$

so if $\delta S = 0$, $*\omega = \frac{-\alpha'}{R^2} d\mu$ and $\omega = \frac{\alpha'}{R^2} * d\mu$. If $\eta = *d\mu$, substituting back into $S(\omega, \mu)$ gives

$$\begin{aligned} S'(\eta) &= \frac{1}{4\pi\alpha'}\int_\Sigma \left(\frac{R^2}{\alpha'}\left(\frac{\alpha'}{R^2}\right)^2 \eta \wedge *\eta + 2\frac{\alpha'}{R^2}\mu\, d * d\mu\right) \\ &= -\frac{1}{4\pi\alpha'}\int_\Sigma \frac{\alpha'}{R^2}\eta \wedge *\eta \end{aligned}$$

[7] More generally, if V is a finite-dimensional real vector space and Λ is a lattice in V, then the dual torus to $T = V/(2\pi\Lambda)$ is $T^\sharp = V^*/(2\pi\alpha'\Lambda^*)$, where V^* is the dual space and Λ^* consists of elements $\gamma \in V^*$ which take integral values on $\Lambda \subset V$.

which is just like the original action (with η replacing ω, $\widetilde{R} = \frac{\alpha'}{R}$ replacing R).

1.3.2.2. *Connection with Theta Functions.* T-duality is also related to the classical theory of θ-functions. Consider a simple theory where $\Sigma = S^1$ and $X = \mathbb{R}/(2\pi R\mathbb{Z})$. (If you like, these are the space-like directions and there is another [inert] time direction, a factor of $\mathbb{R}$.) A string winding around X is like a wound-up rubber band; the higher the winding number, the greater the energy. For simplicity, let's just sum over the semi-classical states, the harmonic maps $x \mapsto 2\pi nRx\colon \mathbb{R}/\mathbb{Z} \to X$, instead of taking the path integral, which involves infinite-dimensional integration over *all* paths. The action (1.2) for this map is

$$\frac{1}{4\pi\alpha'}\int_0^1 \left|\frac{d}{dx}\,(2\pi nRx)\right|^2\, dx = \frac{4\pi^2 n^2 R^2}{4\pi\alpha'} = \frac{\pi\, n^2\, R^2}{\alpha'}\,.$$

The partition function is therefore:

$$Z_R = \sum_{n=-\infty}^{\infty} e^{-\pi\, n^2\, R^2/\alpha'}\,, \tag{1.7}$$

a classical θ-function.

Now the Poisson summation formula says that if f is a function in $\mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions, with Fourier transform $\widehat{f}$, then

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n). \tag{1.8}$$

Take $f(x) = e^{-\pi x^2\, R^2/\alpha'}$. Since $x \mapsto e^{-\pi\, x^2}$ is its own Fourier transform, rescaling shows that $\widehat{f}(s) = \frac{\sqrt{\alpha'}}{R} e^{-\pi s^2\, \alpha'/R^2}$. Thus applying the Poisson summation formula (1.8) to f and keeping (1.7) in mind, we get the famous identity (used in the proof of the functional equation of the Riemann ζ-function):

$$Z_R = \frac{\sqrt{\alpha'}}{R}\, Z_{\widetilde{R}} \quad \text{or} \quad \sqrt{R}\, Z_R = \sqrt{\widetilde{R}}\, Z_{\widetilde{R}}\,, \tag{1.9}$$

where $\widetilde{R} = \alpha'/R$, which is basically a precise form of T-duality.

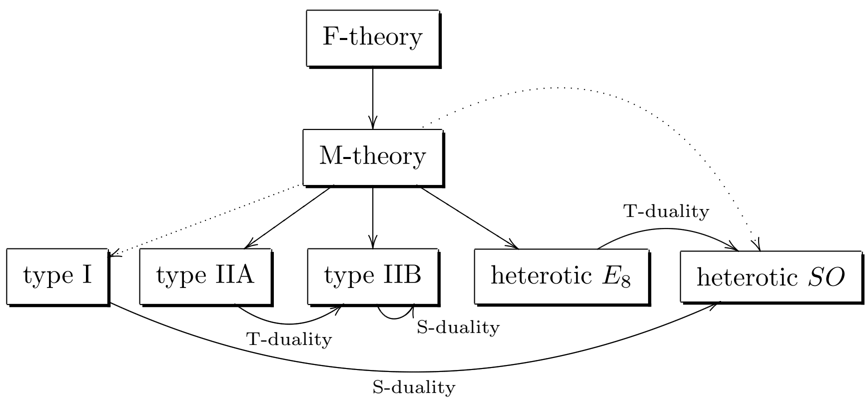
1.3.3. Other Dualities in String Theory. An excellent general survey of dualities in string theory may be found in [**150**]. Here we will be very brief.

1.3.3.1. *S-Duality.* Another important duality in string theory is *S-duality* ("S" for "strong/weak"). This duality is actually an outgrowth of the classical electric-magnetic duality (see Example 1.2), via a suggestion of Dirac [**53**] that the charge of a magnetic monopole (which has never been observed) should be $\hbar c/2$ times the reciprocal of the charge of an electron. Since the charge of an electron is small and easily observable, the charge of a magnetic monopole should be large, which perhaps explains why one has never been observed. S-duality interchanges the strong coupling limit of one string theory with the weak coupling limit of another one.

It has been pointed out [**75**] that S-duality and T-duality are closely linked. S-duality involves the duality between a compact Lie group G and its Langlands dual $G^\vee$. If we choose a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, then the dual space $\mathfrak{h}^*$ can be identified with a Cartan subalgebra in $\mathfrak{g}^\vee$, and we get a pair of dual tori, $\mathfrak{h}/\Lambda$ and $\mathfrak{h}^*/\Lambda^*$, where Λ is the coweight lattice for $\mathfrak{g}$ and the weight lattice for $\mathfrak{g}^\vee$, while Λ^* is the coweight lattice for $\mathfrak{g}^\vee$ and the weight lattice for $\mathfrak{g}$. The T-duality between these tori turns out to reproduce S-duality. Mixing S-duality and T-duality gives a broader family of dualities, sometimes called *U-duality* ("U" for "unified") [**83**].

1.3.3.2. *AdS/CFT Duality.* Another duality which has attracted a lot of attention recently is often called *AdS/CFT duality*. (Here AdS stands for "anti-de Sitter space," a spacetime manifold of constant curvature, and CFT stands for "conformal field theory.") This duality was discovered by Juan Maldacena [**105**], and in general posits an equivalence between gauge theories in dimension d (usually 4) and string theories in a spacetime of dimension $d+1$. There is by now a huge literature on this. For more details, those interested can look at Section 1.4 below.

1.3.3.3. *M-Theory and F-Theory.* There are many other dualities connected with string theory, which fit into various patterns which have been schematized by drawings like:



Superstring theories are (to eliminate certain anomalies) required to be 10-dimensional. The dualities between them seem to involve an 11-dimensional theory, called M-theory, which reduces to 11-dimensional *supergravity* in the low energy limit, and a 12-dimensional theory, called F-theory [**164**].

1.4. ✰ More on S-Duality and AdS/CFT Duality

1.4.1. S-Duality. Since quantum field theories in general, and string theories in particular, are quite complicated, very few things can be computed exactly. For that reason, one of the main calculational tools is *perturbation theory*: expanding in a power series in some parameter, and computing the coefficients. Of course, for this to give useful results, one has to be in a realm where this parameter is small, so that there is hope that the series will converge reasonably well.

The most important dimensionless parameter in string theory is the *string coupling constant* g_S, which measures the intensity of interactions between strings, so it is reasonable to consider perturbation expansions in this parameter, but only if $g_S \ll 1$. However, the interesting feature of string theory is that g_S is not a fixed number; rather, it can be expressed as $e^{\langle\Phi\rangle}$, the exponential of the expectation value of a scalar-valued field Φ called the *dilaton*.

The main feature of S-duality, as opposed to T-duality for instance, is that it exchanges one string theory with a small value of g_S with another with a large value of g_S, by reversing the sign of the dilaton field Φ. T-duality, on the other hand, results in a shift in the dilaton field, but in the form of a translation, so the effect on the coupling constant is less dramatic.

So the main consequence of S-duality is that it makes it possible to probe aspects of string theory not amenable to perturbation theory (i.e., the realm where $g_S \gg 1$) by linking them to the perturbative realm of *another* string theory with $g_S \ll 1$.

1.4.2. AdS/CFT Duality. The discovery of the AdS/CFT duality originally grew out of studies of Yang-Mills theory for $U(N)$-bundles in the limit as $N \to \infty$, and the observation that this theory behaves in ways similar to string theory. A more precise conjecture is that, in the large N limit, supersymmetric Yang-Mills theory on 4-dimensional Minkowski space is dual to type IIB string theory on $\mathrm{AdS}^5 \times S^5$, where AdS^5 is anti-de Sitter space, a 5-dimensional Lorentz manifold of constant curvature. (This is (at least up to coverings) the homogeneous space $SO(4,2)/SO(4,1)$.)

The fact that string theory is connected to Yang-Mills theory is related to something we will discuss in Section 2.2.2 in the next chapter: that D-branes naturally carry bundles (in type IIB these are $U(N)$-bundles) and gauge fields. In fact, open string massless states in type IIB string theory (recall that for physicists, "open" strings are not what mathematicians call "open manifolds"—they have worldsheets with *boundary*, the boundary contained in the D-branes) have an effective Lagrangian that looks a lot like that of Yang-Mills theory. This point of view makes it possible to construct the duality correspondence in the reverse direction, from string theory to gauge theory.

In the AdS/CFT correspondence, the string coupling g_S is supposed to correspond (up to a constant factor of 4π) to a coupling constant g^2_{YM} in Yang-Mills theory. (A factor of $1/g^2_{\mathrm{YM}}$ should have been inserted in front of the Yang-Mills action in Example 1.1 (1); we omitted it there so as not to overburden the reader.) However, the duality is something like S-duality in that it is only supposed to give a reasonable match between the two theories when one is weakly coupled and the other is strongly coupled.

For readers who want to learn more, a good reasonably short survey of the AdS/CFT correspondence, written by its discoverer, may be found in [**104**]. A more comprehensive survey may be found in [**1**].

CHAPTER 2

K-Theory and its Relevance to Physics

2.1. A Quick Review of Topological K-Theory

This section provides a quick review of vector bundles, characteristic classes, and topological K-theory. This is not intended to be a text on these subjects, but just a review, so almost all proofs are omitted, but they are easily accessible in the literature. The material should be quite familiar to topologists. There are many good books on this material, for those who want more details. I would especially recommend [**76**] for most of these topics, as well as [**112, 86**] for vector bundles and characteristic classes and [**6, 94, 128**] for topological K-theory. The book [**85**] covers many of the same topics, but is intended more for physicists.

2.1.1. Vector Bundles. Let X be a locally compact Hausdorff space. A *family of vector spaces over* X is given by a continuous open surjective map $\pi\colon E \to X$, with E locally compact Hausdorff, with scalar multiplication and vector addition maps, $\mathbb{C} \times E \to E$ and $E \times_X E \to E$, satisfying certain obvious axioms, making $E_x = \pi^{-1}(x)$ into a vector space (over $\mathbb{C}$) for each $x \in X$, and so that the vector space structures on the fibers E_x "vary continuously" in x. Such "families of vector spaces over X" form a category, with the morphisms given by commuting diagrams

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

with φ linear on fibers. A *vector bundle over* X is then a family of vector spaces over X, E, which is *locally trivial*, in that there is an open covering $\{U_j\}$ of X with $E|_{U_j} \cong U_j \times \mathbb{C}^n$ in the category of vector spaces over U_j for each j.

Vector bundles *pull back* under continuous maps. If $\pi\colon E \to X$ is a vector bundle over X and if $f\colon Y \to X$ is a continuous map, then $f^*(E)$ is defined by means of the pull-back diagram

$$\begin{tikzcd} f^*(E) \arrow[r, dotted, "\widehat{f}"] \arrow[d, dotted, "f^*\pi"'] & E \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

Here $f^*(E) = \{(y, e) \in Y \times E \mid f(y) = \pi(e)\}$, $f^*\pi$ is projection onto the first factor, and $\widehat{f}$ is projection onto the second factor. We leave it to the reader to check that the local triviality condition is preserved under this operation.

Two useful operations on vector bundles are the *direct sum* and *tensor product*. If E and F are vector bundles over X, then $E \oplus F$ is the vector bundle whose fiber over $x \in X$ is $E_x \oplus F_x$, and $E \otimes F$ is the vector bundle whose fiber over $x \in X$ is $E_x \otimes F_x$. We leave it to the reader to give the (easy) rigorous definition. Note that the rank (fiberwise dimension) of $E \oplus F$ is the sum of those for E and for F, while the rank of $E \otimes F$ is the product of those for E and for F

2.1.2. Classification of Vector Bundles by Cohomology. There are two [equivalent] ways to classify vector bundles (in the category of vector spaces over X): using (Čech) cohomology and using homotopy theory. For simplicity let's take X compact.

The cohomology classification uses *transition functions*. If $E \xrightarrow{\pi} X$ is a vector bundle trivialized by the covering $\{U_j\}$, then on $U_j \cap U_k$ we have *two* trivializations, the one coming from U_j and the one coming from U_k. These need not coincide, but must be related by continuously varying automorphisms of $\mathbb{C}^n$, i.e., by a map $g_{jk}\colon U_j \cap U_k \to GL(n, \mathbb{C})$. The maps g_{jk} satisfy the *cocycle identities*

$$\begin{cases} g_{jk} g_{kj} = 1, \\ g_{jk} g_{k\ell} g_{\ell j} = 1 \quad \text{on } U_j \cap U_k \cap U_\ell. \end{cases}$$

However, if we can choose maps $h_j\colon U_j \to GL(n, \mathbb{C})$ for each j, such that $g_{jk} = h_j h_k^{-1}$ for all j and k, then we can modify the trivialization over U_j by h_j so that all the trivializations match up and the bundle is trivial. In other words, the class of the bundle depends on the class of the cocycle $\{g_{jk}\}$ modulo *coboundaries*, or in fancier language, on a class in $H^1(X, \underline{GL(n, \mathbb{C})})$, where $\underline{GL(n, \mathbb{C})}$ is the sheaf of germs of $GL(n, \mathbb{C})$-valued continuous functions. For $n > 1$, this is *nonabelian cohomology*. (The reader who does not know about sheaves can ignore most of this, and instead think of a more naive view of Čech cohomology, defined as the quotient of the group of cocycle systems $\{g_{jk}\}$ modulo coboundaries, those of the form $g_{jk} = h_j h_k^{-1}$. More exactly, this is the Čech cohomology for a fixed covering $\{U_j\}$, and then one has to pass to a limit as the coverings get finer and finer.)

2.1.3. Classification of Line Bundles. For $n = 1$ (the case of *line bundles*), $GL(1, \mathbb{C}) = \mathbb{C}^\times$ and we have an exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \underline{\mathbb{C}} \xrightarrow{\exp} \underline{\mathbb{C}^\times} \to 1.$$

Since the sheaf $\underline{\mathbb{C}}$ is "fine" and thus has no higher cohomology, we get an exact sequence

$$0 = H^1(X, \underline{\mathbb{C}}) \xrightarrow{\exp} H^1(X, \underline{\mathbb{C}^\times}) \to H^2(X, \mathbb{Z}) \to H^2(X, \underline{\mathbb{C}}) = 0,$$

and thus line bundles are classified by $H^2(X, \mathbb{Z})$. In fact, one can check that one gets an isomorphism of groups

$$\operatorname{Pic} X \cong H^2(X, \mathbb{Z}), \tag{2.1}$$

where Pic, the (topological) Picard group of X, denotes the group of line bundles, with group operation given by fiberwise tensor product over $\mathbb{C}$.

2.1.4. Classification of Vector Bundles by Homotopy Theory. The classification by homotopy theory is based on the fact that every vector bundle E (over a compact base X) is a direct summand in some trivial bundle $X \times \mathbb{C}^N$. Here N can be much larger than the rank n of the bundle. But this means that E can be viewed as continuous way of selecting a rank-n subspace from $\mathbb{C}^N$, or as a map $\varphi\colon X \to \operatorname{Gr}(n, N)$, where $\operatorname{Gr}(n, N)$ is the *Grassmannian* of n-dimensional subspaces in $\mathbb{C}^N$. Furthermore, homotopic maps $X \to \mathrm{Gr}(n, N)$ define isomorphic bundles, and isomorphic bundles give rise to homotopic maps, at least if one takes N sufficiently large. Thus we get a bijection

$$\mathrm{Vect}_n(X) \cong [X, \lim_{N\to\infty} \mathrm{Gr}(n, N)],$$

where $\mathrm{Vect}_n(X)$ is the set of isomorphism classes of rank-n vector bundles over X, and $[X, Y]$ is the set of homotopy classes of maps from X to Y, that is, the set of path components of $\mathrm{Map}(X, Y)$. The space $\mathrm{Gr}(n, N)$ can be identified with the homogeneous space $U(N)/(U(n) \times U(N - n))$, and so $\lim_{N\to\infty} \mathrm{Gr}(n, N)$ is the quotient of $\lim_{N\to\infty} U(N)/U(N - n)$ by the free action of $U(n)$. This space is therefore usually called $BU(n)$. When $n = 1$, this is $\mathbb{CP}^\infty$, whose cohomology ring is a polynomial ring on a single generator in degree 2. Pulling back this generator to X recovers our earlier classification (2.1).

The space $BU(n)$ is a special case of what is called in topology a *classifying space*. A classifying space for a homotopy-invariant functor F is a space BF with the property that $F(X)$ is in natural bijection with homotopy classes of maps $X \to BF$. We will encounter several other examples later in this book.

2.1.5. The Splitting Principle. Attached to vector bundles are certain canonical cohomology classes, the *Chern classes*. There are several ways to define these. One method uses the classification of line bundles via H^2 (recall (2.1)), along with:

LEMMA 2.1 (Splitting Principle). *Given a compact space X and a rank-n vector bundle E over X, there is a compact space Y and a map $f\colon Y \to X$, such that f^* is injective on cohomology* (*one can also arrange for it to be injective on K-theory, to be defined below*) *and such that $f^*(E)$ splits as a direct sum of line bundles*: $f^*(E) \cong L_1 \oplus \cdots \oplus L_n$.

This means that for many purposes, we can always pretend that a vector bundle splits into line bundles. Such a splitting for physicists corresponds to a *symmetry breaking* from $U(n)$ to $U(1)^n$.

2.1.6. Chern Classes.

DEFINITION 2.2. Let X, E, Y, and f be as in Lemma 2.1. Define $c(f^*(E))$ to be $\prod_{j=1}^n (1 + c_1(L_j))$, where c_1 is the class in H^2 attached to a line bundle via (2.1). This has the form $1 + \sum_{j=1}^n c_j(f^*(E))$, where $c_j(f^*E) \in H^{2j}(X, \mathbb{Z})$ is the j-th elementary symmetric function of the $c_1(L_j)$. Then define $c(E)$ so that $f^*(c(E)) = c(f^*(E))$. (Of course one has to check that $c(f^*(E))$ lies in the image of f^* and that the resulting class $c(E)$ is independent of the choice of Y and f.)

An alternate approach is to use the homotopy classification of rank-n vector bundles via maps to

$$BU(n) = \lim_{N\to\infty} \mathrm{Gr}(n, N).$$

One shows $H^*(BU(n), \mathbb{Z})$ is a polynomial ring on classes $c_j \in H^{2j}(BU(n), \mathbb{Z})$, and then if E over X is classified by $f\colon X \to BU(n)$, we define $c_j(E) = f^*(c_j)$.

2.1.7. Connections. There is another approach to Chern classes, more familiar to physicists. Let $\pi\colon E \to X$ be a vector bundle. Put a hermitian metric on E (a smoothly varying family of inner products on the fibers). A *connection* on E is a way of relating one section to another, and is defined by means of a "directional derivative" operator

$$\nabla\colon \Gamma^\infty(X, TX) \times \Gamma^\infty(X, E) \to \Gamma^\infty(X, E)\colon (Y, s) \mapsto \nabla_Y(s)$$

(here Γ^∞ stands for smooth sections, TX for the tangent bundle) satisfying

$$\nabla_Y(f \cdot s) = Yf \cdot s + f \cdot \nabla_Y(s).$$

We can alternatively think of ∇ as an operator

$$\Gamma^\infty(X, E) \to \Gamma^\infty(X, T^*X \otimes E)$$

from sections of E to 1-forms with values in E. As such, it can be iterated, so ∇^2 is an operator from sections of E to 2-forms with values in E. Unlike the exterior derivative d, which satisfies $d^2 = 0$, one need not have $\nabla^2 = 0$, but one can check that all the derivatives in ∇^2 cancel out, so that it is given by a two-form Θ with values in $\operatorname{End}(E)$, called the *curvature* of the connection.

2.1.8. Chern-Weil Theory. More precisely,

$$\Theta(Y, W) = \nabla_Y \nabla_W - \nabla_W \nabla_Y - \nabla_{[Y,W]},$$

and Θ is a 2-form with values in $\operatorname{End}(E)$.

THEOREM 2.3 (Chern-Weil). *The de Rham classes of the coefficients of the characteristic polynomial of*

$$\frac{-1}{2\pi i}\Theta$$

(*viewed as a matrix of* 2*-forms*) *are independent of the choice of connection and lie in the image of* $H^{\text{even}}(X, \mathbb{Z}) \to H^{\text{even}}(X, \mathbb{R})$.

One can then check that these classes are (up to a sign) the images of the $c_j(E)$ in $H^{\text{even}}(X, \mathbb{R})$. A detailed proof may be found in [**57**]. We will content ourselves here with a single example.

EXAMPLE 2.4 (The Hopf line bundle). Identify S^2 with $\mathbb{CP}^1$, the space of 1-dimensional subspaces of $\mathbb{C}^2$. By definition, we have a subspace E_x of $\mathbb{C}^2$ attached to each $x \in \mathbb{CP}^1$, and these vary continuously with x, so they fit together to form a rank-1 complex vector bundle, i.e., a line bundle E over S^2. If $E_x^\perp$ denotes the orthogonal complement to E_x in $\mathbb{C}^2$ (for the usual hermitian inner product on $\mathbb{C}^2$), then the $E_x^\perp$ also fit together to form a line bundle $E^\perp$ over S^2, and clearly $E \oplus E^\perp$ is the trivial rank-2 bundle $S^2 \times \mathbb{C}^2 \to S^2$. Projection from this trivial bundle down to E is given by a continuous map $P\colon S^2 \to M_2(\mathbb{C})$, whose image at each point is a rank-one idempotent. If we view S^2 as the one-point compactification of $\mathbb{C}$, then at a point $z = x + iy \in \mathbb{C}$ this is orthogonal projection onto the span of $(z, 1)$ in $\mathbb{C}^2$, i.e.,

$$P(z) = \frac{1}{|z|^2 + 1}\begin{pmatrix} |z|^2 & z \\ \bar{z} & 1 \end{pmatrix} = \begin{pmatrix} \frac{z\bar{z}}{z\bar{z}+1} & \frac{z}{z\bar{z}+1} \\ \frac{\bar{z}}{z\bar{z}+1} & \frac{1}{z\bar{z}+1} \end{pmatrix},$$

whereas by continuity,

$$P(\infty) = \lim_{z \to \infty} \frac{1}{|z|^2 + 1}\begin{pmatrix} |z|^2 & z \\ \bar{z} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We can define a connection ∇ on E by first taking the usual directional derivative for sections of the trivial bundle, then projecting back into E by means of P. Note that since $P(z)^2 = P(z)$, we have

$$\frac{\partial P}{\partial z}P + P\frac{\partial P}{\partial z} = \frac{\partial P}{\partial z} \quad \text{or} \quad P\frac{\partial P}{\partial z} = \frac{\partial P}{\partial z}(1-P),$$

and similarly with $\frac{\partial P}{\partial \bar{z}}$ in place of $\frac{\partial P}{\partial z}$. Also, for a section f of E, $f = Pf$, so

$$\frac{\partial P}{\partial z}f + P\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} \quad \text{or} \quad (1-P)\frac{\partial f}{\partial z} = \frac{\partial P}{\partial z}f,$$

and similarly with $\frac{\partial P}{\partial \bar{z}}$ in place of $\frac{\partial P}{\partial z}$. So the curvature becomes

$$\begin{aligned}
&\Theta\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) f \\
&= P(z)\frac{\partial}{\partial z}\left(P\frac{\partial f}{\partial \bar{z}}\right) - P(z)\frac{\partial}{\partial \bar{z}}\left(P\frac{\partial f}{\partial z}\right) \\
&= P(z)\frac{\partial P}{\partial z}\frac{\partial f}{\partial \bar{z}} + P(z)\frac{\partial^2 f}{\partial z \partial \bar{z}} - P(z)\frac{\partial P}{\partial \bar{z}}\frac{\partial f}{\partial z} - P(z)\frac{\partial^2 f}{\partial z \partial \bar{z}} \\
&= \left(P(z)\frac{\partial P}{\partial z}\right)\frac{\partial f}{\partial \bar{z}} - \left(P(z)\frac{\partial P}{\partial \bar{z}}\right)\frac{\partial f}{\partial z} \\
&= P\frac{\partial P}{\partial z}(1-P)\frac{\partial f}{\partial \bar{z}} - P\frac{\partial P}{\partial \bar{z}}(1-P)\frac{\partial f}{\partial z} \\
&= P\left(\frac{\partial P}{\partial z}\frac{\partial P}{\partial \bar{z}} - \frac{\partial P}{\partial \bar{z}}\frac{\partial P}{\partial z}\right) f\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\Theta\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \\
&= P\left(\frac{\partial P}{\partial z}\frac{\partial P}{\partial \bar{z}} - \frac{\partial P}{\partial \bar{z}}\frac{\partial P}{\partial z}\right) \\
&= \frac{1}{(1+|z|^2)^5}\begin{pmatrix} |z|^2 & z \\ \bar{z} & 1 \end{pmatrix}\left(\begin{pmatrix} \bar{z} & 1 \\ -\bar{z}^2 & -\bar{z} \end{pmatrix}\begin{pmatrix} z & -z^2 \\ -\bar{z}^2 & -\bar{z} \end{pmatrix} - \begin{pmatrix} z & -z^2 \\ -\bar{z}^2 & -\bar{z} \end{pmatrix}\begin{pmatrix} \bar{z} & 1 \\ -\bar{z}^2 & -\bar{z} \end{pmatrix}\right) \\
&= \frac{1}{(1+|z|^2)^3}\begin{pmatrix} -|z|^2 & -z \\ -\bar{z} & -1 \end{pmatrix}
\end{aligned}$$

and so $\operatorname{Tr}\Theta = -\left(1+|z|^2\right)^{-2} dz \wedge d\bar{z}$. Since $dz \wedge d\bar{z} = -2i dx \wedge dy$,

$$\frac{-1}{2\pi i}\operatorname{Tr}\Theta = \frac{1}{2\pi i}\frac{1}{\left(1+|z|^2\right)^2}\, dz \wedge d\bar{z} = -\frac{1}{\pi\left(1+|z|^2\right)^2} dx \wedge dy,$$

which integrates over $\mathbb{C}$ to

$$-\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\frac{1}{\left(1+r^2\right)^2} r\, dr\, d\theta = -1,$$

which shows $c_1(E) = -1$ (when we identify $H^2(S^2, \mathbb{Z})$ with $\mathbb{Z}$). (Note: some people use a reversed sign convention in which $c_1(E) = 1$.)

2.1.9. The Chern Character. The total Chern class behaves well under direct sum of vector bundles:

$$c(E \oplus E') = c(E)c(E'),$$

but not so well under tensor products. So we introduce another function of the Chern classes, the *Chern character*, that satisfies

$$\mathrm{Ch}(E \oplus E') = \mathrm{Ch}(E) + \mathrm{Ch}(E'), \quad \mathrm{Ch}(E \otimes E') = \mathrm{Ch}(E)\,\mathrm{Ch}(E').$$

For a line bundle L, let

$$\mathrm{Ch}(L) = \exp(c_1(L)) = 1 + c_1(L) + \tfrac{1}{2}c_1(L)^2 + \cdots .$$

For general vector bundles, we use the Splitting Principle (Lemma 2.1) and define

$$\mathrm{Ch}(L_1 \oplus \cdots \oplus L_n) = \sum_{j=1}^{n} \mathrm{Ch}(L_j).$$

Note that the Chern character lives in $H^*(X, \mathbb{Q})$, not $H^*(X, \mathbb{Z})$, since the power series for the exponential function involves denominators.

2.1.10. What is K-Theory? For many purposes (and we will see a case in physics), one wants to be able to add and subtract vector bundles. This is done using (topological) *K-theory.* For X a compact Hausdorff space, we define $K(X)$ to be the *group completion* or *Grothendieck group* of the monoid of isomorphism classes of vector bundles over X. In other words, $K(X)$ is the set of *formal differences* $[E] - [F]$, where E and F are vector bundles over X, and where

$$[E] - [F] = [E'] - [F'] \Leftrightarrow E \oplus F' \oplus G \cong E' \oplus F \oplus G \text{ for some } G. \tag{2.2}$$

This is an abelian group, and it becomes a commutative ring if we let

$$([E] - [F]) \cdot ([E'] - [F']) = [E \otimes E'] + [F \otimes F'] - [E \otimes F'] - [F \otimes E'].$$

The Chern character gives a ring homomorphism $K(X) \to H^{\mathrm{even}}(X, \mathbb{Q})$.

Examples 2.5. We will compute $K(X)$ for $X = S^1$ and S^2. First we need to compute some homotopy groups of the complex Grassmannians. Note that $U(n)$ acts transitively on the unit sphere S^{2n-1} of $\mathbb{C}^n$, with isotropy group $U(n-1)$ at $(0, \cdots, 0, 1)$. So we have a fibration $U(n-1) \to U(n) \to S^{2n-1}$. Since $U(1) = S^1$, the long exact homotopy sequence

$$\begin{aligned} &\pi_2(U(n-1)) \to \pi_2(U(n)) \to \pi_2(S^{2n-1}) = 0 \\ &\qquad \to \pi_1(U(n-1)) \to \pi_1(U(n)) \to \pi_1(S^{2n-1}) \to 0 \end{aligned}$$

shows by induction on n that $\pi_2(U(n)) = 0$ and $\pi_1(U(n)) \cong \mathbb{Z}$ for all n, with the map $\pi_1(U(n-1)) \to \pi_1(U(n))$ an isomorphism for all $n \geq 2$. Now let's compute $\pi_2(\mathrm{Gr}(n, N))$ via the fibration $U(n) \times U(N-n) \to U(N) \to \mathrm{Gr}(n, N)$. We have

$$\begin{aligned} &0 = \pi_2(U(N)) \to \pi_2(\mathrm{Gr}(n, N)) \to \pi_1(U(n) \times U(N-n)) \\ &\qquad \cong \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \cong \pi_1(U(N)) \to \pi_1(\mathrm{Gr}(n, N)) \to 0, \end{aligned}$$

with the map α identified to $(a, b) \mapsto a + b$. So $\mathrm{Gr}(n, N)$ is always simply connected, with $\pi_2(\mathrm{Gr}(n, N)) \cong \mathbb{Z}$, for $n \geq 1$ and $N \geq n + 1$.

(1) Let $X = S^1$. Since the complex Grassmannian $\mathrm{Gr}(n, N)$ is simply connected for all n and N, every vector bundle over X is trivial, and is determined up to isomorphism by its rank (or dimension). Thus the semigroup of equivalence classes

of vector bundles over X can be identified with $\mathbb{N}$, and taking the Grothendieck group introduces virtual bundles of negative dimension. So $K(X) \cong \mathbb{Z}$.

(2) Let $X = S^2$. Since $\pi_2(\mathrm{Gr}(n, N)) \cong \mathbb{Z}$ and the map $\mathrm{Gr}(n, N) \to \mathrm{Gr}(n, N+1)$ induces an isomorphism on π_2, we see $\pi_2(BU(n)) \cong \mathbb{Z}$ for any $n \geq 1$. Furthermore, the map $\pi_2(BU(n-1)) \to \pi_2(BU(n))$ is an isomorphism for all $n \geq 2$. This implies that any vector bundle of rank $n \geq 1$ over S^2 is isomorphic to the direct sum of a line bundle (characterized by its Chern class c_1) and a trivial bundle of rank $n-1$. Thus $K(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, the first $\mathbb{Z}$ given by the rank of the bundle and the second given by its Chern class c_1.

2.1.11. Bott Periodicity. We define *K-theory with compact supports* for locally compact spaces by letting

$$K(X) =_{\mathrm{def}} \ker \iota^* \colon\ K(X^+) \to K(\mathrm{pt}),$$

where $X^+ = X \cup \{\infty\}$ is the one-point compactification of X, and $\iota\colon \mathrm{pt} \to X^+$ is the inclusion of the point at infinity into X^+. Untangling the definitions shows that this is the same as the Grothendieck group of vector bundles on X which are trivialized in a neighborhood of infinity, i.e., which are trivial off a compact set. With the understanding that, if X is already compact, $X^+ = X \amalg \{\infty\}$, this extends the old definition. We let $K^{-j}(X) = K(X \times \mathbb{R}^j)$.

THEOREM 2.6 (Bott Periodicity). *For any locally compact space X, there is a natural isomorphism $K(X) \to K(X \times \mathbb{R}^2)$.*

The proof of this theorem is rather difficult and can be found in [**6, 94, 76, 16, 49**], or other books on topological K-theory. But we should at least explain where the isomorphism comes from. In Example 2.5, we computed that $K(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$. In fact, in Example 2.4, we exhibited a specific line bundle E over S^2 which, together with the trivial bundle, generates $K(S^2)$. The formal difference $b = 1 - [E]$ has virtual dimension 0, and thus defines a generator for $K(\mathbb{R}^2)$ (remember this is K-theory with compact supports, so it is the same as the relative K-group $K(S^2, \{\infty\})$). The map $K(X) \to K(X \times \mathbb{R}^2)$ is given by the (external) product with b, i.e., by $x \mapsto x \times b$. This is an isomorphism when X is a point, simply by our calculations for S^2. One can then show in various ways that it extends to an isomorphism for arbitrary X.

Bott periodicity has the consequence that $K^j(X)$ makes sense whether or not j is non-positive, and we can think of $K^*(X)$ as being $\mathbb{Z}/2$-graded.

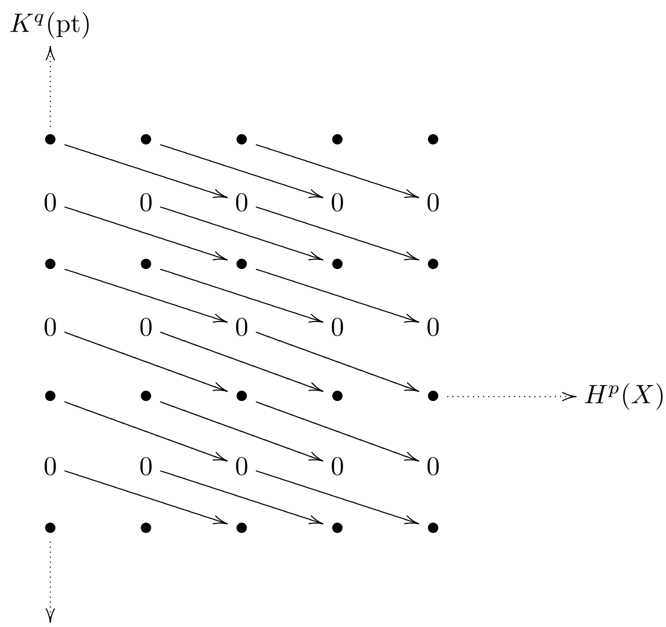
2.1.12. K-Theory and Cohomology. It turns out that the functor $X \rightsquigarrow K^*(X)$ (extended to compact pairs by letting $K^*(X, A) = \ker c_*\colon\ K^*(X/A) \to K^*(\mathrm{pt})$) becomes a cohomology theory on the category of compact (Hausdorff) spaces, or on the category of locally compact spaces and proper maps. In other words, it satisfies the Eilenberg-Steenrod axioms except for the dimension axiom. The theory is $\mathbb{Z}/2$-graded, so there are only two groups, $K = K^0$ and $K^1 = K^{-1}$.

What's the connection with ordinary cohomology? It turns out that the Chern character $K(X) \to H^{\mathrm{even}}(X, \mathbb{Q})$ becomes a *rational isomorphism* of cohomology theories, sending products in $K(X)$ to cup products in cohomology. But the torsion in $K^*(X)$ and $H^*(X, \mathbb{Z})$ can differ. For example, all torsion in $H^*(\mathbb{RP}^n, \mathbb{Z})$ has order 2, whereas $K^*(\mathbb{RP}^n)$ has torsion of order going to infinity as $n \to \infty$.

2.1.13. The Atiyah-Hirzebruch Spectral Sequence. The connection between $K^*(X)$ and $H^*(X,\mathbb{Z})$ is a little more subtle, and is governed by something called the *Atiyah-Hirzebruch spectral sequence*.

THEOREM 2.7 (Atiyah-Hirzebruch). *There is a spectral sequence converging to $K^*(X)$ with $E_2^{p,q} = H^p(X, K^q(\mathrm{pt}))$. Note that $K^q(\mathrm{pt}) = \mathbb{Z}$ for q even, 0 for q odd. The first non-zero differential is $d_3\colon H^p(X,\mathbb{Z}) \to H^{p+3}(X,\mathbb{Z})$, which is equal to the Steenrod operation Sq^3.*

For those unfamiliar with spectral sequences, this basically says that there is an iterative process for computing $K^*(X)$ from $H^*(X,\mathbb{Z})$, where at each stage of the process, one computes the cohomology of the result of the previous stage E_r with regard to a differential d_r (only affecting torsion). More exactly, there is a filtration on $K^*(X)$ for which the associated graded group E_∞ is the limit of groups E_r, where E_{r+1} is the cohomology of E_r with respect to a differential d_r. This iterative process starts with (E_2, d_2), which looks like:



The differential d_r maps $E_r^{p,q}$ to $E_r^{p+r,q+1-r}$, and since $E_r^{p,q} = 0$ for q odd, only the odd differentials $d_3, d_5, \cdots$ can be non-zero. Thus $E_3 = E_2$, for example. Once one has computed E_∞, this computes the K-theory up to iterated extensions (of abelian groups). The fact that differentials only involve torsion has the following consequence:

COROLLARY 2.8. *If X is a finite CW complex and if $H^*(X,\mathbb{Z})$ is torsion free, then $K^*(X)$ is also torsion free, with $K(X) \cong H^{\mathrm{even}}(X,\mathbb{Z})$ and $K^1(X) \cong H^{\mathrm{odd}}(X,\mathbb{Z})$ (as abstract groups).*

Another useful simple consequence of Theorem 2.7 is the following.

COROLLARY 2.9. *If X is a finite CW complex, then the order of the torsion subgroup of $K^0(X)$ is no greater than the order of the direct sum of the torsion subgroups of the $H^{2k}(X,\mathbb{Z})$, and the order of the torsion subgroup of $K^1(X)$ is no* *greater than the order of the direct sum of the torsion subgroups of the* $H^{2k+1}(X, \mathbb{Z})$. *Furthermore, if all torsion in* $H^*(X, \mathbb{Z})$ *is* p*-primary for some prime* p*, then all torsion in* $K^*(X)$ *is* p*-primary.*

However, as already pointed out, the order of the torsion in $K^*(X)$ can differ from that in $H^*(X, \mathbb{Z})$, and it can happen that $H^*(X, \mathbb{Z})$ has torsion but $K^*(X)$ does not.

2.2. K-Theory and D-Brane Charges

2.2.1. An Analogy: Charges in Electromagnetism. Electric charge is *quantized*, i.e., a discrete invariant: all observed charges are integral multiples of the charge of the electron. This was confirmed experimentally in Millikan's famous oil drop experiment of 1900–1913. (Even if one allows for quarks, their charges should be integral multiples of $\frac{1}{3}$ the charge of the electron.) One can try to look for a *topological explanation* of this fact. In 4-dimensional spacetime, the electromagnetic field can be identified with a 2-form F, which is locally given as dA, where A is the potential (combining the electrostatic potential and the 3-dimensional vector potential for magnetism). But this can't be right globally, since $\int_{S^2} F \neq 0$ by Gauss's Law (S^2 linking the worldline of an electron). Dirac [**53**] therefore suggested [in language which when translated into modern terms means] that A should be viewed as a connection on a line bundle with base the complement of the worldlines of the charged particles. In physicist's language, A is then a $U(1)$ gauge field, and F is the field strength (= curvature). Note that $H^2(\mathbb{R}^4 \smallsetminus \mathbb{R}, \mathbb{Z}) \cong \mathbb{Z}$, and by Chern-Weil, $[F]$ is c_1 of the line bundle, up to a constant. So assuming we accept Dirac's idea, charge is quantized!

2.2.2. Chan-Paton Bundles. As we indicated before, D-branes are submanifolds of the spacetime manifold X on which "open" strings are allowed to end. In superstring theories, X is a 10-dimensional Lorentz manifold, often taken to be a product of a Riemannian manifold with $\mathbb{R}$ (representing time). One often talks about D_p-branes or p-branes, the p (with values ≤ 9) representing the dimension of the space-like part of the brane. (So caution: a p-brane is really $(p+1)$-dimensional.)

The D-branes carry *Chan-Paton bundles*. The physical idea behind this is as follows. Suppose n different D-branes coincide, i.e., map to precisely the same submanifold of spacetime. According to the principles of quantum mechanics, it should be possible to "mix" states supported on the different branes, so that there is a local $U(n)$ gauge symmetry. But this is only a *local* symmetry, as the mixing matrix (in $U(n)$) can vary from point to point along the brane. So the brane carries a $U(n)$ gauge field. This analysis is only local, so globally, the brane carries a bundle of dimension n and the gauge field is a connection on the bundle, whose "field strength" is the curvature of the connection. Branes and their Chan-Paton bundles are allowed to coalesce or to split apart, so the rank n of the bundle can vary.

Just as there are antiparticles, there are *antibranes*. The bundles on such branes should be viewed as having *negative* dimension.

2.2.3. Charges and K-Theory. Just as in the case of electromagnetism, the D-branes carry topological charges associated to the nontriviality of the Chan-Paton bundles. In the case of electromagnetism, if X is spacetime with the worldlines of the electrons removed, X admits a line bundle whose class in $\operatorname{Pic} X$ is equivalent to knowledge of the charges. The case of string theory is analogous, but the gauge theories involved are nonabelian, i.e., involve vector bundles of higher rank. Thus physicists [**113, 173, 174, 106, 119**] arrived at the idea that *charges should be classified by K-theory of spacetime.*

2.2.4. The Minasian-Moore Formula. Some of the evidence for this idea comes from the *Minasian-Moore formula* [**113**] for D-brane charges. If we are only dealing with 9-branes (those that fill all of spacetime), then the K-theory charge is just the class $[E]$ of the Chan-Paton bundle E. Similarly, the charge of a 9-antibrane is $-[E]$. For branes W which are proper submanifolds of spacetime X, the embedding of W in X is also relevant. When W and X have spin^c structures (which means one can define spinors and thus a theory of fermions), Minasian and Moore found that the K-theoretic charge should be identified with $f_!([E])$, where $f\colon W \hookrightarrow X$ and $f_!$ is the *Gysin map* in K-theory, a "wrong way" map defined by Atiyah-Singer [**7**].

It is actually somewhat easier to think of brane charges as living in *K-homology*, the homology theory *dual* to K-theory. In this theory, maps go the "right way," so we just compute the K-theory charge in $K_*(W)$ and push it forward under f_*.

2.3. K-Homology and D-Brane Charges

2.3.1. Topological K-Homology. A geometric realization of K-homology was given by Baum and Douglas [**13, 12**]. If X is a compact space, any K-homology class on X may be defined by a "cycle" consisting of:

(1) a compact spin^c manifold W with a map $f\colon W \to X$;
(2) a [virtual] vector bundle E over W.

We add cycles to make an abelian semigroup using disjoint union. Two such cycles are homologous, i.e., define the same K-homology class, if they are related by the equivalence relation generated by:

(1) spin^c bordism, i.e., if M is a compact spin^c manifold with boundary W, if E is a vector bundle over M, and if $f\colon M \to X$, then $(W, E|_W, f|_W) = 0$;
(2) the relation $(W, E_1, f) + (W, E_2, f) = (W, E_1 \oplus E_2, f)$;
(3) "vector bundle modification" (a way of building in Bott periodicity).

2.3.2. Analytical K-Homology. Another realization of K-homology (i.e., another definition of cycles that yields an isomorphic theory) is due to Kasparov, and is based on *generalized elliptic operators* or *Fredholm modules*. This is a special case of Kasparov's *KK-theory* for C^*-algebras, in that $K_*(X)$ (X compact) is given by $KK_*(C(X), \mathbb{C})$. An even-dimensional K-homology cycle on X is given by a $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ with a $*$-representation of $C(X)$, and an odd bounded self-adjoint operator T such that $T^2 - 1$ and $[T, f]$, $f \in C(X)$, are compact. (The typical example is X a compact manifold, $T = D(1 + D^2)^{-1/2}$ for some self-adjoint elliptic first-order partial differential operator, like the Dirac operator, and $\mathcal{H}$ the space of L^2 sections of the vector bundle on which D acts.) The equivalence relation is generated by homotopy, "block addition," and the relation that $(\mathcal{H}, T)$ is trivial if T can be changed by a compact operator so that $T^2 = 1$ and $[T, f] = 0$, $f \in C(X)$. Odd-dimensional cycles in analytic K-homology are defined similarly, except that we drop the grading on $\mathcal{H}$ and the requirement that

T be odd. (An interesting exercise is to convince yourself that if you start with an even-dimensional K-homology cycle and forget the grading on $\mathcal{H}$ so as to get an odd-dimensional cycle, then as an odd cycle, it is trivial.)

2.3.3. Relating the Two Realizations of K-Homology. Now we can explain how the two realizations of K-homology are related. If W is a closed spin^c manifold, it admits a Dirac operator D. If E is a vector bundle over W and $f\colon W \to X$, we form D_E, "Dirac with coefficients in E." This defines a class in the Kasparov model of $K_*(W)$, the dimension given by the dimension of W mod 2. If $f\colon W \to X$, where X is a compact space, we send the class of (W, E, f) (in the Baum-Douglas model of K-homology) to $f_*([D_E])$ in the Kasparov model, and this gives the isomorphism.[1]

2.3.4. Brane Charges in K-Homology. Now we can explain how brane charges are defined in K-homology. If W is a D-brane in X with Chan-Paton bundle E, then if f denotes the inclusion map $W \hookrightarrow X$, (W, E, f) gives a class in $K_*(X)$ via the Baum-Douglas model, provided that W is spin^c. We will see later when this condition needs to be modified, but this condition is usually needed for anomaly cancellation [**65**].

The identification of D-brane charges with K-homology classes is *Poincaré dual* to the identification of these charges with K-theory classes. So the two points of view are equivalent, at least under mild conditions on X.

[1] I am suppressing one little technical point: to get an isomorphism, we need to assume X is homotopically finite. The reason is that Baum-Douglas K-homology is like singular homology, while Kasparov K-homology is like what is called *Steenrod homology*. These two kinds of homology theories agree on finite CW complexes but not on general compact spaces. However, this is not an issue for the spaces we care about for physical applications.

CHAPTER 3

A Few Basics of C^*-Algebras and Crossed Products

3.1. Basics of C^*-Algebras

3.1.1. Basic Definitions and Theorems. *C^*-algebras* will play a big role in the rest of this book. There are a few reasons why they are especially useful for noncommutative geometry and mathematical physics:

- Compared to noncommutative algebras in general, they have a fairly rigid structure, which makes them easier to classify.
- They generalize the notion of algebras of continuous functions (on locally compact Hausdorff spaces).
- They have isometric representations on a Hilbert space, which is required by the axioms of quantum mechanics. In fact, as we shall see, a theorem of Gelfand and Naimark (Theorem 3.4) characterizes C^*-algebras by this property.

An algebra A (say over $\mathbb{C}$) is called a *Banach algebra* if it is equipped with a complete norm (thus making A into a Banach space) with the compatibility condition

$$\|ab\| \leq \|a\| \cdot \|b\| . \tag{3.1}$$

If there is also a conjugate-linear map $a \mapsto a^*$ satisfying

$$(a^*)^* = a, \quad (ab)^* = b^*a^*, \quad \|a^*\| = \|a\| ,$$

then A is called a *Banach $*$-algebra*. Finally A is called a *C^*-algebra* if it is a Banach $*$-algebra satisfying

$$\|a^*a\| = \|a\|^2 . \tag{3.2}$$

If A is a Banach algebra, condition (3.1) shows that A acts continuously on itself by left translation. A Banach algebra need not be unital (have an identity), but if it doesn't, we can make A into an ideal of codimension 1 in a Banach algebra with identity, $\widetilde{A}$. As a vector space, $\widetilde{A} = A \oplus \mathbb{C} \cdot 1$, where 1 is a new element which is an identity element for $\widetilde{A}$. (Thus the multiplication rule in $\widetilde{A}$ is that $(a+\lambda\cdot 1)\cdot(b+\mu\cdot 1) = ab+\mu a+\lambda b+\lambda\mu\cdot 1$.) In general, $\widetilde{A}$ may have many Banach algebra norms compatible with the original norm on A, but one that always works is $\|a+\lambda\cdot 1\|_1 = |\lambda| + \|a\|$, since

$$\begin{aligned}\|(a+\lambda\cdot 1)\cdot(b+\mu\cdot 1)\|_1 &= |\lambda\mu| + \|ab+\mu a+\lambda b\| \\ &\leq |\lambda|\cdot|\mu| + |\mu|\,\|a\| + |\lambda|\,\|b\| + \|a\|\,\|b\| \\ &= \|a+\lambda\cdot 1\|_1\,\|b+\mu\cdot 1\|_1 .\end{aligned}$$

In any Banach algebra A, the *spectrum* of an element a is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda\cdot 1$ is not invertible (in A if A is unital, and otherwise in $\widetilde{A}$). This is always a compact non-empty subset of $\mathbb{C}$, and always contains 0 if A does not have a unit. The *spectral radius* of a is the radius of the smallest disk centered at 0 and containing the spectrum of a. This can be computed as $\lim_{n\to\infty} \|a^n\|^{1/n}$. This fact, called the *spectral radius formula*, follows from the formula for the radius of convergence of a power series and the fact that

$$\begin{aligned}\text{spectral rad}(a) &= \text{the smallest value of } R \text{ such that} \\ &\qquad \text{the series for } (a-\lambda\cdot 1)^{-1} \text{ converges for } |\lambda| > R\,.\end{aligned}$$

But if $z = \lambda^{-1}$, then

$$(a-\lambda\cdot 1)^{-1} = -z(1-za)^{-1} = z\sum_{n=0}^{\infty} a^n z^n,$$

which converges absolutely for $|z| < \lim_{n\to\infty} \|a^n\|^{-1/n}$.

In a C^*-algebra A, if $a = a^*$, or even if a and a^* commute, the norm and spectral radius of a coincide. Indeed, first consider the case where $a = a^*$; then (3.2) shows $\|a^2\| = \|a\|^2$, and iterating, we get $\|a^{2^n}\| = \|a\|^{2^n}$, so that $\lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \|a\|$ and spectral $\text{rad}(a) = \|a\|$ by the spectral radius formula. If a and a^* commute, then $(a^*a)^n = (a^n)^*a^n$, and hence $\|a^n\|^2 = \|(a^*a)^n\|$. Thus

$$\text{spectral rad}(a)^2 = \text{spectral rad}(a^*a) = \|a^*a\| = \|a\|^2\,.$$

Since the definition of the spectrum is purely algebraic and doesn't involve the norm, it follows from this that the norm is determined by the algebraic structure (including the $*$-operation). Thus any $*$-homomorphism (i.e., $*$-preserving homomorphism) between C^*-algebras is norm non-increasing, and any injective $*$-homomorphism is an isometry.

Another important fact about C^*-algebras is that if A is a nonunital C^*-algebra, then we can put a norm on $\widetilde{A}$, the algebra with identity adjoined, to make it a C^*-algebra also. To prove this, first observe that because of (3.2), the action π of A on itself by left multiplication is isometric (since $\|\pi(a)(a^*)\| = \|aa^*\| = \|a\|\|a^*\|$). Since the action extends to $\widetilde{A}$ in an obvious way, we renorm $\widetilde{A}$ by means of

$$\|a+\lambda\cdot 1\| = \|\pi(a+\lambda\cdot 1)\|\,.$$

This gives a C^* norm since if $\varepsilon > 0$,

$$\|\pi(a+\lambda\cdot 1)\|^2 = \sup_{y\in A,\ \|y\|=1} \|ay+\lambda y\|^2,$$

so there exists $y \in A$, $\|y\| = 1$ with

$$\begin{aligned}\|\pi(a+\lambda\cdot 1)\|^2 &\le \|ay+\lambda y\|^2 + \varepsilon \\ &= \|(ay+\lambda y)^*(ay+\lambda y)\| + \varepsilon \\ &= \|y^*(a+\lambda\cdot 1)^*(a+\lambda\cdot 1)y\| + \varepsilon \\ &\le \|(a+\lambda\cdot 1)^*(a+\lambda\cdot 1)y\| + \varepsilon \\ &\le \|\pi\big((a+\lambda\cdot 1)^*(a+\lambda\cdot 1)\big)\| + \varepsilon\,.\end{aligned}$$

PROPOSITION 3.1. *In a C^*-algebra A, if $a = a^*$, then the spectrum of a lies in $\mathbb{R}$. If A is unital and if $uu^* = u^*u = 1$, then the spectrum of u lies in $\mathbb{T}$. Furthermore, the following are equivalent:*

(1) *$a = a^*$ and the spectrum of a lies in $[0, \infty)$.*

(2) $a = b^2$ *for some* $b = b^*$.
(3) $a = y^*y$ *for some* $y \in A$.

The proof of this proposition is a bit tricky and we refer the reader to [**54**] or [**129**]. Elements $a = a^*$ are called *self-adjoint*. Every element is of the form $a + ib$ with a, b self-adjoint. Elements u with $uu^* = u^*u = 1$ (in a unital C^*-algebra) are called *unitary*. The elements A_+ with properties 1–3 above are called *positive*. They span A as a vector space. Relevant for physics is that if A represents a quantum mechanical system, then positive observables are represented by positive elements. One has $\varphi(a) \geq 0$ if $a \in A_+$, for any *state* φ on A (linear functional of norm 1 taking the value 1 at the identity, assuming A is unital).

Sometimes when a C^*-algebra A is nonunital, we will need to refer to the *multiplier algebra* $M(A)$ of A. This is the largest unital C^*-algebra containing A as an essential ideal. ("Essential" means that nothing in $M(A)$ except 0 annihilates A.) This can be defined as

$$M(A) = \{(l, r) \mid l,\, r\colon\ A \to A,\ l(xy) = l(x)y,\ r(xy) = xr(y),\ r(x)y = xl(y)\}\,.$$

Clearly $A \hookrightarrow M(A)$ as an essential 2-sided ideal via $a \mapsto (\lambda(a), \rho(a))$, where $\lambda(a)$ and $\rho(a)$ denote left and right multiplication by a. It is also clear that if A embeds in an algebra B as a 2-sided ideal, then the map $b \mapsto (\lambda(b), \rho(b))$, where $\lambda(b)$ and $\rho(b)$ denote left and right multiplication by b on A, gives a map $B \to M(A)$ which is injective if and only if A is an *essential* ideal in B. The fact that $M(A)$ has the natural structure of a C^*-algebra is proved for example in [**129**, §3.1.2].

3.1.2. Examples of C^*-Algebras.

Examples 3.2.

(1) $A = M_n(\mathbb{C})$, $* =$ conjugate transpose, $\|a\| = \max_{|\xi|=1} |a\,\xi|$. Here $|\cdot|$ is the Euclidean norm on $\mathbb{C}^n$.
(2) $\mathcal{H}$ a Hilbert space, $A = \mathcal{L}(\mathcal{H})$ (bounded linear operators on $\mathcal{H}$),

$$\langle a\,\xi,\, \eta\rangle = \langle \xi,\, a^*\eta\rangle\,, \quad \|a\| = \sup_{\|\xi\|=1} \|a\,\xi\|\,.$$

Case (1) can be identified with the special case where $\mathcal{H}$ is finite dimensional.
(3) $\mathcal{H}$ as above, $A = \mathcal{K}(\mathcal{H})$ (*compact* linear operators[1] on $\mathcal{H}$). This algebra does not have a unit unless $\mathcal{H}$ is finite dimensional. Its multiplier algebra is $\mathcal{L}(\mathcal{H})$.
(4) X locally compact Hausdorff, $A = C_0(X)$ (continuous functions vanishing at infinity). $\|f\| = \sup |f(x)|$, $* =$ complex conjugation. This algebra has a unit exactly when X is compact. The multiplier algebra of $C_0(X)$ is $C^b(X)$, the continuous bounded functions on X. The multiplier algebra can also be identified with $C(\beta X)$, the continuous functions on the *Stone-Čech compactification* of X.

As we will see below, these examples are *universal* in a certain sense.

[1]A bounded operator is called *compact* if it takes bounded sets to pre-compact sets. For operators on a Hilbert space, this is equivalent to being in the norm closure of the operators of finite rank.

3.1.3. Basic Theorems About C^*-Algebras.

THEOREM 3.3 (Gelfand). *Every commutative C^*-algebra A is isometrically $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X. The X is unique up to homeomorphism and obtained from A as the space of maximal (modular) ideals.*[2]

THEOREM 3.4 (Gelfand-Naimark). *Every C^*-algebra embeds $*$-isometrically into $\mathcal{L}(\mathcal{H})$ (as a closed $*$-subalgebra) for some Hilbert space $\mathcal{H}$. One can choose $\mathcal{H}$ finite dimensional if and only if $\dim A$ is finite.*

The proofs may be found in standard books on operator algebras, such as [**54**], [**129**], or [**91**].

3.1.4. Morita Equivalence and Tensor Products. For many purposes, the obvious equivalence relation on C^*-algebras ($*$-isomorphism) is too strong, and one needs something a bit weaker. The correct notion is often *Morita equivalence*, a C^*-algebraic version (due to Rieffel [**136, 137, 138**][3]) of an equivalence relation from noncommutative ring theory. (There are now fine textbook-style treatments of Rieffel's theory in [**103**] and [**135**].) Two rings A and B are Morita equivalent if their categories of left modules are equivalent, i.e., they have the "same" representation theory. *Morita's Theorem* says that this is the case exactly when there is an A-B bimodule ${}_AY_B$ such that the equivalence is implemented by tensoring with Y and the reverse equivalence is implemented by tensoring with the "dual" bimodule. The C^*-algebraic version is similar, but one needs topological control in the form of A- and B-valued inner products on Y satisfying certain nice relations.

3.1.5. Characterization of the Compact Operators. As hinted above, the compact operators $\mathcal{K}(\mathcal{H})$ play a distinguished role in the theory of C^*-algebras. We often write simply $\mathcal{K}$ when $\mathcal{H}$ is separable and infinite dimensional. The following collects together (in more modern language) some basic facts which can be found in standard books on C^*-algebras.

THEOREM 3.5. *For a C^*-algebra A, the following are equivalent:*

(1) *A is Morita equivalent to the scalars $\mathbb{C}$.*
(2) *$A \cong \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

If A is separable, one can add:

(3) *A has, up to unitary equivalence, a unique irreducible representation on a Hilbert space.*

3.1.6. C^* Tensor Products. Suppose A and B are C^*-algebras. Their algebraic tensor product (over $\mathbb{C}$) $A \odot B$ is clearly a $*$-algebra with $(a \otimes b)^* = a^* \otimes b^*$. Often one wants to complete $A \odot B$ to a C^*-algebra. This is not as simple a matter as one might hope, as there are usually many different C^*-algebra norms on $A \odot B$ satisfying the obvious *cross-norm* condition $\|a \otimes b\| = \|a\| \cdot \|b\|$. (The problem is that the cross-norm condition determines the norm on *elementary* tensors but not on linear combinations of elementary tensors.) To avoid dealing with this problem, we will always use $\otimes$ to denote the *spatial tensor product*, which is the completion

[2] An ideal is *modular* when the quotient by this ideal is unital. This is only relevant when A itself is non-unital, since if A has a unit, so does A/I for any proper ideal I.

[3] Rieffel used the term *strong* Morita equivalence, but at the slight risk of confusion with the purely algebraic notion, we've simplified the terminology.

of $A \odot B$ for its obvious $*$-representation on $\mathcal{H}\bar{\otimes}\mathcal{H}'$, where $\mathcal{H}$, $\mathcal{H}'$ are Hilbert spaces on which A, resp., B are faithfully represented, and $\bar{\otimes}$ is the tensor product of Hilbert spaces. (It's a nontrivial result of Turumaru [**163, 162**] that $A \otimes B$ doesn't depend on the choices of the (faithful) representations of A and B.)

Aside from the spatial C^* tensor product $A\otimes B$, which comes from the *minimal* C^* cross-norm on $A\odot B$ (this is a theorem of Takesaki [**159**]), there is also a *maximal tensor product* $A \otimes_{\max} B$, whose $*$-representations correspond to commuting pairs of $*$-representations of A and B. Usually these two products are different, and there may be many "intermediate" tensor products between the two. But there is a unique C^* cross-norm on $A \odot B$ if either A or B is *nuclear*. Commutative C^*-algebras, finite dimensional C^*-algebras, and $\mathcal{K}$ are nuclear; so are C^*-algebras generated by representations of locally compact amenable groups. The class of nuclear C^*-algebras is closed under extensions and inductive limits.

There are two alternate characterizations (found in [**30**]) of Morita equivalence which are often useful.

THEOREM 3.6 (Brown-Green-Rieffel). *If A and B are C^*-algebras, then they are Morita equivalent if and only if they both embed as opposite "full corners" of another C^*-algebra C. A* corner *is a C^* subalgebra of the form pCp, where p is a self-adjoint idempotent in the multiplier algebra of C. It is* full *if CpC is dense in C. The* opposite *corner to pCp is $(1-p)C(1-p)$.*

The terminology here is very suggestive; C can be viewed as an algebra of matrices

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix}, \qquad a \in A,\, b \in B,$$

and A and B can be identified with matrices concentrated in the corners.

THEOREM 3.7 (Brown-Green-Rieffel). *If A and B are separable C^*-algebras, then they are Morita equivalent if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$.*

The equivalence relation that $A \sim B \Leftrightarrow A\otimes\mathcal{K} \cong B\otimes\mathcal{K}$ is usually called "stable isomorphism." It sometimes seems more concrete than Morita equivalence, but it has the one disadvantage that if A and B are stably isomorphic, there may not be any distinguished isomorphism from $A \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$, whereas often in the same situation there is a canonical choice of a Morita equivalence A-B-bimodule.

3.2. K-Theory of C^*-Algebras

This section is again going to be a very quick summary of a subject which is well treated in textbooks, for example, [**16**], [**167**], and [**49**]. We will just present what will be needed for applications later in this book, but the reader should consult one of these references for more details.

3.2.1. K_0 of Rings.

3.2.1.1. *Projective Modules.* Let A be a ring with unit. A *finitely generated projective* A-module P is a direct summand in A^n for some n. (Here we will take our modules to be left modules, but it doesn't really matter; right modules give exactly the same theory.) These modules P are especially nice; for example, $\_ \otimes_A P$ and $\mathrm{Hom}_A(P, \_)$ are exact functors (preserve exact sequences). One can make isomorphism classes of finitely generated projective modules into an abelian monoid (abelian semigroup with a 0 element) $\operatorname{Proj} A$ under direct sum $\oplus$.[4] This is almost never a group since we have addition but not subtraction. But we can convert $\operatorname{Proj} A$ into a group $K_0(A)$ by taking its *group completion* or *Grothendieck group*.

3.2.1.2. K_0 *of Rings.* More precisely, $K_0(A)$ consists of objects $[P]-[Q]$, where P and Q are finitely generated projective A-modules, subject to the relations:

(1)

$$[P] - [Q] = [P'] - [Q'] \Leftrightarrow P \oplus Q' \oplus R \cong P' \oplus Q \oplus R \text{ for some } R. \tag{3.3}$$

(Compare (2.2) in the definition of topological K-theory!)

(2) $([P] - [Q]) + ([P'] - [Q']) = [P \oplus P'] - [Q \oplus Q']$.

When A is commutative, tensor product over A makes $K_0(A)$ into a commutative ring with unit element $[A]$.

There is a close connection between topological K-theory for spaces, defined using vector bundles, and K-theory for rings. This comes from:

THEOREM 3.8 (Serre-Swan). *If X is a compact Hausdorff space and E is a vector bundle over X, then the space $\Gamma(E)$ of continuous sections of E is a finitely generated projective $C(X)$-module. Conversely, every finitely generated projective $C(X)$-module is the space of a sections of a vector bundle. Thus there is a natural isomorphism between $K(X)$ and $K_0(C(X))$.*

More precisely, $E \mapsto \Gamma(E)$ sets up an equivalence of categories between vector bundles over X and finitely generated projective $C(X)$-modules.

3.2.1.3. K_0 *as a Functor.* It is easy to see that $A \mapsto K_0(A)$ is a covariant functor from unital rings to abelian groups. We extend it to *nonunital* rings via

$$K_0(A) = \ker\bigl(q_*\colon\ K_0(\widetilde{A}) \to K_0(\mathbb{Z}) \cong \mathbb{Z}\bigr)\,.$$

Here $\widetilde{A}$, which as an abelian group is $A \oplus \mathbb{Z}\cdot 1$, is the result of adjoining a unit to A, and q is the quotient map. If A is a $\mathbb{C}$-algebra, we could just as well define $\widetilde{A}$ to be $A \oplus \mathbb{C}\cdot 1$, as discussed back in section 3.1.1, and we'd get the same group $K_0(A)$.

It is easy to check that if X is a locally compact Hausdorff space and $A = C_0(X)$, the functions on X vanishing at infinity, then $\widetilde{A}$ (with the second definition, $A \oplus \mathbb{C}\cdot 1$) can be identified with the continuous functions on X that tend to a limit at infinity, or to the continuous functions on the one-point compactification X^+. It follows that $K_0(A)$ as defined above agrees with the K-theory of X with compact supports, as defined back in Section 2.1.11.

An interesting observation, even if we are only interested in unital rings, is that K_0 is functorial under *nonunital* ring homomorphisms. There is one important special case.

PROPOSITION 3.9 (Morita Invariance). *For any ring A, the (nonunital) inclusion $A \hookrightarrow M_n(A)$ defined by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ induces an isomorphism on K_0.*

---

[4]Let's explain why finite generation is important. Suppose for example that $A = \mathbb{C}$. Then finitely generated projective modules are just finite-dimensional vector spaces, which are classified by their dimensions, and $\operatorname{Proj} A \cong \mathbb{N}$. But if instead we took modules which are countably generated, we'd have another element ∞ with the property that $x \oplus \infty = \infty$ for any x. This is "bad" since when we take the group completion, we end up with the trivial group. This observation is often known as the "Eilenberg swindle."

Note that the nonunitality of the map is in fact crucial here, for indeed the only *unital* $*$-homomorphism $\mathbb{C} \to M_n(\mathbb{C})$ induces multiplication by n on K_0, and thus does *not* induce an isomorphism on K_0.

3.2.2. Topological K-Theory.

3.2.2.1. *K_0 of Banach Algebras.* Instead of looking at projective modules, it is sometimes easier to look at idempotents. A finitely generated projective A-module P is determined by an *idempotent* $e = e^2 \in M_n(A)$ (projecting A^n onto P).[5]

K-theory for Banach algebras has somewhat better properties than for general rings. The reason has to do with the following:

Proposition 3.10. *Suppose A is a Banach algebra and two idempotents $e, f \in M_n(A)$ are sufficiently close. Then they are homotopic through idempotents, and the associated projective A-modules $A^n e$ and $A^n f$ are isomorphic.*

Proof. Suppose A is unital (we can always reduce to this case anyway). If e and f are close, then $x = 1 - e - f + 2ef$ will be close to 1 and thus invertible. (Here 1 denotes the identity in $M_n(A)$.) But

$$xf = (1 - e - f + 2ef)f = (1 - e)f - f + 2ef = ef = e(1 - e - f + 2ef) = ex,$$

so $xfx^{-1} = e$. Furthermore, since x is close to 1, it can be joined to 1 by a path of invertible elements x_t with $x_0 = 1$ and $x_1 = x$. Then $x_t f x_t^{-1}$ is a homotopy of idempotents from f to e. $\square$

Corollary 3.11. *On the category of Banach algebras, K_0 is a homotopy functor. That is, homotopic homomorphisms induce the same maps on K_0.*

3.2.2.2. *K_1 of Banach Algebras.* The functor K_0 has a companion functor, called K_1. More precisely, we will be working with *topological* K_1, which is not the same as the K_1 in *algebraic* K-theory, though the two are related. For A a unital Banach algebra, let $GL(A) = \varinjlim GL(n, A)$, where $GL(n, A)$ is the group of invertible elements in $M_n(A)$ and we embed $GL(n, A)$ in $GL(n+1, A)$ via $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. We define $K_1(A)$ to be $GL(A)/GL(A)_0$, the quotient of $GL(A)$ by the connected component of the identity. This is clearly a discrete group.

Proposition 3.12. *If A is a unital Banach algebra, then*

$$K_1(A) = GL(A)/GL(A)_0$$

is an abelian group.

Proof. We just need to show that if a, $b \in GL(n, A)$, then the commutator $aba^{-1}b^{-1}$ lies in $GL(A)_0$. We use the fact that we can stabilize to $GL(2n, A)$, and observe that

$$\begin{pmatrix} aba^{-1}b^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ab & 0 \\ 0 & b^{-1}a^{-1} \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} b^{-1} & 0 \\ 0 & b \end{pmatrix}.$$

We claim each of these matrices is in the connected component of the identity. This follows from the fact that for any invertible u,

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

[5] Incidentally, this explains why projective left modules and projective right modules give the same K-theory — both are determined by idempotent matrices.

is a path in $GL(2n, A)$ from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (when $t = 0$) to $\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix}$ (when $t = \pi/2$). $\square$

It is now obvious that $A \mapsto K_1(A)$ is a homotopy functor from unital Banach algebras to abelian groups. Note that $K_1(\mathbb{C}) = 0$, since $GL(n, \mathbb{C})$ is connected for all n. As before, we extend K_1 to nonunital algebras A by defining

$$K_1(A) = K_1(\widetilde{A}) = \ker\bigl(q_*\colon\ K_1(\widetilde{A}) \to K_1(\mathbb{C}) = 0\bigr)\,.$$

Here $\widetilde{A}$ is the Banach algebra with identity adjoined, as in 3.1.1.

3.2.2.3. *The Exact Sequence Relating K_1 and K_0.*

THEOREM 3.13. *Let A be a Banach algebra with a closed ideal I and quotient A/I. Let $i\colon I \hookrightarrow A$ be the inclusion, $q\colon A \to A/I$ be the quotient map. Then there is a natural exact sequence*

$$K_1(A) \xrightarrow{q_*} K_1(A/I) \xrightarrow{\partial} K_0(I) \xrightarrow{i_*} K_0(A) \xrightarrow{q_*} K_0(A/I)\,.$$

We will omit the proof, which is somewhat tedious, though not especially difficult, but will at least mention the idea behind the construction of the boundary map $K_1(A/I) \xrightarrow{\partial} K_0(I)$. We may suppose A is unital and we have an invertible matrix $\dot{a} \in GL_n(A/I)$ representing a class in $K_1(A/I)$. The "dot" notation here means this is the reduction mod I of a matrix $a \in M_n(A)$, but the key point is that the matrix a need not be invertible. However, if we look at $\begin{pmatrix} \dot{a} & 0 \\ 0 & \dot{a}^{-1} \end{pmatrix}$, then by the proof of Proposition 3.12, this lies in the connected component of the identity in $GL(2n, A/I)$, hence is a product of exponentials, so we can lift this to a matrix $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2n, A)$. The matrix u reduces mod I to $\begin{pmatrix} \dot{a} & 0 \\ 0 & \dot{a}^{-1} \end{pmatrix}$, hence commutes modulo I with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So the formal difference

$$\left[u \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u^{-1}\right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right]$$

lies in $K_0(I)$ and is defined to be $\partial([\dot{a}])$.

If A is a Banach algebra, $C_0(\mathbb{R}^n, A)$, the algebra of A-valued functions on $\mathbb{R}^n$ vanishing at infinity, is again a Banach algebra with pointwise addition and multiplication and norm $\|f\| = \sup_x \|f(x)\|$. When A is a C^*-algebra, this coincides with the C^* tensor product $C_0(\mathbb{R}^n) \otimes A$. An important consequence of Theorem 3.13 is the following:

COROLLARY 3.14. *For any Banach algebra A, $K_1(A) \cong K_0(C_0(\mathbb{R}, A))$.*

PROOF. Note that since K_0 and K_1 are homotopy functors, they must vanish on contractible algebras (algebras A with a homotopy from the identity map $A \to A$ to the zero map $A \to 0 \to A$). Apply Theorem 3.13 to the extension of Banach algebras

$$0 \to C_0((0,1), A) \to C_0((0,1], A) \to A \to 0\,.$$

This shows as desired that $K_1(A) \cong K_0(C_0((0,1), A))$, since $C_0((0,1], A)$ is contractible and thus has vanishing K_0 and K_1. $\square$

3.2.2.4. *Topological K-Theory of Banach Algebras.* One can now extend K_0 to a sequence of homotopy functors on the category of Banach algebras. By Corollary 3.14, if A is a Banach algebra, $K_1(A) \cong K_0(C_0(\mathbb{R}, A))$. So we define

$$K_n(A) = K_0(C_0(\mathbb{R}^n, A))$$

for all $n \in \mathbb{N}$, and this is consistent with our existing definitions. With this definition, the exact sequence of Theorem 3.13 now extends indefinitely to the left.

THEOREM 3.15 (Bott Periodicity). *On the category of Banach algebras, there is a natural isomorphism of functors $K_0 \cong K_2$ which comes from a specific element (the "Bott element") in $K_0(C_0(\mathbb{R}^2))$. Thus $K_n(A)$ really only depends on n modulo 2.*

When A is a commutative C^*-algebra, this is essentially Theorem 2.6 all over again. In fact, the most efficient way to prove Theorem 2.6 is probably to prove Theorem 3.15, though there are also other proofs of Theorem 2.6 that don't generalize to the noncommutative case.

We can now define $K_n(A)$ for n negative using the periodicity, and K_* behaves very much like a *homology theory* on Banach algebras. If we dualize to spaces by letting $A = C_0(X)$ with X locally compact, $A/I = C_0(Y)$ for Y closed in X, then this becomes the long exact *cohomology* sequence in topological K-theory, since $X \rightsquigarrow C_0(X)$ is contravariant.

3.2.2.5. *Stability.* Because of the fact that close idempotents in a Banach algebra are equivalent, one can show that if B is a C^* inductive limit of algebras B_n (this is the completion of the algebraic inductive limit in the obvious C^* norm), then $K_0(B) = \varinjlim K_0(B_n)$. (The key observation is that any idempotent in a matrix algebra over B is close, hence equivalent, to an idempotent in a matrix algebra over B_n for sufficiently large n.) Applying this with $B_n = M_n(A)$ for some other C^*-algebra A, with B_n embedded in B_{n+1} by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, and observing that $\varinjlim M_n(A) = A \otimes \mathcal{K}$, we deduce a *topological* form of Morita invariance:

THEOREM 3.16 (Topological Morita Invariance). *For any C^*-algebra A, the (nonunital) inclusion $A \hookrightarrow A \otimes \mathcal{K}$ defined by $a \mapsto a \otimes e$, e a rank-one self-adjoint projection, induces an isomorphism on all topological K-groups.*

COROLLARY 3.17. *If separable C^*-algebras A and B are Morita equivalent, then they have isomorphic topological K-groups.*

PROOF. This immediately follows from the theorem together with Theorem 3.7. $\square$

REMARK 3.18. In fact the conclusion of Corollary 3.17 is true without the separability hypothesis, but one needs a different method of proof, using a Morita equivalence bimodule to construct the isomorphism of K-groups directly.

3.3. Crossed Products

3.3.1. Group Actions on C^*-Algebras.

DEFINITION 3.19. Let A be a C^*-algebra. We denote by $\operatorname{Aut} A$ the group of $*$-automorphisms of A (algebra automorphisms preserving the $*$-operation). This is a topological group with the topology of pointwise convergence. (When $A = C_0(X)$,

$\operatorname{Aut} A$ is the group of homeomorphisms of X with the compact-open topology.) If G is a locally compact group, an *action of G on A* or *C^* dynamical system* is a continuous homomorphism $\alpha\colon G \to \operatorname{Aut} A$.

Note: While one can also consider the norm topology on $\operatorname{Aut} A$, it is usually too strong to be useful. For example, the left translation action of G on $A = C_0(G)$ is usually not continuous for the norm topology on $\operatorname{Aut} A$. But this action is continuous for the topology of pointwise convergence. Let $\alpha\colon G \to A$ be an action of a locally compact group on a C^*-algebra A. A *covariant pair* of representations of (A, G) consists of the following:

(1) a Hilbert space $\mathcal{H}$,
(2) a strongly continuous unitary representation U of G on $\mathcal{H}$, meaning that $g \mapsto U(g)$ gives a homomorphism from G to the group of unitaries on $\mathcal{H}$, and for any $\xi \in \mathcal{H}$, the map $g \mapsto U(g)\xi$ is continuous $G \to \mathcal{H}$,
(3) a $*$-homomorphism $\varphi\colon A \to \mathcal{L}(\mathcal{H})$ satisfying
$$U(g)\varphi(a)U(g)^* = \varphi(\alpha_g(a)), \quad g \in G, \quad a \in A\,.$$

EXAMPLES 3.20.

(1) $A = \mathbb{C}$, U a [strongly continuous] unitary representation of G. Here A acts just by scalar multiplication, the action of G on A is trivial, and condition (3) is automatic.
(2) More generally, if the action of G on A is trivial, a covariant pair is just a commuting pair (of a unitary representation of G and a $*$-representation of A) on the same Hilbert space.
(3) $A = C_0(G)$, G acting by left translation on A, U the left regular representation of G on $L^2(G)$, φ the action on $C_0(G)$ on $L^2(G)$ by pointwise multiplication. This is a covariant pair since
$$\begin{aligned}\big(U(g)\varphi(a)U(g)^*f\big)(s) &= \big(\varphi(a)U(g)^*f\big)(g^{-1}s)\\ &= a(g^{-1}s)\big(U(g)^*f\big)(g^{-1}s)\\ &= a(g^{-1}s)f(s)\\ &= \big(\varphi(g\cdot a)f\big)(s)\,.\end{aligned}$$
(4) H a closed subgroup of G, $A = C_0(G/H)$, G acting by left translation on A, U an "induced representation" $U = \operatorname{Ind}_H^G \rho$ of G, that is, the representation of G by left translation on the space
$$\mathcal{H} = \{f : G \to V \mid f(gh) = \rho(h)^{-1}f(g), \|f\| \in L^2(G/H)\}.$$
Here ρ is assumed to be a unitary representation of H on V. For simplicity, we've assumed G/H carries a G-invariant measure. (If this is not true, some modular functions need to be inserted.) The action of A on $\mathcal{H}$ is the obvious one by multiplication. This extra structure on U is called a *system of imprimitivity*, and plays a key role in the Mackey Imprimitivity Theorem.

3.3.2. Crossed Products. Given an action $\alpha\colon G \to A$ of a locally compact group on a C^*-algebra, there is a unique C^*-algebra whose $*$-representations are in natural bijection with the covariant pairs of representations of (A, G). It is called the *crossed product* and denoted $A \rtimes_\alpha G$ or $C^*(G, A, \alpha)$. When G is discrete and A is unital, $A \rtimes_\alpha G$ is generated by a copy of A and unitary elements u_g, $g \in G$, such that $u_g\, a\, u_g^* = \alpha_g(a)$. It is the completion of the finite linear combinations $\sum_g u_g\, a_g$, $a_g \in A$, in the greatest C^* norm. When G is not discrete or A is nonunital, one still has a copy of A and unitary elements u_g as above, but they live in the multiplier algebra of the crossed product. The crossed product itself is generated by the products $\varphi \cdot a$, where $\varphi \in C_c(G)$ (viewed as a "smeared out" sum $\int \varphi(g)\, u_g\, dg$ of the elements u_g) and $a \in A$. A detailed introduction to crossed products may be found in [**172**].

EXAMPLES 3.21. The examples that follow correspond precisely to Examples 3.20.

(1) $A = \mathbb{C}$, $A \rtimes G$ is the (maximal) *group C^*-algebra $C^*(G)$ of G*, the largest C^*-algebra completion of the convolution algebra $L^1(G)$.
(2) More generally, if α is trivial, $A \rtimes_\alpha G$ is the (maximal) tensor product $C^*(G) \otimes_{\max} A$.
(3) If $A = C_0(G)$, with G acting by left translation, then we have a classic result:

    THEOREM 3.22 (Stone-von Neumann-Mackey). *If G acts on itself by left translations, then $C_0(G) \rtimes G \cong \mathcal{K}(L^2(G))$.*

(4) If $A = C_0(G/H)$, with G acting by left translation, $C_0(G/H) \rtimes G$ is Morita equivalent to $C^*(H)$, by Rieffel's version of the Mackey Imprimitivity Theorem.

3.3.3. Dual Actions. Suppose G is a locally compact abelian group. Its *dual group* is $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$, the set of continuous group homomorphisms from G into the circle group $\mathbb{T}$. This acquires a locally compact topology via the usual compact-open topology for maps, and a group structure via pointwise multiplication, so that $\widehat{G}$ is also a locally compact abelian group. There is an obvious pairing $\langle\ ,\ \rangle$ between G and $\widehat{G}$, with values in $\mathbb{T}$. *Pontrjagin duality* says that the dual of $\widehat{G}$ is G again, or more precisely, that the natural embedding of G into the dual of $\widehat{G}$ is an isomorphism. Incidentally, local compactness is crucial here. There are non-locally compact metrizable abelian topological groups with no continuous homomorphisms to $\mathbb{T}$ at all, so the "dual" of such a group is trivial and inadequate for recovering G by dualizing again.

EXAMPLES 3.23.

(1) $G = \mathbb{Z}$, $\widehat{G} = \mathbb{T}$. Similarly, if $G = \mathbb{Z}^n$, $\widehat{G} = \mathbb{T}^n$.
(2) *Vector groups* $\mathbb{R}^n$ are self-dual. However it is better to identify the Pontrjagin dual with the dual vector space, since this identification is *natural* and not dependent on choices. (If V is a finite-dimensional vector space over $\mathbb{R}$ and if V^* is its vector space dual, then we identify V^* with $\widehat{V}$ by means of the exponential map. In other words, if $\lambda \in V^*$, the corresponding character $V \to \mathbb{T}$ is $v \mapsto e^{i\lambda(v)}$.)

If G as above acts on a C^*-algebra A, there is a *dual action* $\widehat{\alpha}$ of $\widehat{G}$ on $A \rtimes G$ which when G is discrete is given by $\widehat{\alpha}_\gamma(u_g\, a) = \langle \gamma, g\rangle u_g\, a$. (The action is isometric since $\langle \gamma, g\rangle$ has absolute value 1.) In the general case, if $\varphi \in C_c(G)$ and $a \in A$, $\widehat{\alpha}_\gamma(\varphi \cdot a) = (\gamma\varphi) \cdot a$, i.e., we multiply φ pointwise by γ (viewed as a function on G) and leave a unchanged.

The following result, known as the *Takai Duality Theorem*, generalizes the Stone-von Neumann-Mackey Theorem.

THEOREM 3.24 (Takai [**158**], [**172**, §7.1]). *Suppose α is an action of a locally compact abelian group on a C^*-algebra A. Let $\widehat{\alpha}$ be the dual action on the crossed product. Then*

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G))\,.$$

Furthermore, the double dual action can be identified with $\alpha \otimes \operatorname{Ad}\lambda$, where

$$(\operatorname{Ad}\lambda)(g)(a) = \lambda_g a \lambda_g^*,$$

λ the left regular representation.

CHAPTER 4

Continuous-Trace Algebras and Twisted K-Theory

4.1. Continuous-Trace Algebras and the Brauer Group

4.1.1. Continuous-Trace Algebras.

4.1.1.1. *The Trace Function.* We want to introduce a particularly nice class of C^*-algebras, called algebras with *continuous trace*. Roughly speaking, these are algebras that look something like $C(X) \otimes M_n(\mathbb{C})$ (in the unital case) or $C_0(X) \otimes \mathcal{K}$ (in the nonunital case).

Definition 4.1. Let A be a C^*-algebra. The *spectrum*[1] of A, denoted $\widehat{A}$, is the set of unitary equivalence classes of irreducible $*$-representations of A on Hilbert spaces. This is a topological space with the *Fell topology*, defined by pointwise convergence of matrix coefficients $a \mapsto \langle \pi(a)\, \xi,\, \eta \rangle$. Note that $\widehat{A}$ is not necessarily Hausdorff, and (for non-type I C^*-algebras) is not even necessarily a T_0 space.

Given $a \in A_+$, its *trace function* $\widehat{A} \to [0, \infty]$ is defined by $[\pi] \mapsto \operatorname{Tr} \pi(a)$. (For a positive operator on a Hilbert space, the trace is the sum of the eigenvalues (counted with multiplicities), if this makes sense and converges, or is ∞ if there is any non-discrete spectrum or if the sum doesn't converge. Note that $\operatorname{Tr} \pi(a)$ only depends on the unitary equivalence class of π, since conjugating π by a unitary operator doesn't change the trace.) This function is *lower semi-continuous*, not necessarily continuous.

4.1.1.2. *Fell's Condition.* Fell characterized those C^*-algebras for which there are lots of elements with finite and continuous trace function.

Theorem 4.2 (Fell [**62**]). *Let A be a C^*-algebra. Then the following conditions are equivalent:*

(1) *Elements $a \in A_+$ with finite and continuous trace function are dense.*
(2) *$\widehat{A}$ is Hausdorff, and A has lots of local rank-one projections, in that for every $[\pi] \in \widehat{A}$, there is an element $a \in A_+$ with $\sigma(a)$ a rank-one self-adjoint projection (in the Hilbert space of σ) for every $[\sigma]$ in a neighborhood of $[\pi]$ in $\widehat{A}$.*

One cannot omit the requirement that $\widehat{A}$ be Hausdorff, since the algebra

$$A = \{ f \in C\left([0,1], M_2(\mathbb{C})\right) \mid f(1) \text{ is a diagonal matrix} \}$$

has as spectrum the non-Hausdorff space (a line segment with "two endpoints" on one side) and otherwise satisfies (2) but not (1). The maps $\pi^+ \colon f \mapsto$ $(1,1)$ entry of $f(1)$ and $\pi^- \colon f \mapsto (2,2)$ entry of $f(1)$ are irreducible 1-dimensional

[1]The connection with the use of this term in Section 3.1.1 is that if A is unital and $a \in A$ commutes with a^*, then the spectrum of a in A is the same as the spectrum of the C^*-subalgebra of A generated by a and 1.

representations of A. The other irreducible representations are given by $\pi_t\colon f \mapsto f(t)$, $t < 1$. The elements with continuous trace are not dense, since for $f \in A$, as $t \to 1^-$, $\mathrm{Tr}\,\pi_t(f) \to \mathrm{Tr}\,\pi^+(f) + \mathrm{Tr}\,\pi^-(f)$, while π_t converges to both π^+ and π^- in the Fell topology, so that the trace function of f is not continuous if both of $\pi^\pm(f)$ are non-zero.

4.1.1.3. *The Theory of Dixmier-Douady.* A C^*-algebra A satisfying the equivalent conditions of Theorem 4.2 is called a *C^*-algebra with continuous trace* or a *continuous-trace algebra* (sometimes we will use the abbreviation CT-algebra). The structure of these algebras was investigated by Dixmier and Douady [**55**]. Detailed expositions of their work can be found in [**54**, Ch. 4 and 10], in [**135**], and in [**147**].

THEOREM 4.3 (Dixmier-Douady). *Let X be a second-countable locally compact Hausdorff space, and let A be a separable continuous-trace algebra with spectrum X. Then A is isomorphic to the algebra $\Gamma_0(X, \mathcal{A})$ of sections vanishing at ∞ of a locally trivial bundle $\mathcal{A}$ with fibers isomorphic to $\mathcal{K}$, provided that either*

(1) *A is stable, i.e., $A \cong A \otimes \mathcal{K}$, or*
(2) *X is finite dimensional and each irreducible $*$-representation of A is of dimension $\aleph_0$.*

4.1.1.4. *The Dixmier-Douady Class and Bundle Theory.* Whether or not A is locally trivial in the sense of Theorem 4.3, Dixmier and Douady showed that A has a characteristic class $\delta(A) \in H^3(X, \mathbb{Z})$. This class doesn't change if we replace A by $A \otimes \mathcal{K}$. One may explain this *Dixmier-Douady class* as follows:

Suppose for simplicity that A is locally trivial (otherwise replace A by $A \otimes \mathcal{K}$ and use Theorem 4.3) and comes from a bundle $\mathcal{A}$. This bundle has fibers $\mathcal{K}$ and structure group $\mathrm{Aut}\,\mathcal{K} \cong PU(\mathcal{H}) = U(\mathcal{H})/\mathbb{T}$, where $\dim \mathcal{H} = \aleph_0$. But $U(\mathcal{H})$ is contractible for $\mathcal{H}$ infinite dimensional. So PU has the homotopy type $B\mathbb{T}$, which is a $K(\mathbb{Z}, 2)$ space[2]. And principal PU-bundles over X are classified by

$$[X, BPU] = [X, K(\mathbb{Z}, 3)] = H^3(X, \mathbb{Z})\,.$$

In this way, we see that every stable CT-algebra defines a class in H^3, and every class in H^3 comes from a stable CT-algebra.

4.1.1.5. *Gerbes.* There is an alternative approach to the theory of continuous-trace algebras and the Dixmier-Douady class via the theory of *gerbes*, as discussed in [**122**], [**20**], or [**78**]. The main advantage of this approach is that it meshes well with the theory of the H-flux in string theory. Recall that we pointed out before that this is always given by a class in $H^3(X, \mathbb{Z})$, where X is spacetime. This class is the Dixmier-Douady class of a gerbe, which in turn determines a CT-algebra. A gerbe is precisely what is needed to give a rigorous definition of the Wess-Zumino term (1.3) in the string action, without vague references to "locally defined" differential forms [**66**]. Since the theory of gerbes will not be needed for the rest of the book, we have relegated further discussion to a "starred section," Section 4.3.

4.1.2. The Brauer Group.

[2]This means a space with the homotopy type of a CW complex and with exactly one non-zero homotopy group, namely $\pi_2 \cong \mathbb{Z}$. Such a space is unique up to homotopy equivalence.

4.1.2.1. *Review: The Brauer Group of a Field.* Before we get to the next topic, it's convenient to review some classical algebra. Suppose F is a field (this time in the sense of commutative algebra, not in the sense of physics!). By *Wedderburn's Theorem*, every finite dimensional central simple[3] algebra over F is of the form $M_n(D)$, where D is a division algebra having F as its center. Two such algebras are F-Morita equivalent if and only if the associated division algebras D are F-isomorphic.

The Morita equivalence classes $[A]$ of central simple algebras A over F form an abelian group under the operation $\otimes_F$, with F as identity element and $[A^\circ]$ (A° the same underlying vector space as A, but with the order of multiplication reversed) as the inverse of $[A]$. (The point is that $A \otimes_F A^\circ \cong \operatorname{End}_F(A)$, which is isomorphic to a matrix algebra over F and is thus Morita equivalent to F.) This group is called the *Brauer group* $\operatorname{Br} F$. It can be shown to be isomorphic to $H^2(\operatorname{Gal}(F^s/F), (F^s)^\times)$, F^s the separable closure of F. For purposes of relating this to what will come later, it is better to think of this group as $H^2_{\text{ét}}(\operatorname{Spec} F, \mathbb{G}_m)$. The point (see [**111**]) is that the étale open sets of $\operatorname{Spec} F$ correspond to finite Galois coverings L of F, on which $\mathbb{G}_m$ takes the value $L^\times$, viewed as a module over $\operatorname{Gal}(L/F)$.

4.1.2.2. *The Brauer Group of a Commutative Ring.* Similarly there is a notion of *Brauer group* $\operatorname{Br} R$ when R is a commutative ring [**10**]. This time central simple algebras are replaced by *Azumaya algebras* or *central separable algebras*, R-algebras A with R as center for which A is finitely generated projective as a module over $A \otimes_R A^\circ$. The R-Morita equivalence classes of these algebras again form a Brauer group $\operatorname{Br}(R)$, with the group operation as tensor product over R and $[A]^{-1} = [A^\circ]$. This generalizes the Brauer group for fields.

4.1.2.3. *The Grothendieck-Serre Theorem.* The special case of $\operatorname{Br} R$ with $R = C(X)$ was studied by Grothendieck and Serre, who found an analogue of the Galois cohomology computation of the Brauer group for a field.

THEOREM 4.4 (Grothendieck-Serre). *Let X be a connected finite CW-complex. Then the Brauer group of $C(X)$ can naturally be identified with the torsion subgroup of $H^3(X, \mathbb{Z})$, and the Azumaya algebras over $C(X)$ are all of the form $\Gamma(X, \mathcal{A})$, where $\mathcal{A}$ is a locally trivial bundle of algebras over X with fibers $M_n(\mathbb{C})$ and structure group* $\operatorname{Aut} M_n \cong PGL(n, \mathbb{C})$.

For the proof, see [**72**] or [**49**, Theorem 9.13]. Even though this result at first sight appears quite different from the result from fields, it is actually almost the same. Indeed, consider the short exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \underline{\mathbb{C}} \xrightarrow{\exp} \underline{\mathbb{C}^\times} \to 0.$$

The sheaf $\underline{\mathbb{C}}$ is fine, so it has vanishing higher cohomology. Thus the long exact sequence in sheaf cohomology gives

$$0 = H^2(X, \underline{\mathbb{C}}) \to H^2(X, \underline{\mathbb{C}^\times}) \to H^3(X, \mathbb{Z}) \to H^2(X, \underline{\mathbb{C}}) = 0,$$

and so we can write $H^3(X, \mathbb{Z})$ as $H^2(X, \underline{\mathbb{C}^\times})$, which looks a lot like $H^2_{\text{ét}}(\operatorname{Spec} F, \mathbb{G}_m)$ in the case of the Brauer group of a field.

[3] "Central simple" means the algebra has no non-trivial two-sided ideals and has F as its center.

4.1.2.4. *Green's Topological Brauer Group.* When X is a connected finite CW-complex, the Azumaya algebras over $C(X)$ are precisely the unital CT-algebras with center $C(X)$. Philip Green proposed taking a broader point of view in order to account for the missing non-torsion classes in H^3 and thus to get an even better correspondence with the étale cohomology analogy: allowing X to be locally compact and considering *all* CT-algebras A with spectrum $\widehat{A} \cong X$, viewed as algebras over $C_0(X)$. If one then considers these algebras up to $C_0(X)$-linear (topological) Morita equivalence, they again form a Brauer group $\operatorname{Br} X$ with group operation given by $\otimes_{C_0(X)}$ (topological tensor product over $C_0(X)$), and inversion given by $[A] \mapsto [A^\circ]$. More details may be found in [**135**, §6.1].

THEOREM 4.5 (Green). *Let X be a second-countable locally compact Hausdorff space. Then the Dixmier-Douady class defines an isomorphism $\operatorname{Br} X \cong H^3(X, \mathbb{Z})$. When X is a finite CW-complex, the Grothendieck-Serre Brauer group embeds as the torsion subgroup.*

PROOF. We have already seen that the stable continuous-trace algebras over X are classified by their Dixmier-Douady invariants, and that every class in $H^3(X, \mathbb{Z})$ arises in this way. Furthermore, from Theorem 3.7, stable isomorphism and Morita equivalence coincide for separable C^*-algebras, and it is easy to see that this fact is true "over X" also. So $H^3(X, \mathbb{Z})$ classifies the CT-algebras over X up to $C_0(X)$-linear (topological) Morita equivalence. We just need to check that the group operation in the Brauer group corresponds to addition of Dixmier-Douady invariants, and that $CT(X,H)^\circ \cong_X CT(X,-H)$. To prove this, think of $CT(X,H)$ as $\Gamma_0(X, \mathcal{A})$, where $\mathcal{A}$ is a bundle of algebras with fibers $\cong \mathcal{K}$. Then if $CT(X,H') = \Gamma_0(X, \mathcal{A}')$,

$$CT(X,H) \otimes_X CT(X,H') = \Gamma_0(X, \mathcal{A} \otimes \mathcal{A}'),$$

where $\otimes$ denotes the fiberwise tensor product of algebra bundles. The transition functions for $\mathcal{A} \otimes \mathcal{A}'$ are obtained by tensoring together the transition functions for $\mathcal{A}$ and for $\mathcal{A}'$, so one can see that the Dixmier-Douady class for $\mathcal{A} \otimes \mathcal{A}'$ is the sum of those for $\mathcal{A}$ and $\mathcal{A}'$ separately. $\square$

4.2. Twisted K-Theory

4.2.0.5. *An Analogy: Cohomology with Local Coefficients.* To explain the idea of twisted K-theory, it helps to think of an analogous (but simpler) theory: cohomology with local coefficients. (Čech) cohomology can be identified with the sheaf cohomology of a *constant* sheaf of abelian groups. If we replace this by a *locally constant* sheaf or *local coefficient system*, we get cohomology with local coefficients. For example, an oriented compact n-manifold M satisfies Poincaré duality $H^i(M, \mathbb{Z}) \xrightarrow{\cong} H_{n-i}(M, \mathbb{Z})$. If M isn't orientable, there is a canonical local coefficient system $\underline{\mathbb{Z}}$ that has fiber $H_n(M, M \smallsetminus \{x\})$ at $x \in M$, and we instead have Poincaré duality *with local coefficients* $H^i(M, \underline{\mathbb{Z}}) \xrightarrow{\cong} H_{n-i}(M, \mathbb{Z})$.

4.2.0.6. *Twisted K-Theory.* In a similar way, if an n-manifold has a spin^c structure, it satisfies Poincaré duality in K-theory, $K^i(M) \xrightarrow{\cong} K_{n-i}(M)$. If there is no spin^c structure, we need *twisted K-theory.* The simplest way to define this is, given a class $H \in H^3(M, \mathbb{Z})$, to take the K-theory of a *noncommutative* C^*-algebra, namely the stable continuous-trace algebra $CT(M,H)$ with H as its Dixmier-Douady class. Thus we define $K^{-i}(M,H) = K_i(CT(M,H))$. (The idea of using Azumaya algebras to define twisted K-theory goes back to Donovan and Karoubi [**56**], and the idea of using continuous-trace algebras is due to the author [**139, 141**]. For more recent treatments of twisted K-theory, see [**8, 9**] and [**95**].)

EXAMPLES 4.6.

(1) $M = SU(3)/SO(3)$, where $SO(3)$ sits inside $SU(3)$ as the real-valued special unitary matrices. The Lie group $SO(3)$ has $SU(2)$ as a double cover, and thus is topologically $\mathbb{RP}^3$. We have a long exact homotopy sequence

$$\cdots \xrightarrow{\partial} \pi_n(SO(3)) \to \pi_n(SU(3)) \to \pi_n(SU(3)/SO(3)) \xrightarrow{\partial} \cdots ,$$

and since $SU(3)$ is 2-connected and $\pi_1(SO(3)) \cong \mathbb{Z}/2$ while $\pi_2(SO(3)) = 0$, we see that $\pi_1(M) = 0$ and $\pi_2(M) \cong \mathbb{Z}/2$. By the Hurewicz Theorem and Poincaré duality, $H_2(M,\mathbb{Z}) \cong H^3(M,\mathbb{Z}) \cong \mathbb{Z}/2$, while $H^2(M,\mathbb{Z}) \cong H_3(M,\mathbb{Z}) \cong 0$. The manifold M is non-spin (basically because the adjoint action $SO(3) \to SO(\mathfrak{su}(3)/\mathfrak{so}(3)) \cong SO(5)$ doesn't factor through $\mathrm{Spin}(5)$), and since its second Stiefel-Whitney class[4] in $H^2(M,\mathbb{Z}/2)$ can't be the reduction of an integral class (since $H^2(M,\mathbb{Z}) = 0$), it also does not have a spin^c structure. In fact, this is the simplest example of an oriented manifold without a spin^c structure. The Atiyah-Hirzebruch spectral sequence degenerates at E_2, since Sq^3 acts trivially on $H^*(M,\mathbb{Z})$, so $K^0(M) \cong \mathbb{Z}$ and one has an exact sequence

$$0 \to \mathbb{Z} \to K^1(M) \to \mathbb{Z}/2 \to 0,$$

but this is not enough information to determine whether $K^1(M) \cong \mathbb{Z}$ or $K^1(M) \cong \mathbb{Z} \oplus \mathbb{Z}/2$. For those who know about such things, one can resolve this question using the Hodgkin spectral sequence

$$\mathrm{Tor}_*^{R(SU(3))}\big(R(SO(3)), \mathbb{Z}\big) \Rightarrow K^*(SU(3)/SO(3))$$

[**79**]. (This arises from the isomorphisms $K^*(G/H) \cong K_G^*(G/H \times G)$, $K_G^*(G/H) \cong R(H)$, and $K_G^*(G) \cong \mathbb{Z}$, discussed below in Section 5.1.1, along with the "Künneth Theorem" in equivariant K-theory.) One finds that $R = R(SU(3)) \cong \mathbb{Z}[x,y]$, and $R(SO(3))$ can be identified with the R-module $R/(x-y)$, which has the projective resolution

$$0 \to R \xrightarrow{x-y} R \to M \to 0,$$

so the Tor groups are both isomorphic to $\mathbb{Z}$ and $K^1(M) \cong \mathbb{Z}$. Similarly one can show that $K_0(M) \cong K_1(M) \cong \mathbb{Z}$.

The twisted K-theory $K^*(M,H)$ can also be computed by the Hodgkin spectral sequence, this time as

$$\mathrm{Tor}^*_{R(SU(3))}\big(\widetilde{R}(SO(3)), \mathbb{Z}\big) \Rightarrow K^*(SU(3)/SO(3), H),$$

with $\widetilde{R}(SO(3))$ denoting the "genuine" representations of the double cover $SU(2)$ of $SO(3)$, i.e., those that do not descend to $SO(3)$. It turns out that the result is basically the same as before, i.e., $K^*(M,H)$ is torsion-free, and one has a Poincaré duality isomorphism $K^{\mathrm{even}}(M,H) \cong K_{\mathrm{odd}}(M)$, $K^{\mathrm{odd}}(M,H) \cong K_{\mathrm{even}}(M)$, implemented by the "twisted Dirac operator."

[4] We discussed Chern classes in Chapter 2. The Stiefel-Whitney classes w_j in $(\mathbb{Z}/2)$-cohomology are defined similarly for *real* vector bundles. Here we are talking about w_2 of the tangent bundle, which is the obstruction to a spin structure on an oriented manifold.

(2) If $H = 0$, then $CT(M, 0) = C_0(M) \otimes \mathcal{K}$, and $K^{-i}(M, H) = K_i(C_0(M) \otimes \mathcal{K}) \cong K_i(C_0(M)) \cong K^{-i}(M)$ by Morita invariance. So twisted K-theory generalizes ordinary K-theory.

4.2.0.7. *The Atiyah-Hirzebruch Spectral Sequence.* Recall Theorem 2.7 about how to compute K-theory from ordinary cohomology. Something quite similar is true in the case of twisted K-theory $K^*(X, H)$, $H \in H^3(X, \mathbb{Z})$, except that the differentials in the spectral sequence now involve H as well. (See [**9**] for more details.) The first non-zero differential is $d_3\colon H^p(X, \mathbb{Z}) \to H^{p+3}(X, \mathbb{Z})$, which is equal to the sum of the Steenrod operation Sq^3 and cup product with H.

EXAMPLE 4.7. $X = S^3$, $H = k \neq 0$ (when we identify $H^3(S^3)$ with $\mathbb{Z}$). In this case $\mathrm{Sq}^3 = 0$ but $d_3\colon \mathbb{Z} \cong H^0(X) \to H^3(X) \cong \mathbb{Z}$ is multiplication by k. So we get $K^0(S^3, H) = 0$ and $K^1(S^3, H) \cong \mathbb{Z}/k$.

EXAMPLE 4.8. The Atiyah-Hirzebruch spectral sequence can also be used to compute the twisted K-theory in Example 4.6(1) above. The E_2 term is the same as for the untwisted K-theory, but now d_3 sends the generator of $H^0(M, \mathbb{Z})$ to $H \in H^3(M, \mathbb{Z})$, which has the effect of killing off $E_3^{3,0}$. So then it's clear that $K^*(M, H)$ must be torsion-free.

4.2.0.8. *Twisted K-Theory and String Theory.* So what does any of this have to do with string theory and string duality? Well, what we said before about brane charges being classified by K-theory is not exactly right. This is true when the H-flux is trivial, but not in general. By [**65**], in type II string theory, W_3 of a stable D-brane Y (this is a characteristic class of Y taking values in the 2-torsion subgroup of $H^3(Y, \mathbb{Z})$) must match the restriction of the H-flux. So if this is non-zero, the D-branes do *not* usually have spin^c structures and thus do not define classes in topological K-homology, in the way we discussed earlier (see section 2.3.1). However, they define classes in *twisted* K-homology. We can also dualize by Poincaré duality, since the spacetime manifold is still a spin^c manifold. In general type II string theory, the brane changes take values in $K^*(X, H)$, the Ramond-Ramond charges in the even group in type IIB and in the odd group in type IIA [**173**].[5]

4.3. ☆ The Theory of Gerbes

In this section, we give a brief introduction to gerbes for the interested reader. Anyone who wants to learn more should first read Hitchin's concise introduction, [**78**], and then consult the references quoted there, especially Brylinski's book [**31**].

Roughly speaking, just as a line bundle is a geometric object giving rise to a class in H^2 (the Chern class c_1, or in terms of Chern-Weil theory, the de Rham class of the curvature form of a connection on the bundle), so a gerbe is a geometric object giving rise to a class in H^3. Thus it is natural to try to connect gerbes with continuous-trace algebras and the Dixmier-Douady class.

There are two main theories of gerbes, which are closely related but not exactly the same. One is the theory of Giraud [**67**, §III.2] and Brylinski [**31**], based on categorical language. In this theory, a gerbe is really a sheaf of groupoids. The French word *gerbe* is defined by Larousse as "ensemble de choses en faisceau," a

[5]Because of differences in sign conventions, some authors use $K^*(X, -H)$. The groups $K^*(X, H)$ and $K^*(X, -H)$ are isomorphic (as abstract groups).

collection of things (e.g., flowers) in a bundle (faisceau), but the word has been chosen here since the French word *faisceau* is used for the mathematical concept of "sheaf." Since all of this seems rather abstract, we will instead discuss Murray's theory of "bundle gerbes" [**122, 123**], which are both more concrete and a little closer to what is needed for applications to physics. Since we will not discuss Brylinksi-style gerbes here (although they will reappear implicitly in Section 10.3.3), we will generally abbreviate "bundle gerbe" to "gerbe." It will also be convenient to switch from discussion of principal $\mathbb{C}^\times$-bundles (which are what Murray uses in [**122**]) to complex line bundles, which were considered in Chapter 2.

4.3.1. Bundle Gerbes.

Definition 4.9 (Murray). A *bundle gerbe* over M, or a simply a gerbe over M, is given by a pair (L, Y), where $\pi\colon Y \twoheadrightarrow M$ is a surjective map with local sections (often a locally trivial fiber bundle), and L is a complex line bundle over $Y^{[2]}$, where $Y^{[2]} = \{(y_1, y_2) \mid y_j \in Y, \pi(y_1) = \pi(y_2)\}$. There is one extra piece of structure, which we've suppressed in the notation, namely a "product" on L. This is an isomorphism μ of line bundles over $Y^{[3]} = \{(y_1, y_2, y_3) \mid y_j \in Y, \pi(y_1) = \pi(y_2) = \pi(y_3)\}$,

$$\mu\colon\ \pi_{12}^*(L) \otimes \pi_{23}^*(L) \xrightarrow{\cong} \pi_{12}^*(L),$$

where $\pi_{jk}\colon Y^{[3]} \to Y^{[2]}$ is the projection using coordinates j and k. The multiplication μ is required to satisfy the associative law

$$\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu)$$

on $Y^{[4]}$. (Here $Y^{[n]} = \{(y_1, y_2, \dots, y_n) \mid y_j \in Y, \pi(y_1) = \pi(y_2) = \cdots = \pi(y_n)\}$ for any n.)

One can get an (uninteresting) example of a bundle gerbe by starting with a line bundle K over Y and letting

$$L = \operatorname{Hom}(\pi_1^*(K), \pi_2^*(K)) = \pi_1^*\left(K^{-1}\right) \otimes \pi_2^*(K),$$

with the obvious product

$$\operatorname{Hom}(\pi_1^*(K), \pi_2^*(K)) \otimes \operatorname{Hom}(\pi_2^*(K), \pi_3^*(K)) \to \operatorname{Hom}(\pi_1^*(K), \pi_3^*(K)).$$

Such a gerbe is called *trivial*.

Definition 4.10. Given a bundle gerbe (L, Y) over M as in Definition 4.9, one can define a class in $H^3(M, \mathbb{Z})$, called its *Dixmier-Douady class*, as follows. Choose an open covering $\{U_\alpha\}$ of M, such that $\pi\colon Y \to M$ has a section s_α over each U_α. Let $U_{\alpha,\beta} = U_\alpha \cap U_\beta$; then $(s_\alpha, s_\beta)\colon U_{\alpha,\beta} \to Y^{[2]}$. Define a line bundle $L_{\alpha,\beta} = (s_\alpha, s_\beta)^*(L)$ over $U_{\alpha,\beta}$ for each pair of indices α and β. Then the product μ in Definition 4.9 gives an isomorphism $L_{\alpha,\beta} \otimes L_{\beta,\gamma} \otimes L_{\gamma,\alpha} \xrightarrow{\cong} L_{\alpha,\alpha} \cong \mathbb{C}$ (the trivial line bundle) over $U_{\alpha,\beta,\gamma}$, and thus is multiplication by a function $g_{\alpha,\beta,\gamma}\colon U_{\alpha,\beta,\gamma} \to \mathbb{C}^\times$. (Note that we don't have to specify which two factors we multiply first because of the associative rule for μ.) It is easy to check that $\{g_{\alpha,\beta,\gamma}\}$ is a Čech 2-cocycle and thus defines a class in $H^2(M, \underline{\mathbb{C}^\times}) \cong H^3(M, \mathbb{Z})$, and that the class obtained is independent of all choices made. Murray also shows that vanishing of the Dixmier-Douady class is precisely the condition for the bundle gerbe to be trivial.

Example 4.11. Bundle gerbes arise when one considers the case of a principal G-bundle $\pi\colon Y \twoheadrightarrow M$, G some topological group, and a central extension of topological groups

$$1 \to \mathbb{T} \xrightarrow{\iota} \widetilde{G} \xrightarrow{q} G \to 1.$$

Then the bundle Y over M may or may not lift to a principal $\widetilde{G}$-bundle over M. For example, if $G = SO(n) = \mathrm{Spin}(n)/\{\pm 1\}$ and $\widetilde{G} = \mathrm{Spin}^c(n) = \mathbb{T} \times_{\{\pm 1\}} \mathrm{Spin}(n)$, and if Y is the oriented frame bundle of an oriented Riemannian manifold M, then a lifting of this bundle to a principal $\widetilde{G}$-bundle over M is a spin^c structure on M compatible with the orientation, which may not exist at all, or may not be unique if it does exist.

Out of this data one can construct a bundle gerbe in a canonical way. We let $P \to Y^{[2]}$ be defined by

$$P_{(x,y)} = \{h \in \widetilde{G} \mid x \cdot q(h) = y\},$$

with multiplication defined by the product in $\widetilde{G}$. (For this to make sense, we use the fact that $\mathbb{T} = q^{-1}(1)$ is central.) This is a principal $\mathbb{T}$-bundle over $Y^{[2]}$ and we let $L = P \times_{\mathbb{T}} \mathbb{C}$ be the associated line bundle. This bundle gerbe is trivial exactly when a bundle lifting to a principal $\widetilde{G}$-bundle exists; in fact, a lifting of the bundle gives rise to a trivialization of the gerbe.

Example 4.12 (The gerbe of a continuous-trace algebra). Let X be a second-countable locally compact Hausdorff space, and let $\delta \in H^3(X, \mathbb{Z})$. We can think of δ as a homotopy class of a map $X \to BPU(\mathcal{H})$, $\mathcal{H}$ an infinite-dimensional separable Hilbert space, and let Y be the associated principal PU-bundle over X. Now consider the central extension of groups

$$1 \to \mathbb{T} \to U(\mathcal{H}) \to PU(\mathcal{H}) \to 1.$$

Since $U(\mathcal{H})$ is contractible, every principal $U(\mathcal{H})$-bundle over X is trivial. So if the $PU(\mathcal{H})$-bundle Y over X lifts to a principal $U(\mathcal{H})$-bundle over X, that means Y is a trivial bundle, which means $\delta = 0$. On the other hand, as explained in Example 4.11, there is a "lifting bundle gerbe" associated to this situation, which is trivial exactly when the lifting exists, or in other words, exactly when $\delta = 0$. In fact, one can trace through the definitions and check that the Dixmier-Douady class of the gerbe is the same as for the continuous-trace algebra $CT(X, \delta)$.

4.3.2. Connections and Curvature. Let's start by reviewing an analogous situation. Suppose L is a line bundle over X. This line bundle is classified (up to isomorphism) by a characteristic class $c_1(L) \in H^2(X, \mathbb{Z})$. We can choose a connection ∇ on L (not at all unique), and the curvature form of this connection (divided by $-2\pi i$) is a representative for a de Rham class which is precisely the image of $c_1(L)$ under the map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$.

A similar situation occurs with bundle gerbes over X. Two gerbes are said to be stably isomorphic if they become isomorphic after tensoring both of them with trivial bundle gerbes. Then a gerbe is classified up to stable isomorphism [**124**, Proposition 3.2] precisely by its Dixmier-Douady class $\delta \in H^3(X, \mathbb{Z})$. There is a notion of a connection on the gerbe, as well as of the curvature form of the connection, and the de Rham class of the curvature is equal to the image of δ under the map $H^3(X, \mathbb{Z}) \to H^3(X, \mathbb{R})$.

Definition 4.13. A *gerbe connection* on a bundle gerbe (L, Y) over M is given by a hermitian metric and compatible hermitian connection ∇ on the line bundle L which are also compatible with the multiplication map μ, i.e., such that μ sends the induced connection on $\pi_{12}^*(L) \otimes \pi_{23}^*(L)$ to $\pi_{13}^*(\nabla)$. Murray [**122**] shows that such connections always exist. He also observes that if π_1, $\pi_2\colon Y^{[2]} \to Y$ are the two obvious projections and ω is a differential form on Y with $\pi_1^*\omega = \pi_2^*\omega$, then $\omega = \pi^*(\nu)$ for a unique form ν on the base space M. Similarly, if ω is a differential form on $Y^{[2]}$ and $\pi_{12}^*(\omega) - \pi_{13}^*(\omega) + \pi_{23}^*(\omega) = 0$, then $\omega = \pi_1^*\nu - \pi_2^*\nu$ for some form ν on Y, this time unique only up to the pull-back of a form on M.

We use these observations the following way. Consider the curvature 2-form F of a gerbe connection ∇ (since we are dealing with a line bundle, this is an ordinary 2-form on $Y^{[2]}$). Since μ sends the induced connection on $\pi_{12}^*(L)\otimes\pi_{23}^*(L)$ to $\pi_{13}^*(\nabla)$, $\pi_{12}^*(F) - \pi_{13}^*(F) + \pi_{23}^*(F) = 0$, and F can always be written in the special form $\pi_1^*B - \pi_2^*B$. The 2-form B on Y is called the *curving* form. (It is not necessarily unique, but we assume a specific choice has been made.) Since $dF = 0$ on $Y^{[2]}$, we have $\pi_1^*dB = \pi_2^*dB$, which now means that $dB = \pi^*H$ for some (uniquely defined) closed 3-form H on M. Murray calls $\frac{1}{2\pi i}H$ the *Dixmier-Douady form* of the gerbe with connection, and shows that its de Rham class is the image of the Dixmier-Douady class of the gerbe under the map $H^3(X, \mathbb{Z}) \to H^3(X, \mathbb{R})$. In particular, changing the connection on the gerbe can only change H by an exact form.

4.3.3. Gerbes and the Wess-Zumino Term. Finally, we can explain how gerbes can be used to give a mathematically more satisfying treatment of the Wess-Zumino term in string theory. The ideas of this section are due to Murray [**122**], Gawędzki and Reis [**66**], Carey, Mickelsson, and Murray [**44**], and Carey, Johnson, and Murray [**43**], though by now many other authors have produced other variants of the theory, each with its own advantages.

For simplicity we start with the case of the Wess-Zumino-Witten or WZW model. Let's take G to be a simply connected compact Lie group, e.g., $SU(n)$ with $n \geq 2$. Since it's well known that π_2 of every Lie group is 0, G is 2-connected. As a result, any map φ from a closed 2-manifold Σ into G extends to a map $\widetilde{\varphi}\colon M \to G$, where M is a 3-manifold with boundary (a "handlebody" like a solid torus) with $\partial M = \Sigma$.[6] We also assume that M and Σ are compatibly oriented. As we explained in Section 1.2.0.5, the Wess-Zumino term in the action is a factor of $e^{\int_M \widetilde{\varphi}^*(H)}$, where H is a closed bi-invariant 3-form on G such that $\frac{1}{2\pi i}H$ is integral[7], and this is independent of the choice of M and $\widetilde{\varphi}$, i.e., only depends on Σ and φ. Formally, we can also write this term as $e^{\int_\Sigma \varphi^*(B)}$, except for the fact that the B-field B is not globally defined on G. The advantage of the gerbe point of view is that we can indeed make sense of this, but in the following way. We choose a bundle gerbe $(L, \pi\colon Y \to G)$ over G with a gerbe connection ∇ having Dixmier-Douady form H and curving B. Now B is a global 2-form, but on Y instead of G. Under the map φ, the gerbe pulls back to a gerbe $(L', \pi'\colon Y' \to \Sigma)$ over Σ, where we have

[6] We need the fact that $H_2(G, \mathbb{Z}) = 0$, because if φ mapped $[\Sigma]$ to a non-trivial homology cycle in G, it could not bound the 3-chain defined by M.

[7] In the case of $SU(n)$, a multiple of $\operatorname{Tr}\big(g^{-1}dg\big)^3$ will do.

a commutative diagram

$$\begin{tikzcd} Y' \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi'"] & Y \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\varphi"] & G, \end{tikzcd}$$

but since Σ is only 2-dimensional,

$$d\big(\widetilde{\varphi}^* B\big) = \widetilde{\varphi}^*(dB) = \widetilde{\varphi}^*\big(\pi^*(H)\big) = (\pi')^*\big(\varphi^* H\big) = 0.$$

In other words, $\widetilde{\varphi}^* B$ is closed and the gerbe (L', Y') must be trivial, and there is a line bundle L'' over Y' such that $L' = \pi_1^* L'' \otimes \pi_2^* L''^{-1}$. Now we can choose a hermitian connection ∇'' on L'' inducing ∇', the gerbe connection on (L', Y'). If F'' is its curvature 2-form, then $\omega = \widetilde{\varphi}^* B - F''$ is a closed 2-form on Y' with $\pi_1^*\omega = \pi_2^*\omega$. Thus $\omega = (\pi')^*\nu$ for a unique (necessarily closed) 2-form ν on Σ. The Wess-Zumino term of (Σ, φ) is defined to be

$$WZ(\Sigma, \varphi) = e^{\int_\Sigma \nu}. \tag{4.1}$$

We claim this is independent of any choices and matches the original intuitive notion of $e^{\int_\Sigma \varphi^*(B)}$. Indeed, first consider the rather trivial case where the B-field is globally defined, so that the H-flux H is exact. In this case we don't really need gerbes at all, so we may take $Y = G$, π the identity map, L the trivial bundle with the obvious connection, and B the curving form. (Don't confuse *curving* with *curvature*; the latter is $\pi_1^* B - \pi_2^* B = B - B = 0$ in this case.) In this situation, $\widetilde{\varphi} = \varphi$, $F'' = 0$, and ν is simply $\varphi^* B$. So we recover the naive definition in this case.

Finally, we need to check that we have a definition of the Wess-Zumino term that is independent of all choices. Recall that the forms B on Y and H on G are fixed once and for all. The choices we made were a trivialization of L' (this is the line bundle L'' on Y' such that L' is identified with $\pi_1^* L'' \otimes \pi_2^* L''^{-1}$) and the corresponding connection ∇''. The bundle L'' is not necessarily unique, but if we make another choice (L''', ∇'''), then there is a line bundle K on Σ with a connection ∇_K such that $L''' = L'' \otimes \pi^* K$ and $\nabla''' = \nabla'' \otimes \pi^* \nabla_K$. Thus the form ν above will change by the curvature F_K of ∇_K. It follows that $WZ(\Sigma, \varphi)$, defined as in (4.1), will change by multiplication by $e^{\int_\Sigma F_K}$. However, the curvature form of a line bundle is always $2\pi i$ times an integral form (by Theorem 2.3), so this factor is of the form $e^{2\pi i n}$ for some integer n and thus is equal to 1. This completes the proof that $WZ(\Sigma, \varphi)$ is well defined.

CHAPTER 5

More on Crossed Products and Their K-Theory

5.1. A Categorical Framework

5.1.1. Equivariant K-Theory. For applications to come, we want to study the K-theory of crossed products. The simplest example is the case where G is a compact group (possibly even finite) acting on a C^*-algebra A. In this case there is a classical invariant of the action, the *equivariant K-theory* $K_*^G(A)$.

Definition 5.1. Let G be a compact group and let α be an action of G on a unital C^*-algebra A by automorphisms. We define the *equivariant K-group* $K_0^G(A)$ to be the Grothendieck group of equivalence classes of pairs (P, β), where P is a finitely generated projective A-module and β is an action of G on P compatible with the action α of G on A in the following sense: $\beta_g(a \cdot m) = \alpha_g(a) \cdot \beta_g(m)$. Some thought [**16**, Proposition 11.2.3] shows that this is equivalent to saying that (P, β) is a G-equivariant summand in $A^n = A \otimes \mathbb{C}^n$, for the G-action on $A \otimes \mathbb{C}^n$ given by the tensor product of α with a linear representation of G on $\mathbb{C}^n$, which we can take to be unitary since G is compact.

When $A = C(X)$ and G acts on X, we have

$$K_0^G(C(X)) \cong K_G^0(X),$$

the Grothendieck group of G-vector bundles $E \xrightarrow{p} X$. (These are just like regular vector bundles, except that there is an action of G on E, linear on fibers, making p into a G-equivariant map.)

Just as in the case of ordinary (non-equivariant) K-theory, one can extend K_0^G to a $\mathbb{Z}/2$-periodic homology theory K_*^G on C^*-algebras with G-actions, whether or not the algebras are unital.

We can now summarize some of the elementary properties of equivariant K-theory. These are all rather easy; the only hard foundational theorem is the equivariant version of Bott periodicity. For more details, the best reference is still the original paper of Segal [**148**]. The extension of Segal's results to the noncommutative case may be found in [**16**, Chapter 11] or in [**131**].

- $K_0^G(\mathbb{C}) \cong R(G)$, the *representation ring* of virtual (finite dimensional) representations of G. In other words, $R(G)$ consists of formal differences $[V] - [W]$, where V and W are finite dimensional linear representations of G, and the ring structure comes from the tensor product of representations. The structure of this commutative ring is studied in [**149**]. For example, $R(\mathbb{Z}/p) \cong \mathbb{Z}[t]/(t^p - 1)$, $R(\mathbb{T}) \cong \mathbb{Z}[t, t^{-1}]$ and $R(SU(n+1)) \cong \mathbb{Z}[t_1, \cdots, t_n]$.
- $K_*^G(A)$ is always a module over $R(G)$, for any A. The action comes from the rather obvious formula $[V, \rho] \cdot [P, \beta] = [V \otimes P, \rho \otimes \beta]$, when V is a

finite-dimensional $\mathbb{C}$-vector space with a representation ρ of G on it, P is a finitely generated projective A-module, and β is an action of G on P.

- If G acts trivially on A, then $K_*^G(A) \cong R(G) \otimes_{\mathbb{Z}} K_*(A)$ (as $R(G)$-modules).
- If G acts freely on X, $K_G^*(X) \cong K^*(X/G)$. (However, this does *not* mean that the $R(G)$-module structure on $K_G^*(X)$ has to factor through the augmentation $R(G) \twoheadrightarrow \mathbb{Z}$.)
- If $H \subset G$ are compact groups, A is an H-algebra, and we form $\operatorname{Ind}_H^G A$ as in Section 5.3 below, then $K_*^G(\operatorname{Ind}_H^G A) \cong K_*^H(A)$. When $A = C(X)$, X a compact H-space, this is saying that $K_G^*(G \times_H X) \cong K_H^*(X)$. For example (take $X = \mathrm{pt}$), $K_G^*(G/H) \cong K_H^*(\mathrm{pt}) \cong R(H)$. The relationship between module structures is this. $K_*^H(A)$ is an $R(H)$-module, while $K_*^G(\operatorname{Ind}_H^G A)$ is an $R(G)$-module. However, any finite-dimensional representation of G can be viewed by restriction as a representation of H, so we have a "forgetful map" $R(G) \to R(H)$. In this way, $K_*^H(A)$ can be regarded as an $R(G)$-module, and the isomorphism $K_*^G(\operatorname{Ind}_H^G A) \cong K_*^H(A)$ is an isomorphism of $R(G)$-modules.
- Again, if $H \subset G$ are compact groups, and if A is an G-algebra, then we can simply restrict the action to H and get a "forgetful map" $K_*^G(A) \to K_*^H(A)$, compatible with the map of rings $R(G) \to R(H)$.

Equivariant K-theory turns out to be closely related to K-theory of crossed products.

THEOREM 5.2 (Green-Julg [**90**], [**16**, Theorem 11.7.1]). *Let G be a compact group, α an action of G on a C^*-algebra A. Then there is a natural isomorphism $K_*^G(A) \cong K_*(A \rtimes_\alpha G)$.*

SKETCH OF PROOF. Basically, the idea of the proof is that it is enough to prove this for K_0 with A unital. Then one needs to take a G-equivariant projective A-module (P, β) as in Definition 5.1 and send its class in $K_0^G(A)$ to a class in $K_0(A \rtimes_\alpha G)$. When $P = A \otimes V$, V a finite dimensional G-module, V corresponds to a projection p in $C^*(G)$, and $p(A \rtimes_\alpha G)p$ is a unital subalgebra of the crossed product in which p defines a K_0 class. Thus p also defines a class in $K_0(A \rtimes_\alpha G)$, and one sends $[A \otimes V] \mapsto [p]$. Then one checks that if $A \otimes V$ splits as a direct sum, each summand defines a subprojection of p and thus a class in $K_0(A \rtimes_\alpha G)$. Finally, to construct the map $K_*^G(A) \to K_*(A \rtimes_\alpha G)$, one can show fairly easily [**16**, Proposition 11.2.3] that any pair (P, β) as in Definition 5.1 embeds as a direct summand in some $P = A \otimes V$ as above. The surjectivity depends on the fact that $C^*(G)$, and hence also $A \rtimes_\alpha G$, has an approximate identity consisting of projections associated to finite-dimensional representations of G. This can also be used to construct an inverse to the map $K_*^G(A) \to K_*(A \rtimes_\alpha G)$, so one has injectivity as well. $\square$

5.1.2. Questions About K-Theory of Crossed Products. We have seen that when G is compact, the K-theory of the crossed product $K_*(A \rtimes G)$ is the same as the equivariant K-theory $K_*^G(A)$. But whether or not G is compact, there are two natural questions one can ask about K-theory of crossed products:

QUESTIONS 5.3.

(1) Suppose α_t, $t \in [0,1]$, is a homotopy of actions of G on A. Is $K_*(A \rtimes_{\alpha_t} G)$ independent of t?

(2) Suppose A is K-contractible, i.e., $K_*(A) = 0$, and G acts on A. Does it follow that $K_*(A \rtimes G) = 0$?

The two questions 5.3 turn out to be closely related (and always $2 \Rightarrow 1$).

PROPOSITION 5.4. *An affirmative answer to Question* 5.3 (2) *for a certain locally compact group G implies a positive answer to Question* 5.3 (1) (*for the same group G*).

PROOF. Suppose we are in the situation of Question 5.3 (1). We start by defining an action of G on $C([0,1], A)$ by $\alpha_g(f)(t) = \alpha_t(g)(f(t))$. We have an extension of C^*-algebras

$$0 \to C_0((0,1], A) \to C([0,1], A) \to A \to 0.$$

Take the crossed products by A. We get an extension of C^*-algebras

$$0 \to C_0((0,1], A) \rtimes_\alpha G \to C([0,1], A) \rtimes_\alpha G \to A \rtimes_{\alpha_0} G \to 0.$$

Now apply the affirmative answer to Question 5.3 (2). Since $C_0((0,1], A)$ is contractible, the K-theory of $C_0((0,1], A) \rtimes_\alpha G$ must vanish. So by the long exact K-theory sequence and the Five-Lemma, the projection $C([0,1], A) \rtimes_\alpha G \to A \rtimes_{\alpha_0} G$ induces an isomorphism on K-theory.

But we could have done exactly the same thing with the extension

$$0 \to C_0([0,1), A) \to C([0,1], A) \to A \to 0.$$

Thus the projection $C([0,1], A) \rtimes_\alpha G \to A \rtimes_{\alpha_1} G$ also induces an isomorphism on K-theory. So Question (1) has an affirmative answer. $\square$

We can now get an interesting counterexample to Questions 5.3 using a classic theorem of Smith and its converse as proved by L. Jones:

THEOREM 5.5 (P. Smith [**153**], L. Jones [**88**]; see also [**161**, §III.4]). *Let p be a prime. A finite CW complex F is the fixed set of a cellular action of $\mathbb{Z}/p$ on a finite contractible CW complex X if and only if $H_*(F, \mathbb{Z}/p) \cong H_*(\mathrm{pt}, \mathbb{Z}/p)$.*

Choose F satisfying this condition but with $H_*(F)$ having q-torsion for some other prime q. Then we can arrange also to have q-torsion in $K^*(F)$. (Just make sure none of the differentials in the Atiyah-Hirzebruch spectral sequence can cancel out the q-torsion. This is possible since as q increases, q-torsion only shows up in higher and higher differentials.) Thus $K_*(C_0(F \smallsetminus \mathrm{pt}))$ has nontrivial q-torsion. Note that $C_0(X \smallsetminus \mathrm{pt})$ is contractible. We need one more ingredient:

THEOREM 5.6 (Segal [**149, 148**]). *Let G be a compact abelian group and let $\mathfrak{p}$ be a prime ideal of $R(G)$. Then $\mathfrak{p}$ has a "support," a topologically cyclic subgroup H of G determined up to conjugacy, minimal with the property that $\mathfrak{p}$ lies in the image of $\operatorname{Spec} R(H)$ in $\operatorname{Spec} R(G)$, and if X is a locally compact G-space, the restriction map $K^*_G(X) \to K^*_G(X^H)$ is an isomorphism after localization at $\mathfrak{p}$.*

LEMMA 5.7. *Question* 5.3 (2) *has a negative answer for $G = \mathbb{Z}/p$, p a prime.*

PROOF. We apply Segal's Localization Theorem, Theorem 5.6, together with the theorem of Lowell Jones, Theorem 5.5, in the case $G = \mathbb{Z}/p$, $R(G) = \mathbb{Z}[t]/(t^p - 1)$. Fix a prime $q \neq p$ and let $\mathfrak{p} = (q)$. Then $R(G)/\mathfrak{p} \cong \mathbb{F}_q[t]/(t^p - 1)$ is a direct sum of finite fields of characteristic q and $\mathfrak{p}$ is supported on all of $G = \mathbb{Z}/p$, since the only other possibility would be that $\mathfrak{p}$ lies in the image of $\operatorname{Spec} R(\{1\}) = \operatorname{Spec} \mathbb{Z}$

in $\operatorname{Spec} R(G)$, which is evidently not the case, since $\mathfrak{p}$ does not contain the kernel $(t-1)$ of the augmentation map $R(G) \twoheadrightarrow \mathbb{Z}$. Also note that $R(G)_{\mathfrak{p}} = \mathbb{Z}_{(q)}[t]/(t^p-1)$. Choose a G-space $X \supset F$ as in Theorem 5.5, and choose a basepoint $x \in F$. Then the restriction map

$$K_*^G(C_0(X \smallsetminus \{x\})) = K_G^*(X, x)_{\mathfrak{p}} \to K_G^*(F, x)_{\mathfrak{p}} = K^*(F, x) \otimes_{\mathbb{Z}} R(G)_{\mathfrak{p}}$$

is an isomorphism onto a nonzero group. Thus $A = C_0(X \smallsetminus \{x\})$ is contractible but $K_*(A \rtimes G) \neq 0$, so we have a negative answer to Question 5.3 (2) for $G = \mathbb{Z}/p$. $\square$

We can now "induce" this result to arbitrary groups with torsion.

THEOREM 5.8. *Question* 5.3 (2) *has a negative answer for any locally compact group G containing nontrivial torsion.*

PROOF. If G contains nontrivial torsion, then there is an embedding of $\mathbb{Z}/p$ in G as a closed subgroup for some prime p. Let (X, x) be a pointed $\mathbb{Z}/p$-space as in the proof of Lemma 5.7, and let $(Y, A) = G \times_{\mathbb{Z}/p} (X, \{x\})$. This is a G-space pair, and if G is compact, $K_G^*(Y, A) \cong K_{\mathbb{Z}/p}^*(X, \{x\}) \neq 0$. On the other hand, since X is contractible, the inclusion of $\{x\}$ into X is a homotopy equivalence, and then the inclusion of A into Y is a homotopy equivalence. (Both fiber over $G/(\mathbb{Z}/p)$ and the fibers are homotopy equivalent.) So $K^*(Y, A) = 0$. If G is compact, let $B = C_0(Y \smallsetminus A)$ and we see that B is K-contractible while $K_*(B \rtimes G) \neq 0$ by the Green-Julg Theorem, Theorem 5.2. If G is noncompact, we substitute Theorem 5.11 below in place of Theorem 5.2, and the same argument goes through. $\square$

In fact, with somewhat more care, we can also use the construction of L. Jones to get a negative answer to Question 5.3 (1) as well. (Other constructions were given many years ago by N. C. Phillips, using different ideas. A fairly elementary construction using actions on a UHF algebra, a noncommutative C^*-algebra whose K-theory is not finitely generated, is in [**131**, Example 3.3]. A slightly more complicated example, using actions of a finite cyclic group on S^5, was suggested by G. Segal and is given in [**131**, Example 3.5].) Here is a sketch. Jones [**88**] even shows that we can choose F as before so that there is an action α of $G = \mathbb{Z}/p$ on a disk D^n (n sufficiently large compared the dimension of F) having F as fixed set and which restricts to a free action on the boundary $\partial D^n = S^{n-1}$. Now construct a homotopy α_t of actions of $G = \mathbb{Z}/p$ on the disk D^n as follows. For $0 \leq t \leq 1$, $g \in G$, and $z \in D^n$, let

$$\alpha_t(g)(z) = \begin{cases} 0, & t = 0 \text{ and } z = 0, \\ t \cdot \alpha(g)\left(\frac{z}{t}\right), & 0 < t \leq 1 \text{ and } |z| \leq t, \\ |z| \cdot \alpha(g)\left(\frac{z}{|z|}\right), & |z| > t. \end{cases}$$

(In other words, we rescale α down to the subdisk of radius t around 0, and then extend radially outside the subdisk. Eventually, at $t = 0$, we just have the cone on the restriction of α to S^{n-1}.) This is now a homotopy from the action $\alpha_1 = \alpha$ with F as fixed set to an action α_0 with $\{0\}$ as fixed set. We saw above that $K_{G,\alpha}^*(D^n)_{\mathfrak{p}}$ is not isomorphic to $K_{G,\alpha_0}^*(D^n)_{\mathfrak{p}} = K_{G,\alpha_0}^*(\mathrm{pt})_{\mathfrak{p}} = R(G)_{\mathfrak{p}}$ because it has nontrivial q-torsion, so we have a negative answer to Question 5.3 (1).

5.1.3. The Baum-Connes Conjecture. The questions 5.3 we have been discussing are actually related to a deep mathematical question, the Baum-Connes Conjecture.

THEOREM 5.9 (Meyer-Nest [**110**] or [**49**, §5.3 and §13.1.2]). *Let G be a second-countable locally compact amenable group with no nontrivial compact subgroups. Then the following are equivalent:*

(1) *For all separable G-C^*-algebras A, $K_*(A) = 0$ implies $K_*(A \rtimes G) = 0$.*
(2) *The "Baum-Connes Conjecture with coefficients" (a conjectural way of computing $K_*(A \rtimes G)$) holds for G.*

Because of Theorem 5.8 above, condition (1) does not hold if G has torsion (we'll see another example with $G = \mathbb{T}$ later), which is why we have only stated this theorem when G has no nontrivial compact subgroups. It is possible to drop this condition, but then we need to replace the condition $K_*(A) = 0$ in (1) by the condition that $K_*^H(A) = 0$ for all compact subgroups H of G. The theorem is also valid in the case where G is non-amenable, but then one needs to define the *reduced* crossed product $A \rtimes_r G$ and use this instead of $A \rtimes G$.

5.2. Connes' Thom Isomorphism

The simplest example where "Baum-Connes with coefficients" holds is the case of $G = \mathbb{R}$. This is also the case we will need for applications to T-duality in string theory. We will first state the theorem, due to Connes, and then discuss the idea of the proof.

THEOREM 5.10 (Connes [**45**]). *Let α be an action of $\mathbb{R}$ on a C^*-algebra. Then $K_*(A \rtimes_\alpha \mathbb{R})$ is independent of α (and thus agrees with $K_*(A \otimes C_0(\mathbb{R})) \cong K_{*+1}(A)$, which is what we'd get if α were trivial). More precisely, there is a map $\Phi\colon K_0(A) \to K_1(A \rtimes_\alpha \mathbb{R})$, natural in (A, α), which induces an isomorphism (called* Connes' Thom isomorphism*, by analogy with the Thom isomorphism theorem for vector bundles).*

SKETCH OF PROOF OF CONNES' THEOREM. Connes' original idea for proving this is quite simple. The key step is simply construction of a natural map Φ which reduces to the usual isomorphism $K_0(\mathbb{C}) \to K_1(C^*(\mathbb{R})) \cong K_1(C_0(\mathbb{R}))$. For if this is the case, applying Φ twice gives a natural map $K_0(A) \to K_2((A \rtimes_\alpha \mathbb{R}) \rtimes_{\hat{\alpha}} \mathbb{R}) \cong K_0(A)$ (by Bott Periodicity and Taki Duality, Theorems 3.15 and 3.24) which must be the identity if the action of $\mathbb{R}$ on A is trivial.

Since we can adjoin a unit and use Morita invariance, Proposition 3.9, it's enough to take A unital and consider a class in $K_0(A)$ represented by a self-adjoint projection e. If we can change e in its unitary equivalence class so that e is fixed under α, then the map $\mathbb{C} \to A\colon 1 \mapsto e$ is $\mathbb{R}$-equivariant and induces a map $C_0(\mathbb{R}) = \mathbb{C} \rtimes \mathbb{R} \to A \rtimes_\alpha \mathbb{R}$, which in turn sends the canonical generator of $K_1(C_0(\mathbb{R}))$ to our desired class $\Phi([e])$ in $K_1(A \rtimes_\alpha \mathbb{R})$.

The problem of course is that we may not be able to choose e fixed under α. So the trick is to modify the action up to *exterior equivalence*, which doesn't change the K-theory of the crossed product, so as to make e fixed under the new action.

Two actions α and α' of G on A are said to be *exterior equivalent* if they differ by a unitary-valued cocycle. This means that there is a continuous map $G \to U(M(A))$, $M(A)$ the multiplier algebra of A, with $u_{gh} = u_g \alpha_g(u_h)$ such that $\alpha'_g = \operatorname{Ad} u_g \circ \alpha_g$.

Then there is an action β of G on $M_2(A)$ defined by the formula:

$$\beta_g \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha_g(a) & \alpha_g(b)u_g^* \\ u_g\alpha_g(c) & u_g\alpha_g(d)u_g^* \end{pmatrix}$$

This restricts to α on the upper-left corner and to α' on the lower-right corner, so $A \rtimes_\alpha G$ and $A \rtimes_{\alpha'} G$ are opposite corners in $M_2(A) \rtimes_\beta G$, and are thus Morita equivalent. Hence their K-groups are canonically isomorphic.

To finish the proof, we just have to modify e up to K-theoretic equivalence and the action α up to exterior equivalence to make e fixed under α'. First replace e by a nearby projection so that e is smooth, i.e., $t \mapsto \alpha_t(e)$ is C^∞. Since nearby projections are equivalent in K-theory, this change is harmless.

Then smoothness of e means e lies in the domain of the derivation δ which is the infinitesimal generator of α. We can write δ formally as $i \operatorname{ad} H$. Then replace H by

$$H' = eHe + (1-e)H(1-e) = H + i[\delta(e), e],$$

which commutes with e. It is important to note that H' only differs from H by a *bounded* perturbation. Define α'_t by $\operatorname{Ad}\big(e^{itH'}\big)$, defined by expanding the series, and check that it works. $\square$

5.3. The Pimsner-Voiculescu Sequence

While we're on the topic of the K-theory of crossed products, we want to discuss crossed products by $\mathbb{Z}$. An action α of $\mathbb{Z}$ on A is equivalent to a choice of $\theta = \alpha(1) \in \operatorname{Aut} A$. From θ (or α) we can form an *induced action* of $\mathbb{R}$ on the *induced algebra*, also called the *mapping torus*. Namely, let

$$T_\theta = \{f \in C(\mathbb{R}, A) \mid f(x-1) = \theta^{-1}(f(x))\}.$$

Then $\mathbb{R}$ acts on T_θ by translations: $\alpha_t(f)(x) = f(x-t)$. If θ is trivial, T_θ is simply $A \otimes C(\mathbb{T})$.

More generally, if $G \supset H$ are locally compact groups and α is an action of H on A, we have an action $\operatorname{Ind}\alpha$ of G by left translation on

$$\operatorname{Ind}_H^G A = \{f \in C(G, A) \mid f(gh) = \alpha_{h^{-1}} f(g), \quad \|f\| \in C_0(G/H)\}.$$

THEOREM 5.11 (Green [**172**, §4.3]). *Let $G \supset H$ be locally compact groups and let α be an action of H on a C^*-algebra A. Then $A \rtimes_\alpha H$ and $\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G$ are Morita equivalent. In fact, at least if G and H are second-countable,*

$$\operatorname{Ind}_H^G A \rtimes_{\operatorname{Ind}\alpha} G \cong (A \rtimes_\alpha H) \otimes \mathcal{K}(L^2(G/H)).$$

EXAMPLE 5.12. If $A = \mathbb{C}$, this says $C_0(G/H) \rtimes G$ is Morita equivalent to $C^*(H)$, which is Rieffel's form of Mackey's Imprimitivity Theorem. If in addition $H = 1$, this says $C_0(G) \rtimes G \cong \mathcal{K}(L^2(G))$, which is the Stone-von Neumann-Mackey Theorem, Theorem 3.22.

THEOREM 5.13 (Pimsner-Voiculescu [**16**, Ch. V, §10]). *Let A be a C^*-algebra, $\theta \in \operatorname{Aut} A$. Then the K-groups of $A \rtimes_\theta \mathbb{Z}$ fit into a long exact sequence*

$$\cdots \xrightarrow{\partial} K_*(A) \xrightarrow{1-\theta_*} K_*(A) \to K_*(A \rtimes_\theta \mathbb{Z}) \xrightarrow{\partial} K_{*-1}(A) \xrightarrow{1-\theta_*} \cdots.$$

PROOF. By the Green Imprimitivity Theorem and Morita invariance,

$$K_*(A \rtimes_\theta \mathbb{Z}) \cong K_*(\operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} A \rtimes_{\operatorname{Ind}\theta} \mathbb{R}).$$

Now apply Connes' Thom Isomorphism Theorem and this becomes $K_{*-1}(\operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} A)$. But we have a short exact sequence $0 \to C_0(\mathbb{R}) \otimes A \to \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} A \to A \to 0$, and the boundary map in the long exact K-theory sequence for this extension is $1-\theta_*$, so the theorem follows. $\square$

5.3.1. An Application: Irrational Rotation Algebras. As a typical application of the Pimsner-Voiculescu Theorem, let $A = C(\mathbb{T})$, let $\theta \in \mathbb{R}$, and let $\alpha \in \operatorname{Aut}(\mathbb{T})$ be rotation by $2\pi\theta$, i.e., multiplication by $e^{2\pi i\theta}$. Let $A_\theta = C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$. For obvious reasons, this is called a *rotation algebra*. It is also called a *noncommutative torus*, since when $\theta = 0$, $A_\theta \cong C(\mathbb{T}\times\mathbb{T}) = C(T^2)$. One can show that A_θ is simple if and only if $\theta \notin \mathbb{Q}$, in which case we call it an *irrational rotation algebra*. Note that since $C(\mathbb{T})$ is generated by a single unitary V (the identity map $\mathbb{T} \to \mathbb{T}$), A_θ is generated by two unitaries U and V, with commutation relation

$$UVU^* = e^{2\pi i\theta} V\,.$$

Since α is clearly isotopic to the identity, it acts trivially on $K^*(\mathbb{T})$, so $1-\alpha_* = 0$. That means $K_0(A_\theta)$ and $K_1(A_\theta)$ are both isomorphic to $K^0(S^1) \oplus K^1(S^1) = \mathbb{Z}^2$. More detailed examination of the proof of Connes' Thom Isomorphism Theorem shows that U and V generate $K_1(A_\theta)$.

We'll discuss the relevance of A_θ to string theory later, in Chapter 9.

5.3.2. Another Counterexample. We can also use the Pimsner-Voiculescu Theorem, together with the Takai Duality Theorem, to construct an example of a C^*-algebra A with an action α of $\mathbb{T}$, such that $K_*(A) = 0$ but $K_*(A \rtimes_\alpha \mathbb{T}) \neq 0$. To construct the example, start with the *fermion algebra*, an example of a UHF algebra (which comes up in quantum field theory, incidentally!) $B = \varinjlim M_{2^n}(\mathbb{C})$, where we embed $2^n \times 2^n$ matrices in $2^{n+1}\times 2^{n+1}$ matrices by $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. This is *not* the same embedding as in the Morita invariance theorem, as the embedding here is unital and induces multiplication by 2, not the identity, on $K_0 \cong \mathbb{Z}$. So

$$K_0(B) = \varinjlim(\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \cdots) = \mathbb{Z}\big[\tfrac{1}{2}\big]$$

and $K_1(B) = 0$ (since $K_1\big(M_{2^n}(\mathbb{C})\big) = 0$ for all n). Next, look at $B \otimes \mathcal{K}$, which has the same K-theory but no unit. This algebra has an automorphism θ with $\theta_* = 2$ on K_0, which can be viewed as a "shift" on an infinite tensor product of copies of $M_2(\mathbb{C})$ (see [**48**, §2]).

Now let $A = (B \otimes \mathcal{K}) \rtimes_\theta \mathbb{Z}$. (This turns out (*loc. cit.*) to be the algebra $\mathcal{O}_2 \otimes \mathcal{K}$, where $\mathcal{O}_2$ is the "Cuntz algebra" [**48**] with generators S_1 and S_2 satisfying $S_j^* S_j = 1$, $S_1S_1^* + S_2S_2^* = 1$.) We can compute the K-theory of A from the Pimsner-Voiculescu sequence, which reduces to

$$\begin{tikzcd}
\mathbb{Z}\big[\tfrac{1}{2}\big] \arrow[r, "1-2"] & \mathbb{Z}\big[\tfrac{1}{2}\big] \arrow[r] & K_0(A) \arrow[d] \\
K_1(A) \arrow[u] & 0 \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

Since multiplication by $1-2=-1$ is an isomorphism, it follows that $K_0(A)=K_1(A)=0$. On the other hand, if $\alpha=\widehat{\theta}$, the dual action, then by Takai duality, $A\rtimes_\alpha \mathbb{T}\cong B\otimes\mathcal{K}$, which has nonzero K_0.

REMARK 5.14. Similar analysis using Takai duality (Theorem 3.24) and the Pimsner-Voiculescu sequence (Theorem 5.13) gives a necessary and sufficient condition for a $\mathbb{T}$-algebra A to be a counterexample to Question 5.3 (2), or in other words, to have $K_*(A)=0$ and $K_*(A\rtimes\mathbb{T})\neq 0$. View $K_*^{\mathbb{T}}(A)$ as a $\mathbb{Z}/2$-graded module M_* over $R(\mathbb{T})=\mathbb{Z}[t,t^{-1}]$. Then one needs to have $M_*\neq 0$ but for multiplication by $t-1$ on M_* to be an isomorphism. This can happen even if $A=C_0(X)$ is commutative.

REMARK 5.15. It is perhaps worth pointing out that, via Takai duality (Theorem 3.24) and the Green-Julg Theorem (Theorem 5.2), the Pimsner-Voiculescu sequence is equivalent to a special case of the Hodgkin spectral sequence, which in turn is a form of Künneth Theorem for equivariant K-theory. If we dualize from an action θ of $\mathbb{Z}$ on B to the dual action $\widehat{\theta}$ of the compact group $\mathbb{T}$ on $A=B\rtimes_\theta\mathbb{Z}$, then the Pimsner-Voiculescu sequence relates $K_*(A)\cong K_*(B\rtimes_\theta\mathbb{Z})$ to $K_*^{\mathbb{T}}(A)\cong K_*(A\rtimes_{\widehat{\theta}}\mathbb{T})\cong K_*(B)$. But $K_*(A)$ can be rewritten as $K_*^{\mathbb{T}}(\operatorname{Ind}_{\{1\}}^{\mathbb{T}} A)\cong K_*^{\mathbb{T}}(C(\mathbb{T})\otimes A)$, which the Hodgkin spectral sequence [**79**] computes via a spectral sequence

$$\operatorname{Tor}_p^{R(\mathbb{T})}(K_*^{\mathbb{T}}(C(\mathbb{T})),K_*^{\mathbb{T}}(A))=\operatorname{Tor}_p^{\mathbb{Z}[t,t^{-1}]}(\mathbb{Z},K_*^{\mathbb{T}}(A))\Rightarrow K_*^{\mathbb{T}}(C(\mathbb{T})\otimes A)\cong K_*(A).$$

(See [**142**] for the noncommutative version of the Hodgkin spectral sequence, which is what is needed here if A is noncommutative.) The Tor groups here can be computed from the free resolution

$$0\to\mathbb{Z}[t,t^{-1}]\xrightarrow{t-1}\mathbb{Z}[t,t^{-1}]\to\mathbb{Z}\to 0.$$

Thus the E^2 terms of the spectral sequence are

$$\operatorname{Tor}_p^{\mathbb{Z}[t,t^{-1}]}(\mathbb{Z},K_*^{\mathbb{T}}(A))=\begin{cases}\operatorname{coker}(t-1)\text{ on }K_*^{\mathbb{T}}(A), & p=0,\\ \ker(t-1)\text{ on }K_*^{\mathbb{T}}(A), & p=1,\\ 0, & p\geq 2,\end{cases}\tag{5.1}$$

and since $E^2_{p,q}=0$ when $p\geq 2$, the spectral sequence collapses, giving an exact sequence

$$0\to\operatorname{Tor}_0^{\mathbb{Z}[t,t^{-1}]}(\mathbb{Z},K_q^{\mathbb{T}}(A))\to K_q(A)\to\operatorname{Tor}_1^{\mathbb{Z}[t,t^{-1}]}(\mathbb{Z},K_{q-1}^{\mathbb{T}}(A))\to 0.$$

Substituting (5.1) for the Tor terms gives precisely the Pimsner-Voiculescu sequence, since t acts by θ_*.

CHAPTER 6

The Topology of T-Duality and the Bunke-Schick Construction

6.1. Topological T-Duality

6.1.1. A Key Example.

6.1.1.1. *Topology Change in T-Duality.* We talked about the simplest case of T-duality in Section 1.3.2.1. There, string theory on $X = Z \times T$, where $T = S^1$ is a circle of radius R, corresponds to a dual string theory on $X^\sharp = Z \times T^\sharp$, where $T^\sharp$ is the dual circle with radius $\widetilde{R} = \frac{\alpha'}{R}$. But what if X is fibered by circles, but doesn't split as a product?

The first example of this phenomenon was studied by Alvarez, Alvarez-Gaumé, Barbón, and Lozano in [**3**]. (However, the reader should be cautioned that this paper did not get all of the details exactly right.) Their discovery was generalized 10 years later by Bouwknegt, Evslin, and Mathai in [**22, 23**]. Let's start with the simplest example of a circle fibration, where $X = S^3$, identified with $SU(2)$, and T ($\cong S^1$) is a maximal torus. Then T acts freely on X (say by right translation) and the quotient X/T is $\mathbb{CP}^1 \cong S^2$, with quotient map $p\colon X \to S^2$ the *Hopf fibration.* Assume for simplicity that the B-field on X vanishes.

Let's examine this case in more detail. We have $X = S^3$ fibering over $Z = X/T = S^2$. Think of Z as the union of the two hemispheres $Z^\pm \cong D^2$ intersecting in the equator $Z^0 \cong S^1$. The fibration is trivial over each hemisphere, so we have $p^{-1}(Z^\pm) \cong D^2 \times S^1$, with $p^{-1}(Z^0) \cong S^1 \times S^1$. So the T-dual also looks like the union of two copies of $D^2 \times S^1$, joined along $S^1 \times S^1$.

However, we have to be careful about the *clutching* that identifies the two copies of $S^1 \times S^1$. In the original Hopf fibration, the clutching function $S^1 \to S^1$ winds once around, with the result that the fundamental group $\mathbb{Z}$ of the fiber T dies in the total space X. But T-duality is supposed to interchange "winding" and "momentum" quantum numbers. So $X^\sharp$ has no winding and is just $S^2 \times S^1$.

So what happened to the clutching function? It shows up in the H-flux of the dual!

6.1.1.2. *T-Duality with a B-field.* To explain this, let's go back to Buscher's derivation of T-duality for the sigma-model with maps $x = (x_1, x_0)\colon \Sigma \to X^\sharp = Z \times S^1$, but this time including the Wess-Zumino term. The action now has the form

$$S(x) = \frac{1}{4\pi\alpha'} \int_\Sigma \left(\|\nabla x_1\|_Z^2 \, d\mathrm{vol}_\Sigma + \frac{R^2}{\alpha'} dx_0 \wedge * dx_0 + x^* B \right) . \tag{6.1}$$

When we dualize the S^1, we have to be careful about the part of B that involves this factor.

In our situation, we are starting with a case where $B^{\pm} = \eta^{\pm} \times d\mathrm{vol}_{S^1}$ is a 2-form over $Z^{\pm} \times S^1$, and $dB^{\pm}$ is a volume form on $Z^{\pm} \times S^1$. Note that $dB^{\pm}$, but not $B^{\pm}$, are supposed to agree on $Z^0 \times S^1$.

In terms of the closed 1-form $\omega = dx_0$, the action becomes

$$S(\omega) = \frac{1}{4\pi\alpha'} \int_{\Sigma} \cdots + \frac{R^2}{\alpha'} \omega \wedge *\omega + \omega \wedge x_1^* \beta,$$

where we've left out terms not involving $x_0 \colon \Sigma \to S^1$, since they don't change under T-duality. As in Section 1.3.2.1, introduce the Lagrange multiplier μ to get

$$S(\omega, \mu) = \frac{1}{4\pi\alpha'} \int_{\Sigma} \cdots + \frac{R^2}{\alpha'} \omega \wedge *\omega + \omega \wedge x_1^* \beta + 2\mu \, d\omega,$$

which if we vary μ gives back the original action. But take the variation in ω instead. We set $\delta S = 0$ and get what we had before but with an extra term:

$$\frac{2R^2}{\alpha'} * \omega + x_1^* \beta + 2d\mu = 0.$$

So $*\omega = \frac{-\alpha'}{R^2}\left(d\mu + \frac{1}{2} x_1^* \beta\right)$ and $\omega = \frac{\alpha'}{R^2} * \left(d\mu + \frac{1}{2} x_1^* \beta\right)$. If $\eta = d\mu$, substituting back into $S(\omega, \mu)$ gives

$$E'(\eta) = \frac{-1}{4\pi\alpha'} \int_{\Sigma} \cdots + \frac{\alpha'}{R^2} \left(\eta + \frac{1}{2} x_1^* \beta\right) \wedge * \left(\eta + \frac{1}{2} x_1^* \beta\right),$$

which has the same form as E except that $R \leftrightarrow \frac{\alpha'}{R}$ and η is *shifted* by $\frac{1}{2} x_1^* \beta$. Recall β is not globally defined; the forms $\beta^{\pm}$ differ by a closed 1-form on Z^0.

6.1.1.3. *K-Theory Matching.* Thus when we apply T-duality starting with $X^\sharp = Z \times S^1$ and the H-flux a generator of $H^3(X^\sharp)$, we see the closed 1-form associated to the T-dual is shifted on one hemisphere relative to the another, the shifting associated to a generator of $H^1(Z^0)$. That shows exactly that the clutching map of the dual theory on X corresponds to the identity map $S^1 \to S^1$, and so the dual spacetime X is not $S^2 \times S^1$ but S^3.

We can also explain this in terms of matching of D-brane charges. If the sigma models on X and $X^\sharp$ are to give indistinguishable physics, the D-brane charges in the two theories must live in isomorphic groups.

Thus we want to require $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$. The degree shift comes from interchange of type IIA string theory with type IIB.

Example 6.1 (The Case of $S^2 \times S^1$ and S^3). Let's check this principle of K-theory matching in the case we've been considering, $X = S^3$ fibered by the Hopf fibration over $Z = S^2$. The H-flux on X is trivial, so D-brane changes lie in $K^*(S^3)$, with no twisting. And $K^0(S^3) \cong K^1(S^3) \cong \mathbb{Z}$.

On the T-dual side, we expect to find $X^\sharp = S^2 \times S^1$, also fibered over S^2, but simply by projection onto the first factor. If the H-flux on $X^\sharp$ were trivial, D-brane changes would lie in $K^0(S^2 \times S^1)$ and $K^1(S^2 \times S^1)$, both of which are isomorphic to $\mathbb{Z}^2$, which is *too big*.

On the other hand, we can compute $K^*(S^2 \times S^1, H^\sharp)$ for the class $H^\sharp$ which is k times a generator of $H^3 \cong \mathbb{Z}$, using the Atiyah-Hirzebruch spectral sequence of Section 4.2.0.7. The differential is

$$H^0(S^2 \times S^1) \xrightarrow{k} H^3(S^2 \times S^1),$$

so when $k = 1$, $K^*(S^2 \times S^1, H^\sharp) \cong K^*(S^3) \cong \mathbb{Z}$ for both $* = 0$ and $* = 1$.

6.1.2. Axiomatics. This discussion suggests we should try to develop an axiomatic treatment of the *topological* aspects of T-duality. Note that we are ignoring many things, such as the underlying metric on spacetime and the auxiliary fields.

6.1.2.1. *Axioms for Topological T-Duality.*

(1) We have a suitable class of spacetimes X each equipped with a principal S^1-bundle $X \to Z$. (X might be required to be a smooth connected manifold.)
(2) For each X, we assume we are free to choose any H-flux $H \in H^3(X, \mathbb{Z})$.
(3) There is an involution (map of period 2, up to equivalence) $(X, H) \mapsto (X^\sharp, H^\sharp)$ keeping the base Z fixed.
(4) If $X = Z \times S^1$ and $H = 0$, then $(X^\sharp, H^\sharp)$ is topologically again $(Z \times S^1, 0)$. If $Z = S^2$ and $(X, H) = (S^3, 0)$, then $(X^\sharp, H^\sharp)$ is as in 6.1.
(5) Naturality: If $\varphi\colon Z_1 \to Z$ and if (X, H) is a pair over Z, then if we form the pull-back diagram

$$\begin{tikzcd} X_1 \arrow[r, dotted, "\widehat{\varphi}"] \arrow[d, dotted] & X \arrow[d] \\ Z_1 \arrow[r, "\varphi"] & Z, \end{tikzcd}$$

the T-dual of $(X_1, \widehat{\varphi}^*(H))$ (as a pair over Z_1) is the pull-back of $(X^\sharp, H^\sharp)$ (computed over Z).
(6) $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$.

6.2. The Bunke-Schick Construction

Bunke and Schick [**34**] suggested constructing a theory satisfying these axioms by means of a *universal example*. It is known that (for reasonable spaces Z, say CW complexes) all principal S^1-bundles $X \to Z$ come by *pull-back* from a diagram

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d] & ES^1 \simeq * \arrow[d] \\ Z \arrow[r, dotted] & B\mathbb{T} \simeq K(\mathbb{Z}, 2) \end{tikzcd}$$

Here the map $Z \dashrightarrow K(\mathbb{Z}, 2)$ is unique up to homotopy.

Similarly, every class $H \in H^3(X, \mathbb{Z})$ comes by pull-back from a canonical class via a map $X \dashrightarrow K(\mathbb{Z}, 3)$ unique up to homotopy.

Before we state the Bunke-Schick Theorem, let's begin with a classical theorem from algebraic topology, which will be needed later.

THEOREM 6.2 (Gysin sequence). *Let $X \xrightarrow{p} Z$ be a principal S^1-bundle over a path-connected base Z with Chern class $c \in H^2(Z, \mathbb{Z})$. Then the cohomology groups of X and Z are related by a long exact Gysin sequence*

$$\begin{aligned} \cdots \to H^k(Z, \mathbb{Z}) \xrightarrow{\cup c} H^{k+2}(Z, \mathbb{Z}) \xrightarrow{p^*} H^{k+2}(X, \mathbb{Z}) \\ \xrightarrow{p_!} H^{k+1}(Z, \mathbb{Z}) \xrightarrow{\cup c} H^{k+3}(Z, \mathbb{Z}) \to \cdots . \end{aligned} \tag{6.2}$$

PROOF. Proofs can be found in most standard algebraic topology textbooks, but we just indicate how this follows from the Serre spectral sequence

$$H^p(Z, H^q(S^1, \mathbb{Z})) \Rightarrow H^{p+q}(X, \mathbb{Z}),$$

for those familiar with it. The E_2 terms of the spectral sequence look like

$$\begin{tikzcd}[column sep=small]
q & & & & \\
0 \arrow[u] & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
H^0(Z) \arrow[drr, "d"] & H^1(Z) \arrow[drr, "d"] & H^2(Z) \arrow[drr, "d"] & H^3(Z) & \cdots \\
H^0(Z) & H^1(Z) & H^2(Z) & H^3(Z) & \cdots \arrow[r] & p
\end{tikzcd},$$

where d is the only nonzero differential, $d_2\colon H^p(Z) \to H^{p+2}(Z)$. Since Z and X are path-connected, $H^0(Z, \mathbb{Z}) \cong \mathbb{Z}$ and has a canonical generator, 1. The class $d(1) \in H^2(Z)$ is c, and since d must have the derivation property, it has to be cup product with c. Then we have exact sequences

$$0 \to E_\infty^{p,1} \to H^p(Z, \mathbb{Z}) \xrightarrow{\cup c} H^{p+2}(Z, \mathbb{Z}) \to E_\infty^{p+2,0} \to 0$$

and

$$0 \to E_\infty^{p,0} \xrightarrow{p^*} H^p(X, \mathbb{Z}) \to E_\infty^{p-1,1} \to 0,$$

from which (6.2) follows. The Gysin map $p_!$, also sometimes called *integration along the fibers*, since that's what it amounts to in terms of de Rham cohomology, can be defined to be the composite

$$H^k(X, \mathbb{Z}) \to E_\infty^{k-1,1} \hookrightarrow H^{k-1}(Z, \mathbb{Z}). \qquad \square$$

THEOREM 6.3 (Bunke-Schick [**34**]). *There is a classifying space R, unique up to homotopy equivalence, with a fibration*

$$\begin{tikzcd}
K(\mathbb{Z},3) \arrow[r] & R \arrow[d] \\
 & K(\mathbb{Z},2) \times K(\mathbb{Z},2),
\end{tikzcd} \tag{6.3}$$

and any $(X, H) \to Z$ as in the axioms of 6.1.2.1 *comes by pull-back from*

$$\begin{tikzcd}
X \arrow[r, dotted] \arrow[d] & E \arrow[d] \\
Z \arrow[r, dotted] & R,
\end{tikzcd}$$

with the horizontal maps unique up to homotopy and H pulled back from a canonical class in $H^3(E, \mathbb{Z})$.

Furthermore, the k-invariant of the Postnikov tower (6.3) *characterizing R is the cup-product in*

$$H^4(K(\mathbb{Z},2) \times K(\mathbb{Z},2), \mathbb{Z})$$

of the two canonical classes in H^2. The space E in the fibration

$$\begin{tikzcd} \mathbb{T} \arrow[r] & E \arrow[d, "p"] \\ & R \end{tikzcd}$$

has the homotopy type of $K(\mathbb{Z},3) \times K(\mathbb{Z},2)$.

PROOF. Consider the functor $F\colon Z \rightsquigarrow \{[(X,H)]\}$ assigning to Z the set of all equivalence classes of pairs (X,H) over it, where $X \to Z$ is a principal $\mathbb{T}$-bundle and $H \in H^3(X,\mathbb{Z})$. The functor F takes its values in "pointed sets," since while there is no obvious reason $F(Z)$ must be a group, it certainly has a natural zero element. One can show the functor F must be representable, in the sense that $F(Z)$ is the set of (based) homotopy classes of (based) maps $Z^+ \to R$,[1] where R is some (based) classifying space, by the "abstract nonsense" of the Brown Representability Theorem [**28, 29**]. To get an idea of what R looks like, we can compute $F(S^n)$ by brute force. Clearly $F(S^n) = 0$ for $n < 2$ or for $n > 3$, since a circle bundle over S^n can be nontrivial only if $n = 2$, and the total space of a circle bundle over S^n can have nonzero H^3 only if $n = 2$ or 3. Thus the only non-zero homotopy groups of R can be π_2 and π_3. Since R is simply connected, there is no also difference between based and unbased homotopy classes of maps $S^n \to R$ with $n = 2$ or 3. Furthermore, $\pi_2(R) \cong \mathbb{Z}^2$, since a pair (X,H) over S^2 is classified by two integers, the Chern class of the bundle $X \to S^2$ and the value of H in $H^3(X,\mathbb{Z}) \cong \mathbb{Z}$. Similarly, $\pi_3(R) \cong \mathbb{Z}$, since any pair over S^3 is of the form $(S^3 \times S^1, H \times 1)$, where $H \in H^3(S^3,\mathbb{Z}) \cong \mathbb{Z}$. Since we now know all the homotopy groups of R, Postnikov theory gives a fibration of the form (6.3). The problem is to compute its k-invariant, the class in $H^4(K(\mathbb{Z},2) \times K(\mathbb{Z},2),\mathbb{Z})$ by which (6.3) is pulled back from the universal $K(\mathbb{Z},3)$ fibration

$$\begin{tikzcd} K(\mathbb{Z},3) = \Omega K(\mathbb{Z},4) \arrow[r] & \mathrm{pt} \arrow[d] \\ & K(\mathbb{Z},4). \end{tikzcd}$$

Bunke and Schick also give another way of describing R explicitly, which is also useful. Start with the free loop space

$$E = \Lambda K(\mathbb{Z},3) = \mathrm{Map}(S^1, K(\mathbb{Z},3)),$$

on which $\mathbb{T}$ acts by rotating the domain $S^1 \cong \mathbb{T}$. The "Borel construction" gives a homotopy fibration

$$\begin{tikzcd} E \arrow[r, "p"] & R = E\mathbb{T} \times_{\mathbb{T}} E \arrow[d, "c"] \\ & B\mathbb{T} = K(\mathbb{Z},2). \end{tikzcd} \tag{6.4}$$

[1]Here Z^+ is Z with a disjoint basepoint added; this is simply a device to get around the fact that Z need not have any natural basepoint to begin with.

We can think of c as the Chern class of a circle bundle

$$\begin{tikzcd} \mathbb{T} \arrow[r] & E \arrow[d, "p"] \\ & R. \end{tikzcd} \tag{6.5}$$

(Strictly speaking, this is only up to homotopy. To get a genuine circle bundle, replace E by the homotopy-equivalent space $E \times E\mathbb{T}$ with the diagonal action of $\mathbb{T}$, which is now free with R as quotient.) The free loop space comes with a fibration

$$\begin{tikzcd} \Omega K(\mathbb{Z},3) = K(\mathbb{Z},2) \arrow[r] & E = \Lambda K(\mathbb{Z},3) \arrow[d, "e"] \\ & K(\mathbb{Z},3), \end{tikzcd}$$

where $\Omega K(\mathbb{Z},3)$ is the *based* loop space and e is evaluation of loops at $1 \in S^1$. Since e has a section, given by constant loops, $E \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},3)$ and we have a canonical class $h \in H^3(E,\mathbb{Z})$ (associated to the map e).

We just need to check that these specific E and R have the right properties. Given a pair (X,H) over a base Z, we have a map $Z \dashrightarrow B\mathbb{T}$ classifying $X \to Z$. This comes from an $\mathbb{T}$-equivariant map $X \overset{\mathbb{T}}{\dashrightarrow} E\mathbb{T}$. We also have a map $X \dashrightarrow K(\mathbb{Z},3)$ classifying H. View this as an equivariant map $X \overset{\mathbb{T}}{\dashrightarrow} \Lambda K(\mathbb{Z},3)$ and take the product to get a commuting diagram

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d] & E\mathbb{T} \times E \simeq E \arrow[d, "p"] \\ Z \arrow[r, dotted] & R. \end{tikzcd}$$

It's easy to see that this identifies $(X,H) \to Z$ with the pull-back of $(E,h) \xrightarrow{p} R$. From the fibration

$$\begin{tikzcd} E \arrow[r, "p"] & R \arrow[d, "c"] \\ & K(\mathbb{Z},2) \end{tikzcd}$$

and the fact that $E \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},3)$, we also see $\pi_2(R) \cong \mathbb{Z}^2$ and $\pi_3(R) \cong \mathbb{Z}$, which gives an independent check of the calculation we did previously.

Now let's go back to the problem of computing the k-invariant. If the k-invariant were trivial, we'd have $R \simeq K(\mathbb{Z},2) \times K(\mathbb{Z},2) \times K(\mathbb{Z},3)$, which would be impossible, since it would imply the H-flux H on X is always the pull-back of a class in $H^3(Z,\mathbb{Z})$, which need not be the case. Think of the Hopf fibration $S^3 = X \xrightarrow{p} Z = S^2$. $H^3(S^3,\mathbb{Z}) \cong \mathbb{Z}$, but only the zero class is pulled back from $H^3(S^2,\mathbb{Z}) = 0$. In fact in this example, $p_!\colon H^3(X,\mathbb{Z}) \to H^2(Z,\mathbb{Z})$ is an isomorphism, and we see that the classes $c_1(p)$ and $p_!(H)$ are independent of one another.

Now recall that $H^*(K(\mathbb{Z},2)\times K(\mathbb{Z},2),\mathbb{Z})$ is a polynomial ring on two generators, both of degree 2. We choose one of these generators to be u, the Chern class of

(6.5), and let w be the canonical generator of $H^3(K(\mathbb{Z},3),\mathbb{Z})$. The other generator v of $H^2(R)$ will be specified shortly. The E_2 term of the Serre spectral sequence of (6.3) has the form

$$
\begin{tikzcd}
q \\
0 \arrow[u] & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z}w \arrow[rrrrddd, "d"] & 0 & \mathbb{Z}wu \oplus \mathbb{Z}wv \arrow[rrrddd, "d"] & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots \\
\mathbb{Z} & 0 & \mathbb{Z}u \oplus \mathbb{Z}v & 0 & \mathbb{Z}u^2 \oplus \mathbb{Z}uv \oplus \mathbb{Z}v^2 & \cdots & {} \arrow[r] & p
\end{tikzcd} ,
$$

where d sends w to the k-invariant. Since we know this is non-zero, d is injective and $H^3(R,\mathbb{Z})=0$, while the rank of $H^4(R,\mathbb{Z})$ must be 2.

Now consider the Gysin sequence (6.2) applied to the circle bundle (6.5). This has the form

$$
\begin{aligned}
0 = H^3(R,\mathbb{Z}) \xrightarrow{p^*} H^3(E,\mathbb{Z}) = \mathbb{Z}w \xrightarrow{p_!} H^2(R,\mathbb{Z}) = \mathbb{Z}u \oplus \mathbb{Z}v& \\
\xrightarrow{\cdot u} H^4(R,\mathbb{Z}) \xrightarrow{p^*} H^4(E,\mathbb{Z}) = \mathbb{Z}v^2 \xrightarrow{p_!} H^3(R,\mathbb{Z}) = 0.&
\end{aligned}
$$

This exact sequence shows that $H^4(R,\mathbb{Z}) \cong \mathbb{Z}^2 \oplus (\text{torsion})$. It also shows that $p_!(w) \in H^2(R,\mathbb{Z})$ is killed under multiplication by u. This class cannot be a multiple of u, since for a pair $(X \xrightarrow{p'} Z, H)$, p' is pulled back from $p\colon E \to R$ and H is pulled back from w, and the example of the Hopf fibration showed that $c_1(p')$ and ${p'}_!(H)$ have to be independent of one another. Furthermore, $p_!(w)$ must be primitive (i.e., cannot be a nontrivial multiple of another class), for otherwise ${p'}_!(H)$ would always be a nontrivial multiple of another class, which is not the case in the Hopf fibration example when H is a generator of $H^3(S^3,\mathbb{Z})$. In particular, $H^4(R,\mathbb{Z})$ is torsion-free. So we let $v = p_!(w)$ and see that u and v are now free generators for $H^2(R,\mathbb{Z})$.

Since uv dies in $H^4(R,\mathbb{Z})$, this means (up to a sign, which is a matter of convention) that the k-invariant has to be uv, and the other generator of $H^4(R,\mathbb{Z})$ is u^2. $\square$

THEOREM 6.4 (Topological T-Duality). *The space R has a homotopy automorphism of period 2, called topological T-duality, that comes from interchanging the two copies of $K(\mathbb{Z},2)$ in (6.3). This gives an involution*

$$[(X \xrightarrow{p} Z, H)] \mapsto [(X^\sharp \xrightarrow{p^\sharp} Z, H^\sharp)]$$

on the space of pairs (X,H) over a base Z, with the property that

$$c_1(p) = (p^\sharp)_!(H^\sharp) \quad \text{and} \quad c_1(p^\sharp) = (p)_!(H). \tag{6.6}$$

PROOF. Because of the fibration (6.3) and the fact that its k-invariant uv is symmetric in the two factors, the "flip" automorphism $\sharp$ interchanging the two factors of $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)$ extends to a homotopy automorphism $\sharp$ of R of period 2, and thus to an involutive natural transformation from the functor $F\colon Z \rightsquigarrow \{[(X, H)]\}$ to itself. It remains to verify the stated relationship (6.6) between characteristic classes involving the Gysin map.

We saw from the proof of Theorem 6.3 that the two generators of $H^2(R, \mathbb{Z})$ can be taken to be $u = c_1(p)$ and $v = p_!(w)$, where w is the canonical generator of $H^3(E, \mathbb{Z})$ and p is as in (6.5). These are interchanged under $\sharp$, so (6.6) follows. $\square$

The treatment above shows that it is possible to satisfy all the T-duality axioms of Section 6.1.2.1 above, except perhaps for the last one dealing with twisted K-theory. We will come back to this axiom in the next chapter.

EXAMPLE 6.5 (The Case of $S^2 \times S^1$ and S^3, Revisited). We conclude this chapter by looking again at the case of $S^2 \times S^1$ and S^3 from the beginning of this chapter. Let $a \in H^2(S^2)$, $b \in H^1(S^1)$ be the usual generators. Look at the diagram

$$\begin{tikzcd}
(X, H) = (S^3, 0) \arrow[dr, "p"] & & (X^\sharp, H^\sharp) = (S^2 \times S^1, a \times b) \arrow[dl, "p^\sharp"'] \\
& Z &
\end{tikzcd}$$

We have $c_1(p) = (p^\sharp)_!(H^\sharp) = a$, $c_1(p^\sharp) = p_!(H) = 0$. So indeed T-duality interchanges

$$c_1(p) \leftrightarrow (p^\sharp)_!(H^\sharp), \quad c_1(p^\sharp) \leftrightarrow p_!(H).$$

CHAPTER 7

T-Duality via Crossed Products

7.1. Group Actions on Continuous Trace-Algebras

In Section 6.1.2.1 of the last chapter, we proposed a set of axioms for topological T-duality, and showed via the Bunke-Schick Theorem (Theorem 6.3) that there is an essentially unique way of satisfying the first five axioms. The problem is now to check the final axiom about preservation of twisted K-theory under duality. In this chapter, we will use the K-theory of crossed products to verify the last axiom. In this way, we will arrive at a surprising unification of the three areas dealt with in this book: topology, C^*-algebras, and string duality.

Let's start by explaining the strategy of the method. Suppose X is a spacetime manifold which is a principal S^1-bundle over Z. Then we have a free action of S^1 on X, and thus an action $\mathbb{T} \to \operatorname{Aut} C_0(X)$. In order to use Connes' Thom Isomorphism Theorem, Theorem 5.10, we lift this action to the universal cover $\mathbb{R}$ of $\mathbb{T}$ and think of it as an action $\alpha\colon \mathbb{R} \to \operatorname{Aut}(C_0(X))$ which is trivial on $\mathbb{Z}$. Form the crossed product $A = C_0(X) \rtimes_\alpha \mathbb{R}$. By Connes' Theorem, we have $K_{*+1}(A) \cong K_*(C_0(X)) \cong K^{-*}(X)$. Now suppose we knew for some reason that A is a continuous-trace algebra over some space $X^\sharp$, with Dixmier-Douady invariant $H^\sharp$, and suppose we knew that $X^\sharp$ was also a principal S^1-bundle over Z. Then we'd have

$$K^{*+1}(X^\sharp, H^\sharp) \cong K^*(X),$$

which is the final T-duality axiom assuming $(X^\sharp, H^\sharp)$ is T-dual to $(X, 0)$.

Quite magically, this turns out to work! Actually, there is an advantage to working even more generally, since if the H-flux on X is non-zero, we will need an action of $\mathbb{R}$ on the stable CT-algebra with spectrum X and Dixmier-Douady invariant H, since the K-theory of this algebra is $K^*(X, H)$.

So we need to understand the structure of $\operatorname{Aut} CT(X, H)$ (as a topological group) and actions of $\mathbb{R}$ on $CT(X, H)$. Every $*$-automorphism of $\mathcal{K}(\mathcal{H})$ is given by conjugation by a unitary operator, so $\operatorname{Aut} \mathcal{K} \cong PU(\mathcal{H})$. There is an obvious map $\sigma\colon \operatorname{Aut} CT(X, H) \to \operatorname{Homeo}(X)$ gotten by sending an automorphism to the induced action on the spectrum. If one puts these two facts together, one can see that $\operatorname{Aut} C_0(X, \mathcal{K}) \cong \operatorname{Map}(X, PU) \rtimes \operatorname{Homeo}(X)$. Here $\operatorname{Map}(X, PU)$ can be identified with the kernel of σ. A generalization of this to other stable continuous-trace algebras was proved by Phillips and Raeburn.

Before we get to this, it is convenient to record a very simple but still useful lemma, which connects actions on continuous-trace algebras with actions on bundles. This will be used several times in this chapter.

Lemma 7.1. *Let X be a second-countable locally compact space, let $A = CT(X, H)$ be a stable continuous-trace algebra with spectrum X, and let $\pi\colon B \to X$ be the principal PU-bundle associated to H, so that we can identify A with $\Gamma_0(X, B \times_{PU}$*

$\mathcal{K}$), the algebra of sections vanishing at infinity of the bundle of algebras $B \times_{PU} \mathcal{K}$ associated to B.

(1) *Let $\mathcal{B} = B \times_{PU} PU$ be the "gauge bundle" associated to B, the quotient of $B \times PU$ by the equivalence relation $(u, g) \simeq (u \cdot h, h^{-1}gh)$, $h \in PU$. (Unlike P, this is a bundle of topological groups over X, so its sections have a natural topological group structure.) Then $\operatorname{Aut}_X A \cong \Gamma(X, \mathcal{B})$ and $\operatorname{Aut} A \cong \operatorname{Homeo}_{PU}(B)$, the PU-equivariant homeomorphisms of B.*

(2) *Let $\alpha\colon G \to \operatorname{Homeo} X$ be an action of a topological group on X by homeomorphisms. Then the action of G on X lifts to an action $\widetilde{\alpha}\colon G \to \operatorname{Aut} A$, inducing the action α on the spectrum $\widehat{A} = X$, if and only if the action of G on X lifts to an action of G on the principal PU-bundle $B \to X$ defined by H. (Such an action of G on B is required to commute with the free PU-action on B.) In fact, actions of G on A lifting α can be identified with such actions of G on B by bundle automorphisms, or in other words, by liftings*

$$\begin{tikzcd} & \operatorname{Homeo}_{PU}(B) \arrow[d] \\ G \arrow[r] \arrow[ur, dotted] & \operatorname{Homeo}(X). \end{tikzcd}$$

Proof. Since $PU = \operatorname{Aut} \mathcal{K}$ and A is the algebra of sections of the bundle of algebras $B \times_{PU} \mathcal{K}$ associated to B, a PU-equivariant action of G on B certainly defines an action of G on A. Conversely, B can be recovered from A via Dixmier-Douady theory, so that in fact $\operatorname{Aut} A \cong \operatorname{Homeo}_{PU}(B)$. And $\operatorname{Aut}_X A \cong \ker(\operatorname{Homeo}_{PU}(B) \to \operatorname{Homeo} X)$, the PU-equivariant homeomorphisms of B lifting the identity map $X \to X$. Via the theory of gauge groups and bundles (see, e.g., [**2**, §2]), these can be identified with sections of the gauge bundle $\mathcal{B}$. □

The following special case will be especially relevant for us.

Proposition 7.2. *Let X be a second-countable locally compact space, and let $A = CT(X, H)$ be a stable continuous-trace algebra with spectrum X. Suppose a compact group G acts freely on X. Then the action of G on X lifts to an action of G on A if and only if H is the pull-back of a class in $H^3(X/G, \mathbb{Z})$.*

Proof. Since G is compact and acts freely, the quotient space X/G is Hausdorff and locally compact, and $X \to X/G$ is a principal G-bundle. By the lemma, the action of G on X lifts to an action on A if and only if it lifts to an action of G on the associated principal PU-bundle $B \to X$ defined by H. If there is such a lifting, then we have commuting free actions of G and PU on B, so dividing out by G, we get a principal PU-bundle on X/G pulling back to B. For the other direction, if H is pulled back from a class in $H^3(X/G, \mathbb{Z})$, then B is pulled back from a principal PU-bundle B' on X/G, i.e., we have a pull-back diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "PU"'] & B' \arrow[d, "PU"] \\ X \arrow[r, "G"'] & X/G, \end{tikzcd}$$

and the action of G on the second factor in $B' \times X$ restricts to a PU-equivariant action of G on B. □

7.1.1. The Phillips-Raeburn Theorem.

THEOREM 7.3 (Phillips-Raeburn [**130**], [**135**, Theorem 5.42]). *Let A be a stable CT-algebra with spectrum X and Dixmier-Douady invariant H. Then the image of the map $\sigma\colon \operatorname{Aut} CT(X,H) \to \operatorname{Homeo}(X)$ is precisely the stabilizer of H for the action of homeomorphisms on $H^3(X,\mathbb{Z})$. Let $\operatorname{Aut}_X CT(X,H) = \ker\sigma$. Then there is a natural short exact sequence*

$$1 \to \operatorname{Inn} CT(X,H) \to \operatorname{Aut}_X CT(X,H) \xrightarrow{\rho} H^2(X,\mathbb{Z}) \to 0.$$

Here $\operatorname{Inn} CT(X,H)$ is the group of automorphisms implemented by unitary multipliers. These are precisely the automorphisms exterior equivalent to the identity.

PROOF. Since the Dixmier-Douady class is an invariant of A, certainly any automorphism of A must preserve it, so the image of σ lies in the stabilizer of H. For the surjectivity, we can make use of Lemma 7.1, which identifies $\operatorname{Aut} A$ with $\operatorname{Homeo}_{PU}(B)$. Given $\varphi \in \operatorname{Homeo}(X)$, note that $\varphi^*(B \to X)$ is the bundle classified by $\varphi^*(H)$, so if φ stabilizes H, that means $\varphi^*(B \to X) \cong (B \to X)$. An isomorphism of these bundles then gives an element of $\operatorname{Homeo}_{PU}(B)$ lifting φ.

As we mentioned above, $\operatorname{Aut}_X CT(X,H) = \Gamma(X,\mathcal{B})$, where $\mathcal{B}$ is the gauge bundle, a locally trivial bundle of topological groups with fibers $\cong PU$, while $\operatorname{Inn} CT(X,H)$ consists of sections that lift to the bundle of groups with fibers $\cong U$ given by $\widetilde{\mathcal{B}} = B \times_{PU} U$, where $PU = U/Z(U)$ acts on U by conjugation. But from the short exact sequence of topological groups

$$1 \to \mathbb{T} \to U \to PU \to 1,$$

we get an exact sequence of sheaves of sections

$$1 \to \underline{\mathbb{T}} \to \underline{\widetilde{\mathcal{B}}} \to \underline{\mathcal{B}} \to 1$$

and thus an exact sequence in (nonabelian) sheaf cohomology

$$\Gamma(X,\widetilde{\mathcal{B}}) \to \Gamma(X,\mathcal{B}) \xrightarrow{\rho} H^1(X,\underline{\mathbb{T}}) \cong H^2(X,\mathbb{Z}). \tag{7.1}$$

This yields the exact sequence of the theorem except for the surjectivity of ρ, which can be deduced from a construction related to Proposition 7.4 below. Namely, given $c \in H^2(X,\mathbb{Z})$, form the principal S^1-bundle $p\colon Y \to X$ associated to it via the classification of line bundles, (2.1). Pull H back to $p^*H \in H^3(Y,\mathbb{Z})$ via the bundle projection p. There is now an evident "pull-back action" β of $\mathbb{T}$ on $CT(Y,p^*H)$ (use Lemma 7.1 again), and a simple calculation shows that $CT(Y,p^*H) \rtimes_\beta \mathbb{T} \cong CT(X,H)$. Now by Takai duality (Theorem 3.24), the crossed product $CT(X,H) \rtimes_\alpha \mathbb{Z}$, where $\alpha = \widehat{\beta}$ is the dual action, is isomorphic to $CT(Y,H)$. The fact that taking the spectrum gives the bundle $p\colon Y \to X$ shows that $\rho(\alpha) = c$. $\square$

We can think of the Phillips-Raeburn Theorem as a structure theorem for actions α of $\mathbb{Z}$ on $CT(X,H)$, modulo exterior equivalence. For example, the set of equivalence classes of actions of $\mathbb{Z}$ on $C_0(X,\mathcal{K})$ can be identified with

$$H^2(X,\mathbb{Z}) \rtimes \operatorname{Homeo}(X).$$

The $H^2(X,\mathbb{Z})$ arises as $[X, \operatorname{Aut}\mathcal{K} = PU \simeq K(\mathbb{Z},2)]$. $\rho(\alpha)$ can be identified with the Chern class of the circle bundle $\big(CT(X,H) \rtimes_\alpha \mathbb{Z}\big)\widehat{\ } \to X$. More precisely, we have the following "appendix" to the Phillips-Raeburn Theorem:

PROPOSITION 7.4. *Let A be a separable continuous-trace algebra with spectrum X and Dixmier-Douady invariant H, and let $\alpha\colon \mathbb{Z} \to \operatorname{Aut}_X A$ be a spectrum-fixing action of $\mathbb{Z}$ on A (which we can identify with the single automorphism $\alpha(1)$). Then $A \rtimes_\alpha \mathbb{Z}$ is a continuous-trace algebra whose spectrum Y is the total space of a principal $\mathbb{T}$-bundle $Y \to X$ over X. This bundle is classified by the Phillips-Raeburn invariant $\rho(\alpha) \in H^2(X, \mathbb{Z})$, and the Dixmier-Douady invariant of the crossed product is the pull-back of H to Y.*

PROOF. To show the crossed product has continuous trace with spectrum a principal S^1-bundle over X, it suffices to work locally and consider the case where the Dixmier-Douady invariant is trivial. Stabilizing by tensoring with a copy of $\mathcal{K}$ with trivial action, we can assume then that $A = C_0(X, \mathcal{K})$. Then α corresponds to a map $X \to \operatorname{Aut}\mathcal{K} \cong PU$, and cutting down to a still smaller neighborhood, we can assume that the map is null-homotopic. This implies that the action is inner, since by the homotopy lifting property for the quotient map $U \to PU$, we have a lifting

$$\begin{tikzcd} & & & X \arrow[dl, dotted] \arrow[d] & \\ 1 \arrow[r] & \mathbb{T} \arrow[r] & U \arrow[r] & PU \arrow[r] & 1, \end{tikzcd}$$

which shows that the automorphism is implemented by a unitary multiplier. Thus the action is exterior equivalent to a trivial action, and the crossed product is just $C_0(X, \mathcal{K}) \otimes C^*(\mathbb{Z}) = C_0(X \times \mathbb{T}, \mathcal{K})$. This local calculation proves that the crossed product always has continuous trace, and has spectrum which is a circle bundle over X. That $\rho(\alpha)$ classifies this bundle is basically true by definition. Furthermore, the dual action of $\widehat{\mathbb{Z}} = \mathbb{T}$ implements the bundle projection map $Y = (A \times_\alpha \mathbb{Z})\widehat{\ } \to X$.

There is one more thing to verify, namely that the Dixmier-Douady invariant of $B = A \rtimes_\alpha \mathbb{Z}$ is the pull-back of that of A. For this, it is useful to keep in mind that by Takai Duality, Theorem 3.24, $B \rtimes_{\widehat{\alpha}} \mathbb{T}$ is stably isomorphic to A, hence isomorphic to A if (as we may, without loss of generality) we take A to be stable. By Lemma 7.1, $\widehat{\alpha}$ gives a PU-equivariant action of $\mathbb{T}$ on the PU-bundle associated to the Dixmier-Douady class of B, lifting the free $\mathbb{T}$-action on Y with quotient space X, and the quotient of this action must be precisely the PU-bundle over X associated to the Dixmier-Douady class H of B. Proposition 7.2 says exactly that the Dixmier-Douady class of B is the pull-back of H. $\square$

For our purposes we need a similar structure theorem for actions of $\mathbb{R}$ on $CT(X, H)$, modulo exterior equivalence.

LEMMA 7.5 (Lifting). *Suppose one is given an action α of $\mathbb{R}$ on a (second-countable) connected locally compact space X (of the homotopy type of a finite CW complex). Let A be a stable CT-algebra with spectrum X. Then α lifts to an action $\widetilde{\alpha}$ of $\mathbb{R}$ on A, and the lifted action is unique up to exterior equivalence.*

PROOF OF EXISTENCE IN THE SMOOTH CASE. For most applications to physics, we only need the case where X is a manifold and the action α is smooth. The algebra A corresponds to a locally trivial principal PU-bundle B over X. We may assume this bundle is smooth as well (since every equivalence class of bundles contains a smooth representative). By Lemma 7.1, it suffices to lift the action of $\mathbb{R}$ on the base X to a PU-equivariant action on the bundle B. Now an $\mathbb{R}$-action is essentially the same thing as a vector field, since integrating a vector field gives a smooth $\mathbb{R}$-action, while differentiating a smooth $\mathbb{R}$-action gives a vector field.[1] So we need to lift a vector field on the base to a PU-equivariant "horizontal" vector field on the bundle, and a choice of a *connection* is precisely what is needed for this. Since any smooth bundle admits [lots of] connections, this proves existence. □

Proof of Uniqueness. For the uniqueness, we don't need the smoothness of the action. Let α and α' be two actions on $\mathbb{R}$ on A inducing the same action on the spectrum X. Then $c\colon t \mapsto \alpha'_t(\alpha_t)^{-1}$ is a continuous map $\mathbb{R} \to \operatorname{Aut}_X A$. Since we are assuming that X is homotopically finite, so that $H^2(X, \mathbb{Z})$ is countable and discrete in the natural topology, whereas $\mathbb{R}$ is connected, the Phillips-Raeburn Theorem (Theorem 7.3) shows that c takes values in the inner automorphisms, those induced by unitary multipliers. Furthermore, since α and α' are actions, c is a 1-cocycle with values in $\operatorname{Inn} A$. We need to lift it to a 1-cocycle with values in the unitary group of $M(A)$. This can be done using the method described below in Section 9.2.0.2. The point is that there is a suitable cohomology theory H^*_M for topological groups G and suitably topologized modules. From the short exact sequence

$$1 \to \Gamma(X, \underline{\mathbb{T}}) \to \Gamma(X, \widetilde{\mathcal{B}}) \to \big(\Gamma(X, \mathcal{B})\big)_0 \to 1$$

of $\mathbb{R}$-modules, where $\big(\Gamma(X, \mathcal{B})\big)_0$ denotes the kernel of the map ρ in (7.1), we get an exact sequence

$$H^1_M(\mathbb{R}, \Gamma(X, \widetilde{\mathcal{B}})) \to H^1_M(\mathbb{R}, \big(\Gamma(X, \mathcal{B})\big)_0) \to H^2_M(\mathbb{R}, C(X, \mathbb{T})),$$

which shows we have the required lifting of cocycles provided an obstruction in $H^2_M(\mathbb{R}, C(X, \mathbb{T}))$ vanishes. But in fact the whole group in which the obstruction lives is zero, as follows from Corollary 9.2 and Theorem 9.3. □

Proof in the Case of a Free $\mathbb{T}$-Action. Since the reader might find the above proofs of existence and uniqueness unsatisfying, given that they use ideas from differential geometry and group cohomology for which there is no hint in the original statement of the theorem, we shall give a self-contained proof of the case of the Lemma needed for Theorem 7.6 below. Namely, we assume $p\colon X \to Z$ is a principal $\mathbb{T}$-bundle, and we consider the $\mathbb{R}$-action on X which is trivial on $\mathbb{Z}$ and factors to the given action of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

First consider the case $X = \mathbb{T}$ with the simply transitive free action of $\mathbb{T}$ on it, so Z is just a single point. Thus $A = C(\mathbb{T}) \otimes \mathcal{K}$, and an action of $\mathbb{R}$ on A inducing the given action on the spectrum can be written as a product of the obvious action $\alpha = \tau \otimes 1_{\mathcal{K}}$, where τ is the translation action of $\mathbb{R}$ on $C(\mathbb{T})$, and a 1-cocycle $\{\dot{u}_t\}_{t\in\mathbb{R}}$ for α with values in $C(\mathbb{T}, PU)$, i.e., a map $\dot{u}\colon \mathbb{R} \times \mathbb{T} \to PU$ satisfying the cocycle identity $\dot{u}(t+s, \dot{x}) = \dot{u}(t, \dot{x})\dot{u}(s, \dot{t} + \dot{x})$, necessarily jointly continuous in the two variables. Such a cocycle is determined uniquely by its restriction to $[0, 1] \times \{\dot{0}\}$, which is a continuous path $\dot{\gamma}$ in PU with $\dot{\gamma}(0) = \dot{1}$, and conversely, it's easy to see that any such path extends uniquely to a cocycle. Since the quotient map $U \twoheadrightarrow PU$ is a fibration, any path $\dot{\gamma}$ in PU with $\dot{\gamma}(0) = \dot{1}$ lifts to a path γ in U with $\gamma(0) = 1$, so by the same sort of reasoning, any 1-cocycle lifts to a 1-cocycle with values in $C(\mathbb{T}, U)$. Thus any action of $\mathbb{R}$ on A inducing the translation action on

[1] There is a small additional complication if X is noncompact, since then one has to make sure orbits of the integrated action don't "run off to infinity," but this won't happen in our case since we are lifting a vector field on X which integrates to a genuine action of $\mathbb{R}$.

the spectrum is exterior equivalent to α. Note incidentally that we've shown that the sets of 1-cocycles $Z^1_M(\mathbb{R}, C(\mathbb{T}, PU))$ and $Z^1_M(\mathbb{R}, C(\mathbb{T}, U))$ can be identified with path spaces, and are thus contractible. So this completes the proof in this case.

Next suppose that both the Dixmier-Douady class H of A and the class of the bundle $p\colon X \to Z$ are trivial, i.e., that $A \cong C_0(Z) \otimes C(\mathbb{T}) \otimes \mathcal{K}$. Once we have fixed this isomorphism, i.e., fixed a trivialization of the bundle attached to A as in Lemma 7.1, an action of $\mathbb{R}$ on A inducing the given action on the spectrum can be written as a product of the obvious action $1_{C_0(Z)} \otimes \tau \otimes 1_{\mathcal{K}}$ with a 1-cocycle. Thus the set of all actions inducing the given action on the spectrum can be identified with $C(Z, Z^1_M(\mathbb{R}, C(\mathbb{T}, PU)))$. Since $Z^1_M(\mathbb{R}, C(\mathbb{T}, PU))$ is contractible, this space is also contractible. And since there is a continuous lifting from $Z^1_M(\mathbb{R}, C(\mathbb{T}, PU))$ to $Z^1_M(\mathbb{R}, C(\mathbb{T}, U))$, the lift of the action on X to an action on A is unique up to exterior equivalence.

Now consider the general case. Since X and Z are homotopically finite, we can choose a finite open covering $\{V_j\}$ of Z such that for each j, p and H are trivial when restricted to $p^{-1}(V_j)$. As we have already seen, there is a lifted action of $\mathbb{R}$ on the restriction of A to each $p^{-1}(V_j)$. Furthermore, the set of all such actions is contractible, and constitutes a single exterior equivalence class. By induction on n, we construct a lifting over $p^{-1}(V_1 \cup \cdots \cup V_n)$ and show the set of such liftings (over any $\mathbb{T}$-invariant open subset) is contractible and constitutes a single exterior equivalence class. The result for V_1 starts the induction. Assuming we know the result for n, we just need to patch together a lifting over $p^{-1}(V_1 \cup \cdots \cup V_n)$ and a lifting over $p^{-1}(V_{n+1})$. This is possible since on the overlap set, the set of liftings is contractible, so that we can extend the lifting over $p^{-1}\big(V_{n+1} \cap (V_1 \cup \cdots \cup V_n)\big)$ to a lifting over $p^{-1}(V_{n+1})$ compatible with what we have over $p^{-1}(V_1 \cup \cdots \cup V_n)$. The rest of the inductive step is similar. $\square$

7.2. The Raeburn-Rosenberg Theorem

THEOREM 7.6 (Raeburn-Rosenberg [**134**]). *Let X be a (second countable) locally compact space (of the homotopy type of a finite CW complex). Suppose X is the total space of a principal S^1-bundle $p\colon X \to Z$, and suppose $H \in H^3(X, \mathbb{Z})$. Let $A = CT(X, H)$. Lift the free action of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ on X to a locally free action of $\mathbb{R}$ on X, and then to an action α of $\mathbb{R}$ on A using the Lifting Lemma, Lemma 7.5. Let $B = A \rtimes_\alpha \mathbb{R}$. Then B is also a stable CT-algebra, with spectrum $X^\sharp$ the total space of another principal S^1-bundle $p^\sharp\colon X^\sharp \to Z$. The bundles and Dixmier-Douady classes are related by*

$$c_1(p) = (p^\sharp)_!(H^\sharp) \quad \textit{and} \quad c_1(p^\sharp) = (p)_!(H). \tag{7.2}$$

The algebra $B \rtimes_{\widehat{\alpha}} \mathbb{R}$ is isomorphic to A again.

In fact, if Y is the spectrum of $A \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{Z}$, or equivalently, of $B \rtimes_{\widehat{\alpha}|_{\mathbb{Z}}} \mathbb{Z}$, there is a commuting diagram of principal S^1-bundles

$$\begin{tikzcd} & Y \arrow[dl, "(p^\sharp)^* p"'] \arrow[dr, "p^*(p^\sharp)"] & \\ X^\sharp \arrow[dr, "p^\sharp"'] & & X \arrow[dl, "p"] \\ & Z & \end{tikzcd} \tag{7.3}$$

PROOF. First we need to check that B has continuous trace, with spectrum a principal S^1-bundle over Z. Since the statement is local, it's enough to prove this when $X = Z \times \mathbb{T}$ and H is trivial. But then by the uniqueness statement in the Lifting Lemma, we can assume up to Morita equivalence that we're looking at the abelian case, i.e, at $C_0(Z) \otimes \big(C(\mathbb{R}/\mathbb{Z}) \rtimes \mathbb{R}\big)$, where $\mathbb{R}$ is acting on $\mathbb{R}/\mathbb{Z}$ by translation. By the Green Imprimitivity Theorem, Theorem 5.11, $C(\mathbb{R}/\mathbb{Z}) \rtimes \mathbb{R}$ is Morita equivalent to $C^*(\mathbb{Z}) \cong C(\widehat{\mathbb{Z}}) = C(S^1)$. So the crossed product $C_0(Z \times \mathbb{T}) \rtimes \mathbb{R}$ is Morita equivalent to $C_0(Z \times \mathbb{T})$. *Though note, for purposes of connecting this to T-duality, that the circle factor is really the* ***dual circle*** *to the original one.*

The fact that $B \rtimes_{\widehat{\alpha}} \mathbb{R}$ is isomorphic to A again follows from Takai Duality (Theorem 3.24) and stability.

Similarly, if we restrict the action α on $A = CT(X, H)$ from $\mathbb{R}$ to $\mathbb{Z}$, the action is now trivial on the spectrum, and so by the Phillips-Raeburn Theorem, Theorem 7.3, it has an obstruction class $\rho(\alpha|_{\mathbb{Z}}) \in H^2(X, \mathbb{Z})$, which is the Chern class of a principal S^1-bundle $Y \to X$, where Y is the spectrum of $A \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{Z}$. This algebra also has continuous trace, as one can see by a local calculation. Furthermore,

$$B \rtimes_{\widehat{\alpha}|_{\mathbb{Z}}} \mathbb{Z} = \big(A \rtimes_{\alpha} \mathbb{R}\big) \rtimes_{\widehat{\alpha}|_{\mathbb{Z}}} \mathbb{Z}.$$

We now use the "Packer-Raeburn stabilization trick" [**126**], together with the fact that A was assumed stable to begin with. The reader need not know anything about the details, but this means (given suitable separability hypotheses) that any time one has a crossed product of a stable C^*-algebra A by a locally compact group G (in our case $\mathbb{R}$), with G having a closed normal subgroup N (in our case $\mathbb{Z}$), one can rewrite $A \rtimes G$ as an iterated crossed product $(A \rtimes N) \rtimes G/N$. Hence we can rewrite $B \rtimes_{\widehat{\alpha}|_{\mathbb{Z}}} \mathbb{Z}$ as

$$\Big(\big(A \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{Z}\big) \rtimes_{\beta} \mathbb{T}\Big) \rtimes_{\widehat{\beta}} \mathbb{Z} \cong A \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{Z},$$

with β induced from the original action α of $\mathbb{R}$, and the last statement following from Takai duality for β. So now we get a diamond

$$\begin{tikzcd} & Y \arrow[dl, "p_1^\sharp"'] \arrow[dr, "p_1"] & \\ X^\sharp \arrow[dr, "p^\sharp"'] & & X \arrow[dl, "p"] \\ & Z & \end{tikzcd}.$$

By the universal property of the pull-back, it follows that $p_1 = p^*(p^\sharp)$ and that $p_1^\sharp = (p^\sharp)^*(p)$, proving equations (7.3).

It remains to check equation (7.2). We will check at the same time that the duality $(X, H) \leftrightarrow (X^\sharp, H^\sharp)$ corresponds to the homotopy automorphism of the Bunke-Schick classifying space R, that corresponds to interchanging the two factors of $K(\mathbb{Z}, 2)$ in the Postnikov decomposition (6.3), so that the "analytic T-duality" given by crossed products of C^*-algebras coincides with the topological T-duality of Chapter 6.

The first step is to check that $\rho(\alpha|_{\mathbb{Z}}) = p^*(p_!(H))$. But what is $\rho(\alpha|_{\mathbb{Z}})$? Think of the principal PU-bundle $B \to X$ defined by H. As t goes from 0 to 1, the action of t on $x \in X$ traces out a circle (a fiber of $X \to Z$), so the lift α_t returns to the fiber B_x

of B over the starting point x. Thus α_1 is a PU-bundle automorphism of $B \to X$, and thus is a section of the gauge bundle $\mathcal{B}$ over X. The class $\rho(\alpha|_{\mathbb{Z}})$ is the image of this section in $H^1(X, \underline{\mathbb{T}}) \cong H^2(X, \mathbb{Z})$, a kind of monodromy invariant. This clearly depends on the interaction between the bundle $p\colon X \to Z$ and the class H defining the PU-bundle $B \to X$. So by naturality, $\rho(\alpha|_{\mathbb{Z}})$ must be given by a universal formula involving only H and p, and the only possibility is something of the form $np^*(p_!(H))$, where n is an integer. To check that the formula is exactly $p^*(p_!(H))$, it is enough to know what happens in a single nontrivial example, say when $X = S^2 \times S^1$ and $H = a \times b$, a and b the standard generators of $H^2(S^2)$ and $H^1(S^1)$, respectively. Then $[p] = 0$ and $p^*(p_!(H)) = a \times 1$. Choose $\gamma \in \operatorname{Aut}(C(S^2, \mathcal{K}))$ with $\rho(\gamma) = a$, and view $C(S^2, \mathcal{K})$ as a $\mathbb{Z}$-algebra via γ. Then $A = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} C(S^2, \mathcal{K})$ is an $\mathbb{R}$-algebra with spectrum $X = S^2 \times S^1$, and its Dixmier-Douady invariant is $H = a \times b$, as one can see from the fact that it is obtained by gluing two copies of $C(S^2 \times [0, 1], \mathcal{K})$ together using a nontrivial clutching map. When we restrict $\alpha = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} \gamma$ to $\mathbb{Z}$, we get an action which is pointwise given by γ on each fiber, so $\rho(\alpha|_{\mathbb{Z}}) = \rho(\gamma) \times 1 = a \times 1 = p^*(p_!(H))$.

To complete the proof of (7.2), it suffices to check the "universal example" $X = E$ and $Z = R$, H the canonical class in $H^3(E, \mathbb{Z})$ (in the notation of Chapter 6). More precisely, since E and R are not locally compact but are locally finite CW complexes, we take X and Z to be finite skeleta of E and R and pass to the limit, but we'll mostly ignore this technicality. So take Z to be a sufficiently high finite skeleton of R, X its preimage in E, and H the canonical class in $H^3(E)$, restricted to X. Let $A = CT(X, H)$, an let α be an action of $\mathbb{R}$ on A, with $\mathbb{Z}$ acting trivially on X and with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ acting via the free circle group action on X with quotient space Z. By the formula $\rho(\alpha|_{\mathbb{Z}}) = p^*(p_!(H))$ that we already verified, $C = A \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{Z}$ has spectrum Y an approximation to $K(\mathbb{Z}, 3)$ and Dixmier-Douady class the canonical class in $H^3(Y)$, with $Y \to X$ the canonical principal S^1-bundle over $X \approx E \simeq K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$.

Now consider the diamond (7.3). As far as cohomology through degree 4 is concerned, we can assume that Z looks like R, so $H^2(Z, \mathbb{Z}) = \mathbb{Z}u \oplus \mathbb{Z}v$, where u and v are as in the proof of Theorem 6.3, with relations $u \cdot v = 0$, and $H^3(Z, \mathbb{Z}) = 0$, $H^4(Z, \mathbb{Z}) = \mathbb{Z}u^2 \oplus \mathbb{Z}v^2$. We are assuming that $u = c_1(p)$. Since we are assuming $H^3(Y, \mathbb{Z}) = \mathbb{Z}w$, with w the pull-back of H by $p^*(p^\sharp)$, it follows that $H^\sharp \in H^3(X^\sharp, \mathbb{Z})$ also pulls back to w, and so $H^3(X^\sharp, \mathbb{Z}) \neq 0$. And since $c_1(p^\sharp)$ pulls back under p to the generator of $H^2(X, \mathbb{Z})$, we must have $c_1(p^\sharp) = v + nu$ for some $n \in \mathbb{Z}$. (This is because $u = c_1(p)$ generates the kernel of $p^*\colon H^2(Z) \to H^2(X)$.) But if $n \neq 0$, the kernel of multiplication by $c_1(p^\sharp)$ on $H^2(R)$ is

$$\{xu + yv \mid (xu + yv) \cdot (v + nu) = yv^2 + xnu^2 = 0\} = 0,$$

which by the Gysin sequence for $p^\sharp$ would force $H^3(X^\sharp) = 0$, a contradiction. Thus what we already know about $A \rtimes_{\alpha|_{\mathbb{Z}}} \mathbb{Z}$ implies the formula (7.2). This completes the proof. $\square$

CHAPTER 8

Higher-Dimensional T-Duality via Topological Methods

8.1. Higher-Dimensional T-Duality

We now want to generalize T-duality to the case of spacetimes X "compactified on a higher-dimensional torus," or in other words, equipped with a principal $\mathbb{T}^n$-bundle $p\colon X \to Z$. In the simplest case, $X = Z \times \mathbb{T}^n = Z \times \overbrace{S^1 \times \cdots S^1}^{n}$. We can then perform a string of n T-dualities, one circle factor at a time. A single T-duality interchanges type IIA and type IIB string theories, so this n-dimensional T-duality "preserves type" when n is even and switches it when n is odd. In terms of our set of axioms for topological T-duality, we would therefore expect an isomorphism $K^*(X, H) \cong K^*(X^\sharp, H^\sharp)$ when n is even and $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$ when n is odd.

In the higher-dimensional case, a new problem presents itself: *it is no longer clear that the T-dual should be unique.* In fact, if we perform a string of n T-dualities, one circle factor at a time, it is not clear that the result should be independent of the order in which these operations are done. Furthermore, a higher-dimensional torus does not split as a product in only one way, so in principle there can be a lot of non-uniqueness.

The way out of this difficulty has therefore been to try to organize the information in terms of a *T-duality group*, a discrete group of T-duality isomorphisms potentially involving a large number of spacetimes and H-fluxes. We can think of this group as operating on some big metaspace (sometimes called the "landscape of string theory vacua" or simply "string landscape" [**155**]) of possible spacetimes.

Another difficulty is that there are some spacetimes with H-flux that would appear to have no higher-dimensional T-duals at all, at least in the sense we have defined them so far.

Example 8.1. Consider the case of $X = T^3$, which is a principal $\mathbb{T}^3$-bundle over a point, and $H = \alpha \times \alpha \times \alpha$ the generator of $H^3(X, \mathbb{Z}) \cong \mathbb{Z}$ (α the generator of $H^1(S^1)$). Choose one of the circle factors, say the first one, and dualize. Since $p\colon T^3 \to T^2$ is a trivial bundle, the result of the first T-duality should be $p_1\colon X^{(1)} \to T^2$ with H-flux $H^{(1)}$, where $c_1(p_1) = p_!(H) = \alpha \times \alpha$ and $(p_1)_!(H^{(1)}) = 0$. Thus $X^{(1)}$ is the *Heisenberg nilmanifold* and, from the Gysin sequence, $(p_1)_!\colon H^3(X^{(1)}) \to H^2(T^2)$ is an isomorphism, and thus $H^{(1)} = 0$.

Now we continue with $X^{(1)}$, the Heisenberg nilmanifold, and trivial H-flux. Note that $\pi_1(X^{(1)})$ is nonabelian and $H_1(X^{(1)}, \mathbb{Z}) \cong \mathbb{Z}^2$. If $X^{(1)}$ is to be the total space of a principal S^1-bundle, the base has to be an oriented 2-manifold with vanishing Euler characteristic, which can only be T^2. Since there is no way to split off an S^1-factor from $X^{(1)}$, *there is only one way to write the manifold* $X^{(1)}$ *as the total space of a principal* S^1*-bundle, namely as the way it already arose as an* S^1*-bundle over* T^2. So the only way to T-dualize again is to come back to where we started. In other words, there is a "missing T-dual."

EXAMPLE 8.2. We can also think about the example of $X = T^3$, with $H = \alpha \times \alpha \times \alpha$, as a case of a (trivial) principal $\mathbb{T}^2$-bundle over S^1 (via, say, projection p on the second factor when we write T^3 as $T^2 \times S^1$). But there are no nontrivial principal $\mathbb{T}^2$-bundles over S^1, so the only possible T-dual of this form is just the original space back again, again with the same H-flux. This does not seem right since we expect a nontrivial H-flux to be reflected in nontrivial topology for the dual bundle.

8.2. A Higher-Dimensional Bunke-Schick Theorem

8.2.1. Formulation and Consequences. What we are aiming for is a higher-dimensional version of the Bunke-Schick Theorem, Theorem 6.3. As before, we fix a locally compact base space Z and consider the set of pairs $(X \xrightarrow{\pi} Z, H)$, where X is a principal $\mathbb{T}^n$-bundle over Z and $H \in H^3(X, \mathbb{Z})$. We could show that the set of all such pairs up to isomorphism is a representable functor and comes by pull-back from a universal example:

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d, "\pi"] & E \arrow[d, "p"] \\ Z \arrow[r, dotted] & R, \end{tikzcd}$$

with the horizontal maps unique up to homotopy and H pulled back from a canonical class $h \in H^3(E, \mathbb{Z})$.

The problem is the phenomenon we already mentioned of missing T-duals. For this reason, we add an extra condition: *that* H *restricts to* 0 *on each torus fiber.* For if not, when we restrict to a fiber, we basically come back to Example 8.1 from before.

THEOREM 8.3 (Mathai-Rosenberg [**109**, §5], Bunke-Rumpf-Schick [**33**]). *Let*

$$\mathcal{P} = \{(X \xrightarrow{\pi} Z, H) \mid \pi \text{ a } \mathbb{T}^n\text{-bundle}, \ H \in H^3(X, \mathbb{Z}), \ H|_T = 0 \ \forall \text{ fibers } T \text{ of } \pi\}/\sim .$$

There is a connected classifying space R*, unique up to homotopy equivalence, such that each* $(X, H) \in \mathcal{P}$ *comes by pull-back from a universal example:*

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d, "\pi"] & E \arrow[d, "p"] \\ Z \arrow[r, dotted] & R. \end{tikzcd}$$

Furthermore, the horizontal maps are unique up to homotopy. The space R *has the homotopy groups* $\pi_j(R) = 0$ *for* $j > 3$*,* $\pi_1(R) \cong \mathbb{Z}^{\binom{n}{2}}$*,* $\pi_2(R) \cong \mathbb{Z}^{2n}$*,* $\pi_3(R) \cong \mathbb{Z}$*.*

Let $\widetilde{R}$ *denote the universal cover of* R*. Then* $\widetilde{R}$ *sits in a Postnikov fibration*

$$\begin{tikzcd} K(\mathbb{Z},3) \arrow[r] & \widetilde{R} \arrow[d] \\ & K(\mathbb{Z}^n,2) \times K(\mathbb{Z}^n,2), \end{tikzcd} \tag{8.1}$$

with k-invariant $x_1y_1 + \cdots + x_ny_n$, the x's and y's being the canonical generators of $H^2(K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2)) = H^2\Big((\mathbb{CP}^\infty)^{2n}\Big)$.

REMARK 8.4. Note that the group $O(n, n; \mathbb{Z})$ of integral matrices preserving the form $x_1y_1 + \cdots + x_ny_n$ preserves the k-invariant of (8.1) and thus acts by homotopy automorphisms on $\widetilde{R}$. This is indeed the *T-duality group* studied by physicists. That essentially means that if we restrict to the case where Z is simply connected, we have a good theory of topological T-duality.

However, there is no obvious action of $O(n, n; \mathbb{Z})$ on R itself. This is related to Example 8.2 we saw before of missing T-duals when $Z = S^1$ and $n = 2$.

SKETCH OF THE PROOF. We begin by constructing (explicitly!) the spaces E and R and showing that they have the right universal property. Let

$$E = E\mathbb{T}^n \times \mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3)), \qquad R = E\mathbb{T}^n \times_{\mathbb{T}^n} \mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3)), \tag{8.2}$$

where $E\mathbb{T}^n \to B\mathbb{T}^n = K(\mathbb{Z}^n, 2) \cong (\mathbb{CP}^\infty)^n$ is the universal $\mathbb{T}^n$-bundle, and where Map_0 denotes the set of null-homotopic maps (those giving the trivial class in $H^3(\mathbb{T}^n, \mathbb{Z})$). Note that using Map_0 in place of Map makes R path-connected. (When $n = 1$, $\mathrm{Map}(\mathbb{T}^n, K(\mathbb{Z}, 3))$ is the free loop space of $K(\mathbb{Z}, 3)$, and is already connected.) There is an obvious map

$$c\colon\ R = E\mathbb{T}^n \times_{\mathbb{T}^n} (\cdots) \to E\mathbb{T}^n \times_{\mathbb{T}^n} \mathrm{pt} = B\mathbb{T}^n$$

which corresponds to forgetting the second entry of a pair and just taking the underlying bundle. This map is a fibration with homotopy fiber $\mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3))$, and in fact it is the classifying map for the quotient map $p\colon\ E \twoheadrightarrow R$, which sits in a homotopy pull-back diagram

$$\begin{tikzcd}
E \arrow[r, dotted] \arrow[d, dotted, "p"] & E\mathbb{T}^n \arrow[d] \\
R \arrow[r, "c"] & B\mathbb{T}^n.
\end{tikzcd}$$

In fact we can take $E = E\mathbb{T}^n \times \mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3))$. Then we define $\mathbf{h}\colon\ E \to K(\mathbb{Z}, 3)$ to be given by

$$\mathbf{h}(v, \gamma) = \gamma(0),$$

where $\gamma \in \mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3))$ and $v \in E\mathbb{T}^n$. Clearly any map $Z \to R$ enables us to pull back the canonical pair $(E, \mathbf{h})$ to a pair over Z.

Now let's consider the other direction. Given $(X \xrightarrow{\pi} Z, H)$ representing a class in $\mathcal{P}$, we know $X \xrightarrow{\pi} Z$ is pulled back from $E\mathbb{T}^n \to B\mathbb{T}^n$ via a classifying map f unique up to homotopy. Also H comes from a classifying map $h\colon\ X \to K(\mathbb{Z}, 3)$. Then we fill in the diagram

$$\begin{tikzcd}
 & K(\mathbb{Z},3) & & E \arrow[ll, "\mathbf{h}"'] \arrow[dl] \arrow[d, "p"] \\
X \arrow[ur, "h"] \arrow[urrr, dotted, "\widehat{\varphi}"] \arrow[r, "\widehat{f}"'] \arrow[d, "\pi"] & E\mathbb{T}^n \arrow[d] & & R \arrow[dll, "c"] \\
Z \arrow[r, "f"'] \arrow[urrr, dotted, "\varphi"] & B\mathbb{T}^n. & &
\end{tikzcd}$$

as shown to make it commute and to realize (X, h) as the pull-back of $(E, \mathbf{h})$.

Indeed, we simply define $\widehat{\varphi}$ by $\widehat{\varphi}(x) = (\widehat{f}(x), \gamma)$, where $\gamma \in \mathrm{Map}(\mathbb{T}^n, K(\mathbb{Z}, 3))$ is defined by $\gamma(t) = h(x \cdot t)$. Since the definition of a pair includes the requirement that h be null-homotopic on each torus fiber, γ indeed lies in $\mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3))$. The map $\widehat{\varphi}$ is $\mathbb{T}^n$-equivariant and descends to a map $\varphi\colon Z \to R$ since

$$\widehat{\varphi}(x \cdot t_0) = (\widehat{f}(x \cdot t_0), \gamma') = (\widehat{f}(x) \cdot t_0, t_0^{-1} \cdot \gamma) \text{ with } \gamma'(t) = h((x \cdot t_0) \cdot t) = \gamma(t_0 t).$$

Thus R has the required universal property.

Now we need to check the statements about the homotopy groups. Recall that we had a fibration

$$\mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3)) \to R \to B\mathbb{T}^n = K(\mathbb{Z}^n, 2).$$

So to compute the homotopy groups of R, we just need to compute those of the mapping space $\mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3))$. We have

$$\begin{aligned}
\pi_j(\mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3))) &= \big[(S^j, \mathrm{pt}), (\mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3)), \mathrm{pt})\big] \\
&= \ker\Big([S^j \times \mathbb{T}^n, K(\mathbb{Z}, 3)] \to [\mathbb{T}^n, K(\mathbb{Z}, 3)]\Big) \\
&= \ker\Big(H^3(S^j \times \mathbb{T}^n, \mathbb{Z}) \to H^3(\mathbb{T}^n, \mathbb{Z})\Big) \\
&= \begin{cases} 0, & j > 3, \\ H^2(\mathbb{T}^n, \mathbb{Z}), & j = 1, \\ H^1(\mathbb{T}^n, \mathbb{Z}), & j = 2, \\ \mathbb{Z} & j = 3. \end{cases}
\end{aligned}$$

Thus $\pi_j(\mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3))) \cong \mathbb{Z}^{\binom{n}{2}}$ if $j = 1$, $\mathbb{Z}^n$ if $j = 2$, $\mathbb{Z}$ if $j = 3$, and 0 otherwise. Plugging this data into the long exact homotopy sequence

$$\begin{aligned}
\cdots \to \pi_{j+1}(K(\mathbb{Z}^n, 2)) \xrightarrow{\partial} \pi_j(&\mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3))) \\
&\to \pi_j(R) \to \pi_j(K(\mathbb{Z}^n, 2)) \to \cdots
\end{aligned}$$

gives the calculation of the homotopy groups, since it's easy to see the map $\pi_2(R) \to \pi_2(K(\mathbb{Z}^n, 2))$ is surjective and the only potentially non-zero connecting map ∂ is

$$\pi_2(K(\mathbb{Z}^n, 2)) \to \pi_1(\mathrm{Map}_0(\mathbb{T}^n, K(\mathbb{Z}, 3))).$$

At this point we know $\widetilde{R}$ sits in a Postnikov fibration

$$\begin{tikzcd}
K(\mathbb{Z}, 3) \arrow[r] & \widetilde{R} \arrow[d] \\
 & K(\mathbb{Z}^n, 2) \times K(\mathbb{Z}^n, 2),
\end{tikzcd}$$

and it remains to compute the k-invariant. This can be done using computations with the Serre spectral sequence and reduction to the case $n = 1$, where we already know the result. $\square$

8.2.2. More About the Classifying Space. While we have an explicit description of the classifying space R, it is a somewhat complicated and mysterious space compared to the case of $n = 1$. The reason is that we have *three* non-zero homotopy groups, one of which is π_1. Traditional Postnikov theory doesn't apply to spaces which aren't simply connected, but fortunately one can check that R is *simple*, i.e., $\pi_1(R)$ acts trivially on the other homotopy groups. (Otherwise things would be really complicated.)

In the next chapter we will focus on the first interesting case, $n = 2$. There still is no natural T-duality action on all of R, because of the case of a "missing T-dual" described earlier. But when $n = 2$, $\pi_1(R) \cong \mathbb{Z}$, which means one has maps $\begin{tikzcd} R \arrow[r, dotted, bend left] & S^1 \arrow[l, dotted, bend left] \end{tikzcd}$ inducing isomorphisms on π_1, the map to the right given by the generator of $H^1(R, \mathbb{Z}) \cong [R, S^1]$ and the map to the left given by the inclusion of a loop generating π_1. These fit into a split homotopy fibration

$$\begin{tikzcd} \widetilde{R} \arrow[r] & R \arrow[r] & S^1. \arrow[l, dotted, bend right] \end{tikzcd}$$

More detailed analysis then shows *that if* $n = 2$, *R has the homotopy type of* $\widetilde{R} \times S^1$. Since we know the homotopy type of $\widetilde{R}$ completely, this gives us a complete picture of R. In particular, the T-duality group $O(2, 2; \mathbb{Z})$ can be made to act on R by homotopy automorphisms, but not in any *natural* way.

CHAPTER 9

Higher-Dimensional T-Duality via C^*-Algebraic Methods

9.1. Methodology and a Key Example

9.1.1. Methodology. In this chapter we will go back to T-duality to the case of spacetimes X "compactified on a higher-dimensional torus," but using the C^*-algebraic method introduced in Chapter 7. Again we start with a principal $\mathbb{T}^n$-bundle $p\colon X \to Z$ and an "H-flux" $H \in H^3(X, \mathbb{Z})$. As in the last chapter, we assume that H is trivial when restricted to each $\mathbb{T}^n$-fiber of p. This of course is no restriction if $n = 2$.

Proceeding as before, we want to lift the free action of $\mathbb{T}^n$ on X to an action on the continuous-trace algebra $A = CT(X, H)$. Usually (because of Proposition 7.2) there is no hope to get such a lifting for $\mathbb{T}^n$ itself, so we go to the universal covering group $\mathbb{R}^n$. If $\mathbb{R}^n$ acts on A so that the induced action on $\widehat{A}$ is trivial on $\mathbb{Z}^n$ and factors to the given action of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, then we can take the crossed product $A \rtimes \mathbb{R}^n$ and use Connes' Thom Isomorphism Theorem, Theorem 5.10, to get an isomorphism between $K^{*+n}(X, H)$ and $K_*(A \rtimes \mathbb{R}^n)$.

Under favorable circumstances, we can hope that the crossed product $A \rtimes \mathbb{R}^n$ will again be a continuous-trace algebra $CT(X^\sharp, H^\sharp)$, with $p^\sharp\colon X^\sharp \to Z$ a new principal $\mathbb{T}^n$-bundle and with $H^\sharp \in H^3(X^\sharp, \mathbb{Z})$. If we then act on $CT(X^\sharp, H^\sharp)$ with the dual action of $\widehat{\mathbb{R}}^n$, then by Takai Duality, Theorem 3.24, and stability, we come back to where we started. So we have a topological T-duality between (X, H) and $(X^\sharp, H^\sharp)$. Furthermore, we have an isomorphism

$$K^{*+n}(X, H) \cong K^*(X^\sharp, H^\sharp),$$

as required for matching of D-brane charges under T-duality.

Now what about the problems we identified before, about potential non-uniqueness of the T-dual and "missing" T-duals? These can be explained either by lack of a suitable action of $\mathbb{R}^n$ on $A = CT(X, H)$, or by non-uniqueness of the lift to an action of $\mathbb{R}^n$ on $A = CT(X, H)$, or by failure of the crossed product to be a continuous-trace algebra, or by a combination of the last two situations.

9.1.2. A Key Example. Let's now examine what happens when we try to carry out this program in one of our "problem cases," $n = 2$, $Z = S^1$, $X = T^3$ (a trivial $\mathbb{T}^2$-bundle over S^1), and H the usual generator of $H^3(T^3)$. First we show that there is an action of $\mathbb{R}^2$ on $CT(X, H)$ compatible with the free action of $\mathbb{T}^2$ on X with quotient S^1. We will need the definition of an *induced action* from Section 5.3. We start with an action α of $\mathbb{Z}^2$ on $C(S^1, \mathcal{K})$ which is trivial on the spectrum. This is given by a map $\mathbb{Z}^2 \to C(S^1, \operatorname{Aut}\mathcal{K}) = C(S^1, PU(L^2(\mathbb{T})))$ sending the two generators of $\mathbb{Z}^2$ to the maps $S^1 \to U(L^2(\mathbb{T}))$:

$$\begin{cases} w & \mapsto \text{multiplication by } z, \\ w & \mapsto \text{translation by } w. \end{cases}$$

(Though these unitaries do not commute in U, their images in PU do commute as required.)

Now form $A = \operatorname{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} C(S^1, \mathcal{K})$. This is a C^*-algebra with $\mathbb{R}^2$-action $\operatorname{Ind}\alpha$ whose spectrum (as an $\mathbb{R}^2$-space) is $\operatorname{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} S^1 = S^1 \times \mathbb{T}^2 = X$. We can see that $A \cong CT(X, H)$ via "inducing in stages". Let $B = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} C(S^1, \mathcal{K}(L^2(\mathbb{T})))$ be the result of inducing over the first copy of $\mathbb{R}$. It's clear that $B \cong C(S^1 \times \mathbb{T}, \mathcal{K})$. We still have another action of $\mathbb{Z}$ on B coming from the second generator of $\mathbb{Z}^2$, and $A = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} B$. The action of $\mathbb{Z}$ on B is by means of a map $\sigma\colon\ S^1 \times \mathbb{T} \to PU(L^2(\mathbb{T})) = K(\mathbb{Z}, 2)$, whose value at (w, z) is the product of multiplication by z with translation by w. Thus A is a CT-algebra with Dixmier-Douady invariant $[\sigma] \times c = H$, where $[\sigma] \in H^2(S^1 \times \mathbb{T}, \mathbb{Z})$ is the homotopy class of σ and c is the usual generator of $H^1(S^1, \mathbb{Z})$.

Now that we have an action of $\mathbb{R}^2$ on $A = CT(X, H)$ inducing the free $\mathbb{T}^2$-action on the spectrum X, we can compute the crossed product to see what the associated "T-dual" is. Since $A = \operatorname{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} C(S^1, \mathcal{K})$, we can use the Green Imprimitivity Theorem, Theorem 5.11, to see that

$$A \rtimes_{\operatorname{Ind}\alpha} \mathbb{R}^2 \cong \left(C(S^1, \mathcal{K}) \rtimes_\alpha \mathbb{Z}^2 \right) \otimes \mathcal{K}.$$

Recall from Section 5.3.1 that A_θ is the universal C^*-algebra generated by unitaries U and V with $UV = e^{2\pi i\theta} VU$. So if we look at the definition of α, we see that $A \rtimes_{\operatorname{Ind}\alpha} \mathbb{R}^2$ is the algebra of sections of a bundle of algebras over S^1, whose fiber over $e^{2\pi i\theta}$ is $A_\theta \otimes \mathcal{K}$. Alternatively, it is Morita equivalent to $C^*(\Gamma)$, where Γ is the *discrete Heisenberg group* of strictly upper-triangular 3×3 integral matrices. This calculation suggests that irrational rotation algebras should in some cases be connected to string theory. Indeed, Connes, Douglas, and Schwarz [**46**] have constructed a physical model based on noncommutative tori and related to M-theory. This in turn has led to an explosion of work on string theory in "noncommutative spacetime," which is partially summarized in [**157**].

Put another way, we could argue that we've shown that $C^*(\Gamma)$ is a *noncommutative T-dual* to (T^3, H), both viewed as fibering over S^1. So we have an explanation for the missing T-dual: *we couldn't find it just in the world of topology alone because it's noncommutative.* We will want to see how widely this phenomenon occurs, and also will want to resolve the question of uniqueness of T-duals when $n > 1$.

9.2. Uniqueness of Group Actions

To deal with the uniqueness question, we need to recall the notion of *exterior equivalence* which came up in Section 5.2 for actions of a group G on a C^*-algebra A. We want to answer the question: if α and α' are two actions of a Lie group G (say $\mathbb{R}^2$) on a CT-algebra $A = C(X, H)$, and if α and α' induce the same action of G on X, when are they exterior equivalent?

We will come back to these questions shortly, but first we need a digression on group cohomology.

9.2.0.1. *Cohomology Theory for Locally Compact Groups.* As we have mentioned before, an action of a (second-countable) locally compact group G on a (separable) C^*-algebra A is really a continuous homomorphism from G into the topological group $\operatorname{Aut} A$. This topological group is almost never locally compact, but it is Polish, meaning that the group is second-countable (thus separable and metrizable) and complete in its two-sided uniformity. Since the topologies of G and $\operatorname{Aut} A$ are important for our purposes, it seems natural to study group actions using a cohomology theory for (second-countable) locally compact groups acting on Polish modules. There is one (and only one) such theory satisfying certain obvious axioms, namely Calvin Moore's cohomology theory [**116, 117, 118**], which we will denote by $H^\bullet_M(G, —)$. This is sometimes called *cohomology with Borel cochains*, since it can be computed from the complex of Borel cochains on G with values in A. (Part of the genius of Moore's theory is that using Borel cochains, as opposed to continuous ones, is essential here, since there may not be enough of the latter for topological reasons. Nevertheless, it turns out that any Borel 1-*cocycle* is automatically continuous.) Of course, if G is discrete, then any cocycle for G is automatically continuous (or Borel), so Moore cohomology in this case is just usual cohomology for groups.

We will need the following general theorems about the theory $H^\bullet_M(G, —)$. The first generalizes the well-known fact that for a discrete group G, the group cohomology of G agrees with the topological cohomology of the classifying space BG.

THEOREM 9.1 (Wigner [**171**]). *Let G be a Lie group (with countably many components) and let A be a countable discrete G-module. Then there is a natural isomorphism $H^\bullet_M(G, A) \cong H^\bullet(BG, \mathcal{A})$, where $\mathcal{A}$ is the locally constant sheaf (i.e., local coefficient system) defined by A on the classifying space BG.*

COROLLARY 9.2. *Let G be a vector group and let A be a countable discrete group (with the trivial G-action). Then $H^k_M(G, A) = 0$ for $k > 0$.*

PROOF OF COROLLARY FROM THE THEOREM. If G is a vector group, G is contractible and so BG is contractible. Since the action of G on A is trivial, $\mathcal{A}$ is the constant sheaf A, and so $H^\bullet(BG, \mathcal{A}) \cong H^\bullet(BG, A) \cong H^\bullet(\mathrm{pt}, A)$, which vanishes in positive degrees. $\square$

THEOREM 9.3 (Generalized Van Est [**165, 73**]). *Let G be a vector group and let A be a G-module which is a separable locally convex topological vector space, A^∞ the smooth vectors for the G-action. Then*

$$H^k_M(G, A) \cong H^k_M(G, A^\infty) \cong H^k_{\mathrm{Lie}}(\mathfrak{g}, A^\infty).$$

Here $H^\bullet_{\mathrm{Lie}}(\mathfrak{g}, —)$ is Lie algebra cohomology for the Lie algebra $\mathfrak{g}$ of G. In particular, the cohomology vanishes for $k > \dim G$.

9.2.0.2. *Obstruction Theory for Equivalence of Group Actions.* Now suppose G is a second-countable locally compact group and A is a separable C^*-algebra. Recall that two actions α and α' of G on A are exterior equivalent if $\alpha'_g\alpha_g^{-1} = \operatorname{Ad} u_g$, with $g \mapsto u_g$ a 1-cocycle $G \to U(M(A))$. Here $U(M(A))$ is the unitary group of the multiplier algebra of $M(A)$, with the Polish topology coming from its action on A on the left and the right. Assume the center of $U(M(A))$ is $C(X, \mathbb{T})$ for some second-countable locally compact space X. Then the group $C(X, \mathbb{T})$ is Polish for the compact-open topology on maps $X \to \mathbb{T}$.

THEOREM 9.4 (Rosenberg [**140**]). *In this situation, the actions α and α' are exterior equivalent if and only if two conditions are satisfied:*

- *$\forall g \in G$, we need $\alpha'_g \alpha_g^{-1} \in \operatorname{Inn} A$, i.e., is given by conjugation by a unitary multiplier.*
- *we need vanishing of an additional obstruction in $H^2_M(G, C(X, \mathbb{T}))$.*

PROOF. If the actions are exterior equivalent, then first of all $\alpha'_g \alpha_g^{-1}$ must be inner for all $g \in G$, but also we need a lifting

$$\begin{tikzcd} & & & G \arrow[dl, dotted] \arrow[d, "\alpha'\alpha^{-1}"] & \\ 1 \arrow[r] & C(X, \mathbb{T}) \arrow[r] & U(M(A)) \arrow[r] & \operatorname{Inn}(A) \arrow[r] & 1. \end{tikzcd}$$

From the short exact sequence

$$1 \to C(X, \mathbb{T}) \to U(M(A)) \to \operatorname{Inn}(A) \to 1$$

we obtain a long exact cohomology sequence[1]

$$\cdots \to H^1_M(G, U(M(A))) \to H^1_M(G, \operatorname{Inn}(A)) \xrightarrow{\partial} H^2_M(G, C(X, \mathbb{T})),$$

and one has the desired lifting of the 1-cocycle from $G \to \operatorname{Inn}(A)$ to $G \to U(M(A))$ exactly when $\partial(\alpha'\alpha^{-1}) = 1$ in $H^2_M(G, C(X, \mathbb{T}))$. $\square$

9.2.0.3. *Calculations for Our Example.* Let's compute the obstructions for the example above with $n = 2$, $Z = S^1$, and $X = T^3$. Is the action $\operatorname{Ind} \alpha$ we wrote down the *unique* action (up to exterior equivalence) inducing the free action of $\mathbb{T}^2$ on X?

By the Phillips-Raeburn Theorem, Theorem 7.3,

$$\operatorname{Aut}_X CT(X, H) / \operatorname{Inn} CT(X, H) \cong H^2(X, \mathbb{Z}),$$

which is discrete. Since $\mathbb{R}^2$ is connected and we are considering only continuous actions, it follows that any two actions inducing the same action on the spectrum do indeed differ by a cocycle in $\operatorname{Inn} CT(X, H)$. So the first condition in Theorem 9.4 is satisfied and we need to check the second condition.

This requires computing the cohomology of $C(X, \mathbb{T})$ (as a module over $G = \mathbb{R}^2$). First we reduce to computing the cohomology of $C(X, \mathbb{R})$, using the exact sequences of $\mathbb{R}^2$-modules

$$\begin{aligned} 1 \to C(X, \mathbb{T})_0 \to& C(X, \mathbb{T}) \to H^1(X, \mathbb{Z}) \to 1, \\ 1 \to H^0(X, \mathbb{Z}) \to& C(X, \mathbb{R}) \to C(X, \mathbb{T})_0 \to 1. \end{aligned}$$

(The action of $\mathbb{R}^2$ here of course comes from the action on X.)

By Corollary 9.2, the higher cohomology of the discrete G-modules $H^0(X, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(X, \mathbb{Z}) \cong \mathbb{Z}^3$ must vanish. So by the long exact cohomology sequences for these exact sequences of modules, we find that

$$H^2_M(G, C(X, \mathbb{T})) \cong H^2_M(G, C(X, \mathbb{T})_0) \cong H^2_M(G, C(X, \mathbb{R})).$$

We can now apply Theorem 9.3, getting

$$H^2_M(G, C(X, \mathbb{T})) \cong H^2_M(G, C(X, \mathbb{R})) \cong H^2_{\mathrm{Lie}}(\mathfrak{g}, C^{\mathrm{sm}}(X)),$$

[1]The groups $U(M(A))$ and $\operatorname{Inn}(A)$ are noncommutative, but one still gets an exact sequence in (nonabelian) cohomology going out as far as indicated.

where $C^{\mathrm{sm}}(X)$ denotes functions which are smooth along the G-orbits. (Note that this is not the same as $C^\infty(X)$, as functions in $C^{\mathrm{sm}}(X)$ are smooth in two $\mathbb{T}$-variables but only continuous in the third—the Z coordinate.) By Poincaré duality in Lie algebra cohomology, the cohomology is the same as

$$H^0_{\mathrm{Lie}}(\mathfrak{g}, C^{\mathrm{sm}}(X)) = (C^{\mathrm{sm}}(X))^G = C(S^1).$$

Corollary 9.5. *Two G-actions on $CT(X,H)$ which induce the free action of $\mathbb{T}^2$ on $X = T^3$ with quotient space $Z = S^1$ are exterior equivalent if an obstruction in $C(S^1)$ vanishes.*

We want to put all of this together to complete the classification of G-actions on $CT(X,H)$ which induce the free action of $\mathbb{T}^2$ on $X = T^3$, up to exterior equivalence. Analysis of our construction of such an action shows there was a fair amount of freedom; we could have started instead with the action of $\mathbb{Z}^2$ on $C(S^1, \mathcal{K}(L^2(\mathbb{T})))$ with the two generators of $\mathbb{Z}^2$ going to the maps

$$\begin{cases} w & \mapsto \text{multiplication by } z, \\ w & \mapsto \text{translation by } f(w), \end{cases}$$

for $f\colon \mathbb{T} \to \mathbb{T}$ any map with winding number 1. (If we were to change the winding number, we'd get another Dixmier-Douady class.) And the resulting actions are all mutually inequivalent, even up to exterior equivalence. In fact, there is a close relationship between maps $\mathbb{T} \to \mathbb{T}$ in this construction and the exterior equivalence obstruction in $C(S^1)$. The fact that $H^2_M(G, C(X,\mathbb{T}))$ turns out to be a topological vector space (and thus contractible) means in some sense that "discrete invariants" of the noncommutative T-duals we get this way are unique, even though the group action itself is far from unique.

Further analysis of this example leads to the following classification theorem:

Theorem 9.6 (Mathai-Rosenberg [**107**]). *Let $\mathbb{T}^2$ act freely on $X = T^3$ with quotient $Z = S^1$. Consider the set of all actions of $\mathbb{R}^2$ on algebras $CT(X,H)$ inducing this action on X, with H allowed to vary over $H^3(X,\mathbb{Z}) \cong \mathbb{Z}$. Then the set of exterior equivalence classes of such actions is parameterized by* $\mathrm{Map}(Z,\mathbb{T})$. *The winding number of a map $Z \cong \mathbb{T} \to \mathbb{T}$ can be identified with the Dixmier-Douady invariant H. All these actions are given by the construction above, with f as the "Mackey obstruction map."*

9.3. The General Case

To formulate things in greater generality, we need the following:

Definition 9.7 (Crocker-Kumjian-Raeburn-Williams [**47**]). Let X be a second-countable locally compact G-space, G a second-countable locally compact group. The *equivariant Brauer group* $\mathrm{Br}_G(X)$ is the group of classes for G-equivariant Morita equivalence over X of pairs (A,α), where A is a separable CT-algebra with spectrum X and α is an action of G on A inducing the given G-action on X. The group operation is given by tensor product over X with the product action. Inverses are given by $[A,\alpha]^{-1} = [A^\circ, \alpha^\circ]$. There is an obvious forgetful map $F\colon \mathrm{Br}_G(X) \to \mathrm{Br}(X)$, but in general it is neither injective nor surjective.

Examples 9.8.

(1) $G = \{1\}$, $\mathrm{Br}_G(X) = \mathrm{Br}(X) \cong H^3(X, \mathbb{Z})$.
(2) $G = \mathbb{Z}$ acting trivially. By the Phillips-Raeburn Theorem, Theorem 7.3, $\mathrm{Br}_G(X) \cong H^3(X, \mathbb{Z}) \times H^2(X, \mathbb{Z})$, at least as a set, and one can check that the group structure matches up also, with the forgetful map F corresponding to projection onto the first factor.
(3) G acting freely and properly. Then $\mathrm{Br}_G(X) \cong \mathrm{Br}(X/G)$ via pull-back. In other words, if A is a continuous-trace algebra over X/G, we can pull it back to a continuous-trace algebra over X carrying an obvious G-action, and all elements of $\mathrm{Br}_G(X)$ come from this construction.
(4) $G = \mathbb{R}$, X homotopically finite, the action of G on X arbitrary. Via the Lifting Lemma, Lemma 7.5, $\mathrm{Br}_G(X) \cong \mathrm{Br}(X) \cong H^3(X, \mathbb{Z})$.
(5) $X = \mathrm{pt}$. Then $\mathrm{Br}_G(\mathrm{pt})$ classifies exterior equivalence classes of actions of G on $\mathcal{K}$, which are equivalent to projective unitary representations $G \to PU$. From Theorem 9.4, one can deduce that $\mathrm{Br}_G(\mathrm{pt}) \cong H^2_M(G, \mathbb{T})$ via the *Mackey obstruction* of a projective unitary representation (the cohomology class measuring the obstruction to lifting a projective unitary representation to an ordinary unitary representation).
(6) $X = G/H$ a transitive G-space. Then $\mathrm{Br}_G(G/H) \cong \mathrm{Br}_H(\mathrm{pt}) \cong H^2_M(H, \mathbb{T})$ (note that we've used (5) in the last step), since every G-action on a principal PU-bundle over X must be induced from the action of H on the fiber over the identity coset $1 \cdot H$. (This is analogous to the isomorphism $K_G(G/H) \cong R(H)$ from Section 5.1.1.)

Crocker-Kumjian-Raeburn-Williams [**47**] also provided exact sequences for computing $\mathrm{Br}_G(X)$ in terms of the topology of X and the groups $H^p_M(G, C(X, \mathbb{T}))$ with $p \le 3$. These are what one would expect if $\mathrm{Br}_G(X)$ were a kind of equivariant second cohomology group $\mathcal{H}^2(X; G, \underline{\mathbb{T}})$ à la Grothendieck [**71**, Ch. V], and one had a spectral sequence converging to the equivariant cohomology, with E_2 term $H^p_M(G, H^q(X, \underline{\mathbb{T}}))$.[2] While this doesn't seem to be exactly true, they show that one can check directly whatever would be predicted by such a spectral sequence, so that it doesn't matter.

For simplicity let's take $n = 2$, $G = \mathbb{R}^2$. Then we can analyze $H^p_M(G, C(X, \mathbb{T}))$ as in the case handled earlier, and it must vanish for $p > 2$. From this and [**47**], we obtain:

THEOREM 9.9 ([**107**] based on [**47**]). *Let Z be homotopically finite, X a principal $\mathbb{T}^2$-bundle over Z. Let $G = \mathbb{R}^2$ act on X with $\mathbb{Z}^2$ acting trivially, and inducing the given action of $\mathbb{T}^2$ with quotient space Z. Then there is an exact sequence*

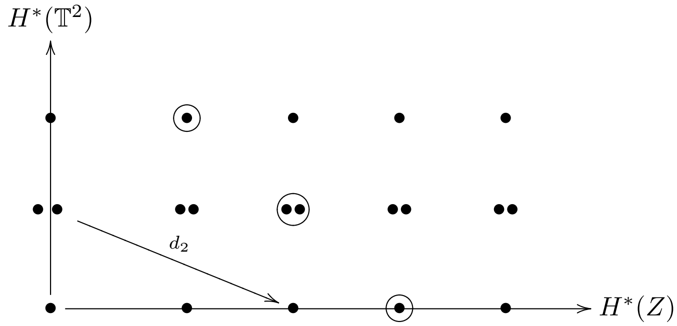
$$H^2(X, \mathbb{Z}) \longrightarrow C(Z) \xrightarrow{\xi} \mathrm{Br}_G(X) \xrightarrow{F} H^3(X, \mathbb{Z}) \longrightarrow 0.$$

Here F is the forgetful map $\mathrm{Br}_G(X) \to \mathrm{Br}(X)$.

We can now put everything together, but first we need to review some more topology. Again, for simplicity, let's take $n = 2$. Let Z be homotopically finite, $p\colon X \to Z$ a principal $\mathbb{T}^2$-bundle over Z. The Serre spectral sequence for p has the

[2]Grothendieck's theory unfortunately doesn't apply directly unless G is discrete, since it doesn't take the topology of G into account.

following form. Here we've circled the terms that contribute to $H^3(X)$.



We thus have an edge homomorphism

$$p_!\colon\ H^3(X,\mathbb{Z}) \to E_\infty^{1,2} \subseteq H^1(Z, H^2(\mathbb{T}^2,\mathbb{Z})) = H^1(Z,\mathbb{Z})$$

which turns out to play a major role.

THEOREM 9.10 (Mathai-Rosenberg [**107**]). *Let $p\colon\ X \to Z$ be a principal $\mathbb{T}^2$-bundle as above, $H \in H^3(X,\mathbb{Z})$. Then we can always find a "generalized T-dual" by lifting the action of $\mathbb{T}^2$ on X to an action of $\mathbb{R}^2$ on $CT(X,H)$ and forming the crossed product. When $p_!H = 0$, we can always do this in such a way as to get a crossed product of the form $CT(X^\sharp, H^\sharp)$, where $(X^\sharp, H^\sharp)$ is a classical T-dual (e.g., as found though the purely topological theory). When $p_!H \neq 0$, the crossed product $CT(X,H) \rtimes \mathbb{R}^2$ can never be even locally stably commutative, and should be viewed instead as a* noncommutative T-dual.

Though we will not go into it in the interests of saving space, the same methodology can also be applied to principal $\mathbb{T}^n$-bundles $p\colon\ X \to Z$ with $n > 2$ [**109**]. The one totally new phenomenon in this case is that if H has a nontrivial restriction to the torus fibers of p, then *there is no lifting at all* of the action of $\mathbb{T}^n$ on X to an action of $\mathbb{R}^n$ on $CT(X,H)$. To prove this, it suffices to consider the case where Z is a point and $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Since $\mathrm{Br}_{\mathbb{R}^n}(\mathbb{R}^n/\mathbb{Z}^n) \cong H^2_M(\mathbb{Z}^n,\mathbb{T}) \cong \mathbb{T}^{\binom{n}{2}}$ by Example 9.8(6), one just needs to compute the Dixmier-Douady invariant of the continuous-trace algebra on $\mathbb{T}^n$ induced from an action of $\mathbb{Z}^n$ on $\mathcal{K}$. Since $H^2_M(\mathbb{Z}^n,\mathbb{T})$ is generated by the images of $H^2_M(\mathbb{Z}^2,\mathbb{T})$ for maps $\mathbb{Z}^2 \to \mathbb{Z}^n$, and since $H^3(\mathbb{T}^2) = 0$, one finds that the Dixmier-Douady invariant always vanishes. This provides additional justification for the restriction in Theorem 8.3 to the case where H restricts to be 0 on each torus fiber.

The exact relationships between the approaches to higher-dimensional T-duality in this chapter and in Chapter 8 have been worked out by Ansgar Schneider in [**146**].

CHAPTER 10

Advanced Topics and Open Problems

10.1. Mirror Symmetry

So far in this book we have dealt with quite general topological spacetimes X. But in fact most of the work in string theory assumes $X = \mathbb{R}^4 \times Y^6$, where the first factor is Minkowski space and Y is a compact 6-manifold with quite special geometry. Supersymmetry considerations (see [**14**, Ch. 9]) make it desirable to take Y to be a *complex manifold*, in fact a *Kähler manifold*, that is, a complex manifold with a hermitian metric such that the complex structure J is parallel ($\nabla J = 0$), and in fact a *Calabi-Yau manifold*, that is, a complex Kähler 3-fold with an everywhere non-zero holomorphic 3-form.

There are many good references on Kähler manifolds, such as [**169, 120, 11**], but we will just recall a few important facts that will be needed in what follows.

First of all, in a Kähler manifold M, the Betti numbers split up as

$$b_n =_{\text{def}} \dim H^n(M, \mathbb{C}) = \sum_{p=0}^{n} h^{p,n-p},$$

where $h^{p,n-p} = \dim\{\text{harmonic } (p, n-p)\text{-forms}\} = \dim H^{n-p}(M, \Omega^p_M)$. We also have $h^{p,q} = h^{q,p}$. In a simply connected Calabi-Yau 3-fold, the Hodge numbers $h^{p,q}$ have to have the specific form:

$$\begin{array}{c|cccc}
q & & & & \\
 & 1 & 0 & 0 & 1 \\
 & 0 & h^{2,1} & h^{1,1} & 0 \\
 & 0 & h^{1,1} & h^{2,1} & 0 \\
q=0 & 1 & 0 & 0 & 1 \\
\hline
 & p=0 & & & \to p
\end{array}$$

Another key fact is that out of the metric g and complex structure J on a Kähler manifold, one can create a 2-form ω via $\omega(X, Y) = g(JX, Y)$. The fact that $\nabla J = 0$ implies that ω is closed, and it is nondegenerate since g is, so it defines a symplectic structure. The theory of Kähler manifolds, as we shall see, involves the interplay of the complex and symplectic structures.

Mirror symmetry begins with the experimental observation (see Figure 10.1) that the set of Hodge numbers of Calabi-Yau 3-folds seems to have a symmetry interchanging $h^{1,1}$ and $h^{2,1}$. (The plot shows $h^{1,1} + h^{2,1}$ plotted against the Euler characteristic $2(h^{1,1} - h^{2,1})$, so mirror symmetry should preserve the vertical coordinate and change the sign of the horizontal coordinate.)

This has a deeper interpretation, since a Kähler symplectic form ω has a de Rham class in $H^{1,1}$, while $H^{2,1}$ can be identified with the tangent space to the set of deformations of the complex structure. (On a general complex manifold,

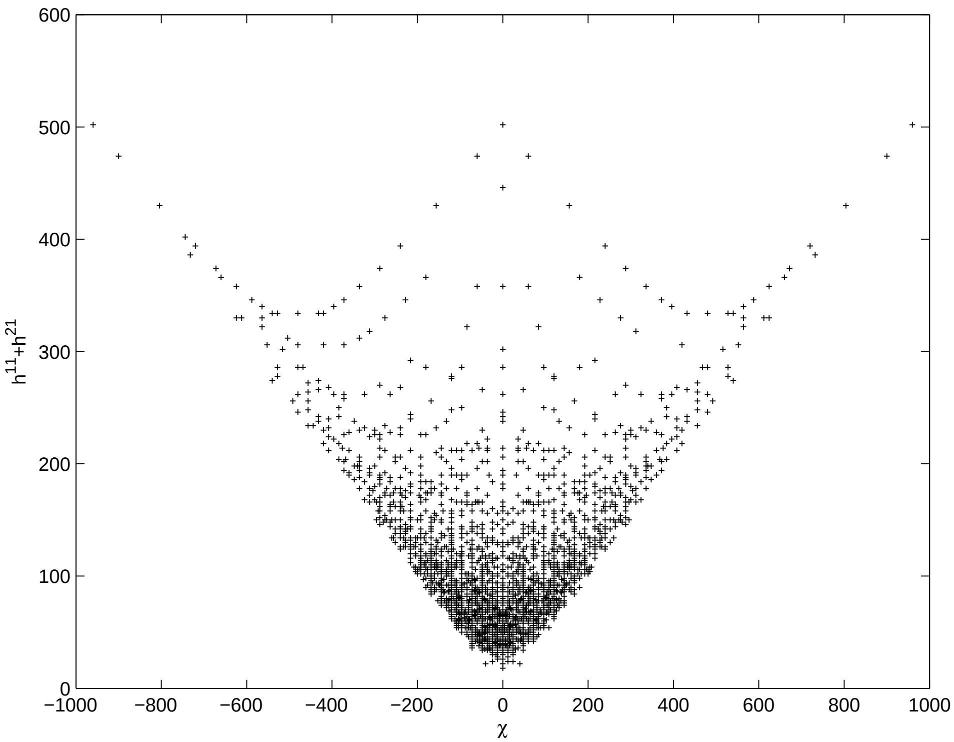


FIGURE 1. A scatter plot of Hodge numbers of 1898 Calabi-Yau 3-folds, from Kreuzer, Schimmrigk, and Skarke [**99**]. The horizontal axis represents the Euler characteristic χ, the vertical axis the sum $h^{1,1} + h^{2,1}$. Approximate reflection symmetry of the figure reflects mirror symmetry.

the tangent space to the set of deformations of the complex structure J can be identified with $H^1(X, T_X)$. Here T_X is the sheaf of sections of the holomorphic tangent bundle, that is, the sheaf of holomorphic vector fields. In the case of a Calabi-Yau 3-fold, since Ω^3_X is trivial, Ω^2_X is dual to Ω^1_M and can thus be identified with T_X, which is why we can replace $H^1(X, T_X)$ by $H^1(X, \Omega^2_X)$, which has dimension $h^{2,1}$.) Thus it is believed that "most" simply connected Calabi-Yau 3-folds M have a *mirror* Calabi-Yau $\widetilde{M}$, such that deformations of the Kähler structure on M correspond to deformations of the complex structure on $\widetilde{M}$, and *vice versa.* (For an exception to this rule, see Problem (1) in Section 10.5 below.)

In fact, we expect IIA string theory on $\mathbb{R}^4 \times M$ to be equivalent to IIB string theory on $\mathbb{R}^4 \times \widetilde{M}$ (when the H-flux vanishes on each). This mirror relationship is expected to be reflected in finer invariants of M and $\widetilde{M}$, since the partition functions for the two string theories reflect invariants of the Kähler structures.

In fact, if the string worldsheet Σ and the spacetime manifold X both have Kähler structures, then holomorphic maps $\Sigma \to X$ are automatically harmonic, i.e., are critical points for the sigma-model action (1.2), and for supersymmetry reasons, these are the solutions that really matter. So the semi-classical partition function is gotten by summing over the holomorphic maps, and this leads to the idea of counting holomorphic curves Σ (of a given genus) on a Calabi-Yau 3-fold Y. (The counting has to be done in an intelligent way, since any holomorphic map $\Sigma \to X$ can be composed with a holomorphic automorphism of Σ.) Such counting functions are called *Gromov-Witten invariants*, and we expect them to be closely related for Y and its mirror $\widetilde{Y}$.

Mirror symmetry for Calabi-Yau 3-folds is believed to be closely related to T-duality, which we have been discussing throughout this book. In fact, Strominger, Yau, and Zaslow (SYZ) [**154**], in a famous paper in 1996, claimed that a simply connected Calabi-Yau 3-fold M should (away from some singular fibers) look like a $\mathbb{T}^3$-fibration over some manifold Z (with *real* dimension 3), and that mirror symmetry should arise from taking the T-dual of this torus fibration (and then filling to get a new Calabi-Yau). The torus fibers T^3 should be special Lagrangian, i.e., Lagrangian[1] for the symplectic structure on M, but also such that the imaginary part of a non-zero holomorphic 3-form on M vanishes on T^3. Since the torus fibers have odd (real) dimension, the T-duality interchanges string theories of types IIA and IIB.

10.2. Other Solutions to the Missing T-Dual Problem

In Section 8.1, we mentioned the problem of missing T-duals in the case of higher-dimensional torus bundles, and we discussed one possible solution [**107, 109, 108**] using noncommutative C^*-algebras. Without going into details, we should point out here that various other solutions to this problem have been proposed. Here are a few of them:

(1) In some situations, we could allow non-principal torus bundles, as well as principal bundles. This does not seem, however, to resolve the difficulties with Examples 8.1 and 8.2.
(2) It has been suggested [**24, 59**] that one should allow not only noncommutative T-duals but nonassociative ones as well. This seems to permit removing the constraint that the H-flux be trivial on each torus fiber of the original $\mathbb{T}^n$-bundle.
(3) Hull [**84**] has proposed allowing spacetimes which are not necessarily manifolds but *T-folds*, or spaces locally modeled on Euclidean space but with patching maps that can be "non-geometric" dualities. These T-folds have been associated to certain noncommutative C^*-algebras by Bouwknegt and Pande [**26**].
(4) Another approach to the problem of missing T-duals has been to try to use "generalized geometry" in the sense of Hitchin [**77**]. The idea of this kind of geometry is to treat the tangent and cotangent bundles of a manifold on an equal basis. Generalized geometry has been used for "Poisson-Lie T-duality" by Hu [**82**] and Kao [**92**], and figures in many other approaches to T-duality, such as [**69, 70**].
(5) Still another approach to topological T-duality has been suggested in [**25**], based on a duality for loop group bundles. So far, this idea has only been applied to the case of 1-dimensional torus bundles, so it remains to be

[1]That means the restriction of the Kähler form ω to T^3 is identically 0.

seen whether it can be generalized to the higher-dimensional case. The idea is this. The approach to topological T-duality in Chapter 6 was based on studying principal G-bundles ($G = PU$) over the total spaces X of principal $\mathbb{T}$-bundles over Z. It would be equivalent to study principal bundles over Z for the group $\Lambda G \rtimes \mathbb{T}$. ($\Lambda G$ is again the free loop space on G, which is a topological group with pointwise operations. $\mathbb{T}$ acts on it by rotation in the domain of a loop.) There is a "hidden symmetry" of $(\Lambda G \rtimes \mathbb{T})$-bundles that comes from reversing the roles of the two copies of $\mathbb{T}$, and this exchanges the momentum and winding quantum numbers, as one would expect for T-duality. In fact, $B(\Lambda G \rtimes \mathbb{T}) \simeq R$, where R is the Bunke-Schick classifying space of Theorem 6.3.

10.3. ☆ Fourier-Mukai Duality

10.3.1. Classical Fourier-Mukai Duality.

10.3.1.1. *Derived Categories.* To link the topological theories we have been discussing with holomorphic geometry, it is necessary to introduce the idea of a *derived category*, as studied by algebraic geometers.

A key notion in algebraic geometry is that of a *coherent sheaf* $\mathcal{F}$ on an algebraic variety X, that is a sheaf of $\mathcal{O}_X$-modules on X which is locally finitely presented, in that each point of X has an open neighborhood on which one has an exact sequence $\mathcal{O}^m \to \mathcal{O}^n \to \mathcal{F} \to 0$. If X is affine with coordinate ring $\mathbb{C}[X]$, a coherent sheaf corresponds to a finitely generated $\mathbb{C}[X]$-module. But considerations of homological algebra make it necessary to work not just with sheaves but with *complexes of sheaves*. These can be packaged together in a *triangulated category*, called the *derived category* $D(X)$. Objects in this category are quasi-isomorphism classes[2] of [bounded] complexes of coherent sheaves on X. The derived functors of certain standard functors on sheaves naturally live on the derived category. The basics of this construction can be found in [**168, 41, 143**] (listed in increasing order of mathematical sophistication).

10.3.1.2. *The Work of Mukai.* In 1981, Mukai [**121**] proved a result which can be viewed as a form of T-duality in holomorphic geometry. It is valid over arbitrary algebraically closed fields, but for simplicity we will just state it for varieties over $\mathbb{C}$. We need to start with a few basic facts about abelian varieties (over $\mathbb{C}$). These are familiar to all algebraic geometers and to experts in several complex variables, but perhaps not to other mathematicians.

By definition, a *complex torus* is a quotient $X = \mathbb{C}^n/\Lambda$, where Λ is a lattice in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Clearly this is a compact complex manifold. When $n = 1$, the theory of elliptic functions shows that every complex torus of complex dimension 1 is a smooth projective variety, called an *elliptic curve*, and can be realized as a subvariety of $\mathbb{CP}^2$ defined by a cubic equation of the form $y^2z = x^3+ax^2z+bxz^2+cz^3$ (in homogeneous coordinates $[x, y, z]$). In higher dimensions, however, not every complex torus has enough meromorphic functions to have an embedding into projective space. One that does is called an *abelian variety*, and is a smooth projective variety.

If X is an abelian variety, $\mathrm{Pic}(X)$, the Picard group of (holomorphic) line bundles on X is both a locally compact topological group (under the tensor product

[2]Quasi-isomorphism is the equivalence relation on chain complexes generated by chain maps which induce isomorphisms on cohomology.

operation) and an algebraic variety. Its identity component, the group of topologically trivial holomorphic line bundles on X, is compact and is a new abelian variety $X^\sharp$, called the *dual* abelian variety. It turns out to be basically the same as the dual torus in the sense we introduced earlier. The product $X \times X^\sharp$ carries a canonical line bundle $\mathcal{P}$, called the *Poincaré bundle*. The simplest way to define it is to use the identification of $X^\sharp$ with a group of line bundles on X. Then the fiber of $\mathcal{P}$ over a point $(x, x^\sharp)$ in $X \times X^\sharp$ is the fiber of $x^\sharp \in X^\sharp \subset \mathrm{Pic}(X)$ at $x \in X$.

THEOREM 10.1 (Mukai [**121**]). *Let X be a complex abelian variety. Consider the obvious diagram*

$$\begin{tikzcd} & X \times X^\sharp \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_{X^\sharp}"] & \\ X & & X^\sharp \end{tikzcd}$$

and define a functor $\mathcal{S}$ from $\mathcal{O}_X$-modules to $\mathcal{O}_{X^\sharp}$-modules by

$$\mathcal{S}(\mathcal{F}) = (\pi_{X^\sharp})_*(\mathcal{P} \otimes \pi_X^*(\mathcal{F})).$$

Then the derived functor $D\mathcal{S}$ of $\mathcal{S}$ gives an equivalence of categories from $D(X)$ to $D(X^\sharp)$.

The functor $D\mathcal{S}$ of Mukai's Theorem is called the *Fourier-Mukai transform*. We need derived categories to define it since the push-forward functor on sheaves (and hence the functor $\mathcal{S}$) is not exact, and derived functors naturally live on the derived category.

The Fourier-Mukai transform, as the name suggests, is related to the classical Fourier transform on a locally compact abelian group G with Pontrjagin dual group $\widehat{G}$. To take the Fourier transform of a function f on G, we pull f back to $G \times \widehat{G}$ and multiply (this is like the tensor product) by the dual pairing $G \times \widehat{G} \to \mathbb{T}$, which is formally analogous to the Poincaré bundle $\mathcal{P}$. Then we push down to a function on $\widehat{G}$ by integrating out in G; this last operation is formally analogous to the (derived) push-forward for sheaves.

10.3.2. Generalized Fourier-Mukai Duality. The theorem of Mukai has led to a big industry and in particular to *Kontsevich's theory of homological mirror symmetry* [**97**]. The expectation was that mirror symmetry would be reflected in Mukai-like equivalences of derived categories. This expectation has now been partially realized:

THEOREM 10.2 (Orlov [**125, 144, 87**]). *Any equivalence of categories from $D(X)$ to $D(Y)$ (X and Y smooth projective varieties) comes from a Mukai-like functor (pull-back, then tensor, then push forward).*

THEOREM 10.3 ([**39**, Corollary 3.1.13 and Proposition 3.1.14]). *If X and Y are simply connected Calabi-Yau 3-folds and if there is an equivalence of categories from $D(X)$ to $D(Y)$, then X and Y have the same Hodge numbers $h^{1,1}$ and $h^{2,1}$.*

THEOREM 10.4 (Bridgeland [**27**], later extended by Kawamata [**96**]; see also [**144, 87**]). *If X and Y are birationally equivalent Calabi-Yau 3-folds, then $D(X)$ and $D(Y)$ are equivalent.*

(Caution: Current evidence [**42**] is that the converse of this is likely to be false; i.e., if X and Y are Calabi-Yau 3-folds, then $D(X)$ and $D(Y)$ could be equivalent without X and Y being birationally equivalent.)

You might wonder what happened to mirror symmetry in all this, since Mukai-like equivalences only occur between Calabi-Yau 3-folds with the *same* Hodge numbers, and mirror symmetry involves *flipped* Hodge numbers $h^{1,1} \leftrightarrow h^{2,1}$. Thus, except perhaps for the rare case when $h^{1,1} = h^{2,1}$, mirror Calabi-Yau 3-folds cannot have equivalent derived categories of coherent sheaves.

The first part of the story is that Mukai duality is related to T-duality between a complex torus and its dual, or more generally, for torus fibrations with *even-dimensional* fibers. (A complex torus has even dimension as a manifold over $\mathbb{R}$.) This kind of T-duality sends types IIA and type IIB back to themselves. Mirror symmetry, on the other hand, comes from T-duality for a torus fibration with odd-dimensional fibers, which interchanges types IIA and type IIB.

The second part of the story comes from an observation of Kontsevich [**97, 98**], namely that string theory of type IIA and string theory of type IIB really involve two *different* derived categories. If X is a Calabi-Yau 3-fold, Type IIB string theory on $\mathbb{R}^4 \times X$ is indeed related to the derived category $D(X)$ of coherent sheaves from algebraic geometry, whereas type IIA string theory on $\mathbb{R}^4 \times X$ is related to a different derived category, the *Fukaya category* $F(X)$ constructed out of the *symplectic* geometry of X and Lagrangian submanifolds. A (very abbreviated) explanation is that in type IIB string theory, the stable D-branes are basically complex subvarieties of X, or something very close to this. (Any subvariety gives a coherent sheaf in a natural way, and a coherent sheaf can be viewed as a generalized complex subvariety.) Thus the D-brane charges take their values in the even-dimensional group $K^0(X)$, and $D(X)$ is the correct object for studying the finer (holomorphic) structure of the D-branes. In type IIA string theory, on the other hand, the stable D-branes are odd-dimensional. Thus the D-brane charges take their values in the odd-dimensional group $K^1(X)$. The Fukaya category, which probes the geometry of Lagrangian submanifolds (which are 3-dimensional), is now relevant.

Conjecture 10.5 (Kontsevich [**97, 98**]). *If X and Y are mirror Calabi-Yau 3-folds, then $D(X)$ is equivalent to $F(Y)$, and $D(Y)$ is equivalent to $F(X)$.*

This is now known in some cases, but making rigorous sense of $F(X)$ has proved difficult.

10.3.3. Noncommutative Extensions.

10.3.3.1. *The Work of Căldăraru and of Ben-Basset, Block, and Pantev.* The success of Fourier-Mukai duality in capturing the algebraic geometry aspects of T-duality raises the question of whether there is something similar to Fourier-Mukai that is related to topological T-duality with H-flux or to the Mathai-Rosenberg noncommutative T-duality between T^3 with nontrivial H-flux and the C^*-algebra of the discrete Heisenberg group.

Some results of this type were obtained by Căldăraru in his thesis [**39, 40**]. He obtained examples of Calabi-Yau 3-folds X with an "elliptic fibration" over a smooth surface S (that is, a map $X \to S$ whose generic fibers are elliptic curves) such that $D(X)$ is equivalent to $D(X', \alpha)$, for X' another Calabi-Yau (also with an elliptic fibration over S) and α an element of the (algebraic) Brauer group of X'. Note that this is highly reminiscent of T-duality with H-flux (since the H-flux can be thought of as a Brauer group element) for torus fibrations with 2-dimensional fibers (since elliptic curves are topologically T^2).

A theory much closer to the results of Chapter 9 was developed by Ben-Basset, Block, and Pantev [**15**]. They start with classical Mukai duality between $D(X)$ and $D(X^\sharp)$, X a complex abelian variety.

Next, since essentially all kinds of deformation theory can be expressed in terms of Hochschild cohomology in degree 2 [**168**, Ch. 9], they observe that the Hochschild cohomology $HH^2(X)$ splits as:

$$HH^2(X) \cong H^0(X, \bigwedge^2 T_X) \oplus H^1(X, T_X) \oplus H^2(X, \mathcal{O}_X).$$

(The definition of Hochschild cohomology for varieties, as well as the derivation of this decomposition, are explained in [**156**] and in [**41**, §6]. For an affine variety, $HH^\bullet(X)$ is simply the Hochschild cohomology of the coordinate ring.)

The three terms $H^0(X, \bigwedge^2 T_X)$, $H^1(X, T_X)$, and $H^2(X, \mathcal{O}_X)$ each have interpretations in terms of different kinds of deformations. $H^1(X, T_X)$ is the tangent space to the *deformations of the complex structure* on X. $H^0(X, \bigwedge^2 T_X)$, or in other words, the space of global alternating holomorphic bivector fields, classifies *holomorphic Poisson structures*.[3] In the case of an abelian variety, T_X is holomorphically trivial, so $\bigwedge^2 T_X$ is just a holomorphically trivial bundle of dimension $\binom{n}{2}$, n the complex dimension of X, and since global holomorphic functions are constant, the integrability condition is automatically satisfied. Poisson structures give rise to deformation quantizations of X, or *Moyal products* or *star-products*. These are noncommutative formal associative multiplications

$$f \star g = fg + \sum_{j=1}^{\infty} P_j(f, g)\hbar^j,$$

where $P_j(f, g)$ is bilinear in f and g and is a differential operator in each, and where we have the familiar "correspondence principle" of quantum mechanics:

$$\lim_{\hbar \to 0} \frac{f \star g - g \star f}{i\hbar} = -i\big(P_1(f, g) - P_1(g, f)\big) = \{f, g\}.$$

In the case of the usual symplectic structure on T^2, the Moyal product can be taken to be the formal series expansion of the multiplication in the irrational rotation algebra A_θ, with θ playing the role of Planck's constant $\hbar$. Finally, $H^2(X, \mathcal{O}_X)$ classifies (holomorphic) *gerbe deformations*, which are reminiscent of CT-algebras. The reason is that classes of holomorphic gerbes (see Section 4.3 and the references quoted there) can be identified with elements of $H^2(X, \mathcal{O}_X^\times)$. From the short exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 1$$

we obtain the long exact cohomology sequence

$$H^2(X, \mathcal{O}_X) \xrightarrow{\exp} H^2(X, \mathcal{O}_X^\times) \xrightarrow{\partial} H^3(X, \mathbb{Z}),$$

in which the connecting map ∂ sends a gerbe to its Dixmier-Douady class. So $H^2(X, \mathcal{O}_X)$ maps to deformations of a gerbe (not changing the Dixmier-Douady class).

[3]Recall that a *Poisson structure* on a manifold is given by skew-symmetric Poisson bracket on functions. Such a bracket has to be of the form $\{f, g\} = \langle df \wedge dg, \Pi\rangle$, for Π an alternating bivector field. An additional integrability condition is required for the bracket to satisfy the Jacobi identity. For compatibility with the complex structure, we also want Π to be holomorphic.

Now let V be the holomorphic tangent space to X at 0. Since the sheaves Ω^*_X and $\bigwedge^* T_X$ are all trivial, we can identify $H^0(X, \bigwedge^2 T_X)$ with $\bigwedge^2 V$ and $H^2(X, \mathcal{O}_X)$ with $\bigwedge^2 V^\sharp$, where $V^\sharp$ is the holomorphic tangent space to the dual torus $X^\sharp$ at 0. So the duality $X \leftrightarrow X^\sharp$ interchanges these two spaces, and a holomorphic Poisson structure Π on X corresponds to an element of $H^2(X^\sharp, \mathcal{O}_{X^\sharp})$ defining a gerbe on $X^\sharp$. We now have the ingredients needed for the following theorem.

THEOREM 10.6 (Ben-Basset-Block-Pantev [**15**]). *Suppose that X is a g-dimensional complex torus equipped with a holomorphic Poisson structure Π. Let X_Π be the corresponding Moyal quantization, and let $X^\sharp_{\Pi^\sharp}$ be the dual $\mathcal{O}_{X^\sharp}$-gerbe on $X^\sharp$. Then there is an equivalence of categories between $D(X_\Pi)$ and $D(X^\sharp_{\Pi^\sharp})$ given by a Fourier-Mukai-like functor.*

10.3.3.2. *The Work of Daenzer, Block, and Block-Daenzer.* Theorem 10.6 has motivated additional work by Daenzer [**50**], Block [**18, 17**], and by Block and Daenzer [**19**]. Daenzer's setting involves topological groupoids and their crossed products. This theory fits well with our C^*-algebraic approach in Chapter 9, and basically gives equivalent results, but generalizing to groups other than tori and to the situation where the base space Z is replaced by a topological stack rather than a topological space. (For motivation as to why one might want to use groupoids and stacks in string theory, see for example [**151, 152**].)

In the papers [**18, 17**], Block uses differential graded categories, or DG-categories, to state and prove a non-formal version of the theorem of Ben-Basset, Block and Pantev. This framework is an attempt to bring the noncommutative geometry of Connes and the approach using derived categories closer together. The approach using DG-categories is motivated by the observation that the structure of a complex manifold X is encoded in its Dolbeault complex, and [**18**, Proposition 2.22] states that from this complex one can define a DG-category which encodes all the structure of the derived category $D(X)$. The paper [**17**] goes on to use this idea to find a substitute for the derived category in the case of noncommutative tori. The work of Block and Daenzer [**19**] then attempts to extend the framework to handle general gerbes with connection. In other words, Block and Daenzer look for an analogue of a Fourier-Mukai transform for gerbes and noncommutative spaces. This uses the machinery of [**18, 17**]. Their final result is that a gerbe with flat $\overline{\partial}$-connection on a torus is dual to a holomorphic noncommutative dual torus, as one might expect from Theorem 10.6.

10.3.3.3. *The Work of Bunke, Schick, Spitzweck, and Thom.* Bunke, Schick, Spitzweck, and Thom [**36**] gave another version of categorical duality related to topological T-duality and noncommutative geometry. The formulation is very powerful but abstract, and involves *Grothendieck topologies* and *stacks*. This work is far beyond the scope of this book, but interested readers should be aware of it. Essentially, they construct a very general version of topological T-duality via *Pontrjagin duality of Picard stacks* over a Grothendieck site $\mathbb{S}$ of (compactly generated) topological spaces. An important role is played by sheaves over $\mathbb{S}$, such as the sheaf $\underline{\mathbb{T}}$ with $\underline{\mathbb{T}}(U) = C(U, \mathbb{T})$, the continuous maps from U into $\mathbb{T}$. The dualizing object for Pontrjagin duality turns out to be $\mathcal{B}\underline{\mathbb{T}}$, the sheafification of the presheaf $U \mapsto BC(U, \mathbb{T})$. In fact any Picard stack A in $\operatorname{Pic}\mathbb{S}$ is an extension of sheaves

$$\mathcal{B}H^{-1}(A) \rightarrowtail A \twoheadrightarrow H^0(A)$$

and Pontrjagin duality reverses the roles of $H^0(A)$ and $H^{-1}(A)$. The idea of [**36**] is that out of a pair $(X \xrightarrow{\pi} Z, H)$ as in Theorem 8.3, π a principal $\mathbb{T}^n$-bundle, first one can view H as classifying a *gerbe with band* $\underline{\mathbb{T}}$, or in other words a map of stacks $G \to X$ which is locally isomorphic to $\mathcal{B}\underline{\mathbb{T}}_{|X} \to \underline{X}$. (This is really fancy language for a PU-bundle over X.) One can try to manufacture a "Picard stack" U out of this data, then take the Pontrjagin dual $\widehat{U} = \underline{\mathrm{Hom}}_{\mathrm{Pic}\,\mathbb{S}}(U, \mathcal{B}\underline{\mathbb{T}})$, and then recover the topological T-dual from this. The Picard stack U is basically a torsor (like a principal homogeneous space) for the Picard stack $(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}^n})|_X$ (the first factor corresponding to the H-flux and the second factor to the $\mathbb{T}^n$-bundle), but for technical reasons it is defined to be an extension of Picard stacks

$$(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}^n})|_X \to U \to \underline{\mathbb{Z}}|_X.$$

This is the object to which one applies Pontrjagin duality. But exact definitions of all the relevant notions are much too complicated for us to give here. Incidentally, the idea of a Picard stack is rather hard to extract from the original sources in the algebraic geometry literature, so a good concise reference is [**170**, Appendix A].

10.4. ✰ Refinements of K-Theory

In this section, we discuss an idea which has recently surfaced in the physics literature, namely that K-theory is not exactly the right mathematical formalism for studying D-brane charges in string theory, and that perhaps certain refinements are needed. This idea may be found, for example, in [**52**] and [**60**, §8], where two specific issues are discussed:

(1) **Problems coming from the Hodge $*$-operator.** The Ramond-Ramond fields in type II string theory are generally assumed to be represented by differential forms. Electric-magnetic duality [**14**, §6.2] relates these forms to their Hodge-$*$ duals. Since the forms couple to the charges, some additional constraints arise, and it is not clear that the K-theoretic classification of charges takes this into account, especially in the case of self-dual fields in middle dimension.

(2) **Problems coming from S-duality.** This area of research has been suggested in [**61, 21, 100, 102, 101**]. While, as we have tried to demonstrate throughout this book, T-duality seems to be consistent with the K-theoretic classification of D-brane charges, it is not clear that the same holds for S-duality. For example, in [**61**], it is argued that studying S-duality on $\mathbb{RP}^3 \times \mathbb{RP}^5 \times S^2$ (with torsion H-flux) leads to a Freed-Witten anomaly on the S-dual. One possibility, discussed in [**100, 102, 101**], is to replace (twisted) K-theory by (twisted) more exotic cohomology theories, such as elliptic cohomology [**160**] or **tmf** ("topological modular forms") [**80**]. These theories are related to K-theory, but are slightly more complicated, and seem to have something to do with "higher orientability" of spacetime or D-branes, that is, with (twisted) String or Fivebrane structures [**145**]. A manifold can be given a String structure, or lifting of the structure group of its tangent bundle to the connective cover of the orthogonal group obtained by killing $\pi_3(O)$, if it is spin and $\frac{1}{2}p_1$, half the first Pontrjagin class, vanishes. (Note that $\frac{1}{2}p_1$ makes sense as an integral, possibly torsion, cohomology class for spin manifolds.) It has a Fivebrane structure if there is a lifting of the structure group of its tangent bundle

to the next connective cover of the orthogonal group obtained by killing $\pi_7(O)$, or if it has a String structure and $\frac{1}{6}p_2$ vanishes. Such structures are related to orientability for certain exotic cohomology theories as mentioned above. (See [**80**] or for example [**4, 5**].) This is a very active area of research described in some of the other lectures at the conference, but it would take us too far afield to go into details.

10.5. Open Problems

We conclude by listing a number of open problems related to some of the ideas we have discussed in this book.

(1) Does the notion of noncommutative T-duals, introduced in Chapter 9, help explain *missing mirrors*, just it explains "missing T-duals"? Missing mirrors come up when one has a simply connected Calabi-Yau Y whose complex structure is infinitesimally rigid, i.e., for which $h^{2,1}(Y) = 0$. In this case there cannot be a mirror Calabi-Yau $\widetilde{Y}$, since one would have to have $h^{1,1}(\widetilde{Y}) = 0$, so that $\widetilde{Y}$ could not be Kähler.

(2) The SYZ construction [**154**] and also other constructions studied by physicists require T-duality for generalized torus fibrations with some isotropy (fibers on which the torus doesn't act freely) or even degenerate fibers (where the torus doesn't act at all). How can one extend the results on topological T-duality to these situations? Some work has been done in this direction already, but only in very special cases. For example, Bunke-Schick [**35**] considered the case of *locally* free actions, and Pande [**127**] considered the case of $\mathbb{T}$-actions with isolated fixed points, known as *Kaluza-Klein monopoles* in the physics literature.

(3) (suggested by Jacques Distler at the conference) In the physics literature, there are reasons for sometimes taking spacetime to be an orbifold instead of a manifold.[4] Can one redo the theory in this book in this generality? A first step would be to show that one can make topological T-duality, at first for S^1-bundles, equivariant for an action of a finite group.

(4) We have discussed how matching of K-theoretic D-brane charges imposes constraints on topological T-duality. Similar considerations should apply to other dualities in string theory, as well as to the AdS/CFT correspondence. Work out the consequences, and see to what extent noncommutative techniques can be used! (My student Stefan Mendez-Diez has been working on this.)

(5) Recently, Echterhoff, Nest and Oyono-Oyono [**58**] gave a definition of "principal noncommutative torus bundles" which includes the noncommutative T-duals we found in Chapter 9, but not all of the classical examples. So ideally, one should try to find a good category for topological T-duality that includes both the category in Theorem 8.3 and the Echterhoff-Nest-Oyono-Oyono category. Then one should try to extend the topological duality theory to this bigger category. What happens with the T-duality group in this context? Does it make any sense in terms of symmetries of the bigger category?

[4]An orbifold is a space with local charts, like a manifold, but where the local model spaces are quotients of a finite-dimensional Euclidean space by an orthogonal action of a finite group.

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Notation and Symbols

α'	Regge slope parameter in string theory, corresponding to the square of the typical string length, page 4
βX	Stone-Čech compactification of X, page 27
δS	variation of S, page 8
∇	directional derivative operator of connection, page 16
Φ	dilaton field in string theory, page 10
$\Gamma(X, E)$	sections of a bundle or sheaf over X, page 16
$\Gamma^\infty(X, E)$	smooth sections of a bundle over X, page 16
ΛX	free loop space on a space X, page 59
Ω^p_M	sheaf of holomorphic p-forms on a complex manifold M, page 85
ΩX	based loop space of a based space X, page 59
π_j	j-th homotopy group, page 18
ρ	Phillips-Raeburn invariant, page 65
Σ	string worldsheet, page 3
$\odot$	uncompleted tensor product of C^*-algebras, page 28
$\otimes$	spatial tensor product of C^*-algebras, page 28
$\bigwedge^p$	pth exterior power, page 91
$\widehat{A}$	spectrum of a C^*-algebra A, page 37
A_+	positive elements in a C^*-algebra A, page 27
A_θ	rotation algebra with unitary generators U and V satisfying $UVU^* = e^{2\pi i\theta}V$, page 53
$A \rtimes_\alpha G$	crossed product of A by an action α of G, page 34
A°	opposite algebra to A, page 39
$\operatorname{Aut} A$	automorphism group of a C^*-algebra A, page 33
$\mathcal{B}$	gauge bundle of a principal bundle, page 64
$\mathcal{B}$	sheafified classifying space, page 92
B	B-field, page 4
BG	classifying space of a topological group G, the quotient EG/G, where EG is (weakly) contractible and G acts freely and properly on it
b_n	n-th Betti number, page 85
Br	Brauer group (of a field, commutative ring, or space), page 39
Br_G	equivariant Brauer group of a G-space, page 81
$\mathbb{C}$	the complex numbers
c	the speed of light in vacuum, page 3
$c(E)$	total Chern class of E, page 15
Ch	Chern character, page 18
$c_j(E)$	j-th Chern class of E, page 15
$C(X)$	continuous functions on X, page 22

$C_0(X)$	continuous functions vanishing at infinity on a locally compact space X, page 27
$C(X, \mathbb{T})$	continuous functions $X \to \mathbb{T}$, page 67
$D(X)$	derived category of coherent sheaves on X, page 88
$[E]$	K-theory class of a vector bundle E, page 18
$f_!$	Gysin map in K-theory, page 22
$\mathbb{F}_q$	the finite field with q elements
$\underline{G}$	sheaf of germs of G-valued continuous functions, G a topological group, page 14
G	Newton's graviational constant, page 3
G_0	connected component of the identity in a topological group G, page 31
$\mathrm{Gal}(E/F)$	Galois group of E over F, page 39
$GL(A)$	stable general linear group of A, page 31
$GL(n, A)$	group of invertible elements in $M_n(A)$, page 31
$\mathbb{G}_m$	group scheme of the multiplicative group, page 39
Gr	Grassmannian, page 14
g_S	string coupling constant, page 10
g_{YM}	Yang-Mills coupling constant, page 11
$\hbar$	Planck's constant divided by 2π, page 3
H	H-flux, page 4
$H^\bullet$	ordinary cohomology, or sometimes Čech cohomology
$H^\bullet_{\text{ét}}$	étale cohomology in algebraic geometry, page 39
$HH^\bullet(X)$	Hochschild cohomology of a variety X, page 91
$H^\bullet_{\mathrm{Lie}}(\mathfrak{g}, V)$	Lie algebra cohomology of a Lie algebra $\mathfrak{g}$ with coefficients in a $\mathfrak{g}$-module V, page 79
$H^\bullet_M(G, A)$	Moore cohomology of a locally compact group G with coefficients in a Polish module A, page 79
$\mathrm{Homeo}(X)$	homeomorphism group of X, page 63
$h^{p,q}$	Hodge number of a complex manifold, page 85
$\mathrm{Ind}_H^G A$	induced algebra, page 52
$\mathrm{Ind}_H^G \rho$	induced representation, page 34
$\mathrm{Inn}(A)$	inner automorphism group of a C^*-algebra A, page 65
J	complex structure on a complex manifold, page 85
$\mathcal{K}(\mathcal{H})$	compact operators on a Hilbert space $\mathcal{H}$ (often the $\mathcal{H}$ is suppressed if $\dim \mathcal{H} = \aleph_0$), page 27
$K(X)$	Grothendieck group of vector bundles on X, page 18
$K^*(X)$	topological K-theory of X, page 19
$K_0(A)$	Grothendieck group of finitely generated projective A-modules, page 30
$K_*(A)$	topological K-theory of A, page 33
$K_*^G(A)$	equivariant topological K-theory of A, page 47
$K(G, n)$	Eilenberg-Mac Lane space with $\pi_n \cong G$, page 38
$KK(A, B)$	Kasparov KK-group, page 22
$\mathcal{L}$	Lagrangian in a physical theory, page 1
$\mathcal{L}(\mathcal{H})$	bounded linear operators on a Hilbert space $\mathcal{H}$, page 27
$\varinjlim$	inductive limit of groups or algebras, page 31
l_P	Planck length, page 3
M_n	the $n \times n$ matrices (over $\mathbb{C}$ if not otherwise stated)
$M(A)$	multiplier algebra of a C^*-algebra A, page 27

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