Rigidity of Transformation Groups in Differential Geometry

Karin Melnick [*]

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Introduction and Framework

A generic Riemannian manifold has no isometries at all. But mathematicians and physicists do not really study such manifolds. The first Riemannian manifolds we learn about, and the most important ones, are Euclidean space, hyperbolic space, and the sphere, followed by quotients of these spaces, and by other symmetric spaces. These spaces go hand in hand with Lie groups and their discrete subgroups.

In this survey, symmetry provides a framework for classification of manifolds with differential-geometric structures. We highlight pseudo-Riemannian metrics, conformal structures, and projective structures. A range of techniques have been developed and successfully deployed in this subject, some of them based on algebra and dynamics and some based on analysis. We aim to illustrate this variety below.

Historical Currents

As with much of modern geometry, this subject has intellectual foundations in the 1872 Erlangen Program of F. Klein. He proposed to study geometries via their transformation groups. Euclidean geometry thus comprises the objects and quantities invariant by translations, rotations, and reflections, while conformal geometry is further enriched with dilations and circle inversions. Klein's classical geometries are homogeneous spaces—their transformation groups act transitively—, and the groups are Lie groups.

The concept of Lie group arose from the search for a theory of symmetries of differential equations.

In the 1930s, É. Cartan laid several foundations of differential geometry with Lie groups: he found a complete set of local differential invariants for many important and interesting geometric structures via his method of *moving frames*. He defined the notion of *G-structure of finite type* and his beautiful theory of *Cartan connections*, in which a manifold is "infinitesimally modeled" on one of Klein's homogeneous spaces.

Several decades later, geometers such as S. Sternberg, V. Guillemin, and S. Kobayashi proved that the local symmetries of any G-structure of finite type form a finite-dimensional Lie pseudogroup. This notion was then greatly extended by Gromov in the 1980s with his *rigid geometric structures*, which arose from his theory of partial differential relations.

Meanwhile, in the study of discrete subgroups of Lie groups, A. Weil, G. Mostow, and, ultimately, G. Margulis discovered remarkable rigidity of lattices in semisimple Lie groups. Among the breakthroughs in these monumental results was the surprisingly powerful application of ergodic theory.

Notions of rigid geometric structures will be explained and partially defined below. The reader could opt to read this section now, to read it later, or to skip it. The article can be well appreciated by just considering the examples of geometric structures given.

Zimmer's Program

In the 1970s and 80s, R. Zimmer developed rigidity theory of Lie groups and their lattices in the context of their actions on manifolds (see [Zim84]). A *lattice* in a Lie group H is a discrete subgroup Γ such that the quotient H/Γ has finite Haar measure.

Let Γ be a lattice in a simple Lie group H; we assume for this subsection and the next that H is noncompact, has finite center, and is not locally isomorphic to $\mathrm{SO}(1,n)$ or $SU(1,n)$, meaning the Lie algebra is not isomorphic to that of either of these groups. Superrigidity in this case says that homomorphisms of Γ to linear Lie groups G are the restrictions of homomorphisms $H \to G$, up to some "precompact noise," and possibly passing to a cover of H. The linear manifestations of these complicated and mysterious objects are thus reduced to those of H—an elementary algebraic matter, solved by Schur's Lemma. (Superrigidity and the remaining results and conjectures of this section are valid for H semisimple, but this entails additional definitions and assumptions.)

Zimmer's Program asserts that, in many cases, homomorphisms from Γ, or H, as above to the group of volume-preserving diffeomorphisms of a compact manifold arise from a short list of algebraic constructions. We illustrate two such actions, along with invariant differential-geometric structures for each. Let $M = G/\Lambda$, for G a connected Lie group and $\Lambda < G$ a cocompact lattice.

1. A homomorphism $\Gamma \to G$ gives an action of Γ on M, preserving the volume determined by the Haar measure.

   For G semisimple, the *Killing form* on the Lie algebra $\mathfrak{g}$ is a nondegenerate bilinear form, invariant by conjugation via the adjoint representation, $\mathrm{Ad}\, G$. It gives rise to a bi-invariant metric on G, of indefinite signature when G is noncompact. This *Cartan-Killing metric* descends to a Γ-invariant pseudo-Riemannian metric on M.

2. Any subgroup $\Gamma \leq \mathrm{Aut}_\Lambda G$, the automorphisms of G normalizing Λ, acts on M. The standard example is $G/\Lambda = \mathbf{R}^n/\mathbf{Z}^n = \mathbf{T}^n$, and $\Gamma = \mathrm{SL}_n(\mathbf{Z})$.

   Any Lie group G has a bi-invariant affine connection ∇_G, defined by declaring left-invariant vector fields to be parallel. This connection descends to M and is Γ-invariant. Although ∇_G is torsion-free only if G is abelian, the existence of a Γ-invariant affine connection on M implies existence of a torsion-free affine connection that is also Γ-invariant.

Pseudo-Riemannian metrics and affine connections are examples of G-structures of finite type. For Γ a lattice in a simple Lie group H as above, an *affine action* of Γ is an action on G/Λ via a homomorphism to $\mathrm{Aut}_\Lambda G \ltimes G$. We can refine the classification claim from Zimmer's Program to say, *infinite actions of Γ or H on a compact manifold M preserving a G-structure of finite type and a volume are affine actions—up to some additional "precompact noise,"* for which we refer to [Fis11] for details.

A famous conjecture of Zimmer is a dimension bound, with no invariant rigid geometric structure assumed: *Let N be the minimal dimension of a nontrivial representation of H. For any compact manifold M of dimension $n < N$, any action of Γ on M by volume-preserving diffeomorphisms preserves a smooth Riemannian metric.* Depending on H, the conclusion can often be strengthened to the action being finite. Affine actions satisfy this conjecture, which can be proved with Zimmer's Cocycle Superrigidity Theorem. Given a volume-preserving action of Γ on any n-dimensional manifold, for any n, Cocycle Superrigidity gives a nontrivial homomorphism $H \to \mathrm{SL}_n(\mathbf{R})$ or a measurable, Γ-invariant Riemannian metric on M. In the case Γ also preserves a G-structure of finite type, Zimmer improved the measurable Riemannian metric to a smooth one.

For decades, researchers struggled to improve the invariant, measurable Riemannian metric in the general case. The breakthrough came in 2016, when Brown–Fisher–Hurtado proved the dimension conjecture for Γ cocompact and H any classical simple Lie group, aside from a few exceptional dimensions for types B_n and D_n. They also proved a dimension bound for smooth actions not necessarily preserving a volume: in this case, $\dim M \geq N - 1$, or the action is finite. A sharp example to have in mind is the action of $\mathrm{SL}_n(\mathbf{R})$, or one of its lattices, on $M = \mathbf{R}\mathbf{P}^{n-1}$. Their proof opens up new avenues for the classification conjecture of Zimmer's Program—see below.

Zimmer-Gromov Program

Gromov generalized the G-structures of finite type in Zimmer's earlier work to the richer *rigid geometric structures*. Inspired in large part by Zimmer's

work, he proved results for a wider class of Lie groups with actions preserving these structures. His vision to some extent parts ways with another, large branch of Zimmer's Program, focused on rigidity of lattices in semisimple Lie groups, which has been demonstrated for a breathtaking range of actions.

Gromov attempted to formulate Zimmer's conjecture about affine actions more broadly, in particular, without assuming a finite, invariant volume. In [DG90], he and D'Ambra state the

Vague General Conjecture (VGC). *All triples* (H, M, ω)*, where* M *is a compact manifold with rigid geometric structure* ω *and* H *is a "sufficiently large" transformation group, are almost classifiable.*

"Large" here often means noncompact. It could also be a transitivity condition, such as having a dense orbit or an ergodic invariant measure. It could be a geometric condition of being *essential*, generally meaning it does not preserve a finer geometric structure subsidiary to ω. They note, "we are still far from proving (or even starting) this conjecture, but there are many concrete results which confirm it." Some such results will be presented below.

Here are examples of actions that join the "almost classification" for large actions of arbitrary groups not necessarily preserving a volume. For a more complete list, see [Gro88, 0.5]

3. $M = G/P$ with P closed—that is, an arbitrary homogeneous space. Assuming G semisimple, an important class are the quotients by *parabolic subgroups*, algebraic subgroups of G for which G/P is a compact projective variety. These G-actions never preserve a finite volume.

    The basic example is projective space $\mathbf{RP}^n$, with $G = \mathrm{PSL}_{n+1}(\mathbf{R})$, and P the stabilizer of a line in $\mathbf{R}^{n+1}$. This is a Klein geometry, in which the invariant notions are lines and intersection.

    Another important parabolic Klein geometry is the round sphere $\mathbf{S}^n$, a homogeneous space of $\mathrm{PO}(1, n + 1) \cong \mathrm{Isom}(\mathbf{H}^{n+1})$. It is the projectivization of the light cone in Minkowski space $\mathbf{R}^{1,n+1}$. It carries an invariant conformal structure, for which the invariant notion is angle.

4. $M = \Gamma\backslash G/P$, where $\Gamma < G$ acts freely, properly discontinuously, and cocompactly on G/P.

    An example is the unit tangent bundle of a compact hyperbolic manifold, where $G \cong \mathrm{PO}(1, n + 1)$ and $P \cong \mathrm{O}(n)$ is the stabilizer of a unit vector in $T\mathbf{H}^{n+1}$. The image of the holonomy representation of $\pi_1(M)$ is Γ. The geodesic flow on M is furnished by a one-parameter subgroup of G centralizing P; it is isometric for a metric obtained by pseudo-Riemannian submersion from the Cartan-Killing metric on $\Gamma\backslash G$.

5. $M = \Gamma\backslash U$, where $U \subset G/P$ is open, and Γ acts freely, properly discontinuously, and cocompactly on U. The example of the conformal Lorentzian Hopf manifold is presented later.

    The spaces in (4) and (5) both carry $(G, \mathbf{X})$-*structures*, which are defined in the next section.

Gromov's first key contribution, based on his theory of partial differential relations, is known as the Frobenius Theorem. At points satisfying a regularity condition, it can produce local transformations of (M, ω) from pointwise, infinitesimal input. The infinitesimal input corresponds to points in an algebraic variety. The Rosenlicht Stratification for algebraic actions on varieties yields a corresponding stratification by local transformation orbits in M. A consequence is the Open-Dense Theorem: if an orbit for local transformations in (M, ω) is dense, then an open, dense subset $U \subseteq M$ has a $(G, \mathbf{X})$-structure.

The Open-Dense Theorem lends support to the VGC. In specific cases, researchers can show that $U = M$, and some have obtained significant theorems in the Zimmer-Gromov Program by this route. But U may not, in general, equal M. Classifying compact $(G, \mathbf{X})$-manifolds compatible with a given geometric structure is a challenging subject unto itself.

A beautiful alternative relating examples 1-2 with example 3 is given by results of Nevo–Zimmer on smooth projective factors. They use Gromov's Stratification and his Representation Theorem to prove: given a connected, simple H preserving a real-analytic rigid geometric structure on a compact manifold M, together with a *stationary measure*, there is i) a smooth, H-equivariant projection of an open, dense

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$U \subseteq M$ to a parabolic homogeneous space H/Q; or *ii*) the Gromov representation of $\pi_1(M)$ contains $\mathfrak{h}$ in its Zariski closure. A stationary measure is a finite measure, here assumed of full support, invariant by convolution with a certain finite measure on H, but not necessarily under the H-action.

Notions of rigid geometric structure

The infinitesimal data of a geometric structure on M inhabits the frame bundle of M, of the appropriate order. The frames at a point $x \in M$ are the linear isomorphisms $T_x M \to \mathbf{R}^n$, where $n = \dim M$. These form a principal $\mathrm{GL}_n(\mathbf{R})$-bundle, denoted $\mathcal{F}M$. For $G \leq \mathrm{GL}_n(\mathbf{R})$, a G-structure is a *reduction* $\mathcal{R}$ of $\mathcal{F}M$ to G—that is, a smooth G-prinicipal subbundle. The frames in $\mathcal{R}$ determine the structure. For example, a (p, q)-semi-Riemannian metric g is a G-structure for $G \cong \mathrm{O}(p, q)$. The frames of $\mathcal{R}$ are the bases of $T_x M$ in which the inner product g_x has the form $-\mathrm{Id}_p \oplus \mathrm{Id}_q$.

The fiber $\mathcal{F}_x M$ comprises the first derivatives of coordinate charts at x. The order-k frame bundle has fibers $\mathcal{F}_x^{(k)} M$ comprising k-jets of coordinate charts φ at x with $\varphi(x) = \mathbf{0}$. Here φ and ψ have the same k-jet at x if $D_{\mathbf{0}}^{(i)}(\varphi \circ \psi^{-1}) = D_{\mathbf{0}}^{(i)} \mathrm{Id}$ for all $1 \leq i \leq k$. The k-jets at $\mathbf{0}$ of diffeomorphisms of $\mathbf{R}^n$ fixing $\mathbf{0}$ form the group $\mathrm{GL}_n^{(k)}(\mathbf{R})$. A G-structure of order k is a reduction of $\mathcal{F}^{(k)} M$ to $G \leq \mathrm{GL}_n^{(k)}(\mathbf{R})$. For example, a linear connection on TM is a reduction $\mathcal{R} \subset \mathcal{F}^{(2)} M$ to $\mathrm{GL}_n(\mathbf{R}) < \mathrm{GL}_n^{(2)}(\mathbf{R})$. Elements of $\mathcal{R}$ are 2-jets of geodesic coordinate charts.

A G-structure is *finite type* if a certain prolongation process stabilizes, which essentially means there is k such that, at all $x \in M$, the $k+i$-frames at x adapted to the structure are determined by the adapted k-frames at x, for all $i \geq 0$ (see [Kob95]). Conformal Riemannian structures, for example, are $\mathrm{CO}(n)$-reductions of $\mathcal{F}M$, where $\mathrm{CO}(n) \cong \mathbf{R}^* \times \mathrm{SO}(n)$. The prolongation gives a reduction of $\mathcal{F}^{(2)} M$ to a bigger group, namely the parabolic subgroup P fixing a point of $\mathbf{S}^n$. É. Cartan found, for $n \geq 3$, the prolongation stabilizes here, and thus any conformal transformation is determined by the 2-jet at a point.

A G-reduction of $\mathcal{F}M$ is the same as an equivariant map $\mathcal{F}M \to \mathrm{GL}_n(\mathbf{R})/G$ (and similarly in higher order). Gromov's *geometric structures of algebraic type* are equivariant maps to algebraic varieties. They are *rigid* if a similar prolongation process stabilizes (see [Gro88]). Some very useful additional flexibility is afforded by varieties which are not necessarily homogeneous. For example, one may add to a G-structure of finite type some vector fields, which correspond to maps to $\mathbf{R}^n$, and the result is again a rigid geometric structure. All rigid geometric structures in this article are understood to be of algebraic type.

For G a Lie group and $P < G$ a closed subgroup, a *Cartan geometry* on M modeled on (G, P) comprises a principal P-bundle $\pi : \hat{M} \to M$ and a $\mathfrak{g}$-valued 1-form ω on $\hat{M}$, called the *Cartan connection*. The Lie group G carries a left-invariant $\mathfrak{g}$-valued 1-form, the *Maurer-Cartan form* ω_G, which simply identifies the left-invariant vector fields on G with $\mathfrak{g}$. The Cartan connection is required to satisfy three axioms, mimicking properties of ω_G. One of them says ω is an isomorphism on each $T_{\hat{x}} \hat{M}$, so it determines a parallelization of $T\hat{M}$ (see [Sha96]).

Essentially all classical rigid geometric structures canonically determine a Cartan geometry, such that the transformations correspond to automorphisms of the Cartan geometry—diffeomorphisms of M lifting to bundle automorphisms of $\hat{M}$ preserving ω. A conformal Riemannian structure in dimension $n \geq 3$, for example, determines a Cartan geometry modeled on $\mathbf{S}^n$, with $G = \mathrm{PO}(1, n+1)$ and P as above. The bundle $\hat{M}$ in this case is the P-reduction of $\mathcal{F}^{(2)} M$ obtained by prolongation of the $\mathrm{CO}(n)$-structure.

Cartan geometries have proven very useful for the study of transformation groups. Fundamental results of the Zimmer-Gromov theory of rigid geometric structures have been translated to this setting. A key asset is the *curvature 2-form*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\hat{M}, \mathfrak{g})$$

Vanishing of Ω over an open $U \subseteq M$ is the obstruction to U having a $(G, \mathbf{X})$-structure with $\mathbf{X} = G/P$.

The notion of $(G, \mathbf{X})$-*structure* was developed by Ehresmann and later by Thurston [Thu97]: on a manifold M, it comprises an atlas of charts to $\mathbf{X}$ with transitions equal to transformations in G (one $g \in G$ on each connected component of the chart overlap).

When $M = \Gamma\backslash\mathbf{X}$ for some $\Gamma < G$, it is called *complete*; quotients $\Gamma\backslash U$ of open subsets $U \subset \mathbf{X}$ are called *Kleinian*.

Isometries of pseudo-Riemannian manifolds

By the classical theorems of Myers and Steenrod, the isometry group of a compact Riemannian manifold is a compact Lie group. Isometries of pseudo-Riemannian manifolds need not act properly; in particular, when the manifold is compact, this group can be noncompact. In this section we describe some important results on isometry groups and illustrate a couple techniques which have been influential. We focus on compact Lorentzian manifolds, mostly because there are as yet few answers to the corresponding questions in higher signature. The term *semi-Riemannian* means Riemannian or pseudo-Riemannian.

Simple group actions: Gauss maps

Let G be a noncompact, simple Lie group. For $\Lambda < G$ a lattice, the G-action on G/Λ is *locally free*, meaning stabilizers are discrete. We sketch an argument of Zimmer, using the Borel Density Theorem, that any isometric G-action on a finite-volume pseudo-Riemannian manifold M is locally free on an open, dense subset. First suppose that G acts ergodically, meaning that any G-invariant measurable function is constant almost everywhere. In fact, any G-invariant measurable map to a *countably separated* Borel space is constant almost everywhere; countably separated means there is a countable collection of Borel subsets B_i such that for any points $x \neq y$, some B_i contains one of $\{x, y\}$ and not the other. This property holds for quotients of algebraic varieties by algebraic actions. Then we can apply this fact to G-equivariant maps from M to G-algebraic varieties, sometimes called *Gauss maps*.

Let σ assign to $x \in M$ the Lie algebra of the stabilizer of x in G, which lies in the disjoint union Gr $\mathfrak{g}$ of the Grassmannians $\mathrm{Gr}(k, \mathfrak{g})$, $k = 0, \ldots, \dim \mathfrak{g}$. The map σ satisfies the equivariance relation $\sigma(g.x) =$ $(\mathrm{Ad}\ g).(\sigma(x))$. We conclude that there is a single G-orbit $\mathcal{O} \subset \mathrm{Gr}\ \mathfrak{g}$ containing $\sigma(x)$ for almost-every x. The volume on M pushes forward to an Ad G-invariant finite volume on $\mathcal{O}$. After passing to Zariski closures, this orbit can be identified with an algebraic homogeneous space. The Borel Density Theorem says that $\overline{\mathcal{O}}$, hence also the Ad G-orbit $\mathcal{O}$, must be a single point. Thus $\sigma(x)$ is an ideal for almost-all $x \in M$, namely, because G is simple, it is 0 or $\mathfrak{g}$.

In general, the metric volume decomposes into ergodic components, and $\sigma(x) \in \{0, \mathfrak{g}\}$ for almost-all x. The points with $\sigma(x) = \mathfrak{g}$ comprise the fixed set of G^0. The fixed set of any nontrivial isometry has null volume (assuming M connected). Thus $\sigma(x) = 0$, and the stabilizer of x is discrete, for all x in a full-volume subset $\Omega \subset M$. It follows from lower semi-continuity of $\dim \sigma(x)$ that Ω is open and dense.

By comparable arguments involving ergodicity and the Borel Density Theorem, Zimmer obtained, for M compact, a Lie algebra embedding $\mathfrak{g} \hookrightarrow \mathfrak{o}(p, q)$ and concluded that, in particular, the real-rank—the dimension of a maximal abelian, $\mathbf{R}$-diagonalizable subgroup of Ad G—satisfies

$$\mathrm{rk}_{\mathbf{R}} G \leq \min\{p, q\} = \mathrm{rk}_{\mathbf{R}} \mathrm{SO}(p, q)$$

Such an embedding for G of real-rank at least two follows from Zimmer's Cocycle Superrigidity; here, with a pseudo-Riemannian metric, a stronger result is obtained by a more elementary proof.

A more general embedding theorem for connected G, not necessarily simple, preserving any rigid geometric structure, not necessarily determining a volume, was proved by Gromov [Gro88]. A version of this generality was proved for compact Cartan geometries by Bader–Frances–Melnick in 2009.

Connected isometry groups of compact Lorentzian manifolds

The group $\mathrm{SL}_2(\mathbf{R})$ with the Cartan-Killing metric is a three-dimensional Lorentzian manifold with isometry group $\mathrm{SL}_2(\mathbf{R}) \times_{\mathbf{Z}_2} \mathrm{SL}_2(\mathbf{R}) \cong \mathrm{SO}^0(2, 2)$. This space has constant negative sectional curvature and is known as anti-de Sitter space, AdS^3. (Together with its higher-dimensional analogues, this space is important in string theory and M-theory.)

Let (M, g) be a compact, connected, Lorentzian manifold and $H \leq \operatorname{Isom}(M, g)$ a noncompact simple group. Zimmer's bound is $\operatorname{rk}_{\mathbf{R}} H \leq 1$; he in fact proved that the identity component $\operatorname{Isom}^0(M, g)$ is locally isomorphic to $SL_2(\mathbf{R}) \times K$, with K compact. Gromov improved Zimmer's result to conclude

$$M \cong (\widetilde{\operatorname{AdS}}^3 \times_f N)/\Gamma$$

where N is a Riemannian manifold, $f : N \to \mathbf{R}^+$ is a warping function, and Γ acts isometrically, freely, and properly discontinuously on the warped product.

The complete determination of connected isometry groups of compact Lorentzian manifolds was simultaneously achieved by Adams–Stuck [AS97a, AS97b] and Zeghib [Zeg98a, Zeg98b]:

Theorem (Adams–Stuck/Zeghib 1997/8). *The identity component of the isometry group of a compact Lorentzian manifold is locally isomorphic to $H \times K \times \mathbf{R}^k$, with K compact, $k \geq 0$, and H one of*

- *$SL_2(\mathbf{R})$*

- *$Heis^m$, a $2m + 1$-dimensional Heisenberg group*

- *an oscillator group, a solvable extension of $Heis^{2m+1}$ by S^1.*

Adams and Stuck proceed by studying the dynamics of a Gauss map $M \to \operatorname{Sym}^2 \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of $\operatorname{Isom}^0 M$. The map sends $x \in M$ to the pullback of the metric on $T_x M$ restricted to the subspace tangent to the G-orbit at x. It is equivariant for the $\operatorname{Ad} G$ representation on $\operatorname{Sym}^2 \mathfrak{g}$. Zeghib works with the average over M of these forms on $\mathfrak{g}$.

The geometry of M when H is a Heisenberg or oscillator group has not been completely described.

Dynamical foliations from unbounded isometries

The anti-de Sitter space AdS^3 can be identified with the unit tangent bundle of the hyperbolic plane. For a hyperbolic surface $\Sigma \cong \Gamma \backslash \mathbf{H}^2$, the quotient $\Gamma \backslash \operatorname{AdS}^3$ is identified with $T^1 \Sigma$, and the geodesic and horocycle flows with the right-action of the diagonal and upper-triangular unipotent subgroups, respectively, of $SL_2(\mathbf{R})$.

The geodesic flow is an Anosov, Lorentz-isometric flow on $\Gamma \backslash \operatorname{AdS}^3$. The weak-stable and -unstable foliations are by totally geodesic surfaces on which the metric is degenerate. These foliations, $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, are the orbits of the two-dimensional upper-triangular and lower-triangular subgroups, respectively (see Figure 1). The horocycle flow is not hyperbolic in the dynamical sense. Nonetheless, it also preserves a foliation by totally geodesic, degenerate surfaces, namely $\mathcal{W}^{cs}$. Zeghib calls this the *approximately stable* foliation of the flow. He proved the following [Zeg99a, Zeg99b]:

Theorem (Zeghib 1999). *Let M be a compact Lorentzian manifold, and suppose that $G = \operatorname{Isom} M$ is noncompact. Any unbounded sequence in G determines, after passing to a subsequence, a foliation by totally geodesic, degenerate hypersurfaces.*

The space of all these approximately stable foliations gives a compactification of G and reflects algebraic properties of G.
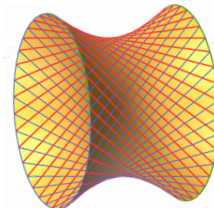

Figure 1: Two-dimensional Anti-de Sitter space with bifoliation by lightlike geodesics (creative commons: bubba on `math.stackexchange.com/q/1606820`)

The construction of such foliations was suggested by Gromov, who provided the following argument. The reader may note the significance of the index $p = 1$, a way in which Lorentzian geometry is "closest" to Riemannian. Let $f_k \to \infty$ in G. For any $x \in M$, we may pass to a subsequence so that $f_k.x$

converges to a point $y \in M$. Equip $M \times M$ with the pseudo-Riemannian metric $g \oplus -g$. The Levi-Civita connection on the product is $\nabla^g \oplus \nabla^g$. The graphs Γ_k of f_k are totally isotropic and totally geodesic in $(M \times M, g \oplus -g)$, and they converge to an n-dimensional, isotropic, totally geodesic submanifold Γ near (x, y).

Because $\{f_k\}$ does not converge, Γ cannot be a graph near (x, y), so it has positive-dimensional intersection with $\{x\} \times M$. This intersection is isotropic and geodesic in (M, g), so coincides with an isotropic geodesic segment γ near y. The projection of $T_{(x,y)}\Gamma$ to $T_y M$ must be orthogonal to $T_y \gamma$, so it is contained in the degenerate hyperplane $T_y \gamma^\perp$. Therefore, $T_{(x,y)}\Gamma$ has nonzero intersection with $T_x M \oplus 0$, which must again must be isotropic; the projection to $T_x M$ is contained in a degenerate hyperplane H. A dimension count implies that the projection equals H. Because Γ is totally geodesic, it projects to the totally geodesic, degenerate hypersurface obtained by exponentiation of H.

A classical theorem of Haefliger says that a compact, simply connected, real-analytic manifold admits no real-analytic, codimension-one foliation. Zeghib used his foliations to give another proof of the following theorem of D'Ambra [D'A88]:

Theorem. *(D'Ambra 1988) Let $n \geq 3$. For (M^n, g) a compact, simply connected, real-analytic Lorentzian manifold, $\mathrm{Isom}(M, g)$ is compact.*

D'Ambra's proof was a *tour de force* of Gromov's theory of rigid geometric structures and belongs to the underpinning of the VGC.

Current Questions

After reading about these wonderful accomplishments of the 1980s and 90s, the reader is likely to have at least two obvious questions: what about nonconnected groups? and, what about higher signature?

Some progress has been made on compact Lorentzian manifolds (M, g) for which $G = \mathrm{Isom}(M, g)$ has infinitely-many components when M is *stationary*— that is, it admits a timelike Killing vector field. The latter property implies that M has closed timelike geodesics, or "time machines." Piccione–Zeghib have

a nice structure theorem for such spaces: G^0 contains a torus $\mathbf{T}^d$, and there is a Lorentzian quadratic form Q on $\mathbf{R}^d$ such that $G/G^0 \leq O(Q)_{\mathbf{Z}}$.

For smooth, three-dimensional, compact Lorentzian manifolds, Frances recently completely classified those with noncompact isometry group, topologically and geometrically. The topological classification says that such a manifold M is, up to a cover of order at most four:

- $\Gamma \backslash \widetilde{\mathrm{SL}}_2(\mathbf{R})$, for Γ a cocompact lattice

- The mapping torus $\mathbf{T}_A^3$ of an automorphism A of $\mathbf{T}^2$.

In the first case, $\widetilde{\mathrm{SL}}_2(\mathbf{R})$ has a left-invariant metric. Geometries of the second type can have infinite, discrete isometry group. A corollary is an improvement of D'Ambra's Theorem to smooth metrics, in the three-dimensional case.

For $p \geq 2$, it is not known which *homogeneous*, compact, (p, q)-pseudo-Riemannian manifolds have noncompact isometry group. Much current work focuses on classifying left-invariant metrics on Lie groups. To illustrate the complexity of higher signature, we indicate Kath–Olbrich's survey on the classification problem for pseudo-Riemannian symmetric spaces [KO08]. While Riemannian symmetric spaces were essentially classified by É. Cartan, and reductive pseudo-Riemannian symmetric spaces by Berger in 1957, a complete classification is not really expected, without restricting to a low index p, or assuming some additional invariant structure.

Conformal Transformations

Compact Lorentzian manifolds are marvelous for the mathematical study of transformation groups, but the possibility of "time machines" makes them less appealing to physicists. Conformal compactifications of Lorentzian manifolds are, on the other hand, quite natural in relativity.

We have seen above that conformal Riemannian transformations are those preserving angles between tangent vectors. For arbitrary signature, we define the *conformal class* of g to be $[g] = \{e^{2\lambda}g\}$, with λ
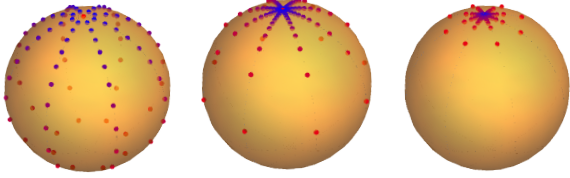
Figure 2: iterates of a conformal transformation of the sphere with source-sink dynamics

ranging over smooth functions on M. The conformal transformations $\mathrm{Conf}(M, [g])$ are those preserving $[g]$. In higher signature, these are the transformations preserving the causal structure—that is, the null cones in each tangent space.

Stereographic projection from a point $\hat{p} \in \mathbf{S}^n$ is a conformal equivalence of $\mathbf{S}^n \backslash \{\hat{p}\}$ with Euclidean space. Thus $\mathbf{S}^n$ is *conformally flat*—locally conformally equivalent to a flat Riemannian manifold. The higher signature analogue of $\mathbf{S}^n$ is a parabolic homogeneous space for $\mathrm{PO}(p+1, q+1)$, called the *Möbius space* $\mathbf{S}^{p,q}$. It is a two-fold quotient of $S^p \times S^q$; the conformal class can be realized by $g_p \oplus -g_q$, where g_p, g_q are the constant-curvature metrics on the respective spheres. These metrics are also conformally flat. The conformal structure of a (p, q)-pseudo-Riemannian metric, $p + q \geq 3$, determines a Cartan geometry modeled on $\mathbf{S}^{p,q}$. It is also a G-structure of finite type, with order 2. Throughout this section, n is an integer greater than or equal 3.

The Ferrand-Obata Theorem

While the isometry group of a compact Riemannian manifold is compact, the conformal group need not be. A dilation of Euclidean space corresponds via stereographic projection to a conformal transformation of $\mathbf{S}^n$ having source-sink dynamics under iteration—see Figure 2. A sequence $\{f_k\}$ of transformations has *source-sink dynamics* if there are points p^+ and p^- in M such that $\{f_k\}$ converges uniformly on compact subsets of $M \backslash \{p^-\}$ to the constant map p^+, and similarly $f_k^{-1} \to p^-$ uniformly on compact subsets of $M \backslash \{p^+\}$. The full conformal group of $\mathbf{S}^n$ is the noncompact simple Lie group $\mathrm{Isom}\, \mathbf{H}^{n+1} \cong$

$\mathrm{PO}(1, n+1)$. This isomorphism comes from the fact that $\mathbf{S}^n$ is the visual boundary in the conformal compactification of $\mathbf{H}^{n+1}$. Being the visual boundary of hyperbolic space is quite a special property. Lichnerowicz conjectured the following striking fact:

Theorem (Ferrand/Obata 1971). *Let (M^n, g) be a compact Riemannian manifold. If $\mathrm{Conf}(M, [g])$ is noncompact, then $(M, [g])$ is globally conformally equivalent to $\mathbf{S}^n$.*

(This theorem is also true for $n = 2$.) Lichnerowicz's original conjecture was actually under the stronger assumption $\mathrm{Conf}^0(M, [g])$ noncompact, and this was proved by Obata [Oba71], using Lie theory and differential geometry. Ferrand proved the theorem in full generality [LF71], using quasiconformal analysis and no Lie theory.

In 1994, Ferrand extended her result to noncompact M and gave a streamlined proof of her original result. She uses conformal capacities to define a conformally invariant function μ on pairs of points. Sometimes it gives a metric on M; in this case, $\mathrm{Conf}(M, [g])$ acts isometrically for μ, and, therefore, properly.

When μ does not define a distance, then Ferrand uses capacities to construct conformally invariant functions on distinct quadruples or triples of points, no three of which are equal, depending whether M is compact or not. In the former case, this function is essentially a log-cross-ratio, which can detect whether distinct points are converging under a sequence of conformal transformations. In the latter case, the function partly extends to the Alexandrov compactification $\hat{M} = M \cup \{\infty\}$ and detects divergence to infinity, as well. With these functions, she reconstructs source-sink dynamics for unbounded sequences in $\mathrm{Conf}(M, [g])$ when it acts nonproperly, and ultimately concludes $(M, [g]) \cong \mathbf{Euc}^n$ or $\mathbf{S}^n$.

PDEs proof, CR analogue

When deforming a metric g to one with desirable properties, a natural choice is to restrict to conformal deformations. The Yamabe Problem is a famous case: on any compact Riemannian manifold (M, g), there is $g' \in [g]$ with constant scalar curvature.

In 1995, Schoen published a different proof of Ferrand's theorem [Sch95], in the wake of his work on the completion of the Yamabe Problem. Given a Riemannian metric g with scalar curvature function S_g and Laplace-Beltrami operator Δ_g, if $fg \in [g]$ has constant scalar curvature c, then $u = f^{\frac{n-2}{4}}$ satisfies

$$-L_g u = c \frac{n-2}{4(n-1)} u^{\frac{n+2}{n-2}}$$

where the conformal Laplacian

$$L_g : u \mapsto \Delta_g u - \frac{n-2}{4(n-1)} S_g u$$

is an elliptic operator satisfying a certain conformal invariance. This equation can be applied to $f(x) = |\mathrm{Jac}_x F|^{2/n}$ if g has constant scalar curvature and $F \in \mathrm{Conf}(M, [g])$. Schoen's arguments are based on the analytic properties of the elliptic operator L_g.

CR, or *Cauchy-Riemann*, structures, model real hypersurfaces in complex vector spaces. On the unit sphere in $\mathbf{C}^{n+1}$, for example, the tangent bundle carries a totally nonintegrable, or *contact*, hyperplane distribution, equal at each $z \in S^{2n+1}$ to the maximal complex subspace of $T_z S^{2n+1} \subset \mathbf{C}^{n+1}$. In general, a CR structure (nondegenerate, of hypersurface type) on an odd-dimensional manifold M comprises a contact hyperplane distribution $\mathcal{D}$ and an almost-complex structure J on $\mathcal{D}$, satisfying a compatibility condition: for any 1-form λ on M with ker $\lambda = \mathcal{D}$, the *Levi form* on $\mathcal{D}$ given by $L_\lambda(u, v) := -d\lambda(u, v)$ is required to obey $L_\lambda(Ju, Jv) = L_\lambda(u, v)$. In this case, L_λ is the imaginary part of a Hermitian form on $\mathcal{D}$. The real part of this Hermitian form is a semi-Riemannian metric B_λ on $\mathcal{D}$.

A different choice of λ gives a metric conformal to B_λ, that is, related by a positive function on M. The CR structure is *strictly pseudoconvex* if any B_λ is positive definite. In this case, there is a subelliptic Laplace operator L_λ on $C^\infty(M)$, sastifying a certain CR invariance. Schoen proved, in analogy with his conformal results, that the automorphisms of a strictly pseudoconvex CR manifold M^{2n+1} act properly, unless M is CR-equivalent to $S^{2n+1} \subset \mathbf{C}^{n+1}$, or to a Heisenberg group Heis^{2n+1} carrying a certain left-invariant CR structure. S. Webster previously proved such a theorem, in 1977, for M compact with noncompact, connected automorphism group, following some analogy with Obata's conformal proof.

Cartan connections proof, rank-one analogues

The CR sphere can be identified as a conformal sub-Riemannian space with the visual boundary of complex hyperbolic space $\mathbf{H}_\mathbf{C}^{n+1}$. The quaternionic hyperbolic space $\mathbf{H}_\mathbf{H}^{n+1}$ and the Cayley hyperbolic plane $\mathbf{H}_\mathbf{O}^2$ have visual boundaries diffeomorphic to spheres, with respective differential-geometric structures invariant by the isometries of the interior. These boundaries of hyperbolic spaces form a family; they are parabolic homogeneous spaces $\mathbf{X} = G/P$ with G simple of $\mathbf{R}$-rank 1—see Table 1. Cartan geometries modeled on one of these homogeneous spaces canonically correspond to certain differential-geometric structures:

$\mathbf{X}$	G	structure
$\partial \mathbf{H}_\mathbf{R}^{n+1}$	$PO(1, n+1)$	conformal Riemannian
$\partial \mathbf{H}_\mathbf{C}^{n+1}$	$PU(1, n+1)$	strictly pseudoconvex CR
$\partial \mathbf{H}_\mathbf{H}^{n+1}$	$PSp(1, n+1)$	contact quaternionic
$\partial \mathbf{H}_\mathbf{O}^2$	F_4^{-20}	contact octonionic

Table 1: rank-one parabolic geometries

Contact quaternionic and contact octonionic structures were introduced by O. Biquard in 2000.

Let M^n be a manifold carrying one of the above structures, and suppose that a group H of transformations acts nonproperly—that is, there are $h_k \to \infty$ in H and points $x_k \to x$ in M such that $h_k.x_k \to y$. Let $\pi : \hat{M} \to M$ be the Cartan bundle. Choose lifts, $\hat{x}_k$ and $\hat{x}$ of x_k and x, respectively, to $\hat{M}$, such that $\hat{x}_k \to \hat{x}$.

Recall that the Cartan connection $\omega \in \Omega^1(\hat{M}, \mathfrak{g})$ determines a parallelization of $T\hat{M}$, invariant by the H-action on $\hat{M}$. It follows that H acts properly on $\hat{M}$; in particular, $\{h_k.\hat{x}_k\}$ diverges. This divergence is necessarily "in the fiber direction"—that is, there exist $p_k \to \infty$ in P such that $\{h_k.\hat{x}_k.p_k^{-1}\}$ converges to some $\hat{y} \in \pi^{-1}(y)$. Such a sequence $\{p_k\}$ is called
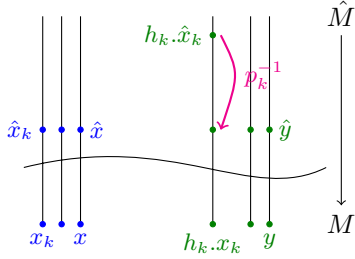
Figure 3: a holonomy sequence for $\{h_k\}$ at x

a *holonomy sequence* for $\{h_k\}$ at x. See Figure 3. (For the initiated: Holonomy sequences are related to P-valued cocycles for the H-action on $\hat{M}$, more precisely, to *placings* of H into P.)

After passing to a subsequence, $\{p_k\}$ has source-sink dynamics on the model space $\mathbf{X}$. Frances used the Cartan connection to prove that $\{h_k\}$ has the same dynamical behavior near x. This led to his simultaneous proof of Ferrand's and Schoen's theorems and of the extension to all structures listed in Table 1: if the automorphism group acts nonproperly, and M is compact, then it is geometrically isomorphic to $\mathbf{X}$. If M is noncompact, then it is isomorphic to the complement of a point in $\mathbf{X}$. This complement is identified with a maximal connected, unipotent subgroup of G, contained in P, via an orbit map. For example, the strictly pseudoconvex CR structure on the Heisenberg group appearing in Schoen's theorem can be defined this way. Frances' proof was a breakthrough, because his basic approach can be applied to a very wide range of problems.

Pseudo-Riemannian analogue?

The Ferrand–Obata Theorem is paradigmatic for the Zimmer–Gromov Program. In [DG90], D'Ambra and Gromov ask whether there is a pseudo-Riemannian analogue, in the case M is compact. Given that pseudo-Riemannian isometry groups can act non-properly, the suitable hypothesis is that the conformal groups acts essentially, meaning it does not preserve any metric in the conformal class. Alekseevsky previously constructed an infinite family of non-conformally flat Lorentzian metrics on $\mathbf{R}^n$ ad-

mitting an essential conformal flow. The answer to D'Ambra and Gromov's question is ultimately "no"; nonetheless, we briefly describe here some positive results holding under substantial assumptions either on the metric or on the conformal group.

Let (M^n, g) be a semi-Riemannian *Einstein* space, meaning the Ricci curvature satisfies

$$\mathrm{Ric}_g = \frac{S_g}{n}g \tag{1}$$

In this case, the scalar curvature S_g is necessarily constant. A notable precedent for the Ferrand–Obata theorem was a result of Nagano–Yano for Riemannian Einstein manifolds. For simplicity, we state the version for M compact: if the conformal group contains a nonisometric flow, then $(M, [g]) \cong (\mathbf{S}^n, [g_{+1}])$. For g Einstein, of any signature, the divergence σ of a conformal vector field satisfies

$$\nabla^2 \sigma = -\frac{S_g}{n(n-1)}\sigma g \tag{2}$$

When M is compact, then σ has extrema, which forces $S_g = 0$ or g definite; thus there is a parallel, isometric vector field, $\nabla \sigma$, or $(M, [g]) \cong (\mathbf{S}^n, [g_{+1}])$. Local normal forms for metrics admitting nontrivial solutions of the PDE (2), due to Brinkmann, rule out the first possibility and also give results for noncompact, complete (M, g), which were obtained by Kerckhove in 1988 and Kühnel–Rademacher in 2009. Also in 2009, Kiosak–Matveev evaluated (2) along null geodesics for a clever proof that any conformal rescaling of an indefinite g which is also Einstein must be constant, if (M, g) is compact or complete.

For (M^n, g) a compact, (p, q)-semi-Riemannian manifold and H a simple group of conformal transformations, Zimmer proved in 1987 that $\mathrm{rk}_{\mathbf{R}} H \leq \min\{p, q\} + 1$. In light of his bound for isometry groups, any H attaining the conformal bound acts essentially. In this case, Bader–Nevo proved in 2002 that H is locally isomorphic to $\mathrm{PO}(p+1, k)$, with $p+1 \leq k \leq q+1$. Frances–Zeghib then strengthened their conclusion in 2005 to say that (M, g) is conformally equivalent to $\mathbf{S}^{p,q}$, up to double cover when $p, q \geq 2$, and up to $\mathbf{Z} \ltimes \mathbf{Z}_2$-covers in the Lorentzian case. These results support a pseudo-Riemannian Lichnerowicz Conjecture when a large simple group

acts. Frances–Melnick proved an analogous theorem when a large, connected, nilpotent group acts in 2010; here, large means of maximal nilpotence degree.

In a recent advance, Pecastaing proved a similar result for a cocompact lattice Γ in a simple Lie group H, exploiting the proof of Zimmer's Conjecture. The standard approach to a Γ-action on a manifold M is via the *suspension* action of H on the diagonal quotient $H \times_\Gamma M$. Brown–Fisher–Hurtado proved key relations on the Lyapunov spectrum of this action in terms of the roots of $\mathfrak{h}$. When Γ acts conformally on M, the Lyapunov spectrum on the vertical distributions, tangent to the M-fibers of the suspension, must be compatible with the spectrum of the standard representation of $CO(p,q)$. This strong restriction leads to the bound $\mathrm{rk}_{\mathbf{R}} H \leq \min\{p,q\}+1$, and to uniform contracting behavior around a point for a sequence in Γ when the bound is attained. Pecastaing then proves that $(M, [g])$ is conformally flat; when $p, q \geq 2$, he obtains Frances–Zeghib's conclusion that $(M, [g]) \cong \mathbf{S}^{p,q}$, up to double covers.

Finally, we come to the counterexamples: in 2012, Frances constructed non-conformally flat metrics on $S^1 \times S^{n-1}$ of all signatures (p, q) with $\min\{p,q\} \geq 2$, admitting an essential conformal flow. We could still dream of a classification of compact pseudo-Riemannian manifolds with essential conformal group, but not of a Ferrand–Obata theorem in higher signature. The CR story has continued on a parallel course: Case–Curry–Matveev constructed compact CR manifolds $(M, \mathcal{D}, J)$ with conformal metric $[B_\lambda]$ of arbitrary higher—that is, non-Lorentzian—signature, which admit an essential CR flow but are not CR-flat. They also constructed a CR analogue of Alekseevsky's noncompact, non-conformally flat, essential Lorentzian examples.

Lorentzian Lichnerowicz Conjecture

We first describe some interesting examples showing that the global conclusion of the Ferrand-Obata Theorem cannot hold in the Lorentzian case. Let $U = \mathrm{Min}^{1,n-1} \backslash \{0\}$. The group $\Lambda = \{2^k \mathrm{Id}_n\}_{k \in \mathbf{Z}}$ acts properly discontinuously and cocompactly on U, commuting with $CO(1, n-1)$. The compact quotient $M = \Lambda \backslash U$ is called the conformal Lorentzian *Hopf*

manifold and is diffeomorphic to $S^1 \times S^{n-1}$. The conformal group is $S^1 \times O(1, n-1)$, which means that M is not conformally equivalent to the double cover of $\mathbf{S}^{1,n-1}$. It is, rather, a quotient of the open, dense image of the conformal embedding of U into $\mathbf{S}^{1,n-1}$, as in example 5. This conformal action is also distinguished from the Möbius space by the special property that it preserves an affine connection, descended from the standard affine connection on $\mathbf{R}^n$; the quotient connection is incomplete.

C. Frances constructed compact, Kleinian $(PO(2, n), \mathbf{S}^{1,n-1})$-spaces, for all $n \geq 3$, of infinitely many distinct topological types, admitting an essential conformal flow. They are quotients of the complement of a finite set of closed isotropic geodesics in $\mathbf{S}^{1,n-1}$ by the action of a Fuchsian group in $SL_2(\mathbf{R})$ embedded in $PO(2, n)$.

The following Lorentzian Lichnerowicz Conjecture has, as yet, neither been proved nor disproved:

Conjecture (LLC). *Let (M^n, g) be a compact Lorentzian manifold, $n \geq 3$. If $Conf(M, [g])$ acts essentially, then (M, g) is conformally flat.*

The recent conformal D'Ambra Theorem of Melnick–Pecastaing verifies the conjecture for (M, g) real-analytic, with finite fundamental group [MP19]:

Theorem (Melnick–Pecastaing 2019). *For (M^n, g) a compact, simply connected, real-analytic Lorentzian manifold, for $n \geq 3$, $Conf(M, [g])$ is compact.*

A compact conformal group always preserves a metric in the conformal class, the average of a given metric in $[g]$ over the orbit. Thus there are no essential conformal groups for (M, g) as in the theorem, and the LLC is vacuously true in this case. Conversely, the LLC together with D'Ambra's theorem would imply the above statement. If $H = Conf(M, [g])$ were noncompact, D'Ambra's Theorem would say H must be essential. The LLC would imply that M is conformally flat. But there are no compact, simply connected, conformally flat Lorentzian manifolds. Indeed, for any manifold M with a $(G, \mathbf{X})$-structure, there is a *developing map* from the universal cover $\delta : \tilde{M} \to \mathbf{X}$. If $\tilde{M}$ is compact, then δ is a covering map, and the $(G, \mathbf{X})$-structure is complete. If $M = \tilde{M}$, this is not

possible with $\mathbf{X} = \mathbf{S}^{1,n-1}$, which has infinite fundamental group.

Other conformal problems

Two basic problems on compact pseudo-Riemannian manifolds, under the auspices of the VGC, are to classify the homogeneous spaces with noncompact conformal group, and to classify the possible connected noncompact conformal groups in general.

In the Lorentzian case, the LLC would indicate that the first problem comprises the analogous classification of isometrically homogeneous Lorentzian spaces, and the classification of compact, conformally flat Lorentzian manifolds with essential conformal group. The first problem was solved by Zeghib in [Zeg98b]. The second problem may be approachable, but we do not currently know that all such $(\mathrm{PO}(2, n-1), \mathbf{S}^{1,n-1})$-spaces must be Kleinian. In higher signature, the first, isometric classification is not known.

For the second problem, we have the real-rank and nilpotence degree bounds mentioned above. For compact Lorentzian manifolds, Pecastaing completed the classification of noncompact, semisimple, conformal groups. They are necessarily of real-rank one or two; symplectic and exceptional groups do not occur.

With regard to conformal pseudo-Riemannian transformations, we are still far from proving, but we are less far from starting the VGC since D'Ambra and Gromov's influential manuscript of 30 years ago.

Projective Transformations

Projective structures are geometrizations of systems of second-order ODEs on $U \subset \mathbf{R}^n$ of the form

$$\frac{\ddot{\gamma}_1 + Q_1(\dot{\gamma})}{\dot{\gamma}_1} = \cdots = \frac{\ddot{\gamma}_n + Q_n(\dot{\gamma})}{\dot{\gamma}_n} \qquad (3)$$

where Q_i are quadratic forms on $\mathbf{R}^n$. Note that the set of solutions is invariant under smooth reparametrization. É. Cartan proved that for such a system, there are torsion-free connections whose geodesics are the solutions. Two such connections ∇

and ∇' have the same geodesics up to reparametrization. They are related, for vector fields X and Y, by:

$$\nabla'_X Y = \nabla_X Y + \nu(X)Y + \nu(Y)X \qquad (4)$$

where ν is a 1-form. A *projective structure* on a manifold M^n can thus be defined as an equivalence class $[\nabla]$ of torsion-free connections according to (4). For $n \geq 2$, these correspond canonically to Cartan geometries modeled on $\mathbf{RP}^n$ and are rigid geometric structures of order 2, formally quite similar to conformal structures. The projective group $\mathrm{Proj}(M, [\nabla])$ comprises the diffeomorphisms of M preserving $[\nabla]$.

Strongly essential projective flows

Consider the parabolic subgroup $P < \mathrm{SL}_{n+1}(\mathbf{R})$ stabilizing a line $\ell \subset \mathbf{R}^{n+1}$. It has Levi decomposition $P \cong \mathrm{GL}_n(\mathbf{R}) \ltimes U$, where the unipotent radical U acts trivially on $\mathbf{R}^{n+1}/\ell$, and thus with trivial differentials on $T_\ell \mathbf{RP}^n$. An isomorphism $U \cong \mathbf{R}^{n*}$ can be defined as follows: given $\xi \in \mathbf{R}^{n*} \cong (\mathbf{R}^{n+1}/\ell)^*$, the element u_ξ is determined by the action on projective lines γ_v through ℓ, as v ranges over $\mathbf{R}^{n+1}/\ell$; it effects the fractional linear reparametrization

$$(u_\xi . \gamma_v)(s) = \gamma_v \left(\frac{s}{1 + \xi(v)s} \right) \qquad (5)$$

Nagano–Ochiai considered a projective flow $\{\varphi^t\}$ on an arbritrary $(M, [\nabla])$ having a fixed point x with $D_x \varphi^t$ trivial for all t. For such a *strongly essential* flow, there is $\xi \in T_x^* M$ such that each geodesic γ_v is reparametrized according to (5) with $t\xi$ in place of ξ. This can be calculated from the ODE (3) or can be seen via the Cartan connection. They then compute that, in a suitable framing along $\gamma_v(s)$, the differential of φ^t is a scalar contraction by $\lambda_s(t) = (1+\xi(v)st)^{-1}$. The flow by φ^t contracts the projective Weyl curvature W at $\gamma_v(s)$ by a factor of $\lambda_s(t)^2$. They conclude that $W_{\gamma_v(s)}$ vanishes for all v and all s. A neighborhood of x in M is thus projectively flat. This 1986 paper was an early application of Cartan connections for rigidity of transformation groups. The technique reappears and evolves in Frances' influential proof of the Ferrand-Obata Theorem for rank-one parabolic geometries from above.

Local flatness results in the presence of strongly essential flows have been proved for other parabolic geometries by Alekseevsky, Frances, Frances–Melnick, Čap–Melnick, and Melnick–Neusser.

Riemannian Levi-Civita connections

The projective analogue of the Lichnerowicz Conjecture would posit that a compact affine manifold (M^n, ∇) with essential projective group is equivalent to $\mathbf{RP}^n$, up to finite covers, with *essential* taken to mean not preserving a connection in $[\nabla]$. Such a statement has not yet received strong endorsement.

For (M^n, ∇) a Riemannian manifold with Levi-Civita connection, however, the study of $\mathrm{Proj}(M, [\nabla])$ is classical—these are the transformations preserving the set of unparametrized geodesics. A Lichnerowicz-type conjecture appears in this context in the paper of Nagano–Ochiai; in fact, they proved that a compact, Riemannian manifold admitting a strongly essential projective flow is equivalent up to finite covers to $\mathbf{RP}^n$.

For metrics admitting essential projective deformations, Levi-Civita found normal forms at generic points. Starting from these, Solodovnikov proved the following theorem in 1969 for real-analytic Riemannian manifolds (M^n, g), with $n \geq 3$. The full theorem was proved by Matveev in 2007 [Mat07].

Theorem (Matveev 2007). *Let $H \leq Proj(M^n, [\nabla^g])$, for M a compact, connected Riemannian manifold, $n \geq 2$, and ∇^g the Levi-Civita connection. If H is connected and does not act by affine transformations of ∇^g, then (M^n, g) is isometric to a sphere of constant curvature, up to finite covers.*

The theorem in fact holds for M noncompact and complete. For M compact, Yano proved in 1952 that $\mathrm{Aff}^0(M, \nabla^g) \cong \mathrm{Isom}^0(M, g)$. The hypothesis that H is essential could thus be replaced with noncompactness. We have another great examplar of the VGC.

Matveev's proof uses techniques from integrable systems as well as the crucial fact, presaged in work of Dini and Liouville, that the projective class of a Levi-Civita connection is a finite-dimensional space. Any two Riemannian metrics g and g' are related by a field L of self-adjoint endomorphisms of TM. The

PDEs on L corresponding to $\nabla^g \sim \nabla^{g'}$ are not linear. However, a "weighted" version is: the metric g' defined for $u, v \in T_x M$ by

$$g'_x(u, v) = \frac{1}{\det L_x} g_x(L_x^{-1} u, v)$$

is projectively equivalent to g if and only if for all $u, v \in T_x M$, for all $x \in M$,

$$g_x((\nabla_u^g L)_x v, w) = \frac{1}{2}(g_x(v, u)d\lambda_x(w) + g_x(w, u)d\lambda_x(v))$$

where $\lambda = \mathrm{tr}\, L$. A *BM-structure* is a smooth field of self-adjoint endomorphisms satisfying this equation. This definition was made in 1979 by Sinjukov and was rediscovered by Bolsinov–Matveev in 2003.

In Matveev's Theorem as stated, H must be connected. He gave a counterexample on the 2-torus, based on a classfication due to Dini: Let f be a nonvanishing function on S^1. For

$$g = \left(f(x) - \frac{1}{f(y)}\right)\left(\sqrt{f(x)}dx^2 + \frac{1}{\sqrt{f(y)}}dy^2\right)$$

the involution exchanging x and y is an essential projective transformation. The reformulation of Matveev's Theorem with H assumed noncompact, however, holds even for H disconnected. Zeghib showed that $|\mathrm{Proj}(M, [\nabla^g])/\mathrm{Isom}(M, g)| \leq 2n$, unless M is isometric to a sphere, in 2016; this bound was subsequently improved to 2 by Matveev.

Pseudo-Riemannian Levi-Civita connections

The definition of BM-structure applies as well to pseudo-Riemannian metrics. In this case, the normal forms for the projectively related metrics are more difficult to analyze. Denote $\mathcal{L}_g$ the space of BM-structures for a metric g. Homothetic metrics are affinely equivalent, so $\dim \mathcal{L}_g \geq 1$ in general.

Let (M^n, g) be a semi-Riemannian Einstein space, not necessarily compact, with $n \geq 3$. Kiosak–Matveev proved in 2009 that if M is geodesically complete and admits essential projective transformations then it is Riemannian and isometric to the sphere,

up to finite covers. They also proved that essential projective transformations of M give nonconstant solutions of the following *Gallot-Tanno equation*:

$$(\nabla^3 \sigma)(X, Y, Z) = c\,(2(\nabla_x \sigma) \cdot g(Y, Z)$$
$$+ (\nabla_Y \sigma) \cdot g(X, Z) + (\nabla_Z \sigma) \cdot g(Y, X)),\ c \in \mathbf{R} \quad (6)$$

where $\nabla = \nabla^g$; namely, $\sigma = \operatorname{tr} L$ for L the corresponding BM-structure. Here is a lovely analogy with the divergence of conformal vector fields and (2).

Now let (M, g) be any compact, semi-Riemannian manifold. In 2010, Matveev–Mounoud proved that (6) has nonconstant solutions only if (M, g) is Riemannian and isometric to a sphere, up to finite covers. This fact was established for Riemannian metrics through work of Gallot, Tanno, and Hiramatu. These two results combine to prove the Projective Lichnerowicz Conjecture (PLC) for compact (M, g) when g is Einstein.

Kiosak–Matveev proved in 2010 that if $\dim \mathcal{L}_g \geq 3$, then there are nonconstant solutions of (6); this was previously proved in the Riemannian case by Solodovnikov. Thus the PLC also holds when (M, g) is compact and $\dim \mathcal{L}_g \geq 3$.

Bolsinov, Matveev, and Rosemann developed normal forms for pseudo-Riemannian metrics admitting essential projective vector fields that led to a proof in 2015 of the PLC for closed Lorentzian manifolds. The function $\sigma = \operatorname{tr} L$ and, more generally, the nonconstant eigenvalues of L, again play an important role. The gradients of the latter functions are coordinate vector fields in the normal forms. The conclusion is ultimately reached by showing that σ satisfies (5). They simultaneously prove the Yano-Obata Conjecture for *c-projective structures* on closed manifolds of dimension at least four. This is the Kähler version of metric projective structures and the PLC. Lacking the space for this interesting topic, we refer to the survey [CEMN], and, for definite Kähler metrics, to papers of Apostolov–Calderbank–Gauduchon.

The space $\mathcal{L}_g$

Let (M, g) be a compact pseudo-Riemannian manifold. Assume $\dim \mathcal{L}_g = 2$. The projective group of $[\nabla_g]$ acts linearly on $\mathcal{L}_g$ by a representation ρ_g.

In 2016 Zeghib showed $\operatorname{Aff}(M, \nabla^g) = \operatorname{Isom}(M, g) = \ker \rho_g$, up to finite index subgroups. The quotient $\bar{H} = \operatorname{Proj}(M, [\nabla^g])/\operatorname{Isom}(M, g)$ acts by fractional linear transformations on $\mathbf{P}(\mathcal{L}_g)$, under a suitable identification with $\mathbf{RP}^1$. He verifies this action has finite kernel, and, by a beautiful analysis, proves that $\bar{\rho}_g(\bar{H})$ lies in an elementary subgroup of $\operatorname{PGL}_2(\mathbf{R})$. Therefore, up to finite index and finite quotients, $\bar{H}$ is isomorphic to a subgroup of $\mathbf{R}$.

Given any torsion-free connection ∇, let $\mathcal{L}$ be the space of semi-Riemannian metrics g with $\nabla^g \sim \nabla$. This is the space of solutions of the *metrizability problem* for ∇, which has been studied as early as the 1880s by Liouville. Let $\mathcal{L}_0$ comprise the symmetric 2-tensors $\eta \in \operatorname{Sym}^2 TM$ for which $(\nabla \eta)_0$, the trace-free component of $\nabla \eta$, vanishes. Then $\mathcal{L}$ can be identified with the open set of nondegenerate elements of $\mathcal{L}_0$. If there is one $g \in \mathcal{L}$, then a neighborhood of g of dimension $\dim \mathcal{L}_0$ is contained in $\mathcal{L}$.

Eastwood–Matveev prolonged the equation $(\nabla \eta)_0 = 0$ and found a nice characterization of the solutions. Let $\hat{M}$ be the Cartan bundle, and P, as usual, the stabilizer of a line ℓ in $\mathbf{R}^{n+1}$. The associated vector bundle $\mathcal{V} = \hat{M} \times_P \operatorname{Sym}^2 \mathbf{R}^{n+1}$ is called a *tractor bundle*. The Cartan connection gives a projectively-invariant connection ∇^ω on $\mathcal{V}$. Modifications of ∇^ω by terms involving the Cartan curvature give other meaningful, invariant connections. For one of these, given explicitly by Eastwood–Matveev, the parallel sections are the solutions of the metrizability equation, via a projection corresponding to $\operatorname{Sym}^2(\mathbf{R}^{n+1}/\ell)$. Those satisfying a nondegeneracy condition give metrics in the projective class.

The PLC for Levi-Civita connections in higher signature is believed to be true, but has yet to be proved in the case $\dim \mathcal{L}_g = 2$. For non-metric connections, a wide range of questions on both projective and affine transformations are open.

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