CHARACTERIZING THE EXISTENCE OF A BOREL COMPLETE EXPANSION

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ABSTRACT. We develop general machinery to cast the class of potential canonical Scott sentences of an infinitary sentence φ as a class of structures in a related language. We show that φ has a Borel complete expansion if and only if S_∞ divides $\mathrm{Aut}(M)$ for some countable model $M \models \varphi$. From this, we prove that for theories T_h asserting that $\{E_n\}$ is a countable family of cross cutting equivalence relations with $h(n)$ classes, if $h(n)$ is uniformly bounded, then T_h is not Borel complete, providing a converse to Theorem 2.1 of [LU23].

1. INTRODUCTION

Back and forth systems are an invaluable tool in understanding the descriptive set theoretic complexity of a class of models. Indeed, within the realm of countable structures, back and forth equivalence is the same as isomorphism. As every structure can be made countable in a forcing extension $\mathbb{V}[G]$ of the the set-theoretic universe $\mathbb{V}$, two arbitrary structures in the same vocabulary are back and forth equivalent if and only if they are potentially isomorphic, i.e., they can become isomorphic in some forcing extension.

Classically, given a countable structure M, there is a 'preferred back and forth system' $\mathcal{F}_\infty := \{(\bar{a}, \bar{b}) \in M^{2n} : (M, \bar{a}) \equiv_{\infty,\omega} (M, \bar{b})\}$ and the data of $(M, \mathcal{F}_\infty)$ is coded into a *Scott sentence* $\varphi_M \in L_{\omega_1,\omega}$, whose countable models N are precisely those isomorphic to M. This assignment can be done canonically, giving a canonical object $\mathrm{css}(M)$. Similarly, a structure M of any size has an associated canonical Scott sentence $\mathrm{css}(M)$, which is a sentence of $L_{\infty,\omega}$ that describes the back-and-forth class of M. Extending on this correspondence, given a sentence $\Phi \in L_{\omega_1,\omega}$, we can look at $\mathrm{CSS}(\Phi)_{\mathrm{sat}} := \{\mathrm{css}(M) : M \models \Phi\}$, which may be a proper class, but is bijective with the class $\mathrm{Mod}(\Phi)/\equiv_{\infty,\omega}$. However, for applications, a possibly larger class $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ is useful. This class consists of sentences $\psi \in \mathbb{V}$, $\psi \in L_{\infty,\omega}$, such that in every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$, if $(\psi \in L_{\omega_1,\omega})^{\mathbb{V}[G]}$, then $\psi = \mathrm{css}(N)$

for some $N \models \Phi$ in $\mathbb{V}[G]$. In [URL17] sentences in $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ are called the *potential canonical Scott sentences* of models of Φ.

The number of potential canonical Scott sentences is relevant to the complexity of $(\mathrm{Mod}_\omega(\Phi), \cong)$, the set of models of Φ with universe ω, modulo isomorphism. In [URL17] it is proved that if there is a Borel reduction $(\mathrm{Mod}_\omega(\Phi), \cong) \leq_B (\mathrm{Mod}_\omega(\Psi), \cong)$, then $|\mathrm{CSS}(\Phi)_{\mathrm{ptl}}| \leq |\mathrm{CSS}(\Psi)_{\mathrm{ptl}}|$ (where we interpret these 'cardinalities' to be ∞ if they are proper classes). This is quite useful, although in the above formulation the elements of $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ are simply sentences of $L_{\infty,\omega}$ and are hard to manipulate.

There are two parts to this paper. In the first, we obtain a natural bijection between $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ and a class of structures in an associated language $L^\flat$. In Section 2 we begin by introducing "sharp" back and forth systems and structures and describe a canonical *flattening* operation that associates an $L^\flat$-structure $\mathbb{B} = (M, \mathcal{F})^\flat$ to each sharp structure $(M, \mathcal{F})$. We show that the flattening operation transforms (infinitary) L-formulas φ into (infinitary) $L^\flat$-formulas $\varphi^\flat$ in a natural way. Using this, we distinguish the class of *Hausdorff* flat structures that correspond to flattenings of the 'preferred' sharp structures $(M, \mathcal{F}_\infty)$. In Section 3 we give a list $\mathrm{Ax}^\flat(L)$ of infinitary $L^\flat$-sentences that axiomatize these flattenings. We show that for a countable flat structure $\mathbb{B}$, we can recover a sharp structure $(M, \mathcal{F})$ whose flattening is isomorphic to $\mathbb{B}$. We show that the class of Hausdorff flat structures is absolute between forcing extensions and we define a canonical class of representatives Hdf^* of the $L^\flat$-isomorphism types of Hausdorff flat structures. We see that this class Hdf^* is naturally bijective with $\mathrm{CSS}(L)_{\mathrm{ptl}}$, so we propose to take Hdf^* as an improved representation of potential canonical Scott sentences. All of this relativizes to models of a given $\Phi \in L_{\omega_1,\omega}$, yielding a class $\mathrm{Hdf}^*(\Phi)$ that is naturally bijective with $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$.

The second part of the paper exploits the fact that the class $\mathrm{Flat}(\varphi)$ of flat models of $\varphi^\flat$ is axiomatized by the sentence $\varphi^\flat \wedge \mathrm{Ax}^\flat(L) \in (L^\flat)_{\omega_1,\omega}$. Non-Hausdorff flat models correspond to flattenings of sharp structures $(M, \mathcal{F})$ where $\mathcal{F}$ is something other than $\mathcal{F}_\infty$, which in turn codes the structure of an expansion M^* of M. In Section 4 we make this correspondence precise and then in Section 5, by applying Morley's theorem to the $(L^\flat)_{\omega_1,\omega}$-sentence $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$, we see that the existence of a Borel complete expansion of a model of φ is equivalent to Hjorth's notion of 'S_∞ divides $\mathrm{Aut}(M)$' for some countable $M \models \varphi$.[1] This connection is indirect, in that we prove that each of these properties is equivalent to $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ having arbitrarily large models,

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[1] In [H01], Hjorth proved that $\mathrm{Aut}(M)$ fails the Topological Vaught Conjecture on analytic sets if and only if S_∞ divides $\mathrm{Aut}(M)$.

which in turn is equivalent to this infinitary theory having Ehrenfeucht-Mostowski models. The following result is proved as Theorem 5.5 in Section 5.

Theorem 1.1. *The following are equivalent for a sentence $\varphi \in L_{\omega_1,\omega}$.*

(1) *φ has a Borel complete expansion (i.e., there is some countable $L' \supseteq L$ and $\psi \in (L')_{\omega_1,\omega}$ such that $\psi \vdash \varphi$ and ψ is Borel complete);*
(2) *$\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ has arbitrarily large models;*
(3) *$\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ admits Ehrenfeucht-Mostowski models;*
(4) *S_∞ divides $\mathrm{Aut}(M)$ for some countable $M \models \varphi$.*

In Section 6 we discuss consequences of Theorem 1.1. We get a distinction between the Borel complexity of first order theories and sentences of $L_{\omega_1,\omega}$. In particular, with Corollary 6.1 we see that every first order theory with an infinite model has a Borel complete expansion, whereas there are are sentences of $L_{\omega_1,\omega}$ (even complete ones) that do not. One example of an infinitary sentence without a Borel complete expansion is the sentence φ_h that is used in the proof of Theorem 6.2. There it is proved that the theory of cross-cutting equivalence relations T_h in the language $\{E_n : n \in \omega\}$ with a uniform, finite bound on the number of E_n-classes is not Borel complete. This is in contrast to Theorem 2.1 of [LU23], where it is proved that the analogous theory with unboundedly many classes is Borel complete.

1.1. **Preliminaries.** Following [URL17], let HC denote the set of hereditarily countable sets. We begin by defining a class of formulas whose restrictions to HC are well behaved.

Definition 1.2. Suppose $\varphi(x)$ is any first order formula of set theory, possibly with a hidden parameter from HC.

- $\varphi(\mathrm{HC}) = \{a \in \mathrm{HC} : (\mathrm{HC}, \in) \models \varphi(a)\}$ and if $\mathbb{V}[G]$ is a forcing extension of $\mathbb{V}$, then $\varphi(\mathrm{HC})^{\mathbb{V}[G]} = \{a \in \mathrm{HC}^{\mathbb{V}[G]} : \mathbb{V}[G] \models \text{'}a \in \varphi(\mathrm{HC})\text{'}\}$.
- $\varphi(x)$ is HC-*forcing invariant* if, for every twice-iterated forcing extension $\mathbb{V}[G][G']$,

$$\varphi(\mathrm{HC})^{\mathbb{V}[G][G']} \quad \cap \quad \mathrm{HC}^{\mathbb{V}[G]} \quad = \quad \varphi(\mathrm{HC})^{\mathbb{V}[G]}$$

The reader is cautioned that when computing $\varphi(\mathrm{HC})^{\mathbb{V}[G]}$, the quantifiers of φ range over $\mathrm{HC}^{\mathbb{V}[G]}$ as opposed to the whole of $\mathbb{V}[G]$. The reason for the double iteration of forcing is to make the notion of a formula $\varphi(x)$ being HC-invariant absolute between forcing extensions. Visibly, the class of HC-forcing invariant formulas is closed under boolean combinations, and

it follows from Lévy Absoluteness that every Σ_1 formula of set theory is HC-invariant, see e.g., Lemma 2.2 of [URL17].

The principal idea we wish to exploit is that every set is potentially in HC. That is, for any set $A \in \mathbb{V}$, there is a forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ such that $A \in HC^{\mathbb{V}[G]}$. With this in mind, for any HC-invariant formula $\varphi(x)$, let

$$\varphi_{\mathrm{ptl}} = \{A \in \mathbb{V} : \mathbb{V}[G] \models \varphi(A) \text{ for some/every } \mathbb{V}[G] \supseteq \mathbb{V} \text{ with } A \in HC^{\mathbb{V}[G]}\}$$

Associated to every HC-invariant $\varphi(x)$ is a *strongly definable* family $X = (X^{\mathbb{V}[G]})$, indexed by collection of forcing extensions $\mathbb{V}[G] \supseteq \mathbb{V}$, where each $X^{\mathbb{V}[G]} = \varphi(HC)^{\mathbb{V}[G]}$. We say X is *strongly definable* if it is strongly definable via some HC-invariant formula. For X strongly definable, we write X_{ptl} for φ_{ptl}, where $\varphi(x)$ is any HC-invariant formula defining X. We call X *short* if X_{ptl} is a set, as opposed to a proper class. For a short X, we let $||X|| := |X_{\mathrm{ptl}}|$, and we write $||X|| = \infty$ when X is not short.

If a strongly definable X is defined by a formula $\varphi(x)$ that is absolute between any forcing extensions, then, using the fact that every set $A \in \mathbb{V}$ is in $HC^{\mathbb{V}[G]}$ for some forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$, X_{ptl} is all sets $A \in \mathbb{V}$ such that $\varphi(A)$ holds, i.e., $A \in X^{\mathbb{V}[G]}$ for some forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$. We abuse notation slightly and refer to this class as X.

As examples, for any countable language L and any $\Phi \in L_{\omega_1,\omega}$, both $\mathrm{Mod}_{HC}(L) = \{M \in HC : M \text{ is an } L\text{-structure}\}$ and $\mathrm{Mod}_{HC}(\Phi) = \{M \in \mathrm{Mod}_{HC}(L) : M \models \Phi\}$ are strongly definable. Here, as being an L-structure and satisfaction are absolute, we write $\mathrm{Mod}(L)$ and $\mathrm{Mod}(\Phi)$ in place of the more cumbersome $(\mathrm{Mod}_{HC}(L))_{\mathrm{ptl}}$ or $(\mathrm{Mod}_{HC}(\Phi))_{\mathrm{ptl}}$.

Working with strongly definable sets, we say a notion holds *persistently* if it holds in every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$. As examples, we say $f : X \to Y$ persistently if, for every $\mathbb{V}[G] \supseteq \mathbb{V}$, $f^{\mathbb{V}[G]}$ is a function with domain $X^{\mathbb{V}[G]}$, taking elements in $Y^{\mathbb{V}[G]}$. *Persistently injective* and *persistently bijective* functions are defined similarly. We record the following fact, which appears as Lemma 2.14 of [URL17].

Lemma 1.3. *Suppose f, A, B are strongly definable with $A, B \subseteq HC$ and $f : A \to B$ is persistently a function. Then f_{ptl} is a class functional from A_{ptl} to B_{ptl}.*

If, in addition, $f : A \to B$ is persistently injective (resp. bijective) then f_{ptl} is injective (resp. bijective).

We also include one new notion that was not discussed in [URL17]. For a persistent function $f : A \to B$ as in Lemma 1.3, we say *f has a*

persistent cross section if there is a strongly definable $g : B \to A$ such that, persistently, $(\forall b \in B) f \circ g(b) = b$.

Lemma 1.4. *Suppose f, A, B are strongly definable with $A, B \subseteq HC$ and $f : A \to B$ is persistently a function. If $g : B \to A$ is a persistent cross section of f, then $f_{\mathrm{ptl}} : A_{\mathrm{ptl}} \to B_{\mathrm{ptl}}$ is surjective.*

Proof. Given $b \in B_{\mathrm{ptl}}$, put $a := g_{\mathrm{ptl}}(b)$. Then $f_{\mathrm{ptl}}(a) = b$, so f_{ptl} is surjective. □

Much of Descriptive Set Theory, or at least the study of Borel reducibility, revolves around analyzing the complexity of equivalence relations, and hence quotients. Call a pair (X, E) a *strongly definable quotient* if both X and E are strongly definable and E is persistently an equivalence relation on X. An *HC-reduction* of (X, E) into another strongly definable quotient (Y, F) is a strongly definable $f \subseteq X \times Y$ such that, persistently,

- The E-saturation of $\mathrm{dom}(f)$ is X. That is, for every $x \in X$, there is an $x' \in X$ and $y' \in Y$ where xEx' holds and $(x', y') \in f$.
- f induces a well-defined injection on equivalence classes. That is, if (x, y) and (x', y') are in f, then xEx' holds if and only if yFy' does.

We say (X, E) *is HC-reducible to* (Y, F), written

$$(X, E) \leq_{HC} (Y, F)$$

if an HC-reduction of (X, E) into (Y, F) exists, and we say that (X, E) and (Y, F) are *HC-bireducible* if if (X, E) is HC-reducible to (Y, F) and (Y, F) is HC-reducible to (X, E). As examples, if X, E, Y, F are Borel subsets of Polish spaces then any Borel reduction is an HC-reduction, as is any absolutely Δ^1_2-reduction, see e.g., [URL17].

A *representation* of a strongly definable quotient (X, E) is a pair f, Z of strongly definable families such that $f : X \to Z$ is persistently surjective and persistently,

$$(\forall a, b \in X)[E(a, b) \leftrightarrow f(a) = f(b)]$$

It is easily checked that if f, Z and f', Z' are two representations of the same (X, E), then $||Z|| = ||Z'||$, so provided a representation exists, we define $||(X, E)||$ to be this common value, which may be ∞. We call (X, E) *short* if $||(X, E)|| < \infty$. The following fact is essentially Proposition 2.18 of [URL17].

Fact 1.5. *Suppose (X, E) and (Y, F) are both represented, strongly definable quotients. If $(X, E) \leq_{HC} (Y, F)$, then $||(X, E)|| \leq ||(Y, F)||$.*

Continuing with our example above, for a countable language L, let $\mathrm{Mod}_\omega(L)$ be the Polish space of L-structures with universe ω. By a theorem of Lopez-Escobar, [LE65], the Borel subsets of $\mathrm{Mod}_\omega(L)$, invariant under the action of S_∞, are of the form $\mathrm{Mod}_\omega(\Phi)$ for some $\Phi \in L_{\omega_1,\omega}$. In [FS89], Friedman and Stanley introduced the concept of Borel reducibility between isomorphism classes of this form. For infinitary Φ, Ψ, possibly in different languages, $(\mathrm{Mod}_\omega(\Phi), \cong)$ is *Borel reducible* to $(\mathrm{Mod}_\omega(\Psi), \cong)$ if there is a Borel $f : \mathrm{Mod}_\omega(\Phi) \to \mathrm{Mod}_\omega(\Psi)$ satisfying $M \cong N$ iff $f(M) \cong f(N)$ for all $M, N \in \mathrm{Mod}_\omega(\Phi)$. They showed that among invariant Borel subsets, there is a maximal class, dubbed *Borel complete*, with respect to Borel reducibility. They showed that e.g., the classes of graphs, linear orders, and trees are Borel complete.

Placing this into our context, there is a representation

$$\mathrm{css} : (\mathrm{Mod}_{HC}(L), \cong) \to \mathrm{CSS}(L)$$

where, for each L-structure $M \in HC$, $\mathrm{css}(M)$ is the *canonical Scott sentence of M*, which is a sentence of $L_{\omega_1,\omega}$ (see Definition 3.1 of [URL17] for a formal definition of this map). The salient property of $\mathrm{css}(M)$ is described by

Fact 1.6. *For any countable language L and any $M, N \in \mathrm{Mod}_{HC}(L)$, $N \models \mathrm{css}(M)$ if and only if $N \cong M$.*

That the set $\mathrm{CSS}(L)$ is strongly definable and that css is a persistent HC-invariant surjection follows from Karp's Completeness Theorem for sentences of $L_{\omega_1,\omega}$ (see e.g., Theorem 3 of [Kei71] and Lemma 3.3 of [URL17]).

For any $\Phi \in L_{\omega_1,\omega}$, the restriction of the css representation gives a representation $\mathrm{CSS}(\Phi)$ of $(\mathrm{Mod}_{HC}(\Phi), \cong)$. Thus, $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$, the class of *potential canonical Scott sentences* is the class of all $\psi \in \mathbb{V}$, where $\psi \in L_{\infty,\omega}$, but for some/every $\mathbb{V}[G] \supseteq \mathbb{V}$ where $(\psi \in L_{\omega_1,\omega})^{\mathbb{V}[G]}$, we have $(\psi \in \mathrm{CSS}(\Phi))^{\mathbb{V}[G]}$.

Note that $\mathrm{CSS}(\Phi)$ is also a representation of $(\mathrm{Mod}_\omega(\Phi), \cong)$, the isomorphism classes of models of Φ with universe ω. We write $||\Phi||$ in place of $||(\mathrm{Mod}_{HC}(\Phi), \cong)|| = |\mathrm{CSS}(\Phi)_{\mathrm{ptl}}|$ (which may be ∞). If $(\mathrm{Mod}_\omega(\Phi), \cong)$ is Borel complete, then Φ is not short, i.e., $||\Phi|| = \infty$. This follows from the fact that there is a proper class of back-and-forth inequivalent graphs.

Note, however, that for some Φ, the class $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ can be strictly larger than $\mathrm{CSS}(\Phi)_{\mathrm{sat}}$, the latter being the class of all $\mathrm{css}(M)$ for $M \in \mathrm{Mod}(\Phi)$. When these two classes are not equal, we say that Φ *is not grounded*. One of the central goals of Section 3 is to get better control of the class of potential canonical Scott sentences of Φ when Φ is not grounded.

2. Back and forth systems, sharp expansions and their flattenings

Fix, for the whole of this section, a countable language L. We begin with the well known definition of a back and forth system, but then we add additional adjectives.

Definition 2.1. Given an arbitrary L-structure M, a (finitary) *back-and-forth system on M* is a non-empty set $\mathcal{F}$ of pairs $(\bar{a}, \bar{b})$ of finite tuples from M with $\lg(\bar{a}) = \lg(\bar{b})$ satisfying:

(1) For $(\bar{a}, \bar{b}) \in \mathcal{F}$, the map $\bar{a} \mapsto \bar{b}$ is q.f.-L-elementary (i.e., for any atomic L-formula $\alpha(\bar{x})$, $M \models \alpha(\bar{a}) \leftrightarrow \alpha(\bar{b})$); and

(2) For all $(\bar{a}, \bar{b}) \in \mathcal{F}$ and singleton $c \in M$, there is $d \in M$ such that $(\bar{a}c, \bar{b}d) \in \mathcal{F}$; and dually for every $d' \in M$ there is $c' \in M$ such that $(\bar{a}c', \bar{b}d') \in \mathcal{F}$.

For an L-structure M of any size, we define the *subsequence maps* as follows. For each $n \in \omega$, all $k \leq n$ and all injective $f : k \to n$, f induces a mapping from $M^n \to M^k$ defined as: for every $\bar{a} = (a_0, \ldots, a_{n-1}) \in M^n$, $\bar{a}\restriction_f$ is the subsequence $(a_{f(0)}, \ldots, a_{f(k-1)})$ of $\bar{a}$. When $k = 0$, we identify M^0 with the empty sequence $\langle \rangle$. Thus, for every $n \in \omega$, taking $f : 0 \to n$ gives $\bar{a}\restriction_f = \langle \rangle$ for every $\bar{a} \in M^n$. As a special case, when $k = n$, $f \in \mathrm{Sym}(\{0, \ldots, n-1\})$ is a permutation and $\bar{a}\restriction_f$ is the associated permutation of $\bar{a}$.

Definition 2.2. Let $\mathcal{F}$ be any back and forth system on an L-structure M.

- We say $\mathcal{F}$ is *downward closed* if, for all $k \leq n$ and all injective $f : k \to n$, whenever $(\bar{a}, \bar{b}) \in \mathcal{F} \cap M^{2n}$, then $(\bar{a}\restriction_f, \bar{b}\restriction_f) \in \mathcal{F}$ as well.
- $\mathcal{F}$ is a *sharp* back and forth system on M if $\mathcal{F}$ is both downward closed and, for every $k \in \omega$, $\mathcal{F} \cap M^{2k}$ is an equivalence relation E_k on M^k for every k. [When $k = 0$, E_0 is the trivial equivalence relation on $M^0 = \{\langle \rangle\}$.]

It is readily verified that if $\mathcal{F}$ is a back-and-forth system on M, then so is its *downward closure* $dc(\mathcal{F}) = \{(\bar{a}\restriction_f, \bar{b}\restriction_f) : (\bar{a}, \bar{b}) \in \mathcal{F}, f : k \to n \text{ injective}\}$. As well, if for every $k \in \omega$ we let E_k be the symmetric and transitive closure of $(dc(\mathcal{F}) \cap M^{2k}) \cup \{(\bar{a}, \bar{a}) : \bar{a} \in M^k\}$, then the union of the set of equivalence relations $\{E_k : k \in \omega\}$ is the smallest sharp back-and-forth system containing $\mathcal{F}$. For the remainder of this paper, we will only consider sharp back-and-forth systems.

We formalize the operations described by expanding the language.

Definition 2.3. For a given countable language L, let $L^\sharp = L \cup \{E_k : k \in \omega\}$, where each E_k is a new $2k$-ary relation symbol. A *sharp structure* $(M, \mathcal{F})$ is an $L^\sharp$ expansion of an L-structure M by interpreting each E_k as $\mathcal{F} \cap M^{2k}$ for some distinguished sharp back-and-forth system $\mathcal{F}$. The class of sharp structures is (first order) elementary, so fix an axiomatization $\mathrm{Ax}^\sharp(L)$ of this class.

Regardless of the size of M, there is a unique largest sharp back and forth system on M, which we denote by $\mathcal{F}_\infty$ and we let $M^\sharp$ denote the canonical sharp expansion $(M, \mathcal{F}_\infty)$. To describe $\mathcal{F}_\infty$ we recall Karp's notion of back-and-forth equivalence (equivalently, $L_{\infty,\omega}$-elementary equivalence) of structures.

Definition 2.4. Fix an L-structure M. For tuples $\bar{a}, \bar{b}$ of the same length, we say $(M, \bar{a})$ *and* $(M, \bar{b})$ *are back-and-forth equivalent* if there is some back-and-system $\mathcal{F}$ containing $(\bar{a}, \bar{b})$. We denote this as $(M, \bar{a}) \equiv_{\infty,\omega} (M, \bar{b})$. This introduces an equivalence relation on each n and we say $\mathrm{tp}_\infty(\bar{a}) = \mathrm{tp}_\infty(\bar{b})$ if $(M, \bar{a})$ and $(M, \bar{b})$ are back-and-forth equivalent.

Back and forth equivalence is the same as $L_{\infty,\omega}$-elementary equivalence, so $\mathrm{tp}_\infty(\bar{a})$ is a set of $L_{\infty,\omega}$-formulas. Moreover, $\mathrm{tp}_\infty(\bar{a})$ is isolated by a single $L_{\infty,\omega}$-formula, namely the $L_{\infty,\omega}$-Scott sentence of $(M, \bar{a})$. It is straightforward to check that

$$\mathcal{F}_\infty := \{(\bar{a}, \bar{b}) : \mathrm{tp}_\infty(\bar{a}) = \mathrm{tp}_\infty(\bar{b})\}$$

is a sharp back-and-forth system, and in fact, $\mathcal{F}_\infty$ contains every sharp back-and-forth system on M.

Another source of examples is found via expansions. Suppose $L^* \supseteq L$ and M^* is an expansion of M. Define $\mathcal{F}^* := \{(\bar{a}, \bar{b}) : (M^*, \bar{a}) \equiv_{\infty,\omega} (M^*, \bar{b})\}$. Then $(M, \mathcal{F}^*)$ is also a sharp expansion of M and $\mathcal{F}^* \subseteq \mathcal{F}_\infty$. In fact, every sharp expansion $(M, \mathcal{F})$ corresponds to an expansion M^* of M, formed by adding a new k-ary predicate for every E_k-class in $\mathcal{F}$. Then $\mathcal{F}$ is the largest sharp back and forth system on M^*.

2.1. **Flattenings of sharp structures.** Here we introduce the central notion of this paper.

Given a sharp structure $(M, \mathcal{F})$ in the language $L^\sharp = L \cup \{E_n : n \in \omega\}$, it is natural to form a 'restricted eq-expansion' by adding new sorts for each of the $2n$-ary equivalence relations E_n. That is, for each n, add a new unary predicate symbol $U_n(z)$ and an n-ary function symbol π_n and let $(M, \mathcal{F})^{eq}$ be

formed by interpreting each predicate U_n as being a name for the E_n-classes, and $\pi_n : M^n \to U_n$ is the canonical projection, i.e., $\pi_n(\bar{a}) = \bar{a}/E_n$.

Note that the subsequence maps $\bar{a} \mapsto \bar{a}\!\restriction_f$ are definable in this language, so we also get induced (unary!) projection maps $P^f_{k,n} : U_n \to U_k$ indexed by injections $f : k \to n$, and these commute with the projection maps π_n, i.e., if $f : k \leq n$ is any injection, then for any $\bar{a} \in M^n$, $P^f_{k,n}\pi_n(\bar{a}) = \pi_k(\bar{a}\!\restriction_f)$.

We now define the *flattening of* $(M, \mathcal{F})$, which naively can be described as the result of passing to the *eq*-structure described above, and then 'forgetting the home sort.' In what follows, L-formulas $\varphi(x_0, \ldots, x_{n-1})$ (either finitary or infinitary) should be thought of as codes for subsets of M^n. Part of the data of $\varphi(x_0, \ldots, x_{n-1})$ includes the n, hence for the following definitions, L-formulas should not be thought of as being closed under adjunction of dummy variables. By contrast, in the $L^\flat$ language defined below, each U_n is unary, with an element of $U_n^\mathbb{B}$ of a flat structure coding an equivalence class of n-tuples from some L-structure. More formally,

Definition 2.5. For any language L, let $\mathrm{QF}_L = \bigcup\{\mathrm{QF}_n : n \in \omega\}$, where QF_n is the set of all (finitary) quantifier-free formulas $\alpha_n(x_0, \ldots, x_{n-1})$, such that every free variable is among $\{x_0, \ldots, x_{n-1}\}$. [Every quantifier-free formula α appears in infinitely many QF_n.] Let

$$L^\flat := \{U_n(z) : n \in \omega\} \cup \{\alpha_n^\flat(z) : \alpha_n \in \mathrm{QF}_n, n \in \omega\} \cup$$

$$\cup \{P^f_{k,n} : f : k \to n \text{ an injection}\} \cup \{\mathbf{c}_{\langle\rangle}\}$$

where each U_n and $\alpha_n^\flat$ are unary predicates, $P^f_{k,n} : U_n \to U_k$ are unary function symbols, and $\mathbf{c}_{\langle\rangle}$ is a constant symbol.

Given any sharp structure $(M, \mathcal{F})$, its *flattening* is the $L^\flat$-structure $\mathbb{B} = (M, \mathcal{F})^\flat$ with universe $\bigcup_{n \in \omega} M^n/E_n$, where each $U_n^\mathbb{B} = M^n/E_n$, $\alpha_n^\flat(\mathbb{B}) = \{\bar{a}/E_n \in U_n : M \models \alpha_n(\bar{a})\}$, each $P^f_{k,n} : U_n^\mathbb{B} \to U_k^\mathbb{B}$ given by $P^f_{k,n}(\bar{a}/E_n) = \bar{a}\!\restriction_f/E_k$, and $\mathbf{c}_{\langle\rangle}$ is the unique element of M^0/E_0.

In the above, the number of free variables of α_n is critical to its flattening. In particular, every $\varphi_n \in \mathrm{QF}_n$ has $\alpha_n^\flat \vdash U_n$, whereas flattenings of elements of QF_{n+1} entail U_{n+1}. L-sentences have flattenings in U_0. Although being part of the signature, we abuse notation slightly and take the domain of $P^f_{k,n}$ inside any $L^\flat$-structure $\mathbb{B}$ to be $U_n^\mathbb{B}$. In any flattening, $\alpha_n^\flat(\mathbb{B})$ and $P^f_{k,n}$ are well defined because $\mathcal{F}$ is a sharp back-and-forth system on M. In particular, if $E_n(\bar{a}, \bar{b})$ holds, then $\mathrm{qftp}_M(\bar{a}) = \mathrm{qftp}_M(\bar{b})$. Note also that $\mathbf{c}_{\langle\rangle}$ is the unique element of $U_0^\mathbb{B}$ and for every quantifier-free L-sentence σ, $\mathbb{B} \models \sigma_0^\flat(\mathbf{c}_{\langle\rangle})$ if and only if $M \models \sigma$.

The correspondence $\alpha \mapsto \alpha^\flat$ extends naturally to arbitrary L-formulas, and even to $L_{\infty,\omega}$-formulas.

Definition 2.6. For every formula $\varphi_n(x_0, \ldots, x_{n-1})$ of $L_{\infty,\omega}$, we recursively define an $(L^\flat)_{\infty,\omega}$-formula $\varphi^\flat(z)$ as follows:

- For $\alpha_n(x_0, \ldots, x_{n-1}) \in \mathrm{QF}_n$, let $\alpha_n^\flat$ be the distinguished $L^\flat$ predicate symbol;
- For formulas $\{\varphi_i(x_0, \ldots, x_{n-1}) : i \in I\}$, $(\bigwedge_{i \in I} \varphi_i)^\flat := \bigwedge_{i \in I} \varphi_i^\flat$;
- $(\neg\varphi)^\flat := \neg(\varphi^\flat)$; and
- $(\exists y \theta(x_0, \ldots, x_{n-1}, y))^\flat := \exists w(U_{n+1}(w) \wedge \theta^\flat(w) \wedge P_{n,n+1}^{id}(w) = z)$.

Let $\mathrm{Fm}^\flat := \{\varphi^\flat(z) : \varphi \in L_{\infty,\omega}\}$, which is a subset of $(L^\flat)_{\infty,\omega}$.
For an $L^\flat$-structure $\mathbb{B}$, $n \in \omega$, and $\mathbf{a} \in U_n^{\mathbb{B}}$, let

$$\flat\text{-tp}_{\mathbb{B}}(\mathbf{a}) := \{U_n(z)\} \cup \{\varphi^\flat(z) \in \mathrm{Fm}^\flat : \mathbb{B} \models \varphi^\flat(\mathbf{a})\}$$

Note that every $L^\flat$-formula (finitary or infinitary) in $\mathrm{Fm}^\flat$ has at most one free variable. If $\varphi^\flat(z) \vdash U_n(z)$ for some $n \geq 1$, it has exactly one free variable. For $n = 0$, since $U_0^{\mathbb{B}} = \{\mathbf{c}_{\langle\rangle}\}$, for any (infinitary) L-sentence σ we identify the (infinitary) $L^\flat$-sentences $\sigma_0^\flat(\mathbf{c}_{\langle\rangle})$ and $\exists z \sigma_0^\flat(z)$. For brevity, we write $\sigma^\flat$ in place of $\sigma^\flat(\mathbf{c}_{\langle\rangle})$ for the flattening of σ. The following Lemma is straightforward.

Lemma 2.7. *For all n and all $\varphi(x_0, \ldots, x_{n-1}) \in L_{\infty,\omega}$, for all L-structures M and for all sharp back-and-forth systems $\mathcal{F} = \{E_n : n \in \omega\}$ on M we have (letting $\mathbb{B}$ denote the flattening $(M, \mathcal{F})^\flat$) for all $\bar{a} \in M^n$:*

$$M \models \varphi(\bar{a}) \quad \text{if and only if} \quad \mathbb{B} \models \varphi^\flat(\bar{a}/E_n)$$

Proof. We prove this by induction on the complexity of φ. For quantifier-free L-formulas, this correspondence is built into the definition of a flattening, and it is routine to see this correspondence is preserved under (infinitary) boolean combinations. To see that it is preserved under quantification, assume the Lemma holds for $\theta(\bar{x}, y)$ with $\lg(\bar{x}) = n$. Fix any $(M, \mathcal{F})$ and choose any $\bar{a} \in M^n$. Let $\mathbb{B}$ denote the flattening and let $\mathbf{a} = \bar{a}/E_n$. On one hand, if $M \models \exists y \theta(\bar{a}, y)$, choose $b \in M$ such that $M \models \theta(\bar{a}, b)$ and let $\mathbf{b} = \bar{a}b/E_{n+1} \in U_{n+1}$. As our inductive hypothesis gives $\mathbb{B} \models \theta^\flat(\mathbf{b})$, we conclude $\mathbb{B} \models \varphi^\flat(\mathbf{a})$. Conversely, choose $\mathbf{b} \in U_{n+1}$ witnessing $\mathbb{B} \models \varphi^\flat(\mathbf{a})$, i.e., $\mathbb{B} \models \theta^\flat(\mathbf{b}) \wedge P_{n,n+1}^{id}(\mathbf{b}) = \mathbf{a}$. As $\mathbb{B} \models \theta^\flat(\mathbf{b})$, by induction there is some $\bar{b} \in M^n$ and $b^* \in M$ such that $\bar{b}b^*/E_{n+1} = \mathbf{b}$ and $M \models \theta(\bar{b}, b^*)$. It follows from the interpretation of $P_{n,n+1}^{id}$ in $\mathbb{B}$ that $\bar{b}/E_n = \mathbf{a}$, which is also equal to $\bar{a}/E_n$. Thus, $(\bar{a}, \bar{b}) \in \mathcal{F}$. As $\mathcal{F}$ is a back-and-forth system, there is $a^* \in M$ such that $(\bar{a}a^*, \bar{b}b^*) \in \mathcal{F}$, so $\bar{a}a^*/E_{n+1} = \bar{b}b^*/E_{n+1} = \mathbf{b}$. By the inductive

hypothesis applied to $\theta(\overline{x}, y)$, we conclude $M \models \theta(\bar{a}, a^*)$, i.e., $M \models \varphi(\bar{a})$, as required. $\qquad\square$

Definition 2.8. An $L^\flat$-structure $\mathbb{B}$ is *Hausdorff* if $\mathrm{Fm}^\flat$ separates points, i.e., distinct elements from $\mathbb{B}$ have distinct $\flat$-types. That is, $\flat\text{-tp}_\mathbb{B}(\mathbf{a}) = \flat\text{-tp}_\mathbb{B}(\mathbf{b})$ implies $\mathbf{a} = \mathbf{b}$.

Lemma 2.9. *For any L-structure M, the flattening of $M^\flat$, the canonical expansion $M^\sharp = (M, \mathcal{F}_\infty)$, is Hausdorff. Conversely, if $(M, \mathcal{F})$ is any sharp expansion of M whose flattening is Hausdorff, then $\mathcal{F} = \mathcal{F}_\infty$.*

Proof. Let $\mathbb{B}$ be the flattening of the canonical $M^\sharp$. To see that $\mathbb{B}$ is Hausdorff, choose any $\mathbf{a} \neq \mathbf{b}$ from $\mathbb{B}$. If $\mathbf{a}, \mathbf{b}$ are in different U_n's they are clearly separated, so assume $\mathbb{B} \models U_n(\mathbf{a}) \wedge U_n(\mathbf{b})$. By the definition of the flattening, choose $\bar{a}, \bar{b} \in M^n$ such that $\bar{a}/E_n = \mathbf{a}$ and $\bar{b}/E_n = \mathbf{b}$. As $\mathbf{a} \neq \mathbf{b}$, $(\bar{a}, \bar{b}) \notin \mathcal{F}_\infty$, so $\text{tp}_\infty(\bar{a}) \neq \text{tp}_\infty(\bar{b})$. Choose any infinitary $\varphi(\overline{x}) \in \text{tp}_\infty(\bar{a}) \setminus \text{tp}_\infty(\bar{b})$. Then $\mathbb{B} \models \varphi^\flat(\mathbf{a}) \wedge \neg\varphi^\flat(\mathbf{b})$, so $\mathbb{B}$ is Hausdorff.

For the converse, recall that for any back-and-forth system $\mathcal{F}$ on M, if $(\bar{a}, \bar{b}) \in \mathcal{F}$, then $\text{tp}_\infty(\bar{a}) = \text{tp}_\infty(\bar{b})$, hence $\mathcal{F} \subseteq \mathcal{F}_\infty$. Now assume $\mathcal{F}$ is any sharp back-and-forth system that is not equal to $\mathcal{F}_\infty$, and we show that its flattening $\mathbb{B}$ cannot be Hausdorff. Since $\mathcal{F}_\infty \not\subseteq \mathcal{F}$, there is some n and $\bar{a}, \bar{b} \in M^n$ with $(\bar{a}, \bar{b}) \in \mathcal{F}_\infty \setminus \mathcal{F}$, i.e., $\text{tp}_\infty(\bar{a}) = \text{tp}_\infty(\bar{b})$, but $\mathbf{a} \neq \mathbf{b}$, where $E_n = \mathcal{F} \cap M^{2n}$, $\mathbf{a} = \bar{a}/E_n$ and $\mathbf{b} = \bar{b}/E_n$. Thus, $\mathbf{a}, \mathbf{b} \in \mathbb{B}$ and for every $\varphi(\overline{x}) \in L_{\infty,\omega}$, $\mathbb{B} \models \varphi^\flat(\mathbf{a}) \leftrightarrow \varphi^\flat(\mathbf{b})$. So $\mathbf{a}$ and $\mathbf{b}$ witness that $\mathbb{B}$ is not Hausdorff. $\qquad\square$

3. Flat structures, the inverse map, and potential canonical Scott sentences

3.1. **Axiomatizing flat structures.** We formalize the notion of a flat structure $\mathbb{B}$. We will see that every flattening of a sharp expansion is a flat structure, and conversely, every *countable* $\mathbb{B}$ is the flattening of some sharp expansion of some countable L-structure. Throughout, we have a fixed (countable) language L, along with the associated language $L^\flat$.

As notation, for any $L^\flat$-structure $\mathbb{B}$ and any $\mathbf{b} \in U_n$, let

$$\Delta_\mathbf{b} := \{\text{quantifier-free } \beta(x_0, \ldots, x_{n-1}) : \mathbb{B} \models \beta^\flat(\mathbf{b})\}$$

In particular, $\Delta_\mathbf{b}$ is a set of finitary L-formulas, all of whose free variables are among $\{x_0, \ldots, x_{n-1}\}$. The definition below lists several conditions that hold of any flattening of a sharp structure $(M, \mathcal{F})$.

Definition 3.1. Let $\mathrm{Ax}^\flat(L)$ denote the following $L^\flat$ axioms,[2] and call an $L^\flat$-structure $\mathbb{B}$ a *flat structure* if $\mathbb{B} \models \mathrm{Ax}^\flat(L)$.

(1) Structural axioms:
   (a) The unary predicates $\{U_n : n \in \omega\}$ partition the universe.
   (b) $U_0 = \{\mathbf{c}_{\langle\rangle}\}$ is a singleton.
   (c) For all $k \leq n$ and injective $f : k \to n$, $P_{k,n}^f$ is a (unary) function: $U_n \to U_k$.
   (d) For all $n \in \omega$, $P_{n,n}^{id} : U_n \to U_n$ (where $id : n \to n$ is the identity) is the identity function.
   (e) For all $n \in \omega$ and all $\mathbf{b} \in U_n$, $\Delta_{\mathbf{b}}$ is a complete quantifier-free type in the variables $(x_0, \ldots, x_{n-1})$ (i.e., there is some L-structure M and some $\bar{a} \in M^n$ such that $M \models \beta(\bar{a})$ for all $\beta \in \Delta_{\mathbf{b}}$ and, for every quantifier-free $\beta(x_0, \ldots, x_{n-1})$, exactly one of $\beta, \neg\beta$ is contained in $\Delta_{\mathbf{b}}$.)

(2) Relational axioms:
   (a) Composition: For all injective $f : k \to n$ and $g : n \to m$, and all $\mathbf{a} \in U_m$, $P^{g \circ f}(\mathbf{a}) = P_{k,n}^f(P_{n,m}^g(\mathbf{a}))$.
   (b) For all injective $f : k \to n$, all $\mathbf{a} \in U_n$, and all quantifier-free $\alpha_k(x_0, \ldots, x_{k-1})$,

$$\alpha_k(x_0, \ldots, x_{k-1}) \in \Delta_{P_{k,n}^f(\mathbf{a})} \quad \text{if and only if} \quad \alpha_k(x_{f(0)}, \ldots, x_{f(k-1)}) \in \Delta_{\mathbf{a}}$$

(3) Equality Axiom: For all $k \leq n$, all injective $f, g : k \to n$ and all $\mathbf{a} \in U_n$, if $(\bigwedge_{i \in k} x_{f(i)} = x_{g(i)}) \in \Delta_{\mathbf{a}}$, then $P_{k,n}^f(\mathbf{a}) = P_{k,n}^g(\mathbf{a})$.

As notation: If $\mathbf{a} \in U_k$ and $\mathbf{b} \in U_n$, $\mathbf{a} \leq \mathbf{b}$ denotes $\mathbf{a} = P_{k,n}^{id}(\mathbf{b})$ (so $k \leq n$).

(4) Amalgamation: For all $k \leq n, m$, if $\mathbf{a} \in U_k$, $\mathbf{b} \in U_n$, $\mathbf{c} \in U_m$, $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \leq \mathbf{c}$, then there is $\mathbf{d} \in U_{n+m-k}$ such that $\mathbf{b} \leq \mathbf{d}$ and $P^v(\mathbf{d}) = \mathbf{c}$, where $v : m \to n + (m - k)$ is defined by $v(i) = i$ for all $i < k$ and $v(i) = n + (i - k)$ for all $k \leq i < m$.
(5) Duplication: $\mathbb{B} \models \sigma^\flat$, where σ is the L-sentence $\forall x_0 \exists x_1 (x_0 = x_1)$, i.e., for every $\mathbf{a} \in U_1$, there is $\mathbf{b} \in U_2$ such that $P^{f_0}(\mathbf{b}) = P^{f_1}(\mathbf{b}) = \mathbf{a}$ for the two functions $f_0, f_1 : 1 \to 2$ and $(x_0 = x_1) \in \Delta_{\mathbf{b}}$.
(6) Constants: For all constant symbols $c \in L$, $\mathbb{B} \models \sigma_c^\flat$, where σ_c is the L-sentence $\exists x_0(x_0 = c)$.
(7) Functions: For all k-ary function symbols $f \in L$, $\mathbb{B} \models \sigma_f^\flat$, where σ_f is the L-sentence $\forall x_0 \ldots \forall x_{k-1} \exists x_k (x_k = f(x_0, \ldots, x_{k-1}))$.

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[2] All but 1a) are first order axioms.

We remark that without additional conditions, the element $\mathbf{d}$ obtained by Amalgamation is not uniquely described.

3.2. **Generalized subsequences.** In preparation of Proposition 3.7, where we recover an L-structure M from a flat structure $\mathbb{B}$, we must show that elements of $\mathbb{B}$ encode tuples from M with duplicate values. We begin with a one-step lemma, whose proof follows immediately from Amalgamation, Duplication and Axiom 2b.

Lemma 3.2. *Let $\mathbb{B}$ be any flat structure. Suppose $m \geq 1$, $j \in m$, and $\mathbf{a} \in U_m$. There is $\mathbf{c} \in U_{m+1}$ such that $\mathbf{a} \leq \mathbf{c}$ and $(x_j = x_m) \in \Delta_{\mathbf{c}}$.*

By iterating Lemma 3.2 n times, we get the following lemma.

Lemma 3.3. *Suppose $f^* : n \to m$ is any function (not necessarily injective). For any flat structure $\mathbb{B}$ and any $\mathbf{a} \in U_m$, there is a unique $\mathbf{c} \in U_{m+n}$ such that $\mathbf{a} \leq \mathbf{c}$ and $(\bigwedge_{i \in n} x_{f^*(i)} = x_{m+i}) \in \Delta_{\mathbf{c}}$.*

Proof. The existence of such a $\mathbf{c}$ follows by iterating Lemma 3.2 n times. As for uniqueness, suppose $\mathbf{c}_1, \mathbf{c}_2 \in U_{m+n}$ both satisfy these conditions. Let $\mathbf{d} \in U_{m+2n}$ be an amalgamation of $\mathbf{c}_1$ and $\mathbf{c}_2$ over $\mathbf{a}$. In particular, $\mathbf{c}_1 \leq \mathbf{d}$, and $\mathbf{c}_2 = P^g_{m+n,m+2n}$, where g is defined as $g(i) = i$ for all $i < m$ and $g(i) = i + n$ if $m \leq i < m + n$. Because of the Equality Axiom, to conclude that $\mathbf{c}_1 = \mathbf{c}_2$, it suffices to show that $(x_{id(i)} = x_{g(i)}) \in \Delta_{\mathbf{d}}$ for all $i < m + n$. We verify this by cases. For $i < m$, since $\mathbf{a} \leq \mathbf{c}_2$ we have $g(i) = i$, hence $(x_i = x_{g(i)}) \in \Delta_{\mathbf{d}}$ trivially. For $m \leq i < m + n$, let $j = i - m$. By hypothesis, since $i = m + j$, $(x_{f^*(j)} = x_i)$ is in both $\Delta_{\mathbf{c}_1}$ and $\Delta_{\mathbf{c}_2}$. Since $\mathbf{c}_1 \leq \mathbf{d}$, we have $(x_{f^*(j)} = x_i) \in \Delta_{\mathbf{d}}$ as well. But $P^g_{m+n,m+2n}(\mathbf{d}) = \mathbf{c}_2$ implies $(x_{g(f^*(j))} = x_{g(i)}) \in \Delta_{\mathbf{d}}$. Because $f^*(j) < m$, $g(f^*(j)) = f^*(j)$, so $(x_{f^*(j)} = x_{g(i)}) \in \Delta_{\mathbf{d}}$. As $\Delta_{\mathbf{d}}$ is a complete q.f. type, we conclude $(x_i = x_{g(i)}) \in \Delta_{\mathbf{d}}$. Thus, as noted above, we conclude $\mathbf{c}_1 = \mathbf{c}_2$ by the Equality Axiom. $\square$

In light of this uniqueness result, we make the following definitions. For the remainder of this subsection, we assume $\mathbb{B}$ is a fixed flat structure, and we work inside $\mathbb{B}$.

Definition 3.4. Suppose $f^* : n \to m$ is any function and $\mathbf{a} \in U_m$.

The f^*-*blowup of* $\mathbf{a}$ is the (unique) $\mathbf{c} \in U_{m+n}$ satisfying $\mathbf{a} \leq \mathbf{c}$ and $(\bigwedge_{i \in n} x_{f^*(i)} = x_{m+i}) \in \mathbf{c}$.

For any integer ℓ, the (injective) ℓ-*shift function* $v_\ell : n \to n + \ell$ is defined by $v_\ell(i) = \ell + i$.

The *generalized projection* $P^{f^*}(\mathbf{a})$ is the element $\mathbf{b} \in U_n$ satisfying $\mathbf{b} = P^{v_m}_{n,m+n}(\mathbf{c})$, where $\mathbf{c}$ is the f^*-blowup of $\mathbf{a}$ and $v_m : n \to m+n$ is the m-shift function.

We show that these generalized projections satisfy statements analogous to the axioms for (hardwired) projections P^f (for injective functions f). We begin with a Generalized Equality criterion.

Lemma 3.5. *Suppose* $f^*, g^* : n \to m$ *and* $\mathbf{a} \in U_m$ *satisfies* $(\bigwedge_{i \in n} x_{f^*(i)} = x_{g^*(i)}) \in \Delta_{\mathbf{a}}$. *Then* $P^{f^*}(\mathbf{a}) = P^{g^*}(\mathbf{a})$.

Proof. Let $\mathbf{c} \geq \mathbf{a}$ denote the f^*-blowup of $\mathbf{a}$. As $f^*(i) = g^*(i)$ for each $i \in n$, it follows from Lemma 3.3 that $\mathbf{c}$ is also the g^*-blowup of $\mathbf{a}$. Thus, by definition, $P^{f^*}(\mathbf{a}) = P^v_{n,m+n}(\mathbf{c}) = P^{g^*}(\mathbf{a})$. $\square$

To show that Generalized Composition holds, we introduce some more notation. Given a sequence $\langle \mathbf{a}_0, \ldots, \mathbf{a}_{\ell-1} \rangle \in \mathbb{B}^\ell$ with $\mathbf{a}_i \in U_{k_i}$, let $s_i = \sum_{j \leq i} k_j$ and let $s = s_{\ell-1}$. For each $i < \ell$, let $f_i : k_i \to s$ be the s_{i-1}-shift function defined by $f_i(j) = s_{i-1} + j$ (where we take $s_{-1} = 0$). A *joint embedding* of $\langle \mathbf{a}_0, \ldots, \mathbf{a}_{\ell-1} \rangle$ is any $\mathbf{d} \in U_s$ such that $P^{f_i}(\mathbf{d}) = \mathbf{a}_i$ for every $i \leq \ell$. $\mathbf{d}$ is not uniquely determined.

It is easily checked that for any sequence $\langle \mathbf{a}_0, \ldots, \mathbf{a}_{\ell-1} \rangle$ and any joint embedding $\mathbf{d} \in U_s$, $\mathbf{a}_0 \leq \mathbf{d}$ and, letting $v_{k_0} : (s - k_0) \to s$ be the k_0-shift function, the 'truncation' $P^{v_{k_0}}(\mathbf{d})$ is a joint embedding of $\langle \mathbf{a}_1, \ldots, \mathbf{a}_{\ell-1} \rangle$. Given this, Generalized Composition follows easily.

Lemma 3.6. *Suppose* $f^* : k \to n$ *and* $g^* : n \to m$ *are arbitrary functions. For any* $\mathbf{a} \in U_m$ *we have* $P^{g^* f^*}(\mathbf{a}) = P^{f^*}(P^{g^*}(\mathbf{a}))$.

Proof. Let $\mathbf{b} \in U_{m+n}$ be the g^*-blowup of $\mathbf{a}$, let $\mathbf{c} \in U_{m+k}$ be the $g^* f^*$-blowup of $\mathbf{a}$, and, as $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \leq \mathbf{c}$, let $\mathbf{d}$ be an amalgamation of $\mathbf{b}$ and $\mathbf{c}$ over $\mathbf{a}$. [In fact, $\mathbf{d}$ is uniquely determined, but we don't need this.] Then $\mathbf{d}$ is a joint embedding of $\langle \mathbf{a}, P^{g^*}(\mathbf{a}), P^{g^* f^*}(\mathbf{a}) \rangle$. Let $\mathbf{e} = P^{v_m}(\mathbf{d})$ be the truncation of $\mathbf{d}$; so by the preceding remarks, $\mathbf{e}$ is a joint embedding of $\langle P^{g^*}(\mathbf{a}), P^{g^* f^*}(\mathbf{a}) \rangle$. We claim that $\mathbf{e}$ is the f^*-blowup of $P^{g^*}(\mathbf{a})$. This observation completes the proof, since the f^*-blowup of $P^{g^*}(\mathbf{a})$ is a joint embedding of $\langle P^{g^*}(\mathbf{a}), P^{f^*}(P^{g^*}(\mathbf{a})) \rangle$; and hence, by comparing truncations, we get $P^{g^* f^*}(\mathbf{a}) = P^{f^*}(P^{g^*}(\mathbf{a}))$.

To prove the claim, since $\mathbf{e}$ is a joint embedding, $P^{g^*}(\mathbf{a}) \leq \mathbf{e}$. It remains to show that $(x_{f^*(i)} = x_{n+i}) \in \Delta_{\mathbf{e}}$ for all $i < k$. Fix $i < k$ and let $j = f^*(i) \in n$. Now, in $\mathbf{b}$, $(x_{g^*(j)} = x_{m+j}) \in \Delta_{\mathbf{b}}$, hence in $\Delta_{\mathbf{d}}$, since $\mathbf{b} \leq \mathbf{d}$. On the other hand, in $\mathbf{c}$, $(x_{g^* f^*(i)} = x_{m+i}) \in \Delta_{\mathbf{c}}$, hence $(x_{g^* f^*(i)} = x_{m+n+i}) \in \Delta_{\mathbf{d}}$ from

the amalgamation. As $\Delta_{\mathbf{d}}$ is a complete type, these two statements imply $(x_{m+j} = x_{m+n+i}) \in \Delta_{\mathbf{d}}$. As $\mathbf{e} = P^{v_m}(\mathbf{d})$ and recalling $j = f^*(i)$, we have $(x_{f^*(i)} = x_{n+i}) \in \Delta_{\mathbf{e}}$, as required. □

3.3. **Reconstructing $(M, \mathcal{F})$ from a countable $\mathbb{B}$.** Throughout this section we fix a countable language L and a countable, flat structure $\mathbb{B}$ in the language $L^{\flat}$. Armed with our results from Subsection 3.2, our index functions $f : k \to m$ are not assumed to be injective, but to ease notation we write f in place of f^*. The goal of this subsection is to prove the following proposition.

Proposition 3.7. *For every countable flat structure $\mathbb{B}$, there is a countable L-structure M and a sharp back-and-forth system $\mathcal{F}$ such that $(M, \mathcal{F})^{\flat} \cong \mathbb{B}$. The sharp structure $(M, \mathcal{F})$ is unique up to $L^{\sharp}$-isomorphism. Moreover, any $L^{\flat}$-isomorphism $\sigma^{\flat} : \mathbb{B}_1 \to \mathbb{B}_2$ of countable flat structures lifts to an $L^{\sharp}$-isomorphism $\Phi^* : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ satisfying $\sigma^{\flat}(\pi_n^1(\bar{a})) = \pi_n^2(\Phi^*(\bar{a}))$ for all n and all $\bar{a} \in M_1^n$.*

Towards the proof of Proposition 3.7, we introduce a new notion.

Definition 3.8. A *cofinal sequence in $\mathbb{B}$* $\langle \mathbf{a}_n : n \in \omega \rangle$ satisfies: For all $n \in \omega$

  (1) $\mathbf{a}_n \in U_n$ and $\mathbf{a}_n \leq \mathbf{a}_{n+1}$; and
  (2) For all $n, k, \ell \in \omega$, for all $\mathbf{b} \in U_k$ and $\mathbf{c} \in U_{\ell}$, if $\mathbf{b} = P_{k,n}^f(\mathbf{a}_n)$ and $\mathbf{b} = P_{k,\ell}^g(\mathbf{c})$, then there is an $m \geq n$ and a function $g' : \ell \to m$ such that $g' \circ g = id \circ f$ and $P^{g'}(\mathbf{a}_m) = \mathbf{c}$.

That is, a cofinal sequence represents a kind of Fraïssé sequence of elements from $\mathbb{B}$. The following lemma is easily achieved by proper bookkeeping.

Lemma 3.9. *Every countable flat structure $\mathbb{B}$ has a cofinal sequence. In fact, any finite sequence $\langle \mathbf{a}_i : i < n \rangle$ satisfying $\mathbf{a}_i \in U_i$ and $\mathbf{a}_i \leq \mathbf{a}_{i+1}$ for each i can be extended to a cofinal sequence.*

Construction 3.10. *Let $\mathbb{B}$ be a countable flat structure and fix a cofinal sequence $\langle \mathbf{a}_n : n \in \omega \rangle$. We construct a countable L-structure M and a sharp back-and-forth system $\mathcal{F} = \{E_n : n \in \omega\}$ such that $(M, \mathcal{F})^{\flat}$ is $L^{\flat}$-isomorphic to $\mathbb{B}$ as follows:*

  • First, as $\mathbf{a}_n \leq \mathbf{a}_{n+1}$, we have $\Delta_{\mathbf{a}_n} \subseteq \Delta_{\mathbf{a}_{n+1}}$, and as each $\Delta_{\mathbf{a}_n}$ describes a complete, quantifier-free type in the variables $(x_0, \ldots, x_{n-1})$, the union $q(x_0, x_1, \ldots) := \bigcup_{n \in \omega} \Delta_{\mathbf{a}_n}$ describes a complete, quantifier-free type in ω variables $X = \{x_i : i \in \omega\}$.

- Next, define an equivalence relation $\sim$ on X as: $x_i \sim x_j$ iff $(x_i = x_j) \in q$. As q is a complete type, $\sim$ indeed is an equivalence relation.
- Define an L-structure M to have universe $X/\sim$. For any quantifier-free L-formula α with n free variables and any $(i_0, \ldots, i_{n-1}) \in \omega^n$, say $M \models \alpha([x_{i_0}], \ldots, [x_{i_{n-1}}])$ if and only if $\alpha(x_{i_0}, \ldots, x_{i_{n-1}}) \in q$. It is readily checked that this is well defined and defines an L-structure.
- Additionally, for each $n \in \omega$, define a *covering map* $cov_n : M^n \to U_n$ as: For any $\bar{a} = (a_0, \ldots, a_{n-1}) \in M^n$ (so each a_j is a $\sim$-class of elements from X), let $cov_n(\bar{a}) = P^h(\mathbf{a}_m)$ for any/all sufficiently large m and any function $h : n \to m$ satisfying $x_{h(j)} \in a_j$ for all $j < n$. Again, it is easily checked that this definition does not depend on our choice of m or h.
- Finally, for each $n \in \omega$, define $E_n(\bar{x}, \bar{y})$ on M^{2n} as $E_n(\bar{a}, \bar{b}) \Leftrightarrow cov_n(\bar{a}) = cov_n(\bar{b})$ and let $\mathcal{F} := \bigcup_{n \in \omega} E_n$.

Remark 3.11. It is easily verified that if $(M, \mathcal{F})$ is a countable, sharp structure and $\mathbb{B} = (M, \mathcal{F})^\flat$ is its flattening, then for any enumeration $\langle a_i : i \in \omega \rangle$ of M, Construction 3.10 is an inverse of the flattening map. That is, taking $\mathbf{a}_n := \pi_n(\langle a_i : i < n \rangle)$, the sequence $\langle \mathbf{a}_n : n \in \omega \rangle$ is cofinal, and taking $cov_n := \pi_n$ for each n, the sharp structure we obtain from Construction 3.10 is $L^\sharp$-isomorphic to $(M, \mathcal{F})$.

Much of the verification that $\mathcal{F}$ is a sharp back-and-forth system and that $(M, \mathcal{F})^\flat \cong \mathbb{B}$ is codified with the following fundamental lemma, which will be used in later sections as well.

Lemma 3.12. *Suppose $n \in \omega$, $\bar{a} \in M^n$, $cov_n(\bar{a}) = \mathbf{b} \in U_n$, and $\mathbf{c} \in U_{n+k}$ with $\mathbf{b} \leq \mathbf{c}$. Then there is a k-tuple $\bar{c} \in M^k$ such that $cov_{n+k}(\bar{a}\bar{c}) = \mathbf{c}$.*

Proof. Arguing by induction on k, it suffices to prove this when $k = 1$. Say $\bar{a} = ([x_{i_0}], \ldots, [x_{i_{n-1}}])$ and choose any $m > \max\{i_0, \ldots, i_{n-1}\}$. Let $f : n \to m$ be given by $f(j) = i_j$. By the definition of cov_n, $P_{n,m}^f(\mathbf{a}_m) = \mathbf{b}$. As $\langle \mathbf{a}_k : k \in \omega \rangle$ is cofinal, there is some $\ell \geq m$ and $h : n+1 \to \ell$ such that $id \circ f = h \circ id$ and $P^h(\mathbf{a}_\ell) = \mathbf{c}$. The commutativity of the functions gives $h(j) = f(j)$ for all $j < n$, so taking $c = [x_{h(n)}]$ we have $cov_{n+1}(\bar{a}c) = \mathbf{c}$, as required. $\square$

Using Lemma 3.12, we now verify that the statements of Proposition 3.7. First, it is easily checked that $\mathcal{F}$ is a sharp back-and-forth system on M. Second, taking $n = 0$ and k arbitrary, we see that $cov_k : M^k \to U_k$ is onto, so $\mathbb{B}$ is isomorphic to the flattening of $(M, \mathcal{F})$, with cov_n playing the role of π_n.

The uniqueness statement follows from the Moreover clause, which we now establish. Fix an $L^\flat$-isomorphism $\sigma^\flat : \mathbb{B}_1 \to \mathbb{B}_2$ between countable flat structures. Choose any countable sharp structures $(M_1, \mathcal{F}_1)$, $(\mathcal{M}_2, \mathcal{F}_2)$ whose flattenings are $\mathbb{B}_1, \mathbb{B}_2$, respectively. For each $\ell = 1, 2$, by possibly replacing $(M_\ell, \mathcal{F}_\ell)$ by an isomorphic copy, by Remark 3.11 we may assume it is obtained from $\mathbb{B}_\ell$ via Construction 3.10, where $cov_n^\ell = \pi_n^\ell$ for each n. To build an $L^\sharp$-isomorphism, consider the set of all partial bijections $\Phi : \bar{a} \to \bar{b}$, where $\bar{a} \in M_1^n$, $\bar{b} \in M_2^n$, and $\sigma^\flat(\pi_n^1(\bar{a})) = \pi_n^2(\bar{b})$. Clearly, every such Φ preserves quantifier-free types, as, letting $\mathbf{c} = \pi_n^1(\bar{a})$, the L-quantifier-free types $\mathrm{qftp}_{M_1}(\bar{a})$ and $\mathrm{qftp}_{M_2}(\bar{b})$ can be read off from $\Delta_{\mathbf{c}}$ and $\Delta_{\sigma^\flat(\mathbf{c})}$.

By Lemma 3.12, this family of partial bijections Φ is a back and forth system. As both M_1 and M_2 are countable, it follows that there is an L-isomorphism $\Phi^* : M_1 \to M_2$ that satisfies $\sigma^\flat(\pi_n^1(\bar{a})) = \pi_n^2(\Phi^*(\bar{a}))$ for every $\bar{a} \in M_1^n$. But this condition implies that Φ^* is also E_n-preserving: If $E_n(\bar{a}, \bar{a}')$ in M_1, then by definition of E_n, we have $\pi_n^1(\bar{a}) = \pi_n^1(\bar{a}')$. From the sentence above, $\pi_n^2(\Phi^*(\bar{a})) = \pi_n^2(\Phi^*(\bar{a}'))$, so $E_n(\Phi^*(\bar{a}), \Phi^*(\bar{a}'))$ holds in M_2 as well. Thus, Φ^* is an $L^\sharp$-isomorphism from $(M_1, \mathcal{F}_1)$ onto $(M_2, \mathcal{F}_2)$.

3.4. Hausdorff flat structures are potential canonical Scott sentences.

Fix a countable language L. The aim of this section is to distinguish a canonical class Hdf* of Hausdorff flat structures in $\mathbb{V}$. We will see that every Hausdorff flat structure is isomorphic [equivalently, back-and-forth equivalent by Lemma 3.28] to exactly one canonical structure, so we can naturally identify Hdf* with isomorphism classes [equivalently, back and forth classes] of Hausdorff flat structures. Further, we will show that Hdf* is naturally bijective with $\mathrm{CSS}(L)_{\mathrm{ptl}}$.

As a preamble, we describe some expansions of L. For each $k \in \omega$, let $L(\bar{c}_k) := L \cup \{c_0, \ldots, c_{k-1}\}$ (so $L_0 = L$) where $\{c_0, \ldots, c_{k-1}\}$ are new constants. For any $L(\bar{c}_k)$-structure $N = (M, \bar{a}) \in HC$, let $\mathrm{css}_k(M, \bar{a})$ denote the canonical Scott sentence of $N = (M, \bar{a})$. For any sentence $\sigma \in (L(\bar{c}_k))_{\infty,\omega}$, let $\hat{\sigma}(\bar{x})$ be the $L_{\infty,\omega}$ formula formed by replacing each c_i by x_i. Note that $\hat{\sigma}(\bar{c}_k)$ is persistently equivalent to σ, i.e., for any forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ and any $L(\bar{c}_k)^{\mathbb{V}[G]}$-structure $N = (M, \bar{a})$, $N \models \sigma$ if and only if $M \models \hat{\sigma}(\bar{a})$.

Definition 3.13. For any language L, call two $L_{\infty,\omega}$-sentences σ and ψ *explicitly contradictory* if $\sigma \wedge \tau$ has no models in any forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$. An $L_{\infty,\omega}$-sentence δ is *persistently valid* if, for every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$, every $M \in \mathrm{Mod}_{\mathbb{V}[G]}(L)$ models δ.

As examples, note that for any $k \in \omega$, it follows from Fact 1.6 that any two canonical Scott sentences $\mathrm{css}_k(M, \bar{a})$ and $\mathrm{css}_k(M', \bar{a}')$ in $(L(\bar{c}_k))_{\omega_1, \omega}$ are explicitly contradictory or equal. Note that σ and τ are explicitly contradictory $(L(\bar{c}_k))_{\infty, \omega}$-sentences if and only if the $L_{\infty, \omega}$-sentence $\forall \bar{x}(\hat{\sigma}(\bar{x}) \to \neg \hat{\tau}(\bar{x}))$ is persistently valid.

We begin in the countable world, i.e., in HC. Recalling Definition 3.1, let $\mathrm{Flat}_{HC}(L) := \mathrm{Mod}_{HC}(\mathrm{Ax}^{\flat}(L))$, the flat $L^{\flat}$-structures in HC. Note that as satisfaction is absolute between transitive models of set theory, the classes $\mathrm{Flat}_{HC}(L)$ and $\mathrm{Mod}_{HC}(L)$ are strongly definable, i.e., are defined by HC-invariant formulas.

Definition 3.14. Let can : $\mathrm{Mod}_{HC}(L) \to \mathrm{Flat}_{HC}(L)$ be the following canonical map. Given $M \in \mathrm{Mod}_{HC}(L)$, $\mathrm{can}(M)$ is the flat structure with universe

$$\{\mathrm{css}_k(M, \bar{a}) : k \in \omega, \bar{a} \in M^k\}$$

where the $L^{\flat}$ symbols are interpreted as: $U_k(\mathrm{can}(M)) = \{\mathrm{css}_k(M, \bar{a}) : \bar{a} \in M^k\}$; $\mathrm{can}(M) \models \alpha^{\flat}(\mathrm{css}_k(M, \bar{a}))$ if and only if $M \models \alpha(\bar{a})$ for each quantifier-free L-formula $\alpha(x_0, \ldots, x_{k-1})$; and for an injective $f : k \to n$, $P_{k,n}^f(\mathrm{css}_n(M, \bar{a})) = \mathrm{css}_k(M, \bar{a} \restriction_f)$.

For any $M \in \mathrm{Mod}_{HC}(L)$, the structure $\mathrm{can}(M)$ is fundamentally an unpacking of the Scott sentence of M, and hence only depends on the isomorphism type of M. As well, the mappings $\mathrm{css}_n(\bar{a}) \mapsto \pi_n(\bar{a})$ give an $L^{\flat}$-isomorphism between $\mathrm{can}(M)$ and the flattening $M^{\flat} = (M, \mathcal{F}_{\infty})^{\flat}$. In more detail:

Lemma 3.15. *If $M, N \in \mathrm{Mod}_{HC}(L)$, then $M \cong N$ if and only if $\mathrm{can}(M) = \mathrm{can}(N)$.*

Proof. Right to left is easy, as taking $k = 0$, $U_0(\mathrm{can}(M)) = U_0(\mathrm{can}(N)) = \{\mathrm{css}(M)\}$. Thus, $M \cong N$ by the salient property of canonical Scott sentences. Now assume $h : M \to N$ is an L-isomorphism. Then, for any $k \in \omega$ and any $\bar{a} \in M^k$, $(M, \bar{a}) \cong (N, h(\bar{a}))$ as $L(\bar{c}_k)$-structures, hence $\mathrm{css}_k(M, \bar{a}) = \mathrm{css}_k(N, h(\bar{a}))$. Thus, for each k we have

$$\{\mathrm{css}_k(M, \bar{a}) : \bar{a} \in M^k\} = \{\mathrm{css}_k(N, \bar{b}) : \bar{b} \in N^k\}$$

so the universes of $\mathrm{can}(M)$ and $\mathrm{can}(N)$ are equal. Similarly, it is easily checked that the $L^{\flat}$-interpretations in $\mathrm{can}(M)$ and $\mathrm{can}(N)$ are equal as well. Thus, $\mathrm{can}(M) = \mathrm{can}(N)$. $\square$

Using Lemma 3.15 and the uniqueness statement of Proposition 3.7, the following map is well-defined.

Definition 3.16. Let $\mathrm{can}^\flat : \mathrm{Flat}_{HC}(L) \to \mathrm{Flat}_{HC}(L)$ be defined as: $\mathrm{can}^\flat(\mathbb{B}) := \mathrm{can}(M)$, where M is any L-structure obtained from Proposition 3.7, i.e., where $(M, \mathcal{F})^\flat \cong \mathbb{B}$ for some $\mathcal{F}$.

Let Hdf^*_{HC} denote the image of $\mathrm{can}^\flat$, which is also the image of can.

This notation is justified by the following lemma.

Lemma 3.17.   *(1) Every $\mathbb{B}' \in \mathrm{Hdf}^*_{HC}$ is Hausdorff.*

(2) For all Hausdorff $\mathbb{B}_1, \mathbb{B}_2 \in \mathrm{Flat}_{HC}(L)$, $\mathbb{B}_1 \cong \mathbb{B}_2$ if and only if $\mathrm{can}^\flat(\mathbb{B}_1) = \mathrm{can}^\flat(\mathbb{B}_2)$.

(3) $\mathrm{can}^\flat : \mathrm{Flat}_{HC}(L) \to \mathrm{Flat}_{HC}(L)$ is a projection onto Hdf^*_{HC}, i.e., $\mathrm{can}^\flat(\mathrm{can}^\flat(\mathbb{B})) = \mathrm{can}^\flat(\mathbb{B})$ for all $\mathbb{B} \in \mathrm{Flat}_{HC}(L)$.

(4) $\mathrm{Hdf}^*_{HC} = \{\mathbb{B} \in \mathrm{Flat}_{HC}(L) : \mathrm{can}^\flat(\mathbb{B}) = \mathbb{B}\}$.

Proof. (1) Fix distinct $\mathbf{a} \neq \mathbf{b} \in \mathbb{B}'$. For definiteness, assume $\mathbf{a}, \mathbf{b} \in U_k(\mathbb{B}')$. Choose any $M \in \mathrm{Mod}_{HC}(L)$ with $\mathrm{can}(M) = \mathbb{B}'$ and suppose $\mathbf{a} = \mathrm{css}_k(M, \bar{a})$ and $\mathbf{b} = \mathrm{css}_k(M, \bar{b})$. Let $\sigma := \mathrm{css}_k(M, \bar{a})$ and $\tau := \mathrm{css}_k(M, \bar{b})$. As these are distinct canonical Scott sentences, σ and τ are explicitly contradictory. Thus, $M \models \forall \bar{x}(\hat{\sigma}(\bar{x}) \to \neg \hat{\tau}(\bar{x}))$. From this, we have $M \models \hat{\tau}(\bar{b}) \wedge \neg \hat{\tau}(\bar{a})$. As $\mathbb{B}' \cong M^\flat$, $\mathbb{B}' \models \hat{\tau}^\flat(\mathbf{b}) \wedge \neg \hat{\tau}^\flat(\mathbf{a})$. Thus, $\mathbb{B}'$ is Hausdorff.

(2) Since for $\ell = 1, 2$, each $\mathbb{B}_\ell$ is Hausdorff, by Lemma 2.9 and Proposition 3.7 we can choose L-structures M_ℓ such that $M_\ell^\flat \cong \mathbb{B}_\ell$. First, assume $\mathrm{can}^\flat(\mathbb{B}_1) = \mathrm{can}^\flat(\mathbb{B}_2)$. Then $\mathrm{can}(M_1) = \mathrm{can}(M_2)$, so $M_1 \cong M_2$ by Lemma 3.15. It follows by the uniqueness of flattenings that $\mathbb{B}_1 \cong \mathbb{B}_2$ as $L^\flat$-structures. Conversely, assume $\mathbb{B}_1 \cong \mathbb{B}_2$. Then, by the uniqueness sentence of Proposition 3.7, $M_1 \cong M_2$, so again by Lemma 3.15, $\mathrm{can}(M_1) = \mathrm{can}(M_2)$. Thus, $\mathrm{can}^\flat(\mathbb{B}_1) = \mathrm{can}^\flat(\mathbb{B}_2)$.

(3) Given $\mathbb{B} \in \mathrm{Flat}_{HC}(L)$, by Proposition 3.7 choose a sharp structure $(M, \mathcal{F})$ such that $(M, \mathcal{F})^\flat \cong \mathbb{B}$. Put $\mathbb{B}' := \mathrm{can}^\flat(\mathbb{B})$, the latter being equal to $\mathrm{can}(M)$. However, since $\mathbb{B}' \cong (M, \mathcal{F}_\infty)^\flat$, $\mathrm{can}^\flat(\mathbb{B}') = \mathrm{can}(M)$ follows immediately. Thus, $\mathrm{can}^\flat(\mathbb{B}') = \mathrm{can}(M) = \mathbb{B}'$, as required.

(4) Choose any $\mathbb{B} \in \mathrm{Hdf}^*_{HC}$. That $\mathrm{can}^\flat(\mathbb{B}) = \mathbb{B}$ is given by (3). Conversely, assume $\mathrm{can}^\flat(\mathbb{B}) = \mathbb{B}$. Then trivially, $\mathbb{B} \in \mathrm{Hdf}^*_{HC}$ by its definition. $\square$

So far, in the HC world, the surjective maps $\mathrm{can} : \mathrm{Mod}_{HC}(L) \to \mathrm{Hdf}^*_{HC}$ and $\mathrm{can}^\flat : \mathrm{Flat}_{HC}(L) \to \mathrm{Hdf}^*_{HC}$ are somewhat dual. However, this similarity will dissipate when we take potentials. This is due not only to the fact that $\mathrm{can}^\flat$ is a projection, but also because there is a preferred homomorphism $j : \mathbb{B} \to \mathrm{can}^\flat(\mathbb{B})$ that we now describe.

Definition 3.18. Suppose $\mathbb{B}, \mathbb{B}' \in \mathrm{Flat}_{HC}(L)$. A *surjective homomorphism* $j : \mathbb{B} \to \mathbb{B}'$ is an onto map such that: (1) for all $\mathbf{a} \in \mathbb{B}$, $\mathbb{B} \models U_k(\mathbf{a})$

iff $\mathbb{B}' \models U_k(j(\mathbf{a}))$; (2) for all quantifier-free L-formula $\alpha(\overline{x})$ $\mathbb{B} \models \alpha^\flat(\mathbf{a})$ iff $\mathbb{B}' \models \alpha_k^\flat(j(\mathbf{a}))$; and (3) j commutes with each $P_{k,n}^f$.

Recall that by Definition 2.6, for any flat structure $\mathbb{B}$ and any $\mathbf{a} \in \mathbb{B}$, $\flat\text{-tp}_\mathbb{B}(\mathbf{a}) = \{\varphi^\flat(z) : \mathbb{B} \models \varphi^\flat(\mathbf{a})\}$. The following Lemma is proved by induction on the complexity of $L_{\infty,\omega}$-formulas.

Lemma 3.19. *Let* $j : \mathbb{B} \to \mathbb{B}'$ *be any surjective homomorphism of flat structures. Then, for any* $\mathbf{a} \in \mathbb{B}$, $\flat\text{-tp}_\mathbb{B}(\mathbf{a}) = \flat\text{-tp}_{\mathbb{B}'}(j(\mathbf{a}))$.

Lemma 3.20. *For any* $\mathbb{B}, \mathbb{B}' \in \text{Flat}_{HC}(L)$ *with* $\mathbb{B}'$ *Hausdorff, there is at most one surjective homomorphism* $j : \mathbb{B} \to \mathbb{B}'$. *In particular, for any* $\mathbb{B} \in \text{Flat}_{HC}(L)$, *there is a unique surjective homomorphism* $j : \mathbb{B} \to \text{can}^\flat(\mathbb{B})$.

Proof. Choose surjective homomorphisms $j, j^* : \mathbb{B} \to \mathbb{B}'$ and choose any $\mathbf{a} \in \mathbb{B}$. Let $\mathbf{b} := j(\mathbf{a})$ and $\mathbf{b}^* := j^*(\mathbf{a})$. To show that $\mathbf{b} = \mathbf{b}^*$, by applying Lemma 3.19 twice, we have $\flat\text{-tp}_{\mathbb{B}'}(\mathbf{b}) = \flat\text{-tp}_{\mathbb{B}'}(\mathbf{b}^*)$. Thus, $\mathbf{b} = \mathbf{b}^*$ since $\mathbb{B}'$ is Hausdorff.

For the second sentence, choose any $\mathbb{B} \in \text{Flat}_{HC}(L)$, and let (M, cov) be from Proposition 3.7 such that the corresponding $(M, \mathcal{F})^\flat \cong \mathbb{B}$. It is easily checked that the map $j : \mathbb{B} \to \text{can}^\flat(\mathbb{B})$ defined by $j(\mathbf{a}) = \text{css}_k(M, \bar{a})$ for some or any $\bar{a} \in M^k$ such that $cov_k(\bar{a}) = \mathbf{a}$ is well defined and is a surjective homomorphism. As $\text{can}^\flat(\mathbb{B})$ is Hausdorff from Lemma 3.17(1), the uniqueness of j follows from the first sentence. $\square$

Lemma 3.20 gives that the function below is well defined.

Definition 3.21. Let cmap : $\text{Flat}_{HC}(L) \to HC$ be the function described as: $\text{cmap}(\mathbb{B})$ is the unique surjective homomorphism $j : \mathbb{B} \to \text{can}^\flat(\mathbb{B})$.

We glean one easy consequence of this. The characterization will be useful in proving both that the class of Hausdorff structures in HC is HC-invariant, and that 'being Hausdorff' is absolute among forcing extensions.

Lemma 3.22. *The following are equivalent for* $\mathbb{B} \in \text{Flat}_{HC}(L)$.

    (1) $\mathbb{B}$ *is Hausdorff;*
    (2) $\text{cmap}(\mathbb{B})$ *is injective; and*
    (3) $\text{cmap}(\mathbb{B})$ *is an* $L^\flat$-*isomorphism.*

Proof. (1) $\Rightarrow$ (2): Choose any $\mathbb{B} \in \text{Flat}_{HC}(L)$ and denote $\text{can}^\flat(\mathbb{B})$ as $\mathbb{B}'$ and $\text{cmap}(\mathbb{B})$ as $j : \mathbb{B} \to \mathbb{B}'$. First, assume $\mathbb{B}$ is Hausdorff and choose $\mathbf{a} \neq \mathbf{b}$ from $\mathbb{B}$. As $\mathbb{B}$ is Hausdorff, $\flat\text{-tp}_\mathbb{B}(\mathbf{a}) \neq \flat\text{-tp}_\mathbb{B}(\mathbf{b})$, so $\flat\text{-tp}_{\mathbb{B}'}(j(\mathbf{a})) \neq \flat\text{-tp}_{\mathbb{B}'}(j(\mathbf{b}))$ by Lemma 3.19. Thus, $j(\mathbf{a}) \neq j(\mathbf{b})$, so j is injective.

$(2) \Rightarrow (3)$ is immediate from the definitions.

$(3) \Rightarrow (1)$: Since $\mathrm{can}^{\flat}(\mathbb{B})$ is Hausdorff by Lemma 3.17(1), (3) implies $\mathbb{B}$ is as well. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $\mathrm{Hdf}_{HC} := \{\mathbb{B} \in \mathrm{Flat}_{HC}(L) : \mathbb{B}$ is Hausdorff$\}$. We consider forcing extensions $\mathbb{V}[G]$ of $\mathbb{V}$.

Lemma 3.23.   *(1)* can*,* $\mathrm{can}^{\flat}$*, and* cmap *are strongly definable and are persistently functions.*

      (2) Both Hdf^{*}_{HC} *and* Hdf_{HC} *are strongly definable sets.*

      (3) The identity map $\mathrm{id} : \mathrm{Hdf}^{*}_{HC} \to \mathrm{Flat}_{HC}$ *is a persistent, strongly definable cross section of* $\mathrm{can}^{\flat} : \mathrm{Flat}_{HC} \to \mathrm{Hdf}^{*}_{HC}$*.*

Proof. (1) We first show can is HC-invariant and is persistently a function. Choose any $M \in \mathrm{Mod}_{HC}(L)$ and let $\mathbb{V}[G] \supseteq \mathbb{V}$ be any forcing extension. As css is strongly definable, for every $\bar{a} \in M^{k}$, we have $\mathrm{css}_{k}(M, \bar{a})^{\mathbb{V}[G]} = \mathrm{css}_{k}(M, \bar{a})^{\mathbb{V}}$. It follows from this that $\mathrm{can}(M)^{\mathbb{V}[G]} = \mathrm{can}(M)^{\mathbb{V}}$. Thus, can is strongly definable. The argument that can is well defined also holds in $V[G]$, hence can is persistently a function.

The argument for $\mathrm{can}^{\flat}$ is similar. Given $\mathbb{B} \in \mathrm{Flat}_{HC}(L)$, note that any $M \in \mathbb{V}$ witnessing $\mathrm{can}^{\flat}(\mathbb{B}) = \mathrm{can}(M)$ also witnesses this in $\mathbb{V}[G]$. Thus, $\mathrm{can}^{\flat}(\mathbb{B})^{\mathbb{V}[G]} = \mathrm{can}^{\flat}(\mathbb{B})^{\mathbb{V}}$. That $\mathrm{can}^{\flat}$ is persistently a function follows by applying Lemma 3.15 in $\mathbb{V}[G]$.

Now, for cmap, choose $\mathbb{B} \in \mathrm{Flat}_{HC}(L)$ and let $j := \mathrm{cmap}(\mathbb{B})$ in $\mathbb{V}$. Then easily, in $\mathbb{V}[G]$ we also have $j : \mathbb{B} \to \mathrm{can}^{\flat}(\mathbb{B})$ is a surjective homomorphism. By applying Lemma 3.20 to j in $\mathbb{V}[G]$ it is the only one, hence $\mathrm{cmap}(\mathbb{B})^{\mathbb{V}[G]} = j$ as well. Thus, cmap is strongly definable and is persistently a function.

(2) For Hdf^{*}_{HC} we use Lemma 3.17(4). Choose $\mathbb{B} \in HC^{\mathbb{V}}$. First, suppose $\mathbb{B} \in (\mathrm{Hdf}^{*}_{HC})^{\mathbb{V}}$ and let $\mathbb{V}[G] \supseteq \mathbb{V}$ be any forcing extension. Then $\mathbb{B} = \mathrm{can}(\mathbb{B})$ in both $\mathbb{V}$ and $\mathbb{V}[G]$, so $\mathbb{B} \in (\mathrm{Hdf}^{*}_{HC})^{\mathbb{V}[G]}$ by applying Lemma 3.17(4) in $\mathbb{V}[G]$. Conversely, assume $\mathbb{B} \in (\mathrm{Hdf}^{*}_{HC})^{\mathbb{V}[G]}$. As $\mathrm{Flat}(L)$, the class of all models of $\mathrm{Ax}^{\flat}(L)$, is absolute between forcing extensions and $\mathbb{B} \in HC^{\mathbb{V}}$, we know $\mathbb{B} \in \mathrm{Flat}_{HC}(L)^{\mathbb{V}}$. By Lemma 3.17(4) in $\mathbb{V}[G]$, $\mathrm{can}^{\flat}(\mathbb{B}) = \mathbb{B}$ in $\mathbb{V}[G]$. As $\mathrm{can}^{\flat}$ is strongly definable, this implies $\mathrm{can}^{\flat}(\mathbb{B}) = \mathbb{B}$ in $\mathbb{V}$ as well. Thus, $\mathbb{B} \in (\mathrm{Hdf}^{*}_{HC})^{\mathbb{V}}$.

Showing that Hdf_{HC} is strongly definable is similar, using Lemma 3.22 in both $\mathbb{V}$ and $\mathbb{V}[G]$. On one hand, if $\mathbb{B} \in \mathrm{Hdf}_{HC}(L)$ in $\mathbb{V}$, then letting $j = \mathrm{cmap}(\mathbb{B})$, $j : \mathbb{B} \to \mathrm{can}^{\flat}(\mathbb{B})$ is injective, both in $\mathbb{V}$ and (from above) in $\mathbb{V}[G]$. Thus, $\mathbb{B} \in \mathrm{Hdf}_{HC}(L)$ in $\mathbb{V}[G]$ as well. Conversely, if $\mathbb{B} \in HC^{\mathbb{V}}$ and $\mathbb{B} \in \mathrm{Hdf}_{HC}(L)^{\mathbb{V}[G]}$. Then $j : \mathbb{B} \to \mathrm{can}^{\flat}(\mathbb{B})$ is injective in both $\mathbb{V}[G]$ and $\mathbb{V}$, so $\mathbb{B} \in \mathrm{Hdf}_{HC}(L)$.

(3) This is obvious by unpacking the definitions. □

It is noteworthy that the verification of Hdf^* being strongly definable used the characterization of it in terms of the fixed points of $\mathrm{can}^\flat$ as opposed to anything about the strongly definable function can. This asymmetry will be magnified in Conclusion 3.31, where we show that $\mathrm{Hdf}^*_{\mathrm{ptl}}$ is the image of $\mathrm{can}^\flat_{\mathrm{ptl}}$, but the image of $\mathrm{can}_{\mathrm{ptl}}$ might be smaller. Indeed, this distinction manifests the difference between $\mathrm{CSS}(L)_{\mathrm{sat}}$ and $\mathrm{CSS}(L)_{\mathrm{ptl}}$ described in the Introduction.

We now leave the HC world and enter into the world of all sets by applying the ptl operator. Note that as 'being a structure' and 'being a flat structure' are absolute, $\mathrm{Mod}(L)_{\mathrm{ptl}} = \mathrm{Mod}(L)$, the class of all L-structures, and similarly, $\mathrm{Flat}(L)_{\mathrm{ptl}}$ is the class of all flat $L^\flat$-structures, which we denote by $\mathrm{Flat}(L)$. In light of Lemma 3.23, and Lemma 1.3 we have the following:

- $\mathrm{can}_{\mathrm{ptl}} : \mathrm{Mod}(L) \to \mathrm{Hdf}^*_{\mathrm{ptl}}$ defined as: $\mathrm{can}_{\mathrm{ptl}}(M) = \mathrm{can}(M)^{\mathbb{V}[G]}$ for some/every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ with $M \in HC^{\mathbb{V}[G]}$;
- $\mathrm{can}^\flat_{\mathrm{ptl}} : \mathrm{Flat}(L) \to \mathrm{Hdf}^*_{\mathrm{ptl}}$ defined as: $\mathrm{can}^\flat_{\mathrm{ptl}}(\mathbb{B}) = \mathrm{can}^\flat(\mathbb{B})^{\mathbb{V}[G]}$ for some/every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$; with $\mathbb{B} \in HC^{\mathbb{V}[G]}$;
- $\mathrm{cmap}_{\mathrm{ptl}} : \mathrm{Flat}_{\mathbb{V}}(L) \to \mathbb{V}$ defined as: $\mathrm{cmap}_{\mathrm{ptl}}(\mathbb{B})$ is the unique $j : \mathbb{B} \to \mathrm{can}^\flat(\mathbb{B})$, as computed in some/every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ with $\mathbb{B} \in HC^{\mathbb{V}[G]}$.

We isolate the following for emphasis.

Lemma 3.24. *The class functional* $\mathrm{can}^\flat_{\mathrm{ptl}} : \mathrm{Flat}(L) \to \mathrm{Hdf}^*_{\mathrm{ptl}}$ *is a (surjective) projection.*

Proof. This follows immediately from Lemmas 1.4 and 3.23. □

Lemma 3.25. *For any flat structure* $\mathbb{B}$, $\mathrm{can}^\flat_{\mathrm{ptl}}(\mathbb{B})$ *is Hausdorff.*

Proof. Fix a flat structure $\mathbb{B} \in \mathbb{V}$ and choose any forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ such that $\mathbb{B} \in HC^{\mathbb{V}[G]}$. By Proposition 3.7, choose $(M, \mathcal{F}) \in HC^{\mathbb{V}[G]}$ such that $(M, \mathcal{F})^\flat \cong \mathbb{B}$. Working in $\mathbb{V}[G]$, put $\mathbb{B}' := \mathrm{can}(M)$, which is equal to $\mathrm{can}^\flat(\mathbb{B})$. By the strong definability of $\mathrm{can}^\flat$ and since $\mathbb{B} \in \mathbb{V}$, $\mathbb{B}' \in \mathbb{V}$ as well. We know that $\mathbb{B}'$ is Hausdorff in $\mathbb{V}[G]$ by Lemma 3.17(1), but we must show $\mathbb{B}'$ is Hausdorff in $\mathbb{V}$. For this, choose any $\mathbf{a} \neq \mathbf{b}$ in $\mathbb{B}'$. We will show that $\flat\text{-tp}_{\mathbb{B}'}(\mathbf{a})$ and $\flat\text{-tp}_{\mathbb{B}'}(\mathbf{b})$ differ on some formula in $\mathbb{V}$. This is obvious unless both $\mathbf{a}, \mathbf{b}$ are in $U_k^{\mathbb{B}'}$ for the same k, so we assume they are.

Working in $\mathbb{V}[G]$, by the definition of $\mathrm{can}(M)$, $\mathbf{a} = \mathrm{css}(M, \bar{a})$ and $\mathbf{b} = \mathrm{css}(M, \bar{b})$ for some tuples $\bar{a}, \bar{b} \in M^k$, i.e., are $L(\bar{c}_k)_{\infty,\omega}$-sentences σ, τ, respectively. As σ and τ are distinct canonical Scott sentences, they are explicitly

contradictory. Thus, the $L_{\infty,\omega}$-sentence $\forall \overline{x}(\hat{\sigma}(\overline{x}) \to \neg \hat{\tau}(\overline{x}))$ is persistently valid, so holds in M. Now, as $\mathbb{B}' = \mathrm{can}(M)$,

$$\mathbb{B}' \models \hat{\sigma}^\flat(\mathbf{a}) \wedge \hat{\tau}^\flat(\mathbf{b})$$

Since $M \models \forall \overline{x}(\hat{\sigma}(\overline{x}) \to \neg \hat{\tau}(\overline{x}))$, $\mathbb{B}' \cong M^\flat$ and Lemma 2.7 imply

$$\mathbb{B}' \models \forall z[\hat{\sigma}^\flat(z) \to \neg \hat{\tau}^\flat(z)]$$

Thus, in $\mathbb{V}[G]$, $\mathbb{B}' \models \neg \hat{\tau}^\flat(\mathbf{a}) \wedge \hat{\tau}^\flat(\mathbf{b})$. Since $\mathbb{B}' \in \mathbb{V}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{B}'$, both $\sigma, \tau \in \mathbb{V}$, as is the syntactic variant $\hat{\tau}^\flat(z) \in \mathbb{V}$. As satisfaction is absolute and $\mathbb{B}' \in \mathbb{V}$, we conclude that $\mathbb{B}' \models \neg \hat{\tau}^\flat(\mathbf{a}) \wedge \hat{\tau}^\flat(\mathbf{b})$ in $\mathbb{V}$ as well. Thus, the formula $\hat{\tau}^\flat(z)$ witnesses that $\flat\text{-tp}_{\mathbb{B}'}(\mathbf{a}) \neq \flat\text{-tp}_{\mathbb{B}'}(\mathbf{b})$ in $\mathbb{V}$. $\square$

Lemma 3.26. *The notion of '$\mathbb{B}$ is Hausdorff' is absolute between forcing extensions. Thus, $\mathrm{Hdf}_{\mathrm{ptl}}$ denotes the class of Hausdorff flat structures.*

Proof. Suppose $\mathbb{V}[G] \supseteq \mathbb{V}$ is any forcing extension and assume $\mathbb{B} \in \mathbb{V}$. That $\mathbb{B}$ Hausdorff in $\mathbb{V}$ implies $\mathbb{B}$ Hausdorff in $\mathbb{V}[G]$ is easy, since for different elements $\mathbf{a} \neq \mathbf{b}$ of $\mathbb{B}$ if $\mathbb{B} \models \varphi^\flat(\mathbf{a}) \wedge \neg\varphi^\flat(\mathbf{b})$ in $\mathbb{V}$, then, as $\varphi^\flat \in \mathbb{V}$, the same holds in $\mathbb{V}[G]$.

For the converse, assume $\mathbb{B} \in \mathbb{V}$ is Hausdorff in $\mathbb{V}[G]$. As $\mathbb{B} \in \mathbb{V}$, both $\mathrm{can}^\flat_{\mathrm{ptl}}(\mathbb{B})$ and $j := \mathrm{cmap}_{\mathrm{ptl}}(\mathbb{B})$ are in $\mathbb{V}$ by Lemma 1.3. Choose a forcing extension $\mathbb{V}[G][H]$ in which $\mathbb{B} \in HC^{\mathbb{V}[G][H]}$. Note that by the first paragraph, $\mathbb{B}$ is Hausdorff in $\mathbb{V}[G][H]$ as well. By definition of $\mathrm{cmap}_{\mathrm{ptl}}$, $j : \mathbb{B} \to \mathrm{can}^\flat_{\mathrm{ptl}}(\mathbb{B})$ is equal to $\mathrm{cmap}(\mathbb{B})$ in $\mathbb{V}[G][H]$. Thus, working in $\mathbb{V}[G][H]$, j is an $L^\flat$-isomorphism by Lemma 3.22.

Now we work in $\mathbb{V}$. Since $j \in \mathbb{V}$ and 'being an isomorphism' is absolute, $\mathbb{B} \cong \mathrm{can}^\flat_{\mathrm{ptl}}(\mathbb{B})$ in $\mathbb{V}$. As $\mathrm{can}^\flat_{\mathrm{ptl}}(\mathbb{B})$ is Hausdorff by Lemma 3.25, so is $\mathbb{B}$. $\square$

Given this absoluteness, we can now recast our definitions as follows.

Definition 3.27. Let Hdf consist of all Hausdorff flat structures $\mathbb{B} \in \mathbb{V}$ and let Hdf^* consist of all flat structures $\mathbb{B} \in \mathbb{V}$ such that $\mathrm{can}^\flat_{\mathrm{ptl}}(\mathbb{B}) = \mathbb{B}$.

The above agrees with our previous definitions in the sense that for any $\mathbb{B} \in \mathrm{Flat}(L)$, $\mathbb{B} \in \mathrm{Hdf}$ (resp. $\mathbb{B} \in \mathrm{Hdf}^*$) if and only if $(\mathbb{B} \in \mathrm{Hdf}_{HC})^{\mathbb{V}[G]}$ (resp. $(\mathbb{B} \in \mathrm{Hdf}^*_{HC})^{\mathbb{V}[G]}$) for some/every forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ with $\mathbb{B} \in HC^{\mathbb{V}[G]}$. That is, $\mathrm{Hdf} = \mathrm{Hdf}_{\mathrm{ptl}}$ and $\mathrm{Hdf}^* = \mathrm{Hdf}^*_{\mathrm{ptl}}$.

Lemma 3.28. *The following are equivalent for Hausdorff flat structures $\mathbb{B}_1, \mathbb{B}_2$ (of any size):*

(1) $\mathbb{B}_1 \equiv_{\infty,\omega} \mathbb{B}_2$;
(2) $\mathbb{B}_1 \cong \mathbb{B}_2$;

(3) There is a unique $L^\flat$-isomorphism $h : \mathbb{B}_1 \to \mathbb{B}_2$;

(4) $\mathrm{can}^\flat{}_{\mathrm{ptl}}(\mathbb{B}_1) = \mathrm{can}^\flat{}_{\mathrm{ptl}}(\mathbb{B}_2)$.

Proof. As (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is obvious, it suffices to prove (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (3). For both parts, choose a forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$ in which both $\mathbb{B}_1, \mathbb{B}_2$ are in $HC^{\mathbb{V}[G]}$.

(1) $\Rightarrow$ (4): Assume $\mathbb{B}_1 \equiv_{\infty,\omega} \mathbb{B}_2$. Then, classically, $\mathbb{B}_1 \cong \mathbb{B}_2$ in $\mathbb{V}[G]$. Thus, by Lemma 3.17(2), $\mathrm{can}^\flat(\mathbb{B}_1) = \mathrm{can}^\flat(\mathbb{B}_2)$ in $\mathbb{V}[G]$. But, by definition of $\mathrm{can}^\flat{}_{\mathrm{ptl}}$, we have $\mathrm{can}^\flat{}_{\mathrm{ptl}}(\mathbb{B}_\ell) = \mathrm{can}^\flat(\mathbb{B}_\ell)^{\mathbb{V}[G]}$ for $\ell = 1, 2$, hence $\mathrm{can}^\flat{}_{\mathrm{ptl}}(\mathbb{B}_1) = \mathrm{can}^\flat{}_{\mathrm{ptl}}(\mathbb{B}_2)$.

(4) $\Rightarrow$ (3): By definition of $\mathrm{can}^\flat{}_{\mathrm{ptl}}$, we have $\mathrm{can}^\flat{}_{\mathrm{ptl}}(\mathbb{B}_\ell) = \mathrm{can}^\flat(\mathbb{B}_\ell)^{\mathbb{V}[G]}$ for $\ell = 1, 2$, hence $\mathrm{can}^\flat(\mathbb{B}_1) = \mathrm{can}^\flat(\mathbb{B}_2)$ in $\mathbb{V}[G]$. Thus, by Lemma 3.17(2), there is some $L^\flat$-isomorphism $h : \mathbb{B}_1 \to \mathbb{B}_2$ in $\mathbb{V}[G]$. We need to show that $h \in \mathbb{V}$, and moreover, there cannot be any other $L^\flat$-isomorphism. For both of these, note that since both $\mathbb{B}_1$ and $\mathbb{B}_2$ are Hausdorff, distinct elements have distinct $\flat$-tp's. Using h, we see that in $\mathbb{V}$, the sets $\{\flat\text{-tp}(\mathbf{a}) : \mathbf{a} \in \mathbb{B}_1\}$ and $\{\flat\text{-tp}(\mathbf{b}) : \mathbf{b} \in \mathbb{B}_2\}$ are equal. Let $h_0 \in \mathbb{V}$ be the unique bijection $h_0 : \mathbb{B}_1 \to \mathbb{B}_2$ that is $\flat$-tp-preserving, i.e., $\flat\text{-tp}(h_0(\mathbf{a})) = \flat\text{-tp}(\mathbf{a})$ for all $\mathbf{a} \in \mathbb{B}_1$. As the uniqueness of h_0 is persistent in forcing extensions, it follows that $h_0 = h$. Thus, $h \in \mathbb{V}$ and is the unique isomorphism between $\mathbb{B}_1$ and $\mathbb{B}_2$. $\square$

From this, we get the following useful corollary.

Corollary 3.29. *Let M be any L-structure and let $M^\flat$ be the canonical flattening of M. Then for any L-structure N,*

$$M \equiv_{\infty,\omega} N \quad \text{if and only if} \quad N^\flat \cong M^\flat$$

Proof. Say $M \equiv_{\infty,\omega} N$. Pass to any forcing extension $\mathbb{V}[G]$ in which both M and N are countable, hence $M \cong N$ in $\mathbb{V}[G]$. As back-and-forth equivalence is absolute, $M^\flat \cong N^\flat$ in $\mathbb{V}[G]$. Pulling back, we have $M^\flat \equiv_{\infty,\omega} N^\flat$ in $\mathbb{V}$. But, as both $M^\flat$ and $N^\flat$ are Hausdorff, there is a unique isomorphism between them by Lemma 3.28.

Conversely, suppose $N^\flat \cong M^\flat$, say via the map h, which is unique by Lemma 3.28. Let $\mathcal{F} = \{(\bar{a}, \bar{b}) \in N^{<\omega} \times M^{<\omega} : h(\pi(\bar{a})) = \pi(\bar{b})\}$, and it suffices to show that $\mathcal{F}$ is a back-and-forth system between N and M. But this follows easily from Lemma 3.12. $\square$

Lemma 3.30. *There is a class bijection $k : \mathrm{CSS}(L)_{\mathrm{ptl}} \to \mathrm{Hdf}^*$ given by $k(\varphi) = \mathrm{can}(M)$, where $M \models \varphi$, $M \in HC^{\mathbb{V}[G]}$ for some forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$ for which $\varphi \in HC^{\mathbb{V}[G]}$.*

Proof. We first show that the functional k is well-defined and that $k(\varphi) \in \mathbb{V}$. For both, given $\varphi \in \mathrm{CSS}(L)_{\mathrm{ptl}}$, say $\mathbb{V}[G]$ and $\mathbb{V}[H]$ are each forcing extensions of $\mathbb{V}$ for which $\varphi \in HC^{\mathbb{V}[G]} \cap HC^{\mathbb{V}[H]}$. Choose countable models $M \models \varphi$ in $\mathbb{V}[G]$ and $N \models \varphi$ in $\mathbb{V}[H]$. Then both M, N are elements of the product forcing $\mathbb{V}[G \times H]$. Working in $\mathbb{V}[G \times H]$, since φ is a canonical Scott sentence, $M \cong N$, hence $\mathrm{can}(M) = \mathrm{can}(N)$ by Lemma 3.15. Let $\mathbb{B}$ denote this common value. Clearly, $\mathrm{can}^{\flat}(\mathbb{B}) = \mathbb{B}$, so $\mathbb{B} \in \mathrm{Hdf}^*$ (in $\mathbb{V}[G \times H]$). Now, by e.g., Lemma 2.5 of [URL17], it follows that $\mathbb{B} \in \mathbb{V}$, hence $\mathbb{B} \in \mathrm{Hdf}^*$ (in $\mathbb{V}$) by the absoluteness of Hdf^*. So put $k(\varphi) := \mathbb{B}$.

To see that k is bijective, we describe its inverse. Choose any $\mathbb{B} \in \mathrm{Hdf}^*$. As $\mathrm{can}^{\flat}{}_{\mathrm{ptl}}(\mathbb{B}) = \mathbb{B}$, $U_0^{\mathbb{B}} = \{\varphi\}$, where $\varphi = \mathrm{css}(M)$ for some/every $M \in HC_{\mathbb{V}[G]}$ for some/every forcing extension $\mathbb{V}[G]$ in which $\mathbb{B} \in HC^{\mathbb{V}[G]}$. Then $k(\varphi) = \mathbb{B}$.

The following summarizes the key relationships. The map between (1) and (2) is given by Lemma 3.28 and the map between (3) and (2) is given by Lemma 3.30.

Conclusion 3.31. *There are natural bijections between three classes:*

(1) $(\mathrm{Hdf}, \cong)$, *the class of isomorphism types of Hausdorff flat structures;*
(2) *The class* Hdf^**; and*
(3) *The class* $\mathrm{CSS}(L)_{\mathrm{ptl}}$ *discussed in* [URL17].

By contrast, the subclass of Hdf^* consisting of all $\mathrm{can}_{\mathrm{ptl}}(M)$ for $M \in \mathrm{Mod}(L)$ is less interesting. In the notation of [URL17], it is bijective with $\mathrm{CSS}(L)_{\mathrm{sat}}$.

3.5. **Relativizing to** $\mathrm{Mod}(\Phi)$. In the whole of the previous subsection, we were working with the class of all L-structures and all flat $L^{\flat}$-structures. However, for applications we want to restrict to e.g., models of a particular sentence of $L_{\omega_1, \omega}$. For a fixed $\Phi \in L_{\omega_1, \omega}$, let $\mathrm{Mod}(\Phi)$ and $\mathrm{Flat}(\Phi)$ denote the class of models of Φ and flat models of $\Phi^{\flat}$, respectively. Similarly, $\mathrm{Hdf}(\Phi)$ denotes the class of Hausdorff flat structures that model $\Phi^{\flat}$ and $\mathrm{Hdf}^*(\Phi)$ is the image of $\mathrm{Flat}(\Phi)$ under $\mathrm{can}^{\flat}{}_{\mathrm{ptl}}$. Owing to the fact that satisfaction is absolute, all of the results in the previous subsection go through, leading to the following relativization of Conclusion 3.31.

Theorem 3.32. *Let* $\Phi \in L_{\omega_1, \omega}$. *There are natural bijections between three classes:*

(1) $(\mathrm{Hdf}(\Phi), \cong)$, *the class of isomorphism types of Hausdorff flat models of* Φ;

(2) *The class* $\mathrm{Hdf}^*(\Phi)$*; and*

(3) *The class* $\mathrm{CSS}(\Phi)_{\mathrm{ptl}}$ *discussed in* [URL17].

In [URL17], a sentence $\Phi \in L_{\omega_1,\omega}$ is *grounded* if $\mathrm{CSS}(\Phi)_{\mathrm{sat}} = \mathrm{CSS}(\Phi)_{\mathrm{ptl}}$. Here, we obtain a more malleable equivalence. Recall that by Proposition 3.7, every *countable* Hausdorff flat structure is isomorphic to $M^\flat$ for some $M \in \mathrm{Mod}(\Phi)$.

Corollary 3.33. *A sentence* $\Phi \in L_{\omega_1,\omega}$ *is grounded if and only if every Hausdorff flat structure (of any size) is isomorphic to* $M^\flat$ *for some* $M \in \mathrm{Mod}(\Phi)$.

4. Flat structures, sharp expansions of a countable M, and $\mathrm{Aut}(M)$

Let M be a countable L-structure with universe ω and let $\varphi_M \in L_{\omega_1\omega}$ be any Scott sentence for M. Among all flattenings $(M, \mathcal{F})^\flat$, we know there is a unique (up to unique isomorphism) countable, Hausdorff flat structure $\mathbb{B}$, namely the canonical flattening $M^\flat$ of M. As $M \models \varphi_M$, it follows that $\mathbb{B} \models \varphi_M^\flat \wedge \mathrm{Ax}^\flat(L)$. We will see that the other countable models of $\varphi_M^\flat \wedge \mathrm{Ax}^\flat(L)$ correspond to closed subgroups of $\mathrm{Aut}(M)$ and we analyze the complexity of this correspondence.

In what follows, for $\Phi \in L_{\omega_1,\omega}$, $\mathrm{Mod}_\omega(\Phi)$ denotes the Polish space of models of Φ with universe ω. Note that $(\mathrm{Mod}_\omega(\varphi_M \wedge \mathrm{Ax}^\sharp(L)), \cong_{L^\sharp})$ is HC-bireducible to $\{$sharp expansions $(M, \mathcal{F})\}/\cong_{L^\sharp}$.

From Proposition 3.7 there is an HC-bireduction between $\mathrm{Mod}_\omega(\varphi_M^\flat \wedge \mathrm{Ax}^\flat(L))/\cong_{L^\flat}$ and $\{$sharp expansions $(M, \mathcal{F})\}/\cong_{L^\sharp}$, so to obtain a correspondence between $\mathrm{Mod}_\omega(\varphi_M^\flat \wedge \mathrm{Ax}^\flat(L))/\cong_{L^\flat}$ and closed subgroups of $\mathrm{Aut}(M)$, it would suffice to find an HC-invariant bijection between sharp expansions $(M, \mathcal{F})$ of M and closed subgroups of $\mathrm{Aut}(M)$. However, such a mapping cannot exist, for the simple reason that closed subgroups of $\mathrm{Aut}(M)$ are rarely elements of HC. Thus, we instead derive a correspondence between sharp expansions and *codes* of closed subgroups of $\mathrm{Aut}(M)$.

4.1. **Codes of closed subgroups.** We begin by defining the codes. It is natural to work with the topological group $S_\infty = \mathrm{Sym}(\omega)$, which we can view as $\mathrm{Aut}(\mathcal{E})$, where $\mathcal{E} := (\omega, =)$.

Definition 4.1. For each $n \in \omega$, let $I_n = \omega^n \times \omega^n$ and let $I = \bigcup\{I_n : n \in \omega\}$. Applying any $\sigma \in S_\infty$ coordinatewise induces a natural map: $I \to I$ (also called σ). For each $(\bar{a}, \bar{b}) \in I$, let $U_{\bar{a},\bar{b}} = \{\sigma \in S_\infty : \sigma(\bar{a}) = \bar{b}\}$.

Then $\{U_{\bar{a},\bar{b}} : (\bar{a}, \bar{b}) \in I\}$ is the standard basis for the topology of pointwise convergence on S_∞. With respect to this topology, for any closed $C \subseteq S_\infty$, let

$$\mathcal{F}(C) = \{(\bar{a}, \bar{b}) \in I : C \cap U_{\bar{a},\bar{b}} \neq \emptyset\}$$

denote the *code of C*. Note that the mapping $C \mapsto \mathcal{F}(C)$ is injective on closed sets and $\mathcal{F}(C) \in HC$ for every $C \subseteq S_\infty$. Moreover, recalling Definitions 2.1 and 2.2, $\mathcal{F}(C)$ is a downward closed back and forth system on $\mathcal{E}$ for every closed $C \subseteq S_\infty$.

We endow $\mathcal{P}(I)$ with the usual Cantor topology, i.e., sets of the form $V_{\bar{a},\bar{b}} := \{A \subseteq I : (\bar{a}, \bar{b}) \in A\}$ and their complements form a sub-basis for the topology. As I is countable, this topology is Polish.

Now suppose $\mathcal{F} \subseteq I$ is a downward closed back and forth system. Construing $\sigma \in S_\infty$ as $\sigma : I \to I$, the inverse operation is given by

$$C_\mathcal{F} = \{\sigma \in S_\infty : \sigma \subseteq \mathcal{F}\}$$

The proof of the following lemma is routine, once one notes that for $\mathcal{F}$ any downward closed back and forth system, if $(\bar{a}, \bar{b}) \in \mathcal{F}$ then there is some $\sigma \in S_\infty$ with $\sigma \subseteq \mathcal{F}$ and $\sigma(\bar{a}) = \bar{b}$.

Lemma 4.2. *The set $\{\mathcal{F}(C) : \emptyset \neq C \subseteq S_\infty$ closed$\}$ is equal to the set of downward closed back and forth systems on $\mathcal{E}$. Specifically, $C_{\mathcal{F}(C)} = C$ for every non-empty closed $C \subseteq S_\infty$, and $\mathcal{F}(C_\mathcal{F}) = \mathcal{F}$ for every downward closed back and forth system $\mathcal{F}$ on $\mathcal{E}$.*

It is easily seen from Lemma 4.2 that the subspace $\{\mathcal{F}(C) : C$ closed in $S_\infty\} \subseteq \mathcal{P}(I)$ is closed, hence is also a Polish space. The correspondence given in Lemma 4.2 explains our interest in Definitions 2.1 and 2.2. Continuing, we are not so much interested in arbitrary closed sets C, but rather closed subgroups $H \leq \mathrm{Aut}(\mathcal{E})$. In the proof of Lemma 4.3, the three clauses of being a subgroup exactly align with the reflexivity, symmetry, and transitivity clauses of being an equivalence relation.

Lemma 4.3. *A closed subset $H \subseteq \mathrm{Aut}(\mathcal{E})$ is a closed subgroup if and only if $\mathcal{F}(H)$ is a sharp back and forth system on the structure $\mathcal{E}$.*

Relativizing Lemma 4.3 to closed subgroups of $\mathrm{Aut}(M)$ for a countable structure M yields the following fundamental result.

Proposition 4.4. *Let $M \in HC$ be any structure. The map $(M, \mathcal{F}) \mapsto \mathcal{F}$ is a persistent, strongly definable bijection between $\{$sharp expansions of $M\}$ and $\{$codes of closed subgroups of $\mathrm{Aut}(M)\}$.*

4.2. **Conjugacy and sharp expansions.** Fix an L-structure M with universe ω. Classically, any expansion M^* of M can be 'seen' by looking at $\mathrm{Aut}(M^*)$, which is a closed subgroup of $\mathrm{Aut}(M)$. Conversely, any closed subgroup $H \leq \mathrm{Aut}(M)$ gives rise to an expansion M_H by letting $L^* = L \cup \{P^{\bar{a}}(\bar{x}) : \bar{a} \in M^{<\omega}\}$ and letting M^* be the expansion of M formed by interpreting each $P^{\bar{a}}$ as the H-orbit of $\bar{a}$ in $M^{\mathrm{lg}(\bar{a})}$. In such an expansion we have $\mathrm{Aut}(M_H) = H$.

Proposition 4.4 gives a different correspondence between sharp expansions and closed subgroups. Given a sharp expansion $(M, \mathcal{F})$ of M, the relevant closed subgroup is

$$\mathrm{Fix}(M, \mathcal{F}) = \{\sigma \in \mathrm{Aut}(M) : (M, \mathcal{F}) \models E_k(\bar{a}, \sigma(\bar{a})) \text{ for all } k, \ \bar{a} \in M^k\}$$

which is a closed subgroup of $\mathrm{Aut}(M, \mathcal{F})$, but is typically not equal to it. The relevance of $\mathrm{Fix}(M, \mathcal{F})$ is largely explained by the following easy lemmas.

Lemma 4.5. *Let M be any L-structure with universe ω and let $\mathcal{F}$ be any sharp back-and-forth system on M. Then, for any $(\bar{a}, \bar{b}) \in \mathcal{F}$ (i.e., $(M, \mathcal{F}) \models E_k(\bar{a}, \bar{b})$) there is some $\sigma \in \mathrm{Fix}(M, \mathcal{F})$ such that $\sigma(\bar{a}) = \bar{b}$.*

Proof. An appropriate σ can be constructed by finite approximations using the back-and-forth property of $\mathcal{F}$. $\square$

Lemma 4.6. *For any sharp back and forth system $\mathcal{F}$ on M, $\mathrm{Fix}(M, \mathcal{F}) = C_{\mathcal{F}}$ from Lemma 4.2.*

Proof. Choose $\sigma \in \mathrm{Fix}(M, \mathcal{F})$. To see $\sigma \subseteq \mathcal{F}$, choose any $\bar{a} \in M^k$. Since $(M, \mathcal{F}) \models E_k(\bar{a}, \sigma(\bar{a}))$, we have $(\bar{a}, \sigma(\bar{a})) \in \mathcal{F}$. Conversely, choose $\sigma \in C_{\mathcal{F}}$. To see that $\sigma \in \mathrm{Aut}(M)$, choose any $(\bar{a}, \bar{b}) \in \sigma$. Since $(\bar{a}, \bar{b}) \in \mathcal{F}$, it follows from Lemma 4.5 that there is an automorphism $\psi \in \mathrm{Aut}(M)$ with $(\bar{a}, \bar{b}) \in \psi$. As this holds for all $(\bar{a}, \bar{b})$ and since $\mathrm{Aut}(M)$ is closed in S_∞, we conclude $\sigma \in \mathrm{Aut}(M)$. To see $\sigma \in \mathrm{Fix}(M, \mathcal{F})$, choose any $\bar{a} \in M^k$. Since $(\bar{a}, \sigma(\bar{a})) \in \mathcal{F}$, we have $(M, \mathcal{F}) \models E_k(\bar{a}, \sigma(\bar{a}))$, as required. $\square$

Definition 4.7. Given an L-structure M with universe ω, let

$$\mathcal{C}_M = \{\mathcal{F}(H) \in \mathcal{P}(I) : H \text{ a closed subgroup of } \mathrm{Aut}(M)\}.$$

Conjugacy induces an equivalence relation $\sim_{\mathrm{conj}}$ on $\mathcal{C}_M$ as: $\mathcal{F}(H_2) \sim_{\mathrm{conj}} \mathcal{F}(H_1)$ if and only if H_2 and H_1 are $\mathrm{Aut}(M)$-conjugate, i.e., $H_2 = \delta H_1 \delta^{-1}$ for some $\delta \in \mathrm{Aut}(M)$.

In light of Lemma 4.2, $\mathcal{C}_M$ is the set of sharp back and forth systems on M. In terms of descriptive set theory, being a closed subspace of $\mathcal{P}(I)$, $\mathcal{C}_M$

is a Polish space. Thus, being an orbit equivalence relation of the action of a Polish group, $\sim_{\mathrm{conj}}$ is an analytic equivalence relation on $\mathcal{C}_M$. We summarize our correspondence with the following. Recall the definitions of $\mathrm{Ax}^\sharp(L)$ and $\mathrm{Ax}^\flat(L)$ from Definitions 2.3 and 3.1, respectively.

Proposition 4.8. *Let M be any L-structure with universe ω and let $\varphi_M \in L_{\omega_1,\omega}$ be any Scott sentence for M. The following three strongly definable quotients are HC-bireducible.*

(1) $(\mathrm{Mod}_\omega(\varphi_M^\flat \wedge \mathrm{Ax}^\flat(L)), \cong_{L^\flat})$;
(2) $(\mathrm{Mod}_\omega(\varphi_M \wedge \mathrm{Ax}^\sharp(L)), \cong_{L^\sharp})$; and
(3) $(\mathcal{C}_M, \sim_{\mathrm{conj}})$.

Proof. The correspondence between (1) and (2) is given by the flattening operation and Proposition 3.7. To get the HC-bireduction between (2) and (3), we use the map $(M, \mathcal{F}) \mapsto \mathcal{F}$ from Proposition 4.4. We show this map preserves quotients in both directions, i.e.

$$(M, \mathcal{F}_1) \cong_{L^\sharp} (M, \mathcal{F}_2) \quad \text{if and only if} \quad \mathcal{F}_1 \sim_{\mathrm{conj}} \mathcal{F}_2$$

To establish this, choose sharp back and forth systems $\mathcal{F}_1, \mathcal{F}_2$ on M. By Lemma 4.6, $H_1 := \mathrm{Fix}(M, \mathcal{F}_1)$ and $H_2 := \mathrm{Fix}(M, \mathcal{F}_2)$ are the closed subgroups of $\mathrm{Aut}(M)$ coded by $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. It is readily checked that for any $\delta \in \mathrm{Aut}(M)$, $H_2 = \delta^{-1}H_1\delta$ if and only if $\mathcal{F}_2 = \delta^{-1}\mathcal{F}_1\delta$ if and only if $\delta : (M, \mathcal{F}_1) \to (M, \mathcal{F}_2)$ is an $L^\sharp$-isomorphism. $\qquad\square$

4.3. **Comparing families $\mathcal{C}_M$ and the completeness of $\mathcal{C}_{(\omega,=)}$.** Here we compare the complexity of quotients of $\mathcal{C}_M$ and $\mathcal{C}_N$ by $\sim_{\mathrm{conj}}$, where M and N are each structures with universe ω, although possibly in different languages. The following Lemma must be very standard.

Lemma 4.9. *Suppose M and N are structures (possibly in different countable languages), each with universe ω, and choose a continuous, surjective homomorphism $f : \mathrm{Aut}(N) \to \mathrm{Aut}(M)$. Then the map $H \mapsto f^{-1}(H)$ is an injection of closed subgroups that respects conjugacy. Thus, the induced mapping $h : (\mathcal{C}_M, \sim_{\mathrm{conj}}) \to (\mathcal{C}_N, \sim_{\mathrm{conj}})$ of the codes given by $h(\mathcal{F}(H)) = \mathcal{F}(f^{-1}(H))$ is an HC-reduction.*

Proof. We first show that the map $H \mapsto f^{-1}(H)$ factors through conjugacy. That is, suppose $H_1, H_2 \in \mathcal{C}_M$ satisfy $H_2 = \delta H_1 \delta^{-1}$ for some $\delta \in \mathrm{Aut}(M)$. As f is onto, choose $\gamma \in \mathrm{Aut}(N)$ such that $f(\gamma) = \delta$. We argue that $f^{-1}(H_2) = \gamma f^{-1}(H_1)\gamma^{-1}$. To see this, choose $\sigma \in f^{-1}(H_1)$. It suffices to

show $\gamma\sigma\gamma^{-1} \in f^{-1}(H_2)$, i.e., that $f(\gamma\sigma\gamma^{-1}) \in H_2$. But this is immediate, as f is a homomorphism and $\delta f(\sigma)\delta^{-1} \in H_2$. Thus, h is well-defined.

To see that h is injective on classes, suppose $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$ in $\mathcal{C}_N$ are conjugate, i.e., $G_2 = \gamma G_1 \gamma^{-1}$ for some $\gamma \in \mathrm{Aut}(N)$. As f is a homomorphism, it follows that $H_2 = f(\gamma)H_1 f(\gamma)^{-1}$, so H_1 and H_2 are conjugate in $\mathcal{C}_M$.

To show HC-invariance, choose a closed $H \leq \mathrm{Aut}(M)$ and a forcing extension $\mathbb{V}[G] \supseteq \mathbb{V}$. Let H' be the closed subgroup of $\mathrm{Aut}(M)^{\mathbb{V}[G]}$ with the same code as H. Since they have the same codes, $H \subseteq H'$ is dense. Because analytic statements are absolute, we have $f^{-1}(H)^{\mathbb{V}}$ is dense in $f^{-1}(H')^{\mathbb{V}[G]}$, so they have the same codes. $\qquad\square$

For this to be interesting, we need to know that $(S_\infty, \mathrm{conj})$ is sufficiently complicated. Recall that we can identify S_∞ with $\mathrm{Aut}(\omega, =)$.

Lemma 4.10. $(\mathrm{Mod}_\omega(Graphs), \cong)$ *is HC-reducible to* $(\mathcal{C}_{(\omega,=)}, \sim_{\mathrm{conj}})$.

Proof. A graph is an $\{R\}$-structure, where R is a symmetric, irreflexive relation. Call an ω-tree $(T, \trianglelefteq)$ *padded* if, for all $a \in T$ and $b \in \mathrm{Succ}(a)$, there is an infinite set $\{b_i : i \in \omega\} \subseteq \mathrm{Succ}(a)$ such that $T_{\geq b_i} \cong T_{\geq b}$ for every i. An easy alteration[3] of the original Friedman-Stanley proof that {Subtrees of $\omega^{<\omega}$} are Borel complete (Theorem 1.1.1 of [FS89]) shows that there is a Borel reduction

$$f : \text{Graphs} \to \text{Padded subtrees of } (\omega \times \omega)^{<\omega}$$

Recall $\mathcal{E} = (\omega, =)$, so $\mathrm{Aut}(\mathcal{E}) = S_\infty$ and $\mathcal{C}_\mathcal{E}$ denotes the codes of closed subgroups of S_∞. Let $g : \{\text{Graphs}\} \to \mathcal{C}_\mathcal{E}$ be the map $g(G) := \mathcal{F}(\mathrm{Aut}(f(G)))$.

We must show that if G, H are graphs with universe ω and the groups $\mathrm{Aut}(f(G)), \mathrm{Aut}(f(H))$ are conjugate by some $\delta \in S_\infty$, then G and H are isomorphic. [The converse is easy.] We verify this by way of the following Claim.

Claim. If $(T, \trianglelefteq)$ is any padded ω-tree, then for any $a, b \in T$, $a \trianglelefteq b$ if and only if every $\sigma \in \mathrm{Aut}(T, \trianglelefteq)$ that fixes b also fixes a.

Proof. Left to right is obvious, as for any ω-tree $(T, \trianglelefteq)$, if $a \trianglelefteq b$, then $a \in \mathrm{dcl}(b)$. For the converse, note that for any $c \in T$ and any $d, d' \in \mathrm{Succ}(c)$ with $T_{\geq d} \cong T_{\geq d'}$, there is $\sigma \in \mathrm{Aut}(T, \trianglelefteq)$ fixing $T\setminus(T_{\geq d}\cup T_{\geq d'})$ with $\sigma(d) = d'$.

Now assume that $a \not\trianglelefteq b$ and let $c = a \wedge b$. There are two cases. First, if $c = b$ (i.e., if $b \trianglelefteq a$) then let $d \in \mathrm{Succ}(b)$ with $d \trianglelefteq a$. As $(T, \trianglelefteq)$ is padded, choose $d' \in \mathrm{Succ}(b)$ with $d' \neq d$ but $T_{\geq d'} \cong T_{\geq d}$. Then any $\sigma \in \mathrm{Aut}(T, \trianglelefteq)$

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[3]Specifically, adding multiplicities to account for the padding.

fixing b with $\sigma(d) = d'$ will move a as well. Second, assume $c \neq b$, hence $c \triangleleft b$. Let $d \in \mathrm{Succ}(c)$ with $d \trianglelefteq a$. Choose $d' \in \mathrm{Succ}(c)$ such that $d' \neq d$ and $d' \ntrianglelefteq b$, but with $T_{\geq d'} \cong T_{\geq d}$. Any automorphism $\sigma \in \mathrm{Aut}(T, \trianglelefteq)$ with $\sigma(d) = d'$ that fixes $T \setminus (T_{\geq d} \cup T_{\geq d'})$ will fix b, but move a. $\square$

To see that this Claim suffices, suppose G, H are graphs with universe ω and $\mathrm{Aut}(f(H)) = \delta \, \mathrm{Aut}(f(G)) \delta^{-1}$. Because of the Borel reduction, it suffices to show that $\delta : (f(G), \trianglelefteq) \cong (f(H), \trianglelefteq)$ is a tree isomorphism. As $\delta \in S_\infty$ and by symmetry, it suffices to show that if $a, b \in f(G)$ and $(f(G), \trianglelefteq) \models a \trianglelefteq b$, then $(f(H), \trianglelefteq) \models \delta(a) \trianglelefteq \delta(b)$. But this is clear: Let $c = \delta(a)$, $d = \delta(b)$, and choose any $\tau \in \mathrm{Aut}(f(H), \trianglelefteq)$ with $\tau(d) = d$. Put $\sigma := \delta^{-1} \tau \delta \in \mathrm{Aut}(f(G), \trianglelefteq)$. Since $\tau(d) = d$, $\delta(\sigma(b)) = \delta(b)$, hence $\sigma(b) = b$. By the Claim, $\sigma(a) = a$ as well, hence $\tau(c) = c$. Thus, $(f(H), \trianglelefteq) \models c \trianglelefteq d$ by the Claim. $\square$

5. Characterizing Borel complete expansions

With the extensive preliminaries out of the way, we are now able to obtain our characterization of sentences of $L_{\omega_1, \omega}$ that have Borel complete expansions. We begin with a classical result of Morley [Mo63] that characterizes which sentences of $L_{\omega_1, \omega}$ have Ehrenfeucht-Mostowski models. The novelty will be that we apply this result to a class of flat structures. The following fact is contained in Chapter 5 of [Ma16].

Fact 5.1. *The following are equivalent for $\psi \in L_{\omega_1, \omega}$.*

(1) *For all $\alpha < \omega_1$, ψ has a model of size at least $\beth_\alpha$;*

(2) *ψ has arbitrarily large models;*

(3) *There is a countable $L^* \supseteq L$, a countable, Skolemized fragment $\mathcal{F}^*$ containing ψ, and a countable L^*-structure $N^* \models \psi$ that is the Skolem hull of a sequence $\langle a_q : q \in \mathbb{Q} \rangle$ of $\mathcal{F}^*$-order indiscernibles; and*

(4) *For all linear orderings $(J, \leq)$, there is a model $N_J \models \psi$ containing a sequence $\langle a_j : j \in J \rangle$ of order indiscernibles.*

For brevity, we say that ψ *admits Ehrenfeucht-Mostowski models* if any of the four conditions hold. Next, we have a group theoretic notion.

Definition 5.2. We say that the topological group H *divides the topological group G* if there is a closed subgroup $G' \leq G$ and a continuous, surjective homomorphism $h : G' \to H$.

Obviously, if $H \leq G$ is closed, then H divides G. We note two useful examples of dividing among automorphism groups.

Lemma 5.3. (1) For N^* chosen as in Fact 5.1(3), $\mathrm{Aut}(\mathbb{Q}, \leq)$ divides $\mathrm{Aut}(N^*)$.

(2) Let $(M, \mathcal{F})$ be any countable sharp structure and let $\mathbb{B} = (M, \mathcal{F})^\flat$ be its flattening. Then $\mathrm{Aut}(\mathbb{B})$ divides $\mathrm{Aut}(M)$.

Proof. (1) Given N^* as in Fact 5.1(3), let $G' \leq \mathrm{Aut}(N^*)$ denote the setwise stabilizer of $\{a_q : q \in \mathbb{Q}\}$. Then, as $\langle a_q : q \in \mathbb{Q}\rangle$ are order indiscernible and $N^* = \mathrm{dcl}(\{a_q : q \in \mathbb{Q}\})$, G' is topologically isomorphic to $\mathrm{Aut}(\mathbb{Q}, \leq)$.

(2) As $\mathrm{Aut}(M, \mathcal{F})$ is a closed subgroup of $\mathrm{Aut}(M)$, it suffices to verify that the homomorphism $h : \mathrm{Aut}(M, \mathcal{F}) \to \mathrm{Aut}(\mathbb{B})$ is a continuous surjection, where h is defined by $h(\sigma)$ being the automorphism of $\mathbb{B}$ satisfying $h(\sigma)(\mathbf{a}) = \pi_k(\sigma(\bar{a}))$ for every $\mathbf{a} \in U_k(\mathbb{B})$ and some $\bar{a} \in \pi_k^{-1}(\mathbf{a})$. Showing that h is well defined amounts to verifying that $\pi_k(\sigma(\bar{a})) = \pi_k(\sigma(\bar{a}'))$ whenever $(\bar{a}, \bar{a}') \in \mathcal{F}$, which is immediate since $\sigma \in \mathrm{Aut}(M, \mathcal{F})$. That h is continuous follows from the fact that for any $\mathbf{a} \in \mathbb{B}$, $h^{-1}(\mathrm{Fix}(\mathbf{a}))$ is equal to the set of $\sigma \in \mathrm{Aut}(M, \mathcal{F})$ fixing the E_k-class $[\bar{a}]_k$. That h is onto follows from the final sentence of Proposition 3.7: Any $\sigma \in \mathrm{Aut}(\mathbb{B})$ lifts to an automorphism of $(M, \mathcal{F})$. $\square$

It is easily verified that the relation of dividing is transitive. What is more surprising is the following, which largely appears as Lemma 2.5 of [H01].

Lemma 5.4. S_∞ divides $\mathrm{Aut}(\mathbb{Q}, \leq)$. For any topological group G, S_∞ divides G if and only if $\mathrm{Aut}(\mathbb{Q}, \leq)$ divides G.

Proof. To see that S_∞ divides $\mathrm{Aut}(\mathbb{Q}, \leq)$, choose a partition $\mathbb{Q} = \sqcup\{D_n : n \in \omega\}$ into dense sets and let $E \subseteq \mathbb{Q}^2$ be the equivalence relation given by $E(a, b)$ iff $\bigwedge_{n \in \omega} D_n(a) \leftrightarrow D_n(b)$. Let $G' = \mathrm{Aut}(\mathbb{Q}, \leq, E)$, which is a closed subgroup of $\mathrm{Aut}(\mathbb{Q}, \leq)$. But it is easily checked that the map $h : G' \to S_\infty$, where $h(\sigma)$ is the element of S_∞ given by $h(\sigma)(n) =$ the (unique) m such that $\sigma(a) \in D_m$ for any/all $a \in D_n$ is a continuous group homomorphism. The fact that h is surjective is by a standard back-and-forth argument.

As $\mathrm{Aut}(\mathbb{Q}, \leq)$ is a closed subgroup of S_∞, it trivially divides it, so the second sentence follows by the transitivity of dividing. $\square$

We can now state and prove our main theorem. Recall that by Definition 3.1 flat $L^\flat$-structures are axiomatized by $\mathrm{Ax}^\flat(L)$, hence $\mathrm{Mod}(\varphi^\flat \wedge \mathrm{Ax}^\flat(L)) = \mathrm{Flat}(\varphi)$ for any infinitary L-sentence φ.

Theorem 5.5. *For a countable language L, the following are equivalent for a sentence $\varphi \in L_{\omega_1,\omega}$.*

> *(1) φ has a Borel complete expansion (i.e., there is some countable $L' \supseteq L$ such that $\mathrm{Mod}_{L'}(\varphi)$ is Borel complete);*
> *(2) $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ has arbitrarily large models;*
> *(3) $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ admits Ehrenfeucht-Mostowski models;*
> *(4) S_∞ divides $\mathrm{Aut}(M)$ for some countable $M \models \varphi$.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) is by Fact 5.1 applied to the $L_{\omega_1,\omega}$-sentence $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$. We prove the equivalence of (1) and (4) by independently showing that each is equivalent to (2).

(1) $\Rightarrow$ (2): Suppose that the class of L'-models of φ is Borel complete. To ease notation, let ψ denote the $(L')_{\omega_1,\omega}$-sentence φ. As the class of L'-structures modeling ψ is Borel complete, $\|\psi\| = |\psi_{\mathrm{ptl}}| = \infty$, i.e., there is a proper class of pairwise back-and-forth inequivalent models of ψ. Thus, $\psi^\flat \wedge \mathrm{Ax}^\flat(L')$ has a proper class of non-isomorphic models by Corollary 3.29. This implies that $\psi^\flat \wedge \mathrm{Ax}^\flat(L')$ has arbitrarily large models. Taking L-reducts, we conclude that $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ has arbitrarily large models as well.

(2) $\Rightarrow$ (1): Suppose $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ has arbitrarily large models, hence admits EM models. Applying Fact 5.1(3) to $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$, choose a a countable model $\mathbb{B}^* \models \varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ in an expanded language $L^* \supseteq L^\flat$, the Skolem hull of a sequence $\langle \mathbf{a}_q : q \in \mathbb{Q} \rangle$ of order indiscernibles. Let $\mathbb{B}$ be the $L^\flat$-reduct of $\mathbb{B}^*$, which is a countable, flat structure. Using Proposition 3.7, let $(M, \mathcal{F})$ be a sharp structure whose flattening is $\mathbb{B}$. So M is a countable L-structure and $M \models \varphi$ since $\mathbb{B} \models \varphi^\flat$.

For definiteness, suppose $\langle \mathbf{a}_q : q \in \mathbb{Q} \rangle$ are from $U_k(\mathbb{B})$. As in the proof of Lemma 5.4, partition $\mathbb{Q} = \sqcup \{D_n : n \in \omega\}$ into dense pieces and, for each n, let $X_n := \bigcup \{cov_k^{-1}(\mathbf{a}_q) : q \in D_n\}$. Let $X^* := \bigcup \{X_n : n \in \omega\} \subseteq M^k$ and let E^* be the equivalence relation on X^* whose classes are X_n. Let $L' = L \cup \{X^*, E^*, R\}$ and let ψ be the same sentence φ, but viewed as an $(L')_{\omega_1,\omega}$-sentence. We can see that ψ is Borel complete directly: Given any directed graph $G = (\omega, r)$, let M_G be the expansion (M, X^*, E^*, R) of M with $R(\bar{a}, \bar{b})$ holding if and only if for some $n, m \in \omega$, $\bar{a} \in X_n$, $\bar{b} \in X_m$, and $G \models r(n, m)$. It is evident that if $M_{G_1} \cong M_{G_2}$, then $G_1 \cong G_2$ as graphs. For the other direction, choose a graph isomorphism $h : G_1 \to G_2$. As each D_n is a countable, dense subset of $(\mathbb{Q}, \leq)$, a back-and-forth argument yields an automorphism $h^* \in \mathrm{Aut}(\mathbb{Q}, \leq)$ encoding h, i.e.,

$$\text{for all } n \in \omega \text{ and for all } q \in \mathbb{Q}, [q \in D_n \Longleftrightarrow h^*(q) \in D_{h(n)}]$$

As $\mathbb{B}$ is the reduct of a Skolem hull indexed by $(\mathbb{Q}, \leq)$, there is an $L^\flat$-automorphism $\sigma \in \text{Aut}(\mathbb{B})$ satisfying $\sigma(\mathbf{a}_q) = \mathbf{a}_{h^*(q)}$ for every $q \in \mathbb{Q}$. By Proposition 3.7 there is an L-automorphism $j : M \to M$ such that for all $\ell \in \omega$ and $\bar{b} \in M^\ell$, $cov_\ell(j(\bar{b})) = \sigma(cov_\ell(\bar{b}))$. In particular, for each $q \in \mathbb{Q}$, j maps $cov_k^{-1}(\mathbf{a}_q)$ onto $cov_k^{-1}(\mathbf{a}_{h^*(q)})$ setwise. Thus, j preserves X^* setwise, and moreover maps each X_n onto $X_{h(n)}$. This means that j also preserves E^*, and by our interpretation of R in both M_{G_1} and M_{G_2}, $j : M_{G_1} \to M_{G_2}$ is an L'-isomorphism.

$(4) \Rightarrow (2)$: Suppose $M \models \varphi$ is countable and S_∞ divides $\text{Aut}(M)$. Choose an expansion M^* of M to an L^*-structure for which there is a continuous surjection $f : \text{Aut}(M^*) \to Aut(\mathcal{E})$ (recall that $\mathcal{E} = (\omega, =)$ and $S_\infty = \text{Aut}(\omega, =)$). By Lemma 4.9, there is an HC-reduction $(\mathcal{C}_\mathcal{E}, \sim_{\text{conj}}) \leq_{HC} (\mathcal{C}_{M^*}, \sim_{\text{conj}})$. Let φ_{M^*} be a Scott sentence for M^*. Composing the above with the HC-reduction from Lemma 4.10 and applying Lemma 4.8 to the right hand side yields an HC-reduction

$$(\text{Mod}_\omega(\text{Graphs}), \cong) \leq_{HC} (\text{Mod}_\omega(\varphi^\flat_{M^*} \wedge \text{Ax}^\flat(L^*)), \cong_{(L^*)^\flat})$$

As $||(\text{Graphs}, \cong)|| = \infty$, it follows from Fact 1.5 that there is a proper class of non-$(L^*)^\flat$-isomorphic models of $\varphi^\flat_{M^*} \wedge \text{Ax}^\flat(L^*)$. Since $\varphi^\flat_{M^*} \wedge \text{Ax}^\flat(L^*) \vdash \varphi^\flat_M \wedge \text{Ax}^\flat(L)$, the latter has arbitrarily large models.

$(2) \Rightarrow (4)$: $\varphi^\flat \wedge \text{Ax}^\flat(L)$ has arbitrarily large models. By Fact 5.1(3), choose $L^* \supseteq L^\flat$ and a countable L^*-structure $\mathbb{B}^* \models \varphi^\flat \wedge \text{Ax}^\flat(L)$ that is the Skolem hull of a sequence $\langle \mathbf{a}_q : q \in \mathbb{Q} \rangle$ of order indiscernibles and let $\mathbb{B}$ be the $L^\flat$-reduct of $\mathbb{B}^*$. By Lemma 5.3(1), $\text{Aut}(\mathbb{Q}, \leq)$ divides $\text{Aut}(\mathbb{B}^*)$, hence also divides $\text{Aut}(\mathbb{B})$, since $\text{Aut}(\mathbb{B}^*) \leq \text{Aut}(\mathbb{B})$ is closed. As $\mathbb{B} \models \varphi^\flat$ is countable, choose $(M, \mathcal{F})$ as in Proposition 3.7. Then $M \models \varphi$ and is countable. By Lemma 5.3(2) $\text{Aut}(\mathbb{B})$ divides $\text{Aut}(M)$, hence $\text{Aut}(\mathbb{Q}, \leq)$ divides $\text{Aut}(M)$ by transitivity. Thus S_∞ divides $\text{Aut}(M)$ by Lemma 5.4. $\qquad \square$

Corollary 5.6. *Suppose $\varphi \in L_{\omega_1, \omega}$ admits Ehrenfeucht-Mostowski models. Then φ has a Borel complete expansion.*

Proof. Apply Fact 5.1 directly to φ and get $L^* \supseteq L$ countable and a countable L^*-structure $M^* \models \varphi$ that is the Skolem hull of an order-indiscernible sequence $\langle a_q : q \in \mathbb{Q} \rangle$. By Lemma 5.3(1), $\text{Aut}(\mathbb{Q}, \leq)$ divides $\text{Aut}(M^*)$, hence also divides $\text{Aut}(M)$, where M is the L-reduct of M^*. Thus, by Lemma 5.4 S_∞ divides $\text{Aut}(M)$, so Theorem 5.5 applies. $\qquad \square$

The converse of Corollary 5.6 does not hold. It follows from Theorem 1.10 of [BFKL16] that if φ_M is the Scott sentence of the Fraïssé limit M of a

class K of finite structures satisfying disjoint amalgamation, then S_∞ divides $\mathrm{Aut}(M)$. Thus, any of the examples of Scott sentences φ_n characterizing $\aleph_n$ in [BKL17] do not admit Ehrenfeucht-Mostowski models, but have Borel complete expansions. Similarly, in [H02] Hjorth shows that for every $\alpha < \omega_1$, there is a Scott sentence ψ_α characterizing $\aleph_\alpha$ with S_∞ dividing the automorphism group of its countable model. So again, each ψ_α does not admit Ehrenfeucht-Mostowski models, yet has a Borel complete expansion.

By contrast, the much older example of Knight in [Kn77] of a complete $L_{\omega_1,\omega}$-sentence φ_{Kn} characterizing $\aleph_1$ is not obtained as a Fraïssé limit. In [H02], Hjorth proves that S_∞ does not divide $\mathrm{Aut}(M)$, the automorphism group of the (unique) countable model of φ_{Kn}. Thus, φ_{Kn} does not have a Borel complete expansion by Theorem 5.5.

6. Applications

We record two instances where the conditions of Theorem 5.5 hold.

Corollary 6.1. *Let L be a countable language.*

(1) *If T is a first-order L-theory with an infinite model, then T has a Borel complete expansion.*
(2) *If $\varphi \in L_{\omega_1,\omega}$ and $||\varphi_{\mathrm{ptl}}|| \geq \beth_{\omega_1}$, then φ has a Borel complete expansion.*

Proof. (1) By Ramsey and compactness, T admits Ehrenfeucht-Mostowski models, so apply Corollary 5.6 to $\varphi := \bigwedge T$.

For (2), it follows from $||\varphi_{\mathrm{ptl}}|| \geq \beth_{\omega_1}$ that $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ has at least $\beth_{\omega_1}$ non-isomorphic models. As $L^\flat$ is countable, for every $\alpha < \omega_1$, there is a model of $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$ of size at least $\beth_\alpha$. So, by applying Fact 5.1 to $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$, there are arbitrarily large models of $\varphi^\flat \wedge \mathrm{Ax}^\flat(L)$. Thus, Theorem 5.5 applies. □

More interesting are cases where the conditions of Theorem 5.5 do not hold. In [LU23], the authors considered families of cross-cutting equivalence relations, each with finitely many classes. More formally, fix the language $L = \{E_n : n \in \omega\}$. For any function $h : \omega \to (\omega - \{0, 1\})$, let $T_h^\forall$ be the universal L-theory asserting that each E_n is an equivalence relation with at most $h(n)$ classes. It is easily checked that $T_h^\forall$ has a model completion T_h, which asserts that each E_n has exactly $h(n)$ classes, and the relations cross-cut, i.e., for any finite $F \subseteq \omega$, if we let $E_F(x, y) := \bigwedge_{n \in F} E_n(x, y)$, then E_F has exactly $\Pi_{n \in F} h(n)$ classes.

In Remark 2.7 of [LU23], the authors prove that if the set of integers $\{h(n) : n \in \omega\}$ is unbounded, then T_h is Borel complete. Here, we use Theorem 5.5 to obtain the converse.

Theorem 6.2. *If the set of integers $\{h(n) : n \in \omega\}$ is bounded, then T_h is not Borel complete.*

Proof. As T_h is first-order, Corollary 6.1 shows we cannot apply Theorem 5.5 to T_h directly. Rather, consider an arbitrary countable model $M \models T_h$. Consider the infinitary formula $E_\infty(x, y) := \bigwedge_{n \in \omega} E_n(x, y)$. Then E_∞ is an equivalence relation. For each $a \in M$, $[a]_\infty := \{b \in M : E_\infty(a, b)\}$, which ostensibly can have any cardinality m, $1 \le m \le \omega$. Let $\varphi_h \in L_{\omega_1, \omega}$ assert:

$$\bigwedge T_h \wedge \forall x \forall y (E_\infty(x, y) \to x = y)$$

It is easily checked that any model $M \models \varphi_h$ has cardinality at most $2^{\aleph_0}$. Moreover, its automorphism group $\mathrm{Aut}(M)$ is easy to analyze.

Claim. For $M \models \varphi_h$, $\mathrm{Aut}(M)$ isomorphically embeds into $\Pi_{n \in \omega} \mathrm{Sym}(h(n))$, where $\mathrm{Sym}(h(n))$ is the group of permutations of $\{0, \ldots, h(n) - 1\}$. Since $\{h_n\}$ is bounded, $\mathrm{Aut}(M)$ has bounded exponent.

Proof. As no two elements of M realize the same E_∞-class, every automorphism of M is determined by how it permutes the classes. As the equivalence relations E_n have $h(n)$ classes, $\mathrm{Aut}(M)$ is isomorphic to a subgroup of $\Pi_{n \in \omega} \mathrm{Sym}(h(n))$. Since $\{h(n)\}$ is bounded, choose an integer K so that $h(n) \le K$ for all $n \in \omega$. As each of the finite groups $\mathrm{Sym}(h(n))$ has exponent dividing $K!$, so does $\mathrm{Aut}(M)$. □

It follows from the Claim that S_∞ cannot divide the automorphism group of any countable model $M \models \varphi_h$, so we conclude from Theorem 5.5 that no expansion of φ_h can be Borel complete.

To finish the proof of Theorem 6.2, we note there is a natural Borel reduction from $\mathrm{Mod}_\omega(T_h)$ to $\mathrm{Mod}_\omega(\psi_h)$ for some expansion $\psi_h \vdash \varphi_h$. Given a countable $M \models T_h$, we simply encode the size of each E_∞-class by unary predicates. In more detail, let $L^* = L \cup \{U_m : 1 \le m \le \omega\}$, where each U_m is a unary predicate, and let ψ_h be the $(L^*)_{\omega_1, \omega}$-sentence asserting φ_h and that the predicates $\{U_m : 1 \le m \le \omega\}$ partition the universe. As it is an expansion of φ_h, ψ_h is not Borel complete.

But easily, T_h and ψ_h are Borel equivalent, since given any countable $M \models T_h$, let $f(M)$ be the L^*-structure with universe M/E_∞, where the equivalence relations E_n are interpreted naturally, and where $U_m([a]_{E_\infty})$ holds if and only if $|[a]_{E_\infty}| = m$ (i.e., we 'color the branches by the number

of realizations of the branch'). Since ψ_h is not Borel complete, neither is T_h.
$\square$

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