Uniformly bounded arrays and mutually algebraic structures

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Abstract

We define an easily verifiable notion of an atomic formula having *uniformly bounded arrays* in a structure M. We prove that if T is a complete L-theory, then T is mutually algebraic if and only if there is some model M of T for which every atomic formula has uniformly bounded arrays. Moreover, an incomplete theory T is mutually algebraic if and only if every atomic formula has uniformly bounded arrays in every model M of T.

1 Introduction

The notion of a mutually algebraic formula was introduced in [1], and the notions of mutually algebraic structures and theories were introduced in [3]. There, many properties were shown to be equivalent to mutual algebraicity, e.g., a structure M is mutually algebraic if and only if every expansion (M, A) by a unary predicate has the non-finite cover property (nfcp) and a complete theory T is mutually algebraic if and only if it is weakly minimal and trivial. Whereas these characterizations indicate the strength of the hypothesis, they do not lead to an easy verification that a specific structure is mutually algebraic. The purpose of this paper is two-fold. Primarily, we obtain equivalents of mutual algebraicity that are easily verifiable. Most notably, we introduce

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the notion of a structure or a theory having *uniformly bounded arrays* and we prove that for structures M in a finite, relational language, M is mutually algebraic if and only if M has uniformly bounded arrays. This result plays a key role in [4], where the authors describe the growth rates of hereditary classes of structures in a finite, relational language. The other purpose is to develop a 'Ryll-Nardzewski characterization' of mutual algebraicity, which is accomplished in Theorem 6.1. A theory T is mutually algebraic if, for every restriction to a finite sublanguage, for every model M and integer n, there are only finitely many quantifier-free n-types over M that support an infinite array.

2 Preliminaries

Let M be any L-structure and let $\varphi(\overline{z})$ be any $L(M)$-formula. We say that $\varphi(\overline{z})$ is *mutually algebraic* if there is an integer k such that for any proper partition $\overline{z} = \overline{x}\hat{\ }\overline{y}$ (i.e., each of $\overline{x}, \overline{y}$ are nonempty) $M \models \forall \overline{x} \exists^{\leq k} \overline{y} \varphi(\overline{x}, \overline{y})$. Then, following [3], a structure M is *mutually algebraic* if every $L(M)$-formula is equivalent to a boolean combination of mutually algebraic $L(M)$-formulas, and a theory T is mutually algebraic if every model of T is a mutually algebraic structure. The following Theorem, which has the advantage of looking only at atomic formulas, follows easily from two known results.

Theorem 2.1. *Let M be any L-structure. Then M is mutually algebraic if and only if every atomic formula $R(\overline{z})$ is equivalent to a boolean combination of quantifier-free mutually algebraic $L(M)$-formulas.*

Proof. First, assume M is mutually algebraic. The fact that every atomic $R(\overline{z})$ is equivalent to a boolean combination of quantifier-free mutually algebraic $L(M)$-formulas is the content of Proposition 4.1 of [2]. For the converse, let $MA^*(M)$ denote the set of $L(M)$-formulas that are boolean combinations of mutually algebraic formulas. This set is clearly closed under boolean combinations, and is closed under existential quantification by Propositon 2.7 of [3]. Thus, if we assume that every atomic formula is in $MA^*(M)$, it follows at once that every $L(M)$-formula is in $MA^*(M)$, hence M is mutually algebraic.

We will obtain a slight strengthening of Theorem 2.1 with Corollary 7.4(2). Whereas Theorem 2.1 placed no assumptions on the language, the main body of results in this paper assume that the underlying language is finite relational. In Section 7 we obtain equivalents to mutual algebraicity for structures in arbitrary languages.

Henceforth, for all results prior to Section 7, assume L has finitely many relation symbols, finitely many constant symbols, and no function symbols.

For L as above, fix an integer n so that every atomic formula is at most n-ary. As every quantifier-free type $p(\overline{w})$ is determined by its family of restrictions $\{p{\restriction}_{\overline{w}_i}\colon \overline{w}_i$ a subsequence of $\overline{w}$ of length at most $n\}$, choose a specific n-tuple $\overline{z} = (z_1, \dots, z_n)$ of distinct variable symbols. Throughout, we concentrate on understanding spaces of quantifier-free types $p(\overline{x})$, where $\overline{x}$ is a non-empty subsequence of $\overline{z}$.[1]

Fix an L-structure M. For a non-empty subsequence $\overline{x} \subseteq \overline{z}$ and a subset $B \subseteq M$, let $\mathrm{QF}_{\overline{x}}(B)$ denote the set of all quantifier-free L-formulas $\varphi(\overline{x}, \overline{b})$ whose free variables are among $\overline{x}$ and $\overline{b}$ is from B. Whereas we require $\overline{x}$ to be a subsequence of $\overline{z}$, there are no limitations on the length of the parameter sequence $\overline{b}$. By looking at the subsets of $M^{\lg(\overline{x})}$ they define, we can construe $\mathrm{QF}_{\overline{x}}(B)$ as a boolean algebra. Let $S_{\overline{x}}(B)$ denote its associated Stone space, i.e., the set of **quantifier-free $\overline{x}$-types over B that decide each** $\varphi \in \mathrm{QF}_{\overline{x}}(B)$. As usual, each of the Stone spaces $S_{\overline{x}}(B)$ are compact, Hausdorff, and totally disconnected when topologized by positing that the sets $\{U_{\varphi(\overline{x},\overline{b})} : \varphi(\overline{x}, \overline{b}) \in \mathrm{QF}_{\overline{x}}(B)\}$, where $U_{\varphi(\overline{x},\overline{b})} = \{p \in S_{\overline{x}}(B) : \varphi(\overline{x}, \overline{b}) \in p\}$, form a basis. Moreover, because L is finite relational, it follows that each $S_{\overline{x}}(B)$ is finite and every $p \in S_{\overline{x}}(B)$ is determined by a single $\varphi(\overline{x}, \overline{b}) \in p$ whenever B is finite.

Definition 2.2. Fix a non-empty $\overline{x} \subseteq \overline{z}$, a subset $B \subseteq M$ and an integer m. An $\overline{x}$-type $p \in S_{\overline{x}}(B)$ *supports an m-array* if there is a **pairwise disjoint** set $\{\overline{d}_i : i < m\}$ of (distinct) realizations of p in M. *p supports an infinite array* if M contains an infinite, pairwise disjoint set of realizations of p. For each **finite** $D \subseteq M$, let $N_{\overline{x},m}(D)$ be the (finite) number of $p \in S_{\overline{x}}(D)$ that support an m-array.

The following definition is central to this paper, and forms the connection with [4]. A local formulation, which relaxes the restriction on the language is given in Section 7.

[1] The presentation of free variables in a type is delicate, owing to the fact that 'mutual algebraicity' is not preserved under adjunction of dummy variables.

Definition 2.3. A structure M in a finite, relational language has *uniformly bounded arrays* if there is an integer $m > 0$ such that for every non-empty $\overline{x} \subseteq \overline{z}$, there is an integer N such that $N_{\overline{x},m}(D) \leq N$ for all finite $D \subseteq M$. When such an N exists, we let $N^{\text{arr}}_{\overline{x},m}$ denote the smallest possible such N.

It is easily seen that the properties described above are elementary. In particular, if m and the (finite) sequence $\langle N^{\text{arr}}_{\overline{x},m} : \overline{x} \subseteq \overline{z} \rangle$ witness that M has uniformly bounded arrays, then the same m and sequence $\langle N^{\text{arr}}_{\overline{x},m} : \overline{x} \subseteq \overline{z} \rangle$ witness that any M' elementarily equivalent to M also has uniformly bounded arrays. Because of this, we say that a complete theory T in a finite, relational language *has uniformly bounded arrays* if some (equivalently all) models M of T have uniformly bounded arrays.

3 Supportive and array isolating types

Throughout this and the next few sections, fix a complete theory T in a finite, relational language L. Also fix an $\aleph_1$-saturated model $\mathbb{M}$ of T, which is a 'monster model' in the sense that all sets of parameters are chosen from $\mathbb{M}$. The reader is reminded that all formulas and types mentioned are quantifier-free.

Definition 3.1. For $\overline{x}$ a subsequence of $\overline{z}$ and B countable, $\text{Supp}_{\overline{x}}(B)$ is the set of all $p \in S_{\overline{x}}(B)$ that support an infinite array. Let $\text{Supp}(B)$ be the disjoint union of the spaces $\text{Supp}_{\overline{x}}(B)$ for all subsequences $\overline{x} \subseteq \overline{z}$.

Lemma 3.2. *Suppose $\overline{x}$ is a subsequence of $\overline{z}$, B is countable, and $p \in S_{\overline{x}}(B)$.*

1. *The type $p \in \text{Supp}_{\overline{x}}(B)$ if and only if for every $m \in \omega$ and every $\theta(\overline{x}, \overline{b}) \in p$, there is an m-array of solutions to $\theta(\overline{x}, \overline{b})$.*

2. *If p has infinitely many solutions, then there is a (possibly empty) proper subsequence $\overline{x}_u \subseteq \overline{x}$ and a realization $\overline{a}$ of p such that $\text{tp}((\overline{a} \setminus \overline{a}_u)/B\overline{a}_u)$ supports an infinite array, where $\overline{a}_u$ is the subsequence of $\overline{a}$ corresponding to $\overline{x}_u$.*

Proof. (1) is easily seen by compactness. For (2), choose an infinite set $\{\overline{a}_i : i \in \omega\}$ of distinct realizations of p. If some infinite subset forms an array, then take $\overline{x}_u = \emptyset$ and p itself supports an infinite array. If this is not the case, then by the Δ-system lemma, there is an infinite $I \subseteq \omega$, a non-empty subsequence $\overline{x}_u$ of $\overline{x}$ and a fixed root $\overline{r}$ such that $(\overline{a}_i)_u = \overline{r}$ and $\overline{a}_i \cap \overline{a}_j = \overline{r}$ for distinct $i, j \in I$. Then $\mathrm{tp}((\overline{a}_i \setminus \overline{r})/B\overline{r})$ supports an infinite array whenever $i \in I$.

It follows from Lemma 3.2(1) that $\mathrm{Supp}_{\overline{x}}(B)$ is a closed, hence compact subspace of $S_{\overline{x}}(B)$. If we endow $\mathrm{Supp}(B)$ with the disjoint union topology (i.e., $U \subseteq \mathrm{Supp}(B)$ is open if and only if $(U \cap \mathrm{Supp}_{\overline{x}}(B))$ is open in $\mathrm{Supp}_{\overline{x}}(B)$ for every $\overline{x} \subseteq \overline{z}$) then $\mathrm{Supp}(B)$ is compact as well.

When our base set is a countable model, $\mathrm{Supp}_{\overline{x}}(M)$ is easily identified.

Lemma 3.3. *If $M \preceq \mathbb{M}$ is countable, then $p \in S_{\overline{x}}(M)$ supports an infinite array if and only if $(x_i \neq a) \in p$ for all $x_i \in \overline{x}$ and all $a \in M$. In particular, if $\overline{c} \cap M = \emptyset$, then $\mathrm{tp}(\overline{c}/M) \in \mathrm{Supp}_{\overline{x}}(M)$.*

Proof. Clearly, if $(x_i = a) \in p$ for any x_i and any $a \in M$, then p does not support a 2-array. For the converse, choose any realization $\overline{c}$ of p with $\overline{c} \cap M = \emptyset$. To show that p supports an infinite array, we employ Lemma 3.2(2). Choose any $\theta(\overline{x}, \overline{h}) \in p$ (so $\overline{h}$ is from M). We will construct an infinite array of solutions to $\theta(\overline{x}, \overline{h})$ inside M. The construction is easy once we note that for any finite subset $F \subseteq M$, $\overline{c}$ is a witness in $\mathbb{M}$ to

$$\exists \overline{x}(\theta(\overline{x}, \overline{h}) \wedge \bigwedge_{x_i \in \overline{x}} \bigwedge_{a \in F} x_i \neq a)$$

As $M \preceq \mathbb{M}$, we have some $\overline{d} \in M^{\mathrm{lg}(\overline{x})}$ realizing $\theta(\overline{x}, \overline{h})$ disjoint from F.

Next, we explore extensions of types $p \in \mathrm{Supp}(B)$. By compactness, it is easily seen that whenever $B \subseteq B'$ are countable, then every $p \in \mathrm{Supp}(B)$ has an extension to some $q \in \mathrm{Supp}(B')$. Abusing notation somewhat, let $\mathrm{Supp}(\mathbb{M})$ denote the set of global types with the property that every restriction to a countable set supports an infinite array. An easy compactness argument shows that every $p \in \mathrm{Supp}(B)$ has a 'global extension' to some $\mathbf{p} \in \mathrm{Supp}(\mathbb{M})$. In general, a type $p \in \mathrm{Supp}(B)$ has many such global extensions, but we focus on when this is unique.

Definition 3.4. A quantifier-free formula $\varphi(\overline{x}, \overline{e})$ is *array isolating* if there is exactly one global type $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ with $\varphi(\overline{x}, \overline{e}) \in \mathbf{p}$. Call a global type $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ *array isolated* if it contains some array isolating formula. Let $\mathrm{AI}_{\overline{x}}(\mathbb{M})$ denote the set of array isolated global $\overline{x}$-types and let $\mathrm{AI}(\mathbb{M})$ be the disjoint union of $\mathrm{AI}_{\overline{x}}(\mathbb{M})$ over all subsequences $\overline{x} \subseteq \overline{z}$. For $\mathbf{p} \in \mathrm{AI}(\mathbb{M})$, $\mathbf{p}|B$ denotes the restriction of $\mathbf{p}$ to a type in $\mathrm{Supp}(B)$.

The following Lemma is immediate. We will get a stronger conclusion in Section 5 under the additional assumption that $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ is finite.

Lemma 3.5. *Suppose $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ is array isolated as witnessed by the array isolating $\varphi(\overline{x}, \overline{e})$. Then any $q \in S_{\overline{x}}(B)$ containing $\varphi(\overline{x}, \overline{e})$ is either equal to $\mathbf{p}|B$ or does not admit an infinite array.*

Proof. If $q \neq \mathbf{p}|B$ supported an infinite array, then any global extension $\mathbf{q} \supseteq q$ would be distinct from $\mathbf{p}$. This contradicts $\varphi(\overline{x}, \overline{e})$ being array isolating.

Definition 3.6. Say that $\mathbf{p} \in \mathrm{AI}_{\overline{x}}(\mathbb{M})$ is *based on B* if $\mathbf{p} \cap \mathrm{QF}_{\overline{x}}(B)$ contains an array isolating formula $\varphi(\overline{x}, \overline{e})$, the interpretation $c^{\mathbb{M}} \in B$ for every constant symbol, and, moreover there is an infinite array $\{\overline{a}_i : i \in \omega\} \subseteq B^{\lg(\overline{x})}$ of pairwise disjoint realizations of $\varphi(\overline{x}, \overline{e})$.

Clearly, if $\mathbf{p}$ is based on B, then it is also based on any $B' \supseteq B$. If B is a model, then the second and third clauses are redundant, that is:

Lemma 3.7. *If $M \preceq \mathbb{M}$, $\mathbf{p} \in \mathrm{AI}_{\overline{x}}(\mathbb{M})$ and $\mathbf{p} \cap \mathrm{QF}_{\overline{x}}(M)$ contains an array isolating formula $\varphi(\overline{x}, \overline{e})$, then $\mathbf{p}$ is based on M.*

Proof. As $M \preceq \mathbb{M}$, every $c^{\mathbb{M}} \in M$. Now, fix an array isolating formula $\varphi(\overline{x}, \overline{e}) \in \mathbf{p} \cap \mathrm{QF}_{\overline{x}}(M)$ and we recursively construct an infinite array of realizations of $\varphi(\overline{x}, \overline{e})$ inside M as follows. First, let $B_0 = \overline{e}$ and let $p_0 = \mathbf{p}|B_0$. As the language L and B_0 are finite, p_0 is isolated by a formula over B_0. As $M \preceq \mathbb{M}$, choose a realization $\overline{a}_0$ of p_0 inside M. Then put $B_1 = B_0 \cup \{\overline{a}_0\}$, let $p_1 = \mathbf{p}|B_1$, and continue for ω steps.

There is a tight analogy between array isolated types $\mathbf{p}$ based on B and strong types over B in a stable theory, but in general they are not equivalent. Indeed, as we are restricting to quantifier free types, a typical restriction $\mathbf{p}|B$ is not even a complete type with respect to formulas with quantifiers. We show that every $\mathbf{p} \in \mathrm{AI}(\mathbb{M})$ is B-definable for any B on which it is based.

Lemma 3.8. *Suppose $\varphi(\overline{x}, \overline{e})$ is an array isolating formula and $\theta(\overline{x}, \overline{y}) \in \mathrm{QF}_{\overline{xy}}(\emptyset)$. There is an integer $m = m(\varphi(\overline{x}, \overline{e}), \theta)$ such that for all $\overline{d} \in \mathbb{M}^{\lg(\overline{y})}$, exactly one of $\varphi(\overline{x}, \overline{e}) \wedge \theta(\overline{x}, \overline{d})$ and $\varphi(\overline{x}, \overline{e}) \wedge \neg\theta(\overline{x}, \overline{d})$ admits an m-array.*

Proof. As $\varphi(\overline{x}, \overline{e})$ admits an infinite array, at least one of the two formulas will as well. However, if such an m did not exist, then for each m there would be a tuple $\overline{d}_m$ such that both $\varphi(\overline{x},\overline{e})\wedge\theta(\overline{x},\overline{d}_m)$ and $\varphi(\overline{x},\overline{e})\wedge\neg\theta(\overline{x},\overline{d}_m)$ admit an m-array. Thus, by the saturation of $\mathbb{M}$, there would be a tuple $\overline{d}^*$ such that both $\varphi(\overline{x},\overline{e})\wedge\theta(\overline{x},\overline{d}^*)$ and $\varphi(\overline{x},\overline{e})\wedge\neg\theta(\overline{x},\overline{d}^*)$ admit infinite arrays, contradicting $\varphi(\overline{x},\overline{e})$ being array isolating.

Definition 3.9. Fix any $\mathbf{p}\in \mathrm{AI}_{\overline{x}}(\mathbb{M})$ and any set B on which it is based. Choose an array isolating formula $\varphi(\overline{x},\overline{e})\in\mathbf{p}\cap\mathrm{QF}_{\overline{x}}(B)$ and an infinite array $\{\overline{a}_i : i\in\omega\}\subseteq B^{\lg(\overline{x})}$ of realizations of $\varphi(\overline{x},\overline{e})$. For any $\theta(\overline{x},\overline{y})\in\mathrm{QF}_{\overline{xy}}(\emptyset)$ let

$$d_{\mathbf{p}}\overline{x}\theta(\overline{x},\overline{y}) := \bigvee_{s\in\binom{2m}{m}}\bigwedge_{i\in s}\theta(\overline{a}_i,\overline{y})$$

where $m=m(\varphi(\overline{x},\overline{e}),\theta)$ is chosen by Lemma 3.8.

Visibly, $d_{\mathbf{p}}\overline{x}\theta(\overline{x},\overline{y})\in\mathrm{QF}_{\overline{y}}(B)$. Its relationship to $\theta(\overline{x},\overline{y})$ and $\mathbf{p}$ is explained by the following Lemma.

Lemma 3.10. *Suppose $\mathbf{p}\in\mathrm{AI}(\mathbb{M})$ is based on a countable set B and $\varphi(\overline{x},\overline{e})$ and $\{\overline{a}_i : i\in\omega\}$ are chosen as in Definition 3.9. The following are equivalent for any $\theta(\overline{x},\overline{y})\in\mathrm{QF}_{\overline{xy}}(\emptyset)$ and any $\overline{d}\in\mathbb{M}^{\lg(\overline{y})}$:*

1. *$\mathbb{M}\models d_{\mathbf{p}}\overline{x}\theta(\overline{x},\overline{d})$;*
2. *$\theta(\overline{x},\overline{d})\in\mathbf{p}$;*
3. *For all countable B', $\mathbf{p}|B'\cup\{\theta(\overline{x},\overline{d})\}$ supports an infinite array; and*
4. *The partial type $\mathbf{p}|B\cup\{\theta(\overline{x},\overline{d})\}$ supports an array of length $m=m(\varphi(\overline{x},\overline{e}),\theta)$.*

Proof. (1) $\Rightarrow$ (2): As $\mathbb{M}\models d_{\mathbf{p}}\overline{x}\theta(\overline{x},\overline{d})$, some m-element subset of $\{\overline{a}_i : i<2m\}$ is an m-array of realizations of $\varphi(\overline{x},\overline{e})\wedge\theta(\overline{x},\overline{d})$. By choice of m, Lemma 3.8 implies that $\varphi(\overline{x},\overline{e})\wedge\theta(\overline{x},\overline{d})$ supports an infinite array, so $\theta(\overline{x},\overline{d})\in\mathbf{p}$.

(2) $\Rightarrow$ (3): Choose any countable B'. If $\theta(\overline{x},\overline{d})\in\mathbf{p}$, then as $\mathbf{p}|B'\cup\{\theta(\overline{x},\overline{d})\}$ is a countable subset of $\mathbf{p}$, it supports an infinite array.

(3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): Assume that $\mathbb{M}\models\neg d_{\mathbf{p}}\overline{x}\theta(\overline{x},\overline{d})$. Then some m-element subset of $\{\overline{a}_i : i<2m\}$ witnesses that $\varphi(\overline{x},\overline{e})\wedge\neg\theta(\overline{x},\overline{d})$ supports an m-array. By Lemma 3.8, $\varphi(\overline{x},\overline{e})\wedge\theta(\overline{x},\overline{d})$ cannot support an m-array. Consequently $p|B\cup\{\theta(\overline{x},\overline{d})\}$ cannot support an m-array (and thus cannot support an infinite array).

4 Free products of array isolated types

Throughout this section, T is a complete theory in a finite, relational language and $\mathbb{M}$ is an $\aleph_1$-saturated model, from which we take our parameters.

In this section we describe how to construct a 'free join' of array isolated types. Suppose $\overline{x}, \overline{y}$ are disjoint, non-empty subsequences of $\overline{z}$, $\mathbf{p}(\overline{x}) \in \mathrm{AI}_{\overline{x}}(\mathbb{M})$, and $\mathbf{q}(\overline{y}) \in \mathrm{AI}_{\overline{y}}(\mathbb{M})$. We show that there is a well-defined $\mathbf{r}(\overline{x}, \overline{y}) \in \mathrm{Supp}_{\overline{xy}}(\mathbb{M})$ constructed from this data. We begin with lemmas that unpack our definitions.

Lemma 4.1. *Suppose $\mathbf{p}(\overline{x}), \mathbf{q}(\overline{y})$ are as above and B is a countable set on which both $\mathbf{p}$ and $\mathbf{q}$ are based. For any $\theta(\overline{x}, \overline{y}, \overline{b}) \in \mathrm{QF}_{\overline{xy}}(B)$ and any $\overline{c}$ realizing $\mathbf{p}|B$,*

$$\theta(\overline{c}, \overline{y}, \overline{b}) \in \mathbf{q}|B\overline{c} \qquad \textit{if and only if} \qquad d_{\mathbf{q}}\overline{y}\theta(\overline{x}, \overline{y}, \overline{b}) \in \mathbf{p}|B$$

Proof. First, assume $\theta(\overline{c}, \overline{y}, \overline{b}) \in \mathbf{q}|B\overline{c}$. Then $\mathbf{q}|B \cup \{\theta(\overline{c}, \overline{y}, \overline{b})\} \subseteq \mathbf{q}$, hence it supports an infinite array. By Lemma 3.10 applied to $\theta(\overline{c}, \overline{y}, \overline{b})$ (i.e., taking $\overline{d} := \overline{c}\overline{b}$),

$$\mathbb{M} \models d_{\mathbf{q}}\overline{y}\theta(\overline{c}, \overline{y}, \overline{b})$$

Taking $\overline{w}$ to be a sequence of variables for $\overline{b}$, since $d_{\mathbf{q}}\overline{y}\theta(\overline{x}, \overline{y}, \overline{w}) \in \mathrm{QF}_{\overline{xw}}(B)$, $\overline{b}$ is from B, and $\overline{c}$ realizes $\mathbf{p}|B$, we conclude that $d_{\mathbf{q}}\overline{y}\theta(\overline{x}, \overline{y}, \overline{b}) \in \mathbf{p}|B$.

The converse is dual, using $\neg\theta$ in place of θ.

Lemma 4.2. *Suppose $\mathbf{p}(\overline{x}), \mathbf{q}(\overline{y})$ are as above and B is a countable set on which both $\mathbf{p}$ and $\mathbf{q}$ are based. For any $\theta(\overline{x}, \overline{y}, \overline{b}) \in \mathrm{QF}_{\overline{xy}}(B)$ and any $\overline{c}, \overline{c}'$ realizing $\mathbf{p}|B$, $\theta(\overline{c}, \overline{y}, \overline{b}) \in \mathbf{q}|B\overline{c}$ if and only if $\theta(\overline{c}', \overline{y}, \overline{b}) \in \mathbf{q}|B\overline{c}'$.*

Proof. By Lemma 4.1, each statement is equivalent to $d_{\mathbf{q}}\theta(\overline{x}, \overline{y}, \overline{b}) \in \mathbf{p}|B$, which does not depend on our choice of $\overline{c}$.

Extending this,

Lemma 4.3. *Suppose $\mathbf{p}(\overline{x}), \mathbf{q}(\overline{y})$ are as above and B is a countable set on which both $\mathbf{p}$ and $\mathbf{q}$ are based. For any $\theta(\overline{x}, \overline{y}, \overline{b}) \in \mathrm{QF}_{\overline{xy}}(B)$ and any $\overline{c}, \overline{c}'$ realizing $\mathbf{p}|B$, for any $\overline{d}$ realizing $\mathbf{q}|B\overline{c}$ and $\overline{d}'$ realizing $\mathbf{q}|B\overline{c}'$, the following three notions are equivalent:*

1. $\mathbb{M} \models \theta(\overline{c}, \overline{d}, \overline{b})$;

2. $\mathbb{M} \models d_{\mathbf{p}}\overline{x}[d_{\mathbf{q}}\overline{y}\theta(\overline{x},\overline{y},\overline{b})]$; *and*

3. $\mathbb{M} \models \theta(\overline{c}', \overline{d}', \overline{b})$.

Proof. Because of the duality in the statements, it suffices to prove (1) $\Leftrightarrow$ (2). First, assume (1) holds. As $\overline{d}$ realizes $\mathbf{q}|B\overline{c}$, we infer $\theta(\overline{c},\overline{y},\overline{b}) \in \mathbf{q}$. So, by Lemma 3.10, $\mathbb{M} \models d_{\mathbf{q}}\overline{y}\theta(\overline{c},\overline{y},\overline{b})$. As $d_{\mathbf{q}}\overline{y}\theta(\overline{x},\overline{y},\overline{b}) \in \mathrm{QF}_{\overline{xy}}(B)$ and $\overline{c}$ realizes $\mathbf{p}|B$, we conclude that $d_{\mathbf{q}}\overline{y}\theta(\overline{x},\overline{y},\overline{b}) \in \mathbf{p}$ and hence $\mathbb{M} \models d_{\mathbf{p}}\overline{x}[d_{\mathbf{q}}\overline{y}\theta(\overline{x},\overline{y},\overline{b})]$. Showing that $(\neg 1)$ implies $(\neg 2)$ is dual, using $\neg\theta$ in place of θ.

We now define the free product of array supporting global types.

Definition 4.4. Suppose $\overline{x},\overline{y}$ are disjoint subsequences of $\overline{z}$, $\mathbf{p} \in \mathrm{AI}_{\overline{x}}(\mathbb{M})$ and $\mathbf{q} \in \mathrm{AI}_{\overline{y}}(\mathbb{M})$. Then the *free product* $\mathbf{r} = \mathbf{p} \times \mathbf{q}$ is defined as

$$\mathbf{r}(\overline{x},\overline{y}) := \{\theta(\overline{x},\overline{y},\overline{b}) \in \mathrm{QF}_{\overline{xy}}(\mathbb{M}) : \mathbb{M} \models d_{\mathbf{p}}\overline{x}[d_{\mathbf{q}}\overline{y}\theta(\overline{x},\overline{y},\overline{b})]\}$$

Because of Lemma 4.3, $\mathbf{r}(\overline{x},\overline{y})$ is also equal to the set of all $\theta(\overline{x},\overline{y},\overline{b}) \in \mathrm{QF}_{\overline{xy}}(\mathbb{M})$ such that for some/every B on which both $\mathbf{p}$ and $\mathbf{q}$ are based and $\overline{b}$ is from B, for some/every $\overline{c}$ realizing $\mathbf{p}|B$ and for some/every $\overline{d}$ realizing $\mathbf{q}|B\overline{c}$ we have $\mathbb{M} \models \theta(\overline{c},\overline{d},\overline{b})$. It is easily seen from this characterization that $\mathbf{r}(\overline{x},\overline{y}) \in \mathrm{Supp}_{\overline{xy}}(\mathbb{M})$.

Next, we show that the free join is symmetric. We begin with a Lemma.

Lemma 4.5. *Suppose $\overline{x},\overline{y}$ are disjoint subsequences of $\overline{z}$, $\mathbf{p}(\overline{x}) \in \mathrm{AI}_{\overline{x}}(\mathbb{M})$, $\mathbf{q}(\overline{y}) \in \mathrm{AI}_{\overline{y}}(\mathbb{M})$. Then for every countable set B on which both types are based, for every $\overline{c}$ realizing $\mathbf{p}|B$, $\overline{d}$ realizing $\mathbf{q}|B$, and for every $\theta(\overline{x},\overline{y},\overline{b}) \in \mathrm{QF}_{\overline{xy}}(B)$ such that $\mathbb{M} \models \theta(\overline{c},\overline{d},\overline{b})$,*

$$\theta(\overline{x},\overline{d},\overline{b}) \in \mathbf{p}|B\overline{d} \qquad \textit{if and only if} \qquad \theta(\overline{c},\overline{y},\overline{b}) \in \mathbf{q}|B\overline{c}$$

Proof. Assume by way of contradiction that $\theta(\overline{x},\overline{d},\overline{b}) \in \mathbf{p}|B\overline{d}$, but $\theta(\overline{c},\overline{y},\overline{b}) \notin \mathbf{q}|B\overline{c}$, with the other direction being dual.

Write θ as $\theta(\overline{x},\overline{y},\overline{w})$. As both $\mathbf{p}$ and $\mathbf{q}$ are based on B, choose array isolating formulas $\varphi(\overline{x},\overline{e}) \in \mathrm{QF}_{\overline{x}}(B)$ and $\psi(\overline{y},\overline{e}') \in \mathrm{QF}_{\overline{y}}(B)$ for $\mathbf{p}$ and $\mathbf{q}$, respectively. Let $m = \max\{m(\varphi(\overline{x},\overline{e}),\theta(\overline{x};\overline{yw})), m(\psi(\overline{y},\overline{e}'),\theta(\overline{y};\overline{xw}))\}$. (Note the different partitions of θ.)

As B is countable, choose infinite arrays $\{\overline{c}_i : i \in \omega\}$ and $\{\overline{d}_j : j \in \omega\}$ for $\mathbf{p}|B$ and $\mathbf{q}|B$, respectively. By Lemma 4.1 we have that $\theta(\overline{x},\overline{d}_j,\overline{b}) \in \mathbf{p}|B\overline{d}_j$

for each j and that $\theta(\overline{c}_i, \overline{y}, \overline{b}) \notin \mathbf{q}|B\overline{c}_i$ for each i. By passing to infinite subsequences, we may additionally assume these sets are pairwise disjoint (i.e., $\overline{c}_i \cap \overline{d}_j = \emptyset$ for all i, j). Choose a number $K >> m$. Form a finite, bipartite graph with universe $C \cup D$, where $C = \{\overline{c}_i : i < K\}$ and $D = \{\overline{d}_j : j < K\}$ with an edge $E(\overline{c}_i, \overline{d}_j)$ if and only if $\mathbb{M} \models \theta(\overline{c}_i, \overline{d}_j, \overline{b})$. We will obtain a contradiction by counting the number of edges in two different ways.

On one hand, because $\theta(\overline{c}_i, \overline{y}, \overline{b}) \notin \mathbf{q}|B\overline{c}_i$, $\mathbf{q}|B \cup \{\theta(\overline{c}_i, \overline{y}, \overline{b})\}$ does not support an infinite array for each $\overline{c}_i$. By our choice of m, it cannot support an array of length m either. Because any m-element subset of D is an array of length m, we conclude that for every $\overline{c}_i$, there are fewer than m many $\overline{d}_j$ such that $\mathbb{M} \models \theta(\overline{c}_i, \overline{d}_j)$. Thus, the number of edges of the graph is bounded above by Km. On the other hand, for any $\overline{d}_j$, as $\mathbf{p}|B \cup \{\theta(\overline{x}, \overline{d}_j, \overline{b})\}$ supports an infinite array, $\mathbf{q}|B \cup \{\neg\theta(\overline{x}, \overline{d}_j, \overline{b})\}$ cannot support an infinite array, hence cannot support an array of size m. Thus, the edge-valence of each $\overline{d}_j$ is at least $(K - m)$, implying that our graph has at least $K(K - m)$ edges. As K is much larger than m, this is a contradiction.

Corollary 4.6. *Suppose $\overline{x}, \overline{y}$ are disjoint subsequences of $\overline{z}$, $\mathbf{p} \in \mathrm{AI}_{\overline{x}}(\mathbb{M})$, and $\mathbf{q} \in \mathrm{AI}_{\overline{y}}(\mathbb{M})$. Then $\mathbf{p} \times \mathbf{q} = \mathbf{q} \times \mathbf{p}$. That is, for any set B on which both $\mathbf{p}, \mathbf{q}$ are based and for any $\theta(\overline{x}, \overline{y}, \overline{w}) \in \mathrm{QF}_{\overline{xyw}}(\emptyset)$, the formulas $d_{\mathbf{p}}\overline{x}[d_{\mathbf{q}}\overline{y}\theta(\overline{x}, \overline{y}, \overline{w})]$ and $d_{\mathbf{q}}\overline{y}[d_{\mathbf{p}}\overline{x}\theta(\overline{x}, \overline{y}, \overline{w})]$ in $\mathrm{QF}_{\overline{w}}(B)$ are equivalent.*

Proof. Choose any $\theta(\overline{x}, \overline{y}, \overline{b}) \in \mathbf{p} \times \mathbf{q}$ with $\overline{b} \in \mathbb{M}^{\lg(\overline{w})}$. Choose any countable set B containing $\overline{b}$ on which both $\mathbf{p}$ and $\mathbf{q}$ are based. Choose $\overline{c}$ realizing $\mathbf{p}|B$ and $\overline{d}$ realizing $\mathbf{q}|B\overline{c}$. By the equivalent definition of $\mathbf{p} \times \mathbf{q}$, $(\overline{c}, \overline{d})$ realizes $\mathbf{p} \times \mathbf{q}$, hence $\mathbb{M} \models \theta(\overline{c}, \overline{d}, \overline{b})$. Thus, by Lemma 4.5, $\overline{c}$ also realizes $\mathbf{p}|B\overline{d}$. Hence $(\overline{c}, \overline{d})$ also realizes $\mathbf{q} \times \mathbf{p}$, so $\theta(\overline{x}, \overline{y}, \overline{b}) \in \mathbf{q} \times \mathbf{p}$ as well.

5 Finitely many mutually algebraic types supporting arrays

We continue our assumption that $\mathbb{M}$ is an $\aleph_1$-saturated model of a complete theory T in a finite, relational language. We begin by recording two consequences of $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ being finite. Note that simply by adding repeated elements to tuples, $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ being finite implies $\mathrm{Supp}_{\overline{x}'}(\mathbb{M})$ finite for all non-empty subsequences $\overline{x}' \subseteq \overline{x}$.

Lemma 5.1. *Fix* $\overline{x} \subseteq \overline{z}$ *and assume that* $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ *is finite. Then* $\mathrm{Supp}_{\overline{x}}(\mathbb{M}) = \mathrm{AI}_{\overline{x}}(\mathbb{M})$ *and, moreover, every* $\mathbf{p} \in \mathrm{AI}_{\overline{x}}(\mathbb{M})$ *is based on every* $M \preceq \mathbb{M}$*.*

Proof. As $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ is always a closed subspace of $S_{\overline{x}}(\mathbb{M})$, it is compact. Thus, if it is finite, every $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ is isolated. For the moreover clause, write $\mathrm{Supp}_{\overline{x}}(\mathbb{M}) = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ and choose array isolating formulas $\varphi_i(\overline{x}, \overline{e}_i)$ for each $\mathbf{p}_i$. By repeated use of Lemma 3.8, let m^* be the maximum of all $m(\varphi_i(\overline{x}, \overline{e}_i), R(\overline{x}, \overline{y}))$ among all $\mathbf{p}_i \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ and all atomic $R \in L$. Then $\mathbb{M}$ is a model of the sentence

> $\exists \overline{w}_1 \ldots \exists \overline{w}_n[\{\varphi_i(\overline{x}, \overline{w}_i) : 1 \leq i \leq n\}$ are pairwise inconsistent and, for all atomic $R \in L$, for all $\overline{z}$, and for all $1 \leq i \leq n$, exactly one of $\varphi_i(\overline{x}, \overline{w}_i) \wedge R(\overline{x}, \overline{z})$ and $\varphi_i(\overline{x}, \overline{w}_i) \wedge \neg R(\overline{x}, \overline{z})$ supports an m^*-array].

Thus, any $M \preceq \mathbb{M}$ also models this sentence. Choose witnesses $\overline{e}_1^*, \ldots \overline{e}_n^*$ from M. Then, for each i, there is a unique type in $S_{\overline{x}}(M)$ containing $\varphi_i(\overline{x}, \overline{e}_i^*)$ and supporting an infinite array. By a second use of $M \preceq \mathbb{M}$, each $\varphi_i(\overline{x}, \overline{e}_i^*)$ array isolates a global type $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$. As $|\mathrm{Supp}_{\overline{x}}(\mathbb{M})| = n$, we conclude that every $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ is array isolated by some $L(M)$-formula. In light of Lemma 3.7, it follows that each $\mathbf{p}$ is based on M.

As a consequence of Lemma 5.1, if $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ is finite and $M \preceq \mathbb{M}$ is countable, then every $p \in S_{\overline{x}}(M)$ that supports an infinite array contains an array isolating formula, hence has a unique supportive extension $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$. Thus, for any extension $q \in S_{\overline{x}}(M\overline{c})$ of p, either $q = \mathbf{p}|M\overline{c}$ or else q does not support an infinite array. In fact, even more is true.

Definition 5.2. A type $q \in S_{\overline{x}}(M\overline{c})$ *has a finite part* if there is some non-empty $\overline{x}' \subseteq \overline{x}$ and some $\theta(\overline{x}', \overline{c}, \overline{h}) \in q$ for which $\mathbb{M} \models \exists^{<\infty} \overline{x}' \theta(\overline{x}', \overline{c}, \overline{h})$.

Lemma 5.3. *Suppose* $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ *is finite and* $M \preceq \mathbb{M}$ *is countable. For any* $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$*, if a type* $q \in S_{\overline{x}}(M\overline{c})$ *extends* $\mathbf{p}|M$ *but* $q \neq \mathbf{p}|M\overline{c}$*, then* q *has a finite part.*

Proof. We argue by induction on $\lg(\overline{x})$, so assume that the statement holds for all proper $\overline{x}' \subsetneq \overline{x}$. By way of contradiction, choose $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$, and $q \in S_{\overline{x}}(M\overline{c})$ with $\mathbf{p}|M \subseteq q$, $q \neq \mathbf{p}|M\overline{c}$, but q has no finite part. We will obtain a contradiction to Lemma 3.5 by showing that q supports an infinite array. Toward that goal, choose any $\theta(\overline{x}, \overline{c}, \overline{h}) \in q$. By Lemma 3.2(1), it suffices to find an infinite array of realizations to $\theta(\overline{x}, \overline{c}, \overline{h})$.

As q has no finite part, by taking $\overline{x}' = \overline{x}$, q has infinitely many realizations. Thus, by Lemma 3.2(2), there is a proper subsequence $\overline{x}_u \subsetneq \overline{x}$ and a realization $\overline{a}$ of q such that $\mathrm{tp}((\overline{a} \setminus \overline{a}_u)/M\overline{c}\overline{a}_u)$ supports an infinite array. Trivially, if $\overline{x}_u$ is empty, then this type is q itself, q supports an infinite array. So we assume $\overline{x}_u$ is non-empty. Write $\overline{x} = \overline{y}\hat{\ }\overline{x}_u$ and for clarity, write $\overline{r}$ for $\overline{a}_u$ and $\overline{b}$ for $(\overline{a} \setminus \overline{r})$, so $\overline{a} = \overline{b}\hat{\ }\overline{r}$.

Let $q^*(\overline{y}) := \mathrm{tp}(\overline{b}/M\overline{c}\overline{r})$. We know that $\overline{b}'\hat{\ }\overline{r}$ realizes q whenever $\overline{b}'$ realizes q^*. Let $\mathbf{p}^*(\overline{y}) \in \mathrm{Supp}_{\overline{y}}(\mathbb{M})$ be any global type extending q^*. As $\mathrm{Supp}_{\overline{y}}(\mathbb{M})$ is finite, by Lemma 5.1 $\mathbf{p}^*$ is based on M. Choose an array isolating formula $\varphi(\overline{y}, \overline{e}) \in \mathbf{p}^*$ with $\overline{e}$ from M, along with an infinite array $\{\overline{b}_i : i \in \omega\} \subseteq M^{\lg(\overline{y})}$ of realizations of $\varphi(\overline{y}, \overline{e})$. As $\varphi(\overline{y}, \overline{e}) \wedge \theta(\overline{y}, \overline{r}, \overline{c}, \overline{h}) \in q^*$, Lemma 3.8 implies that all but finitely many $\overline{b}_i$ realize $\theta(\overline{y}, \overline{r}, \overline{c}, \overline{h})$, so by elimination and reindexing, assume they all do.

On the other hand, let $q_u(\overline{x}_u) := \mathrm{tp}(\overline{r}/M\overline{c})$. As q does not have a finite part, neither does q_u. As $\overline{x}_u$ is a proper subsequence of $\overline{x}$, our inductive hypothesis implies that q_u must support an infinite array. Let $\{\overline{r}_j : j \in \omega\}$ be such an array. Note that as $\mathrm{tp}(\overline{r}_j/M\overline{c}) = \mathrm{tp}(\overline{r}/M\overline{c})$ and $\theta(\overline{b}_i, \overline{x}_u, \overline{c}, \overline{h}) \in q_u$, it follows that $\theta(\overline{b}_i, \overline{r}_j, \overline{c}, \overline{h})$ holds for all $i, j \in \omega$. From this, as both $\{\overline{b}_i : i \in \omega\}$ and $\{\overline{r}_j : j \in \omega\}$ are infinite arrays, it is easy to meld subsequences of these to produce an infinite array $\{\overline{b}_k\overline{r}_k : k \in \omega\}$ of realizations of $\theta(\overline{y}, \overline{x}_u, \overline{c}, \overline{h})$.

Next, we add mutual algebraicity to the discussion of supportive and array isolating types.

Definition 5.4. For $\overline{x} \subseteq \overline{z}$ non-empty, a global, supportive type $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ is *quantifier-free mutually algebraic (QMA)* if $\mathbf{p}$ contains a mutually algebraic formula $\varphi(\overline{x}) \in \mathrm{QF}_{\overline{x}}(\mathbb{M})$. Let $\mathrm{QMA}_{\overline{x}}(\mathbb{M})$ denote the set of QMA types in $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$. Let $\mathrm{QMA}(\mathbb{M})$ be the (finite) disjoint union of the sets $\mathrm{QMA}_{\overline{x}}(\mathbb{M})$.

The goal of this section will be to deduce consequences from $\mathrm{QMA}(\mathbb{M})$ being finite.

Lemma 5.5. *If* $\mathrm{QMA}_{\overline{x}}(\mathbb{M})$ *is finite, then every* $\mathbf{p} \in \mathrm{QMA}_{\overline{x}}(\mathbb{M})$ *is array isolated, i.e.,* $\mathbf{p} \in \mathrm{AI}_{\overline{x}}(\mathbb{M})$.

Proof. Fix any $\mathbf{p} \in \mathrm{QMA}_{\overline{x}}(\mathbb{M})$. Choose a mutually algebraic $\varphi_0(\overline{x}, \overline{e}_0) \in \mathbf{p}$. For each $\mathbf{q} \in \mathrm{QMA}_{\overline{x}}(\mathbb{M})$ distinct from $\mathbf{p}$, choose a formula $\varphi_{\mathbf{q}}(\overline{x}, \overline{e}_{\mathbf{q}}) \in \mathbf{p} \setminus \mathbf{q}$. Then the formula $\varphi_0(\overline{x}, \overline{e}_0) \wedge \bigwedge_{\mathbf{q} \neq \mathbf{p}} \varphi_{\mathbf{q}}(\overline{x}, \overline{e}_{\mathbf{q}})$ array isolates $\mathbf{p}$.

Definition 5.6. Fix any non-empty $\overline{x} \subseteq \overline{z}$. A *partition* $\mathcal{P} = \{\overline{x}_1, \ldots, \overline{x}_r\}$ *of* $\overline{x}$ satisfies (1) each $\overline{x}_i$ non-empty and (2) Every $x \in \overline{x}$ is contained in exactly one $\overline{x}_i$. For $w \subseteq \{1, \ldots, r\}$, let $\overline{x}_w$ be the subsequence of $\overline{x}$ with universe $\bigcup\{\overline{x}_i : i \in w\}$.

For $\overline{c} \in (\mathbb{M})^{\lg(\overline{x})}$, a partition $\mathcal{P}$ of $\overline{x}$ naturally induces a partition $\{\overline{c}_1, \ldots, \overline{c}_r\}$ of $\overline{c}$. For $w \subseteq \{1, \ldots, r\}$, $\overline{c}_w$ is the subsequence of $\overline{c}$ corresponding to $\overline{x}_w$.

Definition 5.7. Fix any $\overline{x} \subseteq \overline{z}$, any countable $M \preceq \mathbb{M}$, and $\overline{c} \in (\mathbb{M} \setminus M)^{\lg(\overline{x})}$. A *maximal mutually algebraic decomposition of* $\overline{c}$ *over* M is a partition $\mathcal{P} = \{\overline{x}_1, \ldots, \overline{x}_r\}$ of $\overline{x}$ for which the induced partition $\{\overline{c}_1, \ldots, \overline{c}_r\}$ of $\overline{c}$ satisfies the following for each $i \in \{1, \ldots, r\}$:

- $\overline{c}_i$ realizes a mutually algebraic formula $\varphi(\overline{x}_i) \in \mathrm{QF}_{\overline{x}_i}(M)$; but
- For any proper extension $\overline{x}_i \subsetneq \overline{u} \subseteq \overline{x}$, the subsequence $\overline{d}$ of $\overline{c}$ induced by $\overline{u}$ does not realize any mutually algebraic formula $\psi(\overline{u}) \in \mathrm{QF}_{\overline{u}}(M)$.

Lemma 5.8. *For any* $\overline{x} \subseteq \overline{z}$ *and every countable* $M \preceq \mathbb{M}$*, every* $\overline{c} \in (\mathbb{M} \setminus M)^{\lg(\overline{x})}$ *admits a unique maximal mutually algebraic decomposition over* M*.*

Proof. First, by Lemma 3.3, both $\mathrm{tp}(\overline{c}/M)$ and $\mathrm{tp}(\overline{c}'/M)$ for any subsequence $\overline{c}' \subseteq \overline{c}$ support infinite arrays. Next, as every formula $\varphi(x)$ in one free variable is mutually algebraic, every singleton $c \in \overline{c}$ realizes a mutually algebraic formula. For each $x \in \overline{x}$, choose a subsequence $\overline{x}_i$ of $\overline{x}$ containing x such that $\overline{c}_i$ realizes a mutually algebraic formula in $\mathrm{QF}_{\overline{x}_i}(M)$ and is *maximal* i.e., there is no proper extension $\overline{x}' \supsetneq \overline{x}_i$ for which $\overline{c}'$ realizes a mutually algebraic formula in $\mathrm{QF}_{\overline{x}'}(M)$. Clearly, $\{\overline{x}_1, \ldots, \overline{x}_r\}$ covers $\overline{x}$. The fact that it is a partition follows from the fact that if $\overline{x}_i, \overline{x}_j$ are not disjoint and $\varphi(\overline{x}_i)$, $\psi(\overline{x}_j)$ are each mutually algebraic, then their conjunction $(\varphi \wedge \psi)(\overline{x}_i\overline{x}_j)$ is mutually algebraic as well (see e.g. Lemma 2.4(6) of [2]).

It is easily checked that if $\{\overline{c}_1, \ldots, \overline{c}_r\}$ is a maximal mutually algebraic decomposition of $\overline{c}$ over M, then for any $w \subseteq \{1, \ldots, r\}$, the subset $\{\overline{c}_i : i \in w\}$ is a maximal, mutually algebraic decomposition of $\overline{c}_w$ over M.

We are now able to state and prove the following.

Proposition 5.9. *Suppose that* $\mathrm{QMA}(\mathbb{M})$ *is finite. Then, for every subsequence* $\overline{x} \subseteq \overline{z}$*, every* $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ *is equal to a free product* $\mathbf{q} = \mathbf{p}_1 \times \cdots \times \mathbf{p}_r$ *of types from* $\mathrm{QMA}(\mathbb{M})$*. In particular, each* $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ *is finite.*

Proof. As the whole of $\mathrm{QMA}(\mathbb{M})$ is finite, choose a finite D such that for every $\overline{x} \subseteq \overline{z}$ and every $\mathbf{q} \in \mathrm{QMA}_{\overline{x}}(\mathbb{M})$, there is a mutually algebraic formula $\gamma(\overline{x}) \in \mathbf{q} \cap \mathrm{QF}_{\overline{x}}(D)$. Choose a countable $M \preceq \mathbb{M}$ with $D \subseteq M$.

We prove the Proposition by induction on $\overline{x}$, i.e., we assume the Proposition holds for all proper subsequences $\overline{x}'$ of $\overline{x}$ and prove the result for $\overline{x}$. To base the induction, first note that if $x \in \overline{z}$ is a singleton, then as every formula $\varphi(x)$ is mutually algebraic, every $\mathbf{q} \in \mathrm{Supp}_x(\mathbb{M})$ is also in $\mathrm{QMA}_x(\mathbb{M})$, which we assumed was finite.

Now, suppose $\overline{x}$ is a subsequence of $\overline{z}$, $\lg(\overline{x}) \geq 2$, and the Proposition holds for every proper subsequence $\overline{x}'$ of $\overline{x}$. In particular, as $\mathrm{Supp}_{\overline{x}'}(\mathbb{M})$ is finite, Lemma 5.1 implies that each $\mathbf{q} \in \mathrm{Supp}_{\overline{x}'}(\mathbb{M})$ is based on M and is also in $\mathrm{AI}(\mathbb{M})$.

Choose any $\mathbf{q}^* \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$. Towards showing that $\mathbf{q}^*$ is a free product of types from $\mathrm{QMA}_{\overline{x}}(\mathbb{M})$, choose any $\overline{c} \in (\mathbb{M} \setminus M)^{\lg(\overline{x})}$ realizing $\mathbf{q}^*|M$. Suppose the partition $\mathcal{P} = \{\overline{x}_1, \ldots, \overline{x}_r\}$ of $\overline{x}$ yields the maximal, mutually algebraic decomposition $\overline{c}_1\hat{}\ldots\hat{}\overline{c}_r$ of $\overline{c}$ over M.

There are now two cases. First, if $r = 1$, then $\mathrm{tp}(\overline{c}/M)$ contains a mutually algebraic formula, so $\mathbf{q}^* \in \mathrm{QMA}_{\overline{x}}(\mathbb{M})$ and we are finished. So assume $r \geq 2$. As notation, for each $1 \leq j \leq r$, let $\overline{w}_j$ be the subsequence $\overline{x}_1 \ldots \overline{x}_{j-1}\overline{x}_{j+1} \ldots \overline{x}_r$ of $\overline{x}$ and let $\overline{d}_j$ be the corresponding subsequence of $\overline{c}$. As each $\overline{d}_j$ is a proper subsequence of $\overline{c}$, our inductive hypothesis implies that $\mathrm{tp}(\overline{d}_j/M)$ contains an array isolating formula. Let $\mathbf{q}_j$ be the (unique) global extension of $\mathrm{tp}(\overline{d}_j/M)$ to $\mathrm{AI}_{\overline{w}_j}(\mathbb{M})$. By our inductive hypothesis again, each $\mathbf{q}_j = (\mathbf{p}_1 \times \ldots \mathbf{p}_{j-1} \times \mathbf{p}_{j+1} \times \ldots \mathbf{p}_r)$. As each $\mathbf{p}_j$ is also array isolated, by iterating Lemma 4.5 finitely often it follows that there is a unique supportive type $\mathbf{r}^*(\overline{x})$ which is equal to $\mathbf{q}_j(\overline{w}_j) \times \mathbf{p}_j(\overline{x}_j)$ for every $1 \leq j \leq r$.

In light of the characterization of free products following Definition 4.4, in order to conclude that $\mathbf{q}^* = \mathbf{q}_r \times \mathbf{p}_r = \mathbf{r}^*$, it suffices to prove the following Claim.

Claim. $\overline{d}_r$ realizes $\mathbf{q}_r|M\overline{c}_r$.

Assume this were not the case. We obtain a contradiction by showing that the whole of $\mathrm{tp}(\overline{c}_1, \ldots, \overline{c}_r/M)$ contains a mutually algebraic formula $\varphi(\overline{x})$. As $\mathrm{tp}(\overline{d}_r/M) = \mathbf{q}_r|M$, but $\mathrm{tp}(\overline{d}_r/M\overline{c}_r) \neq \mathbf{q}_r|M\overline{c}_r$, Lemma 5.3 allows us to choose a maximal subsequence $\overline{c}_u$ of $\overline{d}_r$ and $\delta_r(\overline{x}_u, \overline{c}_r, \overline{b}_r) \in \mathrm{tp}(\overline{c}_u/M\overline{c}_r)$ ($\overline{b}_r$ from M) with only finitely many solutions. As $\mathrm{tp}(\overline{c}_i/M)$ is mutually algebraic for each i, it follows from the maximality of u that there is a nonempty subset $w \subseteq \{1, \ldots, r-1\}$ such that $\overline{c}_u = \overline{c}_w$. To ease notation, say

$w = \{1, \ldots, s\}$ for some $s \leq r-1$. We argue that $s = r-1$. If this were not the case, then $\{\overline{c}_1, \ldots, \overline{c}_s, \overline{c}_r\}$ would be a maximal mutually algebraic decomposition over M of the subsequence $\overline{c}_1\,\hat{}\,\ldots\,\hat{}\,\overline{c}_s\,\hat{}\,\overline{c}_r$ whose corresponding variables $\overline{x}_1 \ldots \overline{x}_s \overline{x}_r$ form a proper subsequence of $\overline{x}$. Thus, by our inductive hypothesis, $\mathrm{tp}(\overline{c}_1 \ldots \overline{c}_s \overline{c}_r/M)$ would equal $(\mathbf{p}_1 \times \cdots \times \mathbf{p}_s \times \mathbf{p}_r)|M$, which is contradicted by the formula $\delta_r(\overline{x}_1, \ldots, \overline{x}_s, \overline{x}_r, \overline{b}_r) \in \mathrm{tp}(\overline{c}_1 \ldots \overline{c}_s \overline{c}_r/M)$. Thus, we conclude that $s = r-1$. Hence, $\delta_r(\overline{w}_r, \overline{c}_r, \overline{b}_r)$ has only finitely many solutions.

Next, choose any $j < r$. Now the presence of the formula δ_r implies that $\mathbf{q}^* \neq \mathbf{r}^*$, hence $\mathbf{q}^* \neq \mathbf{q}_j \times \mathbf{p}_j$. From this, it follows that $\mathrm{tp}(\overline{d}_j/M\overline{c}_j) \neq \mathbf{q}_j|M\overline{c}_j$. So, arguing just as above but replacing r by j throughout, we conclude there is a formula $\delta_j(\overline{x}, \overline{b}_j) \in \mathrm{tp}(\overline{c}/M)$ for which $\delta_j(\overline{w}_j; \overline{c}_j \overline{b}_j)$ has only finitely many solutions.

Thus, if we choose a mutually algebraic formula $\gamma_j \in \mathrm{tp}(\overline{c}_j/D)$ for each $j \leq r$, we conclude that the formula

$$\varphi(\overline{x}_1, \ldots, \overline{x}_r, \overline{b}_1 \ldots \overline{b}_r) = \bigwedge_{j \leq r} \left(\gamma_j(\overline{x}_j) \wedge \delta_j(\overline{w}_j, \overline{x}_j, \overline{b}_j)\right)$$

is mutually algebraic with free variables $\overline{x}$ and is in $\mathrm{tp}(\overline{c}/M)$. This contradicts our assumption that $\mathrm{tp}(\overline{c}/M)$ was not mutually algebraic. This completes the proof of the Claim as well as the Proposition.

Conclusion 5.10. *If* $\mathrm{QMA}(\mathbb{M})$ *is finite, then so is* $\mathrm{Supp}(\mathbb{M})$. *Moreover, for any* $\overline{x} \subseteq \overline{z}$, *any countable* $M \preceq \mathbb{M}$, *and any* $\overline{c} \in (\mathbb{M} \setminus M)^{\lg(\overline{x})}$, $\mathrm{tp}(\overline{c}/M)$ *is determined by the maximal mutually algebraic partition* $\mathcal{P} = \{\overline{x}_1, \ldots, \overline{x}_r\}$ *and the corresponding set* $\{\mathbf{p}_1, \ldots, \mathbf{p}_r\}$ *of* $\mathrm{QMA}_{\overline{x}_i}(\mathbb{M})$ *types.*

6 Mutual algebraicity and unbounded arrays

The whole of this section is devoted to the statement and proof of Theorem 6.1. It can be construed as a kind of 'Ryll-Nardzewski theorem' for Stone spaces of quantifier-free types.

Theorem 6.1. *Suppose* T *is a complete theory in a finite, relational language, all of whose atomic formulas have free variables among* $\overline{z}$, *and let* $\mathbb{M}$ *be an* $\aleph_1$*-saturated model of* T. *The following are equivalent.*

1. *T has uniformly bounded arrays;*

2. *For all subsequences* $\overline{x} \subseteq \overline{z}$, $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ *is finite;*

3. *For all subsequences* $\overline{x} \subseteq \overline{z}$, *every global supportive type* $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ *is array isolated;*

4. *Whenever* $M \preceq N$ *are models of* T, *for all subsequences* $\overline{x} \subseteq \overline{z}$, *there are only finitely many types in* $S_{\overline{x}}(M)$ *realized in* $(N \setminus M)^{\lg(\overline{x})}$*;*

5. *For all models* M *and all subsequences* $\overline{x} \subseteq \overline{z}$, *only finitely many types in* $S_{\overline{x}}(M)$ *both contain a mutually algebraic formula and support an infinite array; and*

6. T *is mutually algebraic.*

Proof. We begin by showing that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). Fix any non-empty subsequence $\overline{x} \subseteq \overline{z}$. The key observation for showing (2) $\Leftrightarrow$ (3) is that if X is any compact, Hausdorff space, then X is finite if and only if every element $a \in X$ is isolated. Suppose that (2) holds. To establish (3), note that $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ as a subspace of $S_{\overline{x}}(\mathbb{M})$, the Stone space of all quantifier free types is closed, and hence compact. As (2) implies it is finite as well, every $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ must be isolated in the subspace, hence array isolated.

Verifying that (3) $\Rightarrow$ (2) uses the converse of this. Fix $\overline{x} \subseteq \overline{z}$. Applying (3) to the model $\mathbb{M}$ yields that every element of $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ is isolated. As $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$ is compact and Hausdorff, it must be finite.

(2) $\Rightarrow$ (4) is easy. Assume (2). It suffices to prove (4) for all countable $M \preceq N$. As $\mathbb{M}$ is $\aleph_1$-saturated, we may assume $N \preceq \mathbb{M}$. But now, by Lemma 3.3, for any $\overline{c} \in (N \setminus M)^{\lg(\overline{x})}$, $\mathrm{tp}(\overline{c}/M)$ supports an infinite array. As any $p \in \mathrm{Supp}_{\overline{x}}(M)$ extends to some $\mathbf{p} \in \mathrm{Supp}(\mathbb{M})$, there are only finitely many such types.

Next, suppose (2) fails. Choose $\overline{x} \subseteq \overline{z}$ and a countable, infinite $Y \subseteq \mathrm{Supp}_{\overline{x}}(\mathbb{M})$. For each pair $\mathbf{p} \neq \mathbf{q}$, choose a formula $\varphi_{\mathbf{pq}}(\overline{x}, \overline{e}_{\mathbf{pq}}) \in \mathbf{p} \setminus \mathbf{q}$. Choose a countable $M \preceq \mathbb{M}$ containing $\{\overline{e}_{\mathbf{pq}} : \mathbf{p} \neq \mathbf{q} \in Y\}$. Thus, $\{\mathbf{p}|M : \mathbf{p} \in Y\}$ is a countably infinite set of types, each of which support an infinite array. As $\mathbb{M}$ is $\aleph_1$-saturated, each such $\mathbf{p}|M$ is realized by some $\overline{c} \in \mathbb{M}^{\lg(\overline{x})}$. That $\overline{c} \cap M = \emptyset$ follows from the fact that $\mathbf{p}|M$ supports an infinite array.

Continuing on, we consider (2) $\Rightarrow$ (5). It suffices to prove this for M countable, and we may assume $M \preceq \mathbb{M}$. As any $p \in S_{\overline{x}}(M)$ supporting an infinite array has an extension to $\mathrm{Supp}_{\overline{x}}(\mathbb{M})$, (5) follows from (2).

(5) $\Rightarrow$ (2) is immediate from Conclusion 5.10.

(2) $\Rightarrow$ (1). By (2), for each $\overline{x} \subseteq \overline{z}$, let $N(\overline{x}) := |\mathrm{Supp}_{\overline{x}}(\mathbb{M})|$. As (2) $\Rightarrow$ (3), every $p \in \mathrm{Supp}(\mathbb{M})$ is array isolated. For each $\overline{x} \subseteq \overline{z}$ and each $p \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$, choose an array isolating formula $\varphi_{\mathbf{p}}(\overline{x}, \overline{e}_{\mathbf{p}}) \in \mathbf{p}$ and let $D \subseteq \mathbb{M}$ be finite and contain all $\overline{e}_{\mathbf{p}}$ for all $\mathbf{p} \in \mathrm{Supp}(\mathbb{M})$. It is easily seen that for every $\overline{x} \subseteq \overline{z}$ and every countable $D \subseteq B \subseteq \mathbb{M}$, $S_{\overline{x}}(B)$ has exactly $N(\overline{x})$ types that support an infinite array. Moreover, each such $q \in \mathrm{Supp}_{\overline{x}}(B)$ has a unique restriction $q|D \in \mathrm{Supp}_{\overline{x}}(D)$ and a unique extension $\mathbf{q} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$.

Towards finding an appropriate m as in Definition 2.3, fix $\overline{x} \subseteq \overline{z}$ and partition each atomic $R(\overline{z}) \in L$ as $R(\overline{x}, \overline{w})$. For each $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ with array isolating formula $\varphi_{\mathbf{p}}(\overline{x}, \overline{e}_{\mathbf{p}}) \in \mathbf{p}$, let $m(\mathbf{p}, \overline{x})$ be the maximum of the $2|L|$ numbers $m(\varphi_{\mathbf{p}}(\overline{x}, \overline{e}_{\mathbf{p}}), \pm R(\overline{x}, \overline{w}))$ obtained by Lemma 3.8. That is, apply the Lemma $2|L|$ times, once for each $R \in L$, and once for each $\neg R$ for $R \in L$.

The point is that if B is countable, $D \subseteq B \subseteq \mathbb{M}$, and $q \in S_{\overline{x}}(B)$ contains some $\varphi(\overline{x}, \overline{e}_{\mathbf{p}})$ and supports an $m(\mathbf{p}, \overline{x})$-array, then for every $R(\overline{x}, \overline{b})$, $R(\overline{x}, \overline{b}) \in q$ if and only if $\varphi_{\mathbf{p}}(\overline{x}, \overline{e}_{\mathbf{p}}) \wedge R(\overline{x}, \overline{b})$ supports an $m(\mathbf{p}, \overline{x})$-array if and only if $\varphi_{\mathbf{p}}(\overline{x}, \overline{e}_{\mathbf{p}}) \wedge R(\overline{x}, \overline{b})$ supports an infinite array if and only if $R(\overline{x}, \overline{b}) \in \mathbf{p}$. Thus, $q \subseteq \mathbf{p}$.

On the other hand, let $\theta(\overline{x}) := \bigwedge\{\neg\varphi_{\mathbf{p}}(\overline{x}, \overline{e}_{\mathbf{p}}) : \mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})\}$. Since there is no $q \in S_{\overline{x}}(D)$ with $\theta \in q$ that supports an infinite array, compactness yields an integer $m^*(\overline{x})$ such that no $q \in S_{\overline{x}}(D)$ with $\theta \in q$ supports an $m^*(\overline{x})$-array. Clearly, for any $B \supseteq D$, no $q \in S_{\overline{x}}(B)$ with $\theta \in q$ could support an $m^*(\overline{x})$-array either.

Choose an integer m that is greater than all $m^*(\overline{x})$ and all $m(\mathbf{p}, \overline{x})$ for $\overline{x} \subseteq \overline{z}$ and $\mathbf{p} \in \mathrm{Supp}_{\overline{x}}(\mathbb{M})$. Combining the statements above, we see that for any countable $B \supseteq D$ and any $\overline{x} \subseteq \overline{z}$, exactly $N(\overline{x})$ types in $S_{\overline{x}}(B)$ support m-arrays. Thus, $\mathbb{M}$ (and hence T by elementarity) has uniformly bounded arrays.

(1) $\Rightarrow$ (2) is also easy. Assume T has uniformly bounded arrays. Choose m and $\langle N^{\mathrm{arr}}_{\overline{x},m} : \overline{x} \subseteq \overline{z}\rangle$ from the definition. To establish (2), we claim that $|\mathrm{Supp}_{\overline{x}}(\mathbb{M})| \leq N^{\mathrm{arr}}_{\overline{x},m}$ for each $\overline{x} \subseteq \overline{z}$. To see this, fix $\overline{x} \subseteq \overline{z}$ and assume by way of contradiction there is a finite $Y \subseteq \mathrm{Supp}_{\overline{x}}(\mathbb{M})$ with $|Y| > N^{\mathrm{arr}}_{\overline{x},m}$. For each pair $\mathbf{p} \neq \mathbf{q}$ from Y, choose some $\varphi_{\mathbf{pq}}(\overline{x}, \overline{e}_{\mathbf{pq}}) \in \mathbf{p} \setminus \mathbf{q}$ and let $D \subseteq \mathbb{M}$ be finite, containing all these $\overline{e}_{\mathbf{pq}}$. Thus, the set $\{\mathbf{p}|D : \mathbf{p} \in Y\}$ are distinct elements of $S_{\overline{x}}(D)$. As each restriction $\mathbf{p}|D$ supports an infinite array, each supports an m-array, contradicting our definition of $N^{\mathrm{arr}}_{\overline{x},m}$.

Thus, conditions (1)–(5) are equivalent.

(6) $\Rightarrow$ (5): Suppose T is mutually algebraic. Then by Theorem 3.3 of [3], T is weakly minimal, trivial, has nfcp, and moreover, any expansion of

any model of T by unary predicates also has an nfcp theory. By way of contradiction, fix M, $\overline{x} \subseteq \overline{z}$, and an infinite set of distinct mutually algebraic types $\{p_i : i \in \omega\} \subseteq \mathrm{Supp}_{\overline{x}}(M)$. As there are only finitely many permutations of $\overline{x}$, we may assume that for distinct $i, j \in \omega$, no permutation of a realization of p_i realizes p_j. As the language is finite relational, choose the set $\mathcal{F}$ of L-formulas as in Proposition A.2. By the pigeon-hole principle and relabelling, choose an L-formula $\varphi(\overline{x}, \overline{w}) \in \mathcal{F}$ such that for every $i \in \omega$, there is $\overline{h}_i \in M^{\lg(\overline{w})}$ such that $\varphi(\overline{x}, \overline{h}_i) \in p_i$ and $\varphi(\overline{x}, \overline{h}_i)$ is mutually algebraic. Note that since T has nfcp, there is an M-definable formula $\mu(\overline{w})$ such that for any $\overline{h} \in M^{\lg(\overline{w})}$, $M \models \mu(\overline{h})$ if and only if $\varphi(\overline{x}, \overline{y})$ is mutually algebraic.

Let $\mathbb{M} \succeq M$ be $|M|^+$-saturated. We will obtain a contradiction by constructing an expansion $\mathbb{M}^+ = (\mathbb{M}, U, V, W)$ by unary predicates that has the finite cover property. From the saturation of $\mathbb{M}$, choose k-tuples $\{\overline{a}_i^j : i \in \omega, j \leq i\}$ (where $k = \lg(\overline{x})$) such that $\overline{a}_i^j$ realizes $p_i(\overline{x})$ but $\mathrm{acl}(M\overline{a}_i^j) \cap \mathrm{acl}(M\overline{a}_{i'}^{j'}) = M$ unless $i = i'$ and $j = j'$. Let $A = \bigcup\bigcup\{\overline{a}_i^j : i \in \omega, j \leq i\}$ and let $B = \{b_i^j : i \in \omega, j \leq i\}$ be the subset of A consisting of the 'first coordinates' i.e., $b_i^j = (\overline{a}_i^j)_0$ for all i, j.

Let $\mathbb{M}^+$ be the expansion of $\mathbb{M}$ by interpreting $U^{\mathbb{M}^+} = A$, $V^{\mathbb{M}^+} = B$, and $W^{\mathbb{M}^+} = M$. Let

$$P(\overline{x}) := \bigwedge_{x \in \overline{x}} U(x) \wedge \exists \overline{w} \left[\bigwedge_{w \in \overline{w}} W(w) \wedge \mu(\overline{w}) \wedge \varphi(\overline{x}, \overline{w}) \right]$$

Note that $\mathbb{M}^+ \models P(\overline{a}_i^j)$ for all $i \in \omega$, $j \leq i$. Also, if $b_i^j \in B$ and $\mathbb{M}^+ \models P(b_i^j, a_2, \ldots a_k)$, then by mutual algebraicity, $\{a_2, \ldots, a_k\} \subseteq \mathrm{acl}(M\overline{a}_i^j)$, so $(b_i^j, a_2, \ldots, a_k)$ is a permutation of $\overline{a}_i^j$. Let $\overline{x}' = (x_2, \ldots, x_k)$, $\overline{y}' = (y_2, \ldots, y_k)$, and put

$$E(x, y) := \exists \overline{x}' \exists \overline{y}' [P(x, \overline{x}') \wedge P(y, \overline{y}') \wedge (\forall \overline{w} \in W) \bigwedge_{R \in L} R(x, \overline{x}', \overline{w}) \leftrightarrow R(y, \overline{y}', \overline{w})]$$

E is an $\mathbb{M}^+$-definable equivalence relation on $B = V^{\mathbb{M}^+}$, and $\mathbb{M}^+ \models E(b_i^j, b_{i'}^{j'})$ if and only if $i = i'$. Thus, E has arbitrarily large finite classes, which contradicts $Th(\mathbb{M}^+)$ having nfcp.

Remark 6.2. The implication (6) $\Rightarrow$ (5) above really relies on counting *quantifier-free* mutually algebraic types that support infinite arrays. As an example, $Th(\mathbb{Z}, S)$ is mutually algebraic, but there are infinitely many mutually algebraic formulas $\varphi_n(x, y)$, each of which support infinite arrays. Take $\varphi_n(x, y) := \exists z_0 \ldots \exists z_n[(x = z_0) \wedge (y = z_n) \wedge \bigwedge_{i=0}^{n-1} S(z_n, z_{n+1})]$.

(2–5) $\Rightarrow$ (6): We assume all of (2–5) and prove that T is mutually algebraic by way of Theorem 2.1. Choose an $\aleph_1$-saturated $\mathbb{M} \models T$. As $\mathrm{QMA}(\mathbb{M})$ is finite, choose a finite $D \subseteq \mathbb{M}$ so that every $\mathbf{p} \in \mathrm{QMA}(\mathbb{M})$ contains a mutually algebraic formula in $\mathrm{QF}(D)$, and choose a countable $M \preceq \mathbb{M}$ with $D \subseteq M$. Fix $\overline{x} \subseteq \overline{z}$ and let

> $\mathcal{B}_{\overline{x}} :=$ {all $L(M)$-formulas $\varphi(\overline{x})$ that are equivalent to boolean combinations of **quantifier-free** mutually algebraic $\alpha(\overline{x}')$ for some $\overline{x}' \subseteq \overline{x}$}.

Note that since $y = y$ is mutually algebraic and $\varphi(\overline{x}) := \alpha(\overline{x}') \wedge \bigwedge_{y \in \overline{x} \setminus \overline{x}'} y = y$ is equivalent to $\alpha(\overline{x}')$, we can construe every quantifier-free, mutually algebraic $\alpha(\overline{x}')$ as an element of $\mathcal{B}_{\overline{x}}$.

Claim 1. If $\overline{c}, \overline{c}' \in \mathbb{M}^{\lg(\overline{x})}$ satisfy $\varphi(\overline{c}) \leftrightarrow \varphi(\overline{c}')$ for every $\varphi \in \mathcal{B}_{\overline{x}}$, then $\mathrm{tp}(\overline{c}/M) = \mathrm{tp}(\overline{c}'/M)$.

Proof. First, as $x = m$ is mutually algebraic for each $x \in \overline{x}$ and $m \in M$, it suffices to show this for all $\overline{c}, \overline{c}' \in (\mathbb{M} \setminus M)^{\lg(\overline{x})}$. In this case, as $\overline{c}, \overline{c}'$ agree on all quantifier-free, mutually algebraic $L(M)$-formulas $\alpha(\overline{x}_u)$ for $\overline{x}_u \subseteq \overline{x}$, it follows that $\overline{c} = \overline{c}_1 \dots \overline{c}_r$ and $\overline{c}' = \overline{c}'_1 \dots \overline{c}'_r$ have corresponding maximal mutually algebraic decompositions over M, and moreover, by our choice of D, $\mathrm{tp}(\overline{c}_i/M) = \mathbf{p}|M$ if and only if $\mathrm{tp}(\overline{c}'_i/M) = \mathbf{p}|M$ for every $1 \leq i \leq r$ and every $\mathbf{p} \in \mathrm{QMA}(\mathbb{M})$. Thus, $\mathrm{tp}(\overline{c}/M) = \mathrm{tp}(\overline{c}'/M)$ by Conclusion 5.10.

Claim 2. Every $\theta(\overline{x}) \in \mathrm{QF}_{\overline{x}}(M)$ is equivalent to some $\varphi(\overline{x}) \in \mathcal{B}_{\overline{x}}$.

Proof. Fix $\theta(\overline{x}) \in \mathrm{QF}_{\overline{x}}(M)$. We first show that for every $\overline{e} \in \theta(\mathbb{M})$, there is some $\delta_{\overline{e}}(\overline{x}) \in \mathcal{B}_{\overline{x}}$ such that $\mathbb{M} \models \delta_{\overline{e}}(\overline{e})$ and

$$\mathbb{M} \models \forall \overline{x}(\delta_{\overline{e}}(\overline{x}) \rightarrow \theta(\overline{x})).$$

To see this, let $\Gamma(\overline{x}) := \{\delta(\overline{x}) \in \mathcal{B}_{\overline{x}} : \mathbb{M} \models \delta(\overline{e})\}$. By Claim 1, every $\overline{e}'$ realizing $\Gamma(\overline{x})$ also realizes $\theta(\overline{x})$, so the existence of $\delta_{\overline{e}}(\overline{x})$ follows by compactness and the $\aleph_1$-saturation of $\mathbb{M}$. Let $\Delta(\overline{x}) = \{\delta_{\overline{e}}(\overline{x}) : \overline{e} \in \theta(\mathbb{M})\}$. A second application of compactness and saturation implies that some finite $\Delta_0 \subseteq \Delta$ satisfies

$$\mathbb{M} \models \forall \overline{x}(\theta(\overline{x}) \rightarrow \bigvee \Delta_0(\overline{x}))$$

so take $\varphi(\overline{x}) := \bigvee \Delta_0(\overline{x})$.

That $T = Th(M)$ is mutually algebraic now follows immediately from Claim 2 and Theorem 2.1 applied to M.

7 Identifying mutually algebraic structures and theories in arbitrary languages

In this section, we use Theorem 6.1 to deduce local tests for mutual algebraicity without regard to the size of the language, nor the completeness of the theory. We begin by noting, either by Lemma 2.10 of [3] or deducing it from Lemma 2.1, that 'mutual algebraicity' is an elementary property, i.e., if M is mutually algebraic, then any elementarily equivalent M' is mutually algebraic as well.

Lemma 7.1. *Suppose $L_0 \subseteq L$ are finite relational languages, M is an L-structure, and M_0 is its reduct to an L_0-structure.*

1. *If $D \subseteq M$ is finite, every L_0-type $p \in \mathrm{Supp}^{L_0}_{\overline{x}}(D)$ has an extension to an L-type $q \supseteq p$ with $q \in \mathrm{Supp}^{L}_{\overline{x}}(D)$.*

2. *If M has uniformly bounded arrays, then so does M_0.*

Proof. (1) Choose an infinite array $\{\overline{a}_i : i \in \omega\} \subseteq (M_0)^{\lg(\overline{x})}$ of realizations of p. As both D and L are finite, there are only finitely many L-types in $S_{\overline{x}}(D)$, so there is an infinite subsequence $\{\overline{a}_i : i \in I\}$ such that $\mathrm{tp}_M(\overline{a}_i/D) = \mathrm{tp}_M(\overline{a}_j/D)$ for all $i, j \in I$. Then $\mathrm{tp}_M(\overline{a}_i)$ for any $i \in I$ is as required.

(2) Assume that M has uniformly bounded arrays. Let $\mathbb{M}$ be an $\aleph_1$-saturated model of $Th(M)$ and let $\mathbb{M}_0$ be the reduct of $\mathbb{M}$ to L_0. So $\mathbb{M}_0$ is an $\aleph_1$-saturated model of $Th(M_0)$. Suppose every L-atomic formula has free variables among $\overline{z}$. By applying Theorem 6.1 to $Th(M)$, $\mathrm{Supp}^{L}_{\overline{x}}(\mathbb{M})$ is finite for all subsequences $\overline{x} \subseteq \overline{z}$, and by a second application of Theorem 6.1 to $Th(M_0)$, it suffices to establish the following Claim:

Claim. For each $\overline{x} \subseteq \overline{z}$, $|\mathrm{Supp}^{L_0}_{\overline{x}}(\mathbb{M}_0)| \leq |\mathrm{Supp}^{L}_{\overline{x}}(\mathbb{M})|$.

Proof. Fix $\overline{x} \subseteq \overline{z}$, and assume that $\mathrm{Supp}^{L}_{\overline{x}}(\mathbb{M}) = \{\mathbf{q}_i : i < N\}$. In order to establish the Claim, it suffices to prove that for every finite $D \subseteq \mathbb{M}_0$, every L_0-type $p \in \mathrm{Supp}^{L_0}_{\overline{x}}(D)$ is a subset of some $\mathbf{q}_i$. To see this, choose any finite $D \subseteq \mathbb{M}_0$ and any $p \in \mathrm{Supp}^{L_0}_{\overline{x}}(D)$. By (1), there is some L-type $q \supseteq p$ with $q \in \mathrm{Supp}^{L}_{\overline{x}}(D)$. Since q has an extension to some $\mathbf{q}_i \in \mathrm{Supp}^{L}_{\overline{x}}(\mathbb{M})$, it follows that $p \subseteq \mathbf{q}_i$ for some $i < N$.

Definition 7.2. For L an arbitrary language, fix any atomic L-formula $R(\overline{z})$. Let $L_R := \{R, =\}$, which is visibly finite relational. Given an L-structure M, let M_R denote the reduct of M to an L_R-structure. We say R *has uniformly bounded arrays in* M if the L_R-structure M_R has uniformly bounded arrays.

Theorem 7.3. *The following are equivalent for an L-structure M in an arbitrary language L:*

1. *M is mutually algebraic;*
2. *Every atomic $R(\overline{z})$ has uniformly bounded arrays in M;*
3. *For every atomic $R(\overline{z})$, the reduct M_R is mutually algebraic.*

Proof. The equivalence of (2) and (3) follows by applying Theorem 6.1 to each of the (finite relational) L_R-theories $Th(M_R)$.

For (3) $\Rightarrow$ (1), in order to prove that M is mutually algebraic, by Theorem 2.1, it suffices to prove that every atomic $R(\overline{z})$ is equivalent to a boolean combination of mutually algebraic formulas. Fix an atomic $R(\overline{z})$. By (3) and Theorem 2.1 applied to M_R, $R(\overline{z})$ is equivalent to a boolean combination of quantifier-free mutually algebraic $L_R(M)$-formulas. As a mutually algebraic formula in M_R is also mutually algebraic in M, the result follows.

Finally, assume (1). To obtain (2), fix an atomic $R(\overline{z})$. By Theorem 2.1, choose a finite set $\{\varphi_1(\overline{x}_1, \overline{e}_1), \dots, \varphi_k(\overline{x}_k, \overline{e}_k)\}$ of mutually algebraic, quantifier-free L-formulas for which $R(\overline{z})$ is equivalent in M to some boolean combination (so each $\overline{x}_i$ is a subsequence of $\overline{z}$). Expand L to L', adding new $\lg(\overline{x}_i)$-ary relation symbols U_i and let M' be the definitional expansion interpreting each U_i as $\varphi_i(M, \overline{e}_i)$. Let $L_0 = \{U_1, \dots, U_k\}$, $L_0^R = L_0 \cup \{R\}$, and let M_0, M_0^R be the reducts of M' to L_0 and L_0^R, respectively. Note that M_0 and M_0^R have the same quantifier-free definable sets, and that the reduct of M_0^R to L_R is M_R.

As each L_0-atomic formula is mutually algebraic, it follows from Theorem 2.7 of [3] that M_0 is mutually algebraic. As L_0 is finite relational, by applying Theorem 6.1 to $Th(M_0)$, M_0 has uniformly bounded arrays. Since M_0 and M_0^R have the same quantifier-free definable sets, we conclude that M_0^R also has uniformly bounded arrays. As L_0^R is finite relational, it follows from Lemma 7.1 that M_R has uniformly bounded arrays as well.

The following Corollary now follows easily. Clause 2 is a slight strengthening of Theorem 2.1.

Corollary 7.4. *Let L be an arbitrary language.*

1. *The reduct of a mutually algebraic L-structure is mutually algebraic.*
2. *If M is mutually algebraic, then every atomic $R(\overline{z})$ is equivalent to a boolean combination of mutually algebraic, quantifier-free L_R-formulas.*

Proof. (1) Let M be any mutually algebraic L-structure, let $L_0 \subseteq L$ be arbitrary, and let M_0 be the reduct of M to L_0. Fix any atomic $R(\overline{z}) \in L_0$. Applying Theorem 7.3 to M gives M_R mutually algebraic. As this holds for all atomic $R \in L_0$, a second application of Theorem 7.3 implies M_0 is mutually algebraic.

(2) is also by Theorem 7.3.

Finally, we consider incomplete theories. The following Corollary follows immediately from Theorem 7.3, as by definition, an incomplete theory T is mutually algebraic if and only if every model $M \models T$ is mutually algebraic.

Corollary 7.5. *A possibly incomplete theory T in an arbitrary language is mutually algebraic if and only if for every $M \models T$, every atomic $R(\overline{z})$ has uniformly bounded arrays in M.*

A A basis of mutually algebraic formulas

Lemma A.1. *Let T be an arbitrary L-theory, $\mathrm{lg}(\overline{x}) = k$, and suppose that*

$$\theta(\overline{x}) := \bigwedge_{i \in S} \alpha_i(\overline{x}_i) \wedge \bigwedge_{j \in U} \neg\beta_j(\overline{x}_j)$$

is mutually algebraic and supports an infinite array, with each $\alpha_i(\overline{x}_i), \beta(\overline{x}_j)$ mutually algebraic and each $\overline{x}_i, \overline{x}_j$ a subsequence of $\overline{x}$. Then there is a subset $S_0 \subseteq S$ of size at most k such that $\bigcup_{i \in S_0} \overline{x}_i = \overline{x}$ and $\theta^*(\overline{x}) = \bigwedge_{i \in S_0} \alpha_i(\overline{x}_i)$ is mutually algebraic.

Proof. Form a maximal sequence $\langle i_0, \ldots, i_{m-1} \rangle$ from S such that for each $j < m$, $\overline{x}_{i_j}$ is properly partitioned by $\bigcup_{t<j} \overline{x}_{i_t}$, i.e., $\overline{x}_{i_j} \cap \bigcup_{t<j} \overline{x}_{i_t} \neq \emptyset$ and $\overline{x}_{i_j} \setminus \bigcup_{t<j} \overline{x}_{i_t} \neq \emptyset$. As $\mathrm{lg}(\overline{x}) = k$, $m \leq k$. Take $S_0 = \{i_j : j < m\}$, let $\overline{x}_m = \bigcup_{i \in S_0} \overline{x}_i$. By iterating Lemma 2.4(6) of [2], $\theta^*(\overline{x}_m) := \bigwedge_{i \in S_0} \alpha_i(\overline{x}_i)$ is mutually algebraic, so to complete the proof it suffices to prove that $\overline{x}_m = \overline{x}$.

Suppose this were not the case, i.e., write $\overline{x} = \overline{x}_m \hat{}\, \overline{y}$ with $\overline{y} \neq \emptyset$. Choose an infinite array $\{\overline{a}_n : n \in \omega\}$ of realizations of $\theta(\overline{x})$. Let $\overline{b} = \overline{a}_0 \restriction_{\overline{y}}$ and, for every $n \in \omega$, let $\overline{c}_n = (\overline{a}_n \restriction_{\overline{x}_m}) \hat{}\, \overline{b}$. It suffices to prove the following Claim, as it contradicts $\theta(\overline{x})$ being mutually algebraic.

Claim. $\theta(\overline{c}_n)$ holds for cofinitely many n.

Towards the Claim, we first show that $\alpha_i(\overline{c}_n \restriction_{\overline{x}_i})$ holds for every $i \in S$ and $n \in \omega$. To see this, fix $i \in S$. By the maximality of the sequence defining S_0, either $\overline{x}_i \subseteq \overline{x}_m$ or $\overline{x}_i \cap \overline{x}_m = \emptyset$. If $\overline{x}_i \subseteq \overline{x}_m$, then $\alpha_i(\overline{c}_n \restriction_{\overline{x}_i})$ holds because $\overline{c}_n \restriction_{\overline{x}_i} = \overline{a}_n \restriction_{\overline{x}_i}$ and $\overline{a}_n$ realizes $\theta(\overline{x})$. On the other hand, if $\overline{x}_i \cap \overline{x}_m = \emptyset$, then $\overline{c}_n \restriction_{\overline{x}_i} = \overline{b} \restriction_{\overline{x}_i} = \overline{a}_0 \restriction_{\overline{x}_i}$ and $\overline{a}_0$ realizes $\theta(\overline{x})$.

To finish the proof of the Claim, it suffices to show that for any $j \in U$, $\neg\beta_j(\overline{c}_n \restriction_{\overline{x}_j})$ holds for cofinitely many n. Write $\overline{x}_j = \overline{x}' \hat{}\, \overline{y}'$ with $\overline{x}' = \overline{x}_j \cap \overline{x}_m$ and $\overline{y}' = \overline{x}_j \setminus \overline{x}_m$, and again we split into cases. If $\overline{y}' = \emptyset$, i.e., $\overline{x}_j \subseteq \overline{x}_m$, then $\overline{c}_n \restriction_{\overline{x}_j} = \overline{a}_n \restriction_{\overline{x}_j}$, so $\neg\beta_j(\overline{c}_n \restriction_{\overline{x}_j})$ since $\theta(\overline{a}_n)$. If $\overline{x}' = \emptyset$ then $\overline{c}_n \restriction_{\overline{x}_j} = \overline{b} \restriction_{\overline{x}_j} = \overline{a}_0 \restriction_{\overline{x}_j}$, so again $\neg\beta_j(\overline{c}_n \restriction_{\overline{x}_j})$ since $\theta(\overline{a}_0)$. Finally, if both $\overline{x}'$ and $\overline{y}'$ are non-empty, $\overline{x}' \hat{}\, \overline{y}'$ is a proper partition of $\overline{x}_j$. Let $\overline{b}' = \overline{b} \restriction_{\overline{y}'}$. As $\beta_j(\overline{x}_j)$ is mutually algebraic, choose an integer s such that $\exists^{\leq s} \overline{x}' \beta_j(\overline{x}', \overline{b}')$. As $\{\overline{c}_n \restriction_{\overline{x}'} : n \in \omega\}$ are disjoint, there are at most s n's for which $\beta_j(\overline{c}_n \restriction_{\overline{x}'}, \overline{b}')$ holds, hence $\neg\beta_j(\overline{c}_n \restriction_{\overline{x}_j})$ holds for cofinitely many n.

The following Proposition reaps the benefit of a finite, relational language.

Proposition A.2. *Suppose M is mutually algebraic in a finite, relational language with every atomic formula having free variables among $\overline{z}$. There is a finite set $\mathcal{F} = \{\varphi_i(\overline{x}_i, \overline{w}_i) : i < m\}$ of quantifier-free L-formulas such that for every $\overline{x} \subseteq \overline{z}$ and every $p \in \mathrm{Supp}_{\overline{x}}(M)$ that contains a mutually algebraic formula, there is some $\varphi_i \in \mathcal{F}$ and $\overline{e}_i \in M^{\lg(\overline{w}_i)}$ such that $\varphi_i(\overline{x}, \overline{e}_i) \in p$ and is mutually algebraic.*

Proof. First, by Theorem 2.1, there is a finite set $\mathcal{B} = \{\delta_i(\overline{x}_i, \overline{e}_i) : i < n\}$ of mutually algebraic $L(M)$-formulas such that every atomic $R \in L$ is equivalent to a boolean combination of formulas from $\mathcal{B}$. It follows that for every $\overline{x} \subseteq \overline{z}$, every $\gamma(\overline{x}) \in \mathrm{QF}_{\overline{x}}(M)$ is also equivalent to a boolean combination of $\mathcal{B}$-formulas. Let $k = \lg(\overline{z})$ and let $\mathcal{B}_k$ denote the (finite) set of $\leq k$-conjunctions of formulas from $\mathcal{B}$, i.e., $\mathcal{B}_k = \{\bigwedge \mathcal{B}_0 : \mathcal{B}_0 \subseteq \mathcal{B}$ and $|\mathcal{B}_0| \leq k\}$. Let

$$\mathcal{F} = \{\varphi(\overline{x}, \overline{w}) : \overline{x} \subseteq \overline{z} \text{ and } \varphi(\overline{x}, \overline{h}) \in \mathcal{B}_k \text{ for some } \overline{h} \in M^{\lg(\overline{w})}\}$$

To see that $\mathcal{F}$ is as desired, fix $\overline{x} \subseteq \overline{z}$, $p \in \mathrm{Supp}_{\overline{x}}(M)$ containing a mutually algebraic formula $\gamma(\overline{x})$, and a realization $\overline{a}$ of p. Write $\gamma(\overline{x})$ as a disjunction $\bigvee \theta_\ell$, where each θ_ℓ is a conjunction of $\mathcal{B}$-formulas and their negations. Let $\theta(\overline{x})$ be one of the conjuncts for which $\theta(\overline{a})$ holds. Then $\theta(\overline{x}) \in p$ and since $\theta(\overline{x}) \vdash \gamma(\overline{x})$, $\theta(\overline{x})$ is mutually algebraic. Write

$$\theta(\overline{x}) = \bigwedge_{i \in S} \delta_i(\overline{x}_i, \overline{e}_i) \wedge \bigwedge_{j \in U} \neg\delta_j(\overline{x}_j, \overline{e}_j)$$

with each $\delta_i(\overline{x}_i, \overline{e}_i), \delta_j(\overline{x}_j, \overline{e}_j) \in \mathcal{B}$, hence mutually algebraic. Apply Lemma A.1 to $\theta(\overline{x})$, obtaining $S_0 \in [S]^{\leq n}$ as there. Thus, $\varphi(\overline{x}, \overline{h}) := \bigwedge_{i \in S_0} \delta_i(\overline{x}_i, \overline{e}_i) \in p$ and is mutually algebraic. Visibly, $\varphi(\overline{x}, \overline{h}) \in \mathcal{B}_k$, so $\varphi(\overline{x}, \overline{w}) \in \mathcal{F}$ as required.

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