The rise and fall of uncountable models

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Recall: Every counterexample Φ to Vaught's Conjecture is scattered.

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If a countable fragment $\Delta \subseteq L_{\omega_1,\omega}$ is sufficiently nice, then Morley proved:

Fact (Poor-man's compactness)

Suppose $T \subseteq \Delta$ is:

- *finitely satisfiable; and,*
- *For every valid disjunction $\bigvee \Gamma \in \Delta$, $T \models \theta$ for some $\theta \in \Gamma$.*

THEN T is satisfiable.

Suppose Φ is a counterexample to VC. Call a sentence large if it is satisfied in uncountably many countable models.

Definition

A minimal counterexample is a counterexample Φ such that for every $\Psi \in L_{\omega_1,\omega}$, exactly one of $\Phi \wedge \Psi$, $\Phi \wedge \neg\Psi$ is large.

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Choose a sufficiently nice, countable fragment Δ such that $\Phi \in \Delta$ and for every $\Phi' \models \Phi$, if $\Phi' \in \Delta$, then there is some $\Psi \in \Delta$ such that both $\Phi' \wedge \Psi$ and $\Phi' \wedge \neg\Psi$ are large.

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Using 'Poor-man's compactness' construct a perfect tree $\{T_\eta : \eta \in 2^\omega\}$ of satisfiable, pairwise contradictory subsets of Δ. [Each finite approximation $\bigwedge T_\nu$ will be large. Dovetail 'contradictory' and 'deciding $\bigvee \Gamma_i$' along each branch.]

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- Each T_Δ is Δ-complete.
- Thus: There is a unique Δ-prime model $M_\Delta \models T_\Delta$.
- Furthermore: $\Delta_1 \subseteq \Delta_2$ implies $T_{\Delta_1} \subseteq T_{\Delta_2}$, so
- There is a Δ_1-elementary map $f_{1,2} : M_{\Delta_1} \to M_{\Delta_2}$.

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To show N is uncountable, at each α, choose $\Delta_{\alpha+1}$ to contain the (small) Scott sentence Φ_α of M_α. As $M_\alpha, M_{\alpha+1}$ disagree on Φ_α, $M_{\alpha+1}$ properly extends M_α.

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Baldwin: If T is a complete, first-order counterexample, then $I(T,\aleph_1) = 2^{\aleph_1}$. [Pf: T is not ω-stable! Look at spectra.]

Upshot: Models of size $\aleph_1$ of any counterexample are abundant.

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- $\{U_n : n \in \omega\}$ partition A^2;
- If $U_n(a,b)$, then $f_m(a,b) = a$ for all $m \geq n$; and
- With respect to 'smallest substructure' $\mathfrak{A}$ has no independent subset of size 3.

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Facts:

- K_0 has countably many isomorphism types and satisfies disjoint amalgamation.
- There is a Fraïsse limit M. Its Scott sentence θ has a proper elementary extension.
- Every model of θ is locally finite with respect to 'smallest substructure'.
- Every model of θ has no independent subset of size 3.

Thus: θ has models N of size $\aleph_1$, yet every such N is maximal.

The point: Let X be any uncountable set, and $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ a locally finite closure relation on X. Then:

- X has an independent subset of size 2.
- If X has a proper, uncountable cl-closed $Y \subseteq X$, then X has an independent subset of size 3.

Variant: Let $L^h = L \cup \{U, V, \pi\}$ and K_0^h are all 2-sorted, finite $\mathscr{B}$ satisfying:

- The reduct of $U(\mathscr{B})$ to L is an element of K_0;
- $V(\mathscr{B})$ has no structure;
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Then K_h has a Fraïsse limit M_h with Scott sentence θ_h.
Moreover:

- $V(M_h)$ is absolutely indiscernible (every permutation of $V(M_h)$ extends to an automorphism of M_h);
- $\pi : U(M_h) \to V(M_h)$ is onto;
- Every model of θ_h of size $\aleph_1$ is maximal (note: $|U(N)| \geq |V(N)|$).

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Theorem (Baldwin-Friedman-Koerwien-L)

If there is a cx to VC Φ, then there is a cx to VC Φ^* with the property that every model of cardinality $\aleph_1$ is maximal.

Previously, Hjorth proved that if a cx to VC exists, then there is one with no model of size $\aleph_2$.