Unique decomposition in classifiable theories

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1 Introduction

By a classifiable theory we shall mean a theory which is superstable, without the dimensional order property, which has prime models over pairs. In order to define what we mean by unique decomposition, we remind the reader of several definitions and results. We adopt the usual conventions of stability theory and work inside a large saturated model of a fixed classifiable theory T; for instance, if we write $M \subseteq N$ for models of T, M and N we are thinking of these models as elementary submodels of this fixed saturated models; so, in particular, M is an elementary submodel of N. Although the results will not depend on it, we will assume that T is countable to ease notation.

We do adopt one piece of notation which is not completely standard: if T is classifiable, $M_0 \subseteq M_i$ for $i = 1, 2$ are models of T and M_1 is independent from M_2 over M_0 then we write $M_1 \oplus_{M_0} M_2$ for the prime model over $M_1 \cup M_2$.

Definition 1.1 *1. If $M \subseteq N$ are models of T then $M \subseteq_{na} N$ if whenever $\varphi(x) \in L(M)$ such that $\varphi(N) \setminus M$ is non-empty and $F \subseteq M$ is any finite set then $\varphi(M) \setminus acl(F)$ is non-empty.*

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2. *We write $M \subseteq_{\aleph_1} N$ and say that M is a relatively $\aleph_1$-saturated substructure of N if, whenever A and B are countable subsets of M and N respectively, there is B' in M with the same type as B over A.*

3. *If $M_0 \subseteq M \subseteq N$ then M is an M_0-component of N if the weight of $tp(M/M_0)$ is 1 and M is maximal with respect to domination over M_0 i.e. if $M \subseteq X \subseteq N$ and M dominates X over M_0 then $M = X$.*

The following Theorem from [1] explains the importance of components in classifiable theories.

Theorem 1.2 *If T is classifiable and N is a model of T with $M \subseteq_{na} N$ then N is prime and minimal over any maximal M-independent collection of M-components of N.*

Keeping the notation of the previous Theorem, we will call such a maximal collection of M-components an M-component decomposition of N or simply a component decomposition of N if the intended M is clear from context. The question we wish to address in this paper is, under what general circumstances is the component decomposition unique?

Definition 1.3 *Let T be a classifiable theory.*

1. *If $M \subseteq_{na} N$ then we say that N has a unique decomposition over M if whenever $\mathcal{C}_1$ and $\mathcal{C}_2$ are component decompositions of N over M there is a bijection $f : \mathcal{C}_1 \to \mathcal{C}_2$ such that for $C \in \mathcal{C}_1$, C is isomorphic to $f(C)$ over M; we say that these two component decompositions are M-isomorphic.*

2. *A substructure notion for T, $\subseteq_*$, is a relation between pairs of models of T such that*
   - *(a) if $M \subseteq_* N$ then $M \subseteq_{na} N$ and,*
   - *(b) if $M_0 \subseteq_* M_i$ for $i = 1, 2$, M_1 and M_2 are independent over M_0 and*
     $N = M_1 \oplus_{M_0} M_2$ then $M_i \subseteq_* N$ for $i = 1, 2$.

3. *We say that a classifiable theory T has unique decompositions with respect to a substructure notion $\subseteq_*$ if whenever $M \subseteq_* N$ are models of T then N has a unique decomposition over M*

Remark: $\subseteq_{na}$ is a substructure notion for any superstable theory; see the appendices of [2].

Example: $T = Th(Z, +)$ does not have unique decompositions with respect to $\subseteq_{na}$. To see this, suppose that N is a saturated model of T of cardinality greater than $2^{\aleph_0}$ and M is any countable submodel. The M-components of N are of the form $\langle M, a\rangle$ for any element $a \in N \setminus M$. Let J be a maximal M-independent collection of representatives of cosets of the connected component not realized in M; note that since M is countable, J is of cardinality $2^{\aleph_0}$. Fix $a \in J$. Let I be a maximal M-independent collection of realizations of the connected component in N; I has cardinality greater than $2^{\aleph_0}$.

The first component decomposition is the collection of subgroups generated by M and b for any $b \in I \cup J$. The second component decomposition is the collection of subgroups generated by M and $b \in (a + I) \cup J$ where $a + I = \{a + c : c \in I\}$. Simply by considering the size of I, one sees that the two component decompositions are not M-isomorphic.

On the other hand, in Chapter XIII of [3], the following Theorem is proved.

Theorem 1.4 *If T is classifiable and non-multidimensional then T has unique decompositions with respect to relative $\aleph_1$-saturation.*

To make the statement of the following Proposition easier, let us say that for a classifiable theory, if $M \subseteq_{na} N$ then we say that N *has unique components over* M if whenever
$M \subseteq M_i, P \subseteq N$, $N = M_1 \oplus_M P = M_2 \oplus_M P$ and M_i is an M-component of N then M_1 is isomorphic to M_2 over M.

Proposition 1.5 *Suppose that T is a classifiable theory.*

1. *If $\subseteq_*$ is a substructure notion then T has unique decompositions with respect to $\subseteq_*$ iff whenever $M \subseteq_* N$, N has unique components over M.*

2. *Suppose that $M \subseteq_{na} N$ and N has unique components over M then N has a unique decomposition over N.*

Proof: Suppose that T has unique decompositions with respect to $\subseteq_*$; we adopt the notation of the paragraph preceding the Proposition. Let N' be the prime model over $|P|^+$-many P-independent copies of N over P. Now N' has two decompositions over M; one which involves $|P|^+$-many M-independent copies of M_1 and one which involves the same number of M-independent copies of M_2. It follows then, since T has unique decompositions, that M_1 is isomorphic to M_2 over M.

To prove the other direction and the second part of the Proposition, we make use of the following fact:

Fact 1.6 *Suppose that X is a closed subset of some pregeometry and I and J are two bases of X. Then there is a bijection $f : I \to J$ such that for any $a \in I$, $J \cup \{a\} \setminus \{f(a)\}$ is a basis for X and for any $b \in J$, $I \cup \{b\} \setminus \{f^{-1}(b)\}$ is a basis for X.*

So if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two decompositions of N over M then by applying the fact, there is a bijection $f : \mathcal{C}_1 \to \mathcal{C}_2$ such that for any $C \in \mathcal{C}_1$, $\{C\} \cup \mathcal{C}_2 \setminus \{f(C)\}$ is a decomposition of N. If P is the prime model over $\mathcal{C}_2 \setminus \{f(C)\}$ inside N then since $N = C \oplus_M P = f(C) \oplus_M P$, by assumption, C is isomorphic to $f(C)$ over M which proves that N has a unique decomposition over M. □

2 The main theorem

Suppose that $M \subseteq N$. We now describe a game between two players. For the first move, Player A fixes a countable subset of M, C, and then chooses a countable set $A_1 \subseteq N$ and Player B responds by choosing $B_1 \subseteq M$ and a C-elementary map $f_1 : A_1 \to B_1$. After the nth play of the game, Player I will have chosen a countable set $A_n \subseteq N$ and Player B will have chosen $B_n \subseteq M$ and a C-elementary map $f_n : A_n \to B_n$. For the (n+1)$^{\text{st}}$ move then Player A chooses a countable subset of N, $A_{n+1} \supseteq A_n$ and Player B responds with B_{n+1} and an elementary map $f_{n+1} : A_{n+1} \to B_{n+1}$ extending f_n. If Player B can always make a legal move then B wins. If Player B has a winning strategy for this game then we write $M \subseteq^* N$.

For a arbitrary model N, an $\subseteq^*$-substructure is in general quite large. However, if M is $\aleph_1$-saturated then $M \subseteq^* N$ for any model N of which it is an elementary submodel. Moreover, we will see later in this section that models of classifiable, shallow theories have reasonably small $\subseteq^*$-substructures. The main theorem of this section is

Theorem 2.1 *Suppose that T is a countable, classifiable theory and $M \subseteq^* N$. Then N has a unique decomposition over M.*

Proof: By Proposition 1.5, it suffices to prove that if $\overline{M}$ and $\overline{M}'$ are two P_0-components of N such that $P_0 \subseteq^* N$ and there is P, $P_0 \subseteq P \subseteq N$ such that $N = \overline{M} \oplus_{P_0} P = \overline{M}' \oplus_{P_0} P$ then $\overline{M} \cong \overline{M}'$ over P_0.

So fix $\overline{M}, \overline{M}', P_0, P$ and N and elements $a \in \overline{M}$ and $a' \in \overline{M}'$ which dominate $\overline{M}$ and $\overline{M}'$ over P_0 respectively. Choose $M_0 \subseteq_{na} P_0$ such that a and a' are independent from P_0 over M_0. Let M and M' be the M_0-components dominated over M_0 by a and a' inside $\overline{M}$ and $\overline{M}'$ respectively. Since under these circumstances $\overline{M} = M \oplus_{M_0} P_0$ and $\overline{M}' = M' \oplus_{M_0} P_0$, it suffices to prove the following technical lemma

Lemma 2.2 *Suppose $N = M \oplus_{M_0} P = M' \oplus_{M_0} P$ where M_0 is a countable model,*
$M_0 \subseteq_{na} M, M' \subseteq N$ and $M_0 \subseteq_{na} P_0 \subseteq^* P \subseteq N$. Then $M \oplus_{M_0} P_0 \cong_{P_0} M' \oplus_{M_0} P_0$.

Before we begin the proof, we remind the reader of the following terminology.

Definition 2.3 *Suppose that $M \subseteq_{na} N$ where N is a model of T. Then a* tree decomposition *of N over M consists of a tree I, with ordering , of height at most ω and a family of models M_η for $\eta \in I$ such that*

1. *if $\langle\rangle$ is the root of I then $M_{\langle\rangle} = M$,*
2. *whenever $\eta \quad \nu$ in I, $M_\eta \subseteq_{na} M_\nu \subseteq_{na} N$ and $wt(M_\nu/M_\eta) = 1$,*
3. *for any $\eta \in I$, $\{M_\nu : \nu^- = \eta\}$ is independent over M_η where ν^- represents the predecessor of ν, and*
4. *if $\eta \quad \nu \quad \mu$ in I then $M_\mu/M_\nu \perp M_\eta$.*

Such a decomposition is called a countable decomposition *if M_η is countable for all $\eta \in I$.*

Any tree of models which satisfies the last two conditions of the above definition is called a *normal tree of models*. One important consequence of being a normal tree of models is that the indexed family of models is independent with respect to the underlying tree ordering.

The following is proved in [1].

Theorem 2.4 *If T is classifiable and $M \subseteq_{na} N$ then N is prime and minimal over any maximal tree decomposition over M.*

Proof of technical Lemma 2.2:

We first produce a tree I with ordering and, for $\eta \in I$, countable models $C_\eta, C'_\eta, D_\eta, D'_\eta$ and P_η such that

1. if $\eta \quad \mu$ then $C_\eta \subseteq_{na} C_\mu \subseteq M$, $C'_\eta \subseteq_{na} C'_\mu \subseteq M'$ and $P_\eta \subseteq P$; if $\langle\rangle$ is the root of I then $C_{\langle\rangle} = C'_{\langle\rangle} = P_{\langle\rangle} = M_0$,

2. $\{C_\eta : \eta \in I\}$ and $\{C'_\eta : \eta \in I\}$ are normal trees of models, M is prime over C_I and M' is prime over C'_I,

3. M_η is the prime model over $C_\eta D_\eta P_\eta$ as well as the prime model over $C'_\eta D'_\eta P_\eta$,

4. C_η is domination equivalent to C'_η over M_{η^-},

5. $D_\eta = C_{I_\eta}$ for some countable, downward closed subset of $\{\mu \in I : \mu \; / \; \eta\}$ and

6. $D'_\eta \subseteq M'$ and C'_η is independent from D'_η over C'_{η^-}.

The above data is produced by starting with countable decompositions of M and M' and then working upwards, by induction, to produce the required models. In fact, $\{C_\eta : \eta \in I\}$ and $\{C_\eta : \eta \in I\}$ can be decompositions of M and M' respectively except that possibly, for an $\eta \in I$ which has no successors, the weight of $tp(C_\eta/C_{\eta^-})$ may not be one due to the presence of non-trivial types; similarly for the C'_η's. The D_η and D'_η play a purely auxillary role simply to guarantee the existence of M_η.

Now pick $P^*_\eta \subseteq P_0$ and an M_0-elementary map $f_\eta : P_\eta \to P^*_\eta$ inductively such that if $\eta \quad \nu$ then f_ν extends f_η. We can do this because $P_0 \subseteq^* P$.

Now pick inductively C^*_η, D^*_η and an elementary map g_η, extending f_η such that g_η fixes C_η and D_η and $g_\eta(C'_\eta) = C^*_\eta$, $g_\eta(D'_\eta) = D^*_\eta$ and if $\nu \quad \eta$

then g_ν extends g_η. Let M^*_η be the prime model over $C_\eta D_\eta P^*_\eta$ (equivalently the prime model over $C^*_\eta D^*_\eta P^*_\eta$). Let $h_\eta = g_\eta \quad {}_{C'_\eta}$.

We will show that $\bigcup\{h_\eta : \eta \in I\}$ is elementary. Since $\{C'_\eta : \eta \in I\}$ is a normal tree and h_η is elementary for each η, it suffices to show that for every $\nu \in I$, $\{C^*_\eta : \eta^- = \nu\}$ is independent over C^*_ν. To see this, note that $\{C_\eta : \eta^- = \nu\}$ is independent over M^*_ν. Now for any η such that $\eta^- = \nu$, C_η is domination equivalent to C'_η over M_ν. Since g_η is elementary, C_η is domination equivalent to C^*_η over M^*_ν. It follows then that $\{C^*_\eta : \eta^- = \nu\}$ is independent over M^*_ν. But C'_η is independent from M_{η^-} over C'_{η^-} and so we conclude that $\{C^*_\eta : \eta^- = \nu\}$ is independent over C^*_ν.

Let M^* be prime over C^*_I and contained in $\hat{N} = M \oplus_{M_0} P_0$. It is clear that M^* is independent from P_0 over M_0 and so it suffices to show that if $C^*_I \subseteq M^* \subseteq \hat{M} \subseteq \hat{N}$ and $\hat{M}$ is independent from P_0 over M_0 then $M^* = \hat{M}$. If not, by standard arguments, one can find a $b \in \hat{N}$ and $\eta \in I$ such that $tp(b/C^*_\eta)$ is orthogonal to $C^*_{\eta^-}$, or η is the root of I, and b is independent from M^* over C^*_η.

Let's handle the case when η is the root of I first. In this case, b is independent from $\overline{C}^*_\eta = \{C^*_\mu : \mu^- = \eta\}$ over M_0. Moreover, $b\overline{C}^*_\eta$ is independent from P_0 over M_0. Since $\overline{C}^*_\eta$ is domination equivalent to $\{C_\mu : \mu^- = \eta\}$ over M_0 and the latter dominates M over M_0, if follows that b is independent from MP_0 over M_0 which implies that $b \in M_0$ which is a contradiction.

So now we are assuming that η is not the root of I. By construction, C^*_η is independent from $M^*_{\eta^-}$ over $C^*_{\eta^-}$. We know that C_η is independent from $\{C_\mu : \mu \;/\; \eta\}P_0$ over $M^*_{\eta^-}$. Now since C_η is domination equivalent to C^*_η over $M^*_{\eta^-}$, we get by transitivity that C^*_η is independent from $\{C_\mu : \mu \;/\; \eta\}P_0$ over $C^*_{\eta^-}$. Let N_1 be contained in N and prime over $\{C_\mu : \mu \quad \eta\} \cup M^*_\eta \cup P_0$ and N_2 be contained in N and prime over $\{C_\mu : \mu \;/\; \eta\} \cup C_\eta \cup P_0$. Let N_η be prime over $M^*_\eta P_0$.

Suppose for a moment that $tp(c/C^*_\eta)$ is any type orthogonal to $C^*_{\eta^-}$. From above we get that c is independent from $\{C_\mu : \mu \;/\; \eta\}P_0$ over C^*_η and so, in particular,

$$c \mathop{\downarrow}_{C^*_\eta} N_\eta \quad \text{and} \quad c \mathop{\downarrow}_{N_\eta} N_2$$

We apply this to obtain that $b\overline{C}^*_\eta$ is independent from N_2 over N_η and $b\overline{C}^*_\eta$ is independent from N_η over C^*_η where $\overline{C}^*_\eta = \{C^*_\mu : \mu^- = \eta\}$. Since b and $\overline{C}^*_\eta$ are independent over C^*_η, it follows that $\overline{C}^*_\eta$ is independent from bN_2 over

N_η. $\overline{C}^*_\eta$ and N_1 are domination equivalent over N_η so it follows that b is independent from $\hat{N}$ over N_2; b is independent from N_2 over C^*_η so $b \in C^*_\eta$ which is a contradiction. □

Corollary 2.5 *Any countable, classifiable theory T has unique decompositions with respect to $\subseteq^*$.*

The following corollary is interesting and appears to be new.

Corollary 2.6 *If T is superstable without the dimensional order property then the class of $\aleph_1$-saturated models of T has unique decompositions; that is, if $M \subseteq N$ are both $\aleph_1$-saturated models of T then N has a unique decomposition over M.*

Proof: By our earlier remark, any $\aleph_1$-saturated model is a $\subseteq^*$-substructure of any model of which it is a substructure; by the main Theorem, the result follows immediately. □

Definition 2.7 *Suppose that $\mathcal{A} = \{M_i : i \in I\}$ and $\mathcal{B} = \{N_j : j \in J\}$ are increasing families of models which are independent with respect to trees I and J respectively. We say that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as labelled trees via a system of maps $\{f_i : i \in I\}$ if there is an order isomorphism $\nu : I \to J$ so that $f_i : M_i \to N_{\nu(i)}$ is an isomorphism and $f_i \restriction_{M_j} = f_j$ whenever $j \leq i$.*

Definition 2.8 *Suppose that N has a countable decomposition $\mathcal{P} = \{N_\eta : \eta \in I\}$.*

1. *We say that N is homogeneous with respect to $\mathcal{P}$ if whenever J and J' are countable downward closed subsets of I so that $\{N_\eta : \eta \in J\}$ is isomorphic to $\{N_\eta : \eta \in J'\}$ via a collection of elementary maps $\mathcal{F}$ then for any countable downward closed K, $J \subseteq K \subseteq I$ then there is a downward closed K', $J' \subseteq K' \subseteq I$ so that $\{N_\eta : \eta \in K\}$ is isomorphic to $\{N_\eta : \eta \in K'\}$ as labelled trees via a collection of elementary maps which extends $\mathcal{F}$.*

2. *If M is prime over N_J for some $J \subseteq I$ then $M \subseteq^{\mathcal{P}}_{\aleph_1} N$ if whenever there is countable, downward closed $J' \subseteq J$ and countable I', $J' \subseteq I' \subseteq I$ there is a countable I'', $J' \subseteq I'' \subseteq J$ such that $\{N_\eta : \eta \in I'\}$ is isomorphic to $\{N_\eta : \eta \in I''\}$ as labelled trees via a collection of elementary maps which contains the identity maps on N_η for all $\eta \in J'$.*

Proposition 2.9 *1. If N is any model with decomposition $\mathcal{P}$ then there is a model M, $|M| \leq 2^{\aleph_0}$, such that $M \subseteq^{\mathcal{P}}_{\aleph_1} N$.*

2. If N is homogeneous with respect to $\mathcal{P}$ and $M \subseteq^{\mathcal{P}}_{\aleph_1} N$ then $M \subseteq^* N$.

Proof: The first is a routine union of chains argument and the second is straightforward remembering that in N, any countable set is constructible over a countable, downward closed part of $\mathcal{P}$. □

In [2], the following is critical.

Theorem 2.10 *Suppose T is a countable, classifiable theory, $M \subseteq_{na} N_i$ and $wt(N_i/M) = 1$ for $i = 1, 2$, P is homogeneous with respect to $\mathcal{P}$ and $M \subseteq^{\mathcal{P}}_{\aleph_1} P$. Then if $N_1 \oplus_M P \cong_P N_2 \oplus_M P$ then $N_1 \cong_M N_2$.*

Proof: The proof is identical to the proof to Lemma 2.2. □

Theorem 2.11 *Suppose that T is a countable, classifiable theory of depth d. Then any model N has a $\subseteq^*$-substructure of size at most $_d$; in fact, if d is an infinite successor ordinal then there is a $\subseteq^*$-substructure of size $_{d-1}$.*

Proof: Fix any countable decomposition of N, $\mathcal{P} = \langle M_\eta : \eta \in I \rangle$. We now label the nodes of I by induction on depth; remember that I is well-founded. For a node η of depth 0, label it by the isomorphism type of the chain of models $\langle M_{\eta\, n} : n \leq l(\eta) \rangle$. Note that there are at most $_1$ many such labels; we call these labels of depth zero. For the sake of induction, say that M_η is the witness for its label for any η of depth zero.

Now suppose we have labelled all nodes of depth less than α and have assigned witnesses for every such label. Fix a node η of depth α. Define a function f from the set of labels of depth less than α to $\omega \cup \{\omega, \omega_1\}$ where, for a label L, $f(L)$ is the number of immediate successors of η with label L if this number is countable and ω_1 otherwise. f will be the label for η and will be a label of depth α. To obtain a witness for the label f, take M_η together the union of all the witnesses of labels L which appear countably often immediately above η and $\aleph_1$-many witnesses for labels L which appear uncountably often immediately above η. It is easy to check by induction that the witness for the label η will have size at most $_\alpha$ if α is finite and not zero, or if α is an infinite limit ordinal. Otherwise, the witness will have size at most $_{\alpha-1}$.

In the end, let $\mathcal{P}_\prime = \langle M_\eta : \eta \in I_0 \rangle$ be the witness for the label of the root node and let M be prime over $\mathcal{P}_\prime$. It is straightforward to show that $M \subseteq^* N$. $\square$

Remark: In fact, in the last Theorem, the M we found is slightly more than an L_{∞,ω_1}-substructure.

3 An example

This section is devoted to the proof of the following

Proposition 3.1 *There is a countable, classifiable theory T of depth 3 which fails to have unique decompositions with respect to relative $\aleph_1$-saturation.*

We shall describe the theory and a standard model of the theory simultaneously. To begin with, there is an index set P which we will treat as one sort and a cover of this set Q which we will treat as a separate sort. Let π represent a surjective map from Q to P. Moreover, there will be a free action of P on each fibre of π.

In the standard model of the theory in question, we do the following: let G be the free group on some infinite set of generators I. Let $P = I \cup I^{-1}$ where I^{-1} is the set of inverses of the elements in I in G. Let $Q = P \times G$ and let π be the first projection from Q to P. For the action of P on Q, let τ be defined by $\tau(p, (p', g)) := (p', pg)$.

Now there will be two additional sorts, $\hat{P}$ and $\hat{Q}$ which are covers of P and Q respectively; ρ_P and ρ_Q will be surjective maps from $\hat{P}$ to P and $\hat{Q}$ to Q respectively. Furthermore, elements of $\hat{P}$ and $\hat{Q}$ will be "coloured" by elements of $2^{\aleph_0}$. Finally there will be an action $\cdot$ of $\hat{P}$ on $\hat{Q}$.

In the standard model, these sorts are realized as follows: $\hat{P} = P \times 2^{\aleph_0}$ and $\hat{Q} = Q \times 2^{\aleph_0}$. ρ_P and ρ_Q are the first projections onto P and Q respectively. We define $\cdot$ by $(p, \eta) \cdot (q, \mu) := (\tau(p, q), \eta + \mu)$ where the addition in the second component is occurring co-ordinatewise modulo 2 in $2^{\aleph_0}$. We note then that the two actions are compatible in the sense that

$$\rho_Q(\hat{p} \cdot \hat{q}) = \tau(\rho_P(\hat{p}), \rho_Q(\hat{q}))$$

holds in the standard model.

To obtain the "colours" on $\hat{P}$ and $\hat{Q}$ we introduce predicates U_η and V_η for $\eta \in 2^{<\omega}$. In the standard model, for $\eta \in 2^{<\omega}$, $U_\eta = \{(p, \mu) \in \hat{P} : \eta \subseteq \mu\}$ and $V_\eta = \{(q, \mu) \in \hat{Q} : \eta \subseteq \mu\}$. For any model of the theory so far described, we see that we can define the colour of an element of $\hat{P}$ or $\hat{Q}$ as follows: if $x \in \hat{P}$ then $c_P(x) = \mu$ iff $x \in U_{\mu\restriction n}$ for all $n \in \omega$. Similarly, if $x \in \hat{Q}$ then $c_Q(x) = \mu$ iff $x \in V_{\mu\restriction n}$ for all $n \in \omega$. We record the following relationship between the colours that holds in the standard model (and any model of its theory):

$$c_Q(\hat{p} \cdot \hat{q}) = c_P(\hat{p}) + c_Q(\hat{q})$$

Note also that in a saturated model of the theory described so far the maps c_P and c_Q are onto $2^{\aleph_0}$.

The theory T will be the theory of the standard model describe above with sorts $P, Q, \hat{P}$ and $\hat{Q}$ together with the functions $\pi, \rho_P, \rho_Q, \tau$ and $\cdot$ and predicates U_η and V_η for every $\eta \in 2^{<\omega}$. It is left to the reader to verify that this theory is classifiable and of depth 3.

We will construct models of T, N and a weight one extension N' such that for any model $N_0 \subseteq N$ of cardinality less than $_2$ there are models M_1 and M_2 extending N_0, independent from N over N_0 such that N' is prime (in fact algebraic) over $M_i \cup N$ for $i = 1, 2$ but such that M_1 and M_2 are not isomorphic over N_0. This will be enough to show the failure of unique decompositions with respect to relative $\aleph_1$-saturation.

We start by fixing some notation and terminology. For any model M of T and any $p \in P(M)$, we call the set $\{c_P(\hat{p}) : \hat{p} \in \hat{P}(M), \rho_P(\hat{p}) = p\}$, the colours of p. Similarly, for any $q \in Q(M)$, we call the set $\{c_Q(\hat{q}) : \hat{q} \in \hat{Q}(M), \rho_Q(\hat{q}) = q\}$, the colours of q. Let $X_0 = 2^{<\omega}$ and let B be a basis of $2^{\aleph_0}/X_0$ considered as a vector space over F_2. It is easy to construct N so that $P(N)$ contains $\{c_X : X \subseteq B\}$ and that the colours of c_X is exactly the subspace of $2^{\aleph_0}$ generated by X_0 together with $\bigcup X$. For $Q(N)$, we simply fix a b_X in the π-fibre above c_X and it is easy to arrange that the colours of b_X are X_0. The rest of N is filled out by virtue of the action of P on Q via τ and by the action of $\hat{P}$ on $\hat{Q}$.

For N' we add a single element a to $P(N)$ and arrange that in N' the colours of a are X_0. Now fix an element b in the π-fibre above a and arrange that in N', the colours of b are X_0. Now let N' be the closure of N, a and b under the actions τ and $\cdot$.

Now fix any elementary submodel N_0 of N of cardinality less than $_2$. For some $X \subseteq B$, $c_X \notin N_0$. Let M_1 be the closure under the actions of N_0, a and b and let M_2 be the closure under the actions of N_0, a and $b' = \tau(c_X, b)$. It is easy to see that either of M_1 or M_2 together with N generates N'. However, the colours of b' are not the colours of any point in M_1 and so M_1 is not isomorphic to M_2 over N_0.

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