A simpler axiomatization of the Shelah-Spencer almost sure theories

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Abstract

We give an explicit AE-axiomatization of the almost sure theories of sparse random graphs $G(n, n^{-\alpha})$ of Shelah-Spencer. In the process we give a method of constructing extensions of graphs whose 'relative dimension' is negative, but arbitrarily small. We describe the existentially closed and locally finite models of the theory and produce types of dimension zero. We offer a useful characterization of forking and generalize results about stability and the Dimensional Order Property (DOP) that were known for graphs to arbitrary relational languages.

1 Introduction

Fix an irrational α satisfying $0 < \alpha < 1$. Shelah and Spencer [7] proved that class $G(n, n^{-\alpha})$ of finite (symmetric) graphs with edge probability $n^{-\alpha}$ satisfies a 0-1 law. That is, the almost sure theory T^α of first-order sentences in the language of graphs is complete. Motivated by some unrelated problems in model theory, Baldwin and Shi [3] exhibited a class $\mathbf{K}_\alpha$ of finite graphs and an associated notion $\leq$ of strong substructure such that a $(\mathbf{K}_\alpha, \leq)$-generic object exists (see Definition 5.9) and proved that its theory T_α is stable.

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Following this, Baldwin and Shelah [1] proved the spectacular result that the two theories are equal, i.e. $T^\alpha = T_\alpha$. They also gave an explicit Π_3 (i.e., AEA) axiomatization of the common theory.[1]

Their investigations into this theory continued in [2], where they proved that T_α has the dimensional order property (DOP) and does not have the finite cover property.

In this paper we give a unifying treatment to all of these results in a more general setting. Instead of working with probability measures or generic models, we start from scratch. We fix a finite, relational language L, define a class of finite structures $\mathbf{K}_\alpha$, and describe a Π_2 (AE) theory S_α. We give a bare-bones proof that every L-formula is S_α-equivalent to a boolean combination of 'extension formulas'. It follows easily that S_α is complete, and, when the language is restricted to graphs S_α is equal to both T_α and T^α. *A posteriori* our results allow for many simplifications of the combinatorics and probabalistic methods occurring in [7, 8, 2, 3].

Our method is to establish two finiteness results that follow from the dimension function and the notion of 'granularity' and combine these with existence results (Lemma 4.1 and Proposition 4.2) that are obtained by analyzing good rational approximations of irrational numbers. The construction in Lemma 4.1 is an extension of an idea of Ikeda [5], who used the existence of a good rational approximation of α to establish the nonsuperstability of a general class of structures that includes the Shelah-Spencer graphs.

Using these results, in Section 5 we prove that the theory S_α admits elimination of quantifiers down to certain 'extension formulas.' The completeness of S_α and the fact that S_α is equal to both T^α and T_α follow easily. It is noteworthy that this method bypasses most of the probabalistic complexity of [7].

In Section 6 we describe many different models of S_α and argue that the generic model, while it is unique, is not a 'typical' model of S_α. We are aided by knowing that the class of models of S_α is closed under unions of chains. We characterize the existentially closed models of S_α and show that any model of S_α can be extended to a larger model into which no nonempty finite structure embeds strongly. We also construct an e.c. model $\mathfrak{M}$ in which $\mathrm{acl}(\{a\}) = M$ for every $a \in M$. We define the notion of local finiteness, which is a salient feature of the generic model. To illustrate the point that this does not characterize the generic model, we construct large families of

[1] Somewhat later, Spencer [8] offered a different (but similar) AEA axiomatization.

nonisomorphic countable, locally finite models of S_α.

Finally, in Section 7 we give short, unified proofs that S_α is stable, unsuperstable, has the Dimensional Order Property (DOP) and weak elimination of imaginaries, but does not have the finite cover property (fcp). Some of these facts were known in the special case of graphs (see e.g., [3] and [2]). As well, we offer a useful characterization of forking in models of S_α.

2 Preliminaries

Let L be a finite, relational language with at least one relation symbol of arity at least two. We work exclusively in the class $\mathbf{K}$ of all *symmetric, irreflexive* L-structures, i.e., $\mathfrak{A} \in \mathbf{K}$ if and only if for every $R \in L$ of arity n and every $\bar{a} \in A^n$, if $\bar{a} \in R^{\mathfrak{A}}$ then the elements of $\bar{a}$ are without repetition and $\pi(\bar{a}) \in R^{\mathfrak{A}}$ for every permutation π of $\{0, \ldots, n-1\}$. Hence, for any $\mathfrak{A} \in \mathbf{K}$, $R^{\mathfrak{A}}$ can be thought of as a set of n-element *subsets* of A.

Fix, for the whole of this paper, a set $\{\alpha_R : R \in L\} \subseteq (0,1)$ of irrational numbers such that $\sum_{R\in L} \alpha_R n_R$ is never a positive integer for any sequence $\langle n_R : R \in L\rangle$ of integers. For each positive integer m, define the *granularity* $Gr(m)$ to be the smallest positive value of $\sum \alpha_R n_R - k$, where k is an integer satisfying $0 < k < m$ and $\langle n_R : R \in L\rangle$ is a sequence of nonnegative integers. (For a fixed m there are only finitely many 'candidates' hence the minimum is achieved.)

For a finite $\mathfrak{A} \in \mathbf{K}$, let

$$\delta(\mathfrak{A}) = |A| - \sum_{R\in L} \alpha_R e_R(\mathfrak{A})$$

where $|A|$ denotes the cardinality of $\mathfrak{A}$ and $e_R(\mathfrak{A})$ denotes the number of *subsets* of A that are included in $R^{\mathfrak{A}}$. Let $\overline{\mathbf{K}}_\alpha$ denote the class of all $\mathfrak{A} \in \mathbf{K}$ for which $\delta(\mathfrak{A}') \geq 0$ for all finite substructures $\mathfrak{A}' \subseteq \mathfrak{A}$. We denote the class of finite structures in $\overline{\mathbf{K}}_\alpha$ by $\mathbf{K}_\alpha$. To simplify notation, we include the *empty structure* as an element of $\mathbf{K}_\alpha$.

As notation, if $\mathfrak{A}, \mathfrak{B} \in \mathbf{K}$ are finite and $\mathfrak{A} \subseteq \mathfrak{B}$ we write $\delta(\mathfrak{B}/\mathfrak{A})$ for $\delta(\mathfrak{B}) - \delta(\mathfrak{A})$. We say $\mathfrak{A}$ *is a strong substructure of* $\mathfrak{B}$, written $\mathfrak{A} \leq \mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\delta(\mathfrak{A}'/\mathfrak{A}) \geq 0$ for all $\mathfrak{A} \subseteq \mathfrak{A}' \subsetneq \mathfrak{B}$.

Definition 2.1 The theory S_α is the smallest set of sentences insuring that if $\mathfrak{M} \models S_\alpha$, then

1. $\mathfrak{M} \in \overline{\mathbf{K}}_\alpha$, i.e., every finite substructure of $\mathfrak{M}$ is an element of $\mathbf{K}_\alpha$; and

2. For all $\mathfrak{A} \leq \mathfrak{B}$ from $\mathbf{K}_\alpha$, every embedding $f : \mathfrak{A} \to \mathfrak{M}$ extends to an embedding $g : \mathfrak{B} \to \mathfrak{M}$.

Note that since the empty structure is a strong substructure of every $\mathfrak{B} \in \mathbf{K}_\alpha$, (2) implies that every element of $\mathbf{K}_\alpha$ embeds into every model of S_α.

Definition 2.2 Let n be any positive integer. A set $\{\mathfrak{B}_i : i < n\}$ of finite elements of $\mathbf{K}$ is *disjoint over* $\mathfrak{A}$ if $\mathfrak{A} \subseteq \mathfrak{B}_i$ for each i and $B_i \cap B_j = A$ for all $i < j < n$. If $\{\mathfrak{B}_i : i < n\}$ is disjoint over $\mathfrak{A}$, then a structure $\mathfrak{D}$ is a *join* of $\{\mathfrak{B}_i : i < n\}$ if the universe $D = \bigcup\{B_i : i < n\}$ and $\mathfrak{B}_i \subseteq \mathfrak{D}$ for all i i.e., $R^{\mathfrak{B}_i} \subseteq R^{\mathfrak{D}}$ for all $i \in I$ and all $R \in L$. A join $\mathfrak{D}$ is called a *free join*, which we denote by $\bigoplus_{i<n} \mathfrak{B}_i$, if there are no additional relations, i.e, $R^{\mathfrak{D}} = \bigcup\{R^{\mathfrak{B}_i} : i < n\}$ for all $R \in L$.

The following computational Lemma is straightforward.

Lemma 2.3 *1. If $\{\mathfrak{B}, \mathfrak{C}\}$ are disjoint over $\mathfrak{A}$ and $\mathfrak{D}$ is any join of $\{\mathfrak{B}, \mathfrak{C}\}$, then $\delta(\mathfrak{D}/\mathfrak{B}) \leq \delta(\mathfrak{C}/\mathfrak{A})$. Furthermore, equality holds if $\mathfrak{D}$ is a free join, while $\delta(\mathfrak{D}/\mathfrak{B}) + \alpha_R \leq \delta(\mathfrak{C}/\mathfrak{A})$ whenever $R^{\mathfrak{D}} \neq R^{\mathfrak{B}} \cup R^{\mathfrak{C}}$.*

2. For any $n \in \omega$, if $\{\mathfrak{B}_i : i < n\}$ is disjoint over $\mathfrak{A}$ and $\mathfrak{D} = \bigoplus_{i<n} \mathfrak{B}_i$ is their free join, then $\delta(\mathfrak{D}/\mathfrak{A}) = \sum_{i<n} \delta(\mathfrak{B}_i/\mathfrak{A})$. In particular, if $\mathfrak{A} \leq \mathfrak{B}_i$ for each $i < n$, then $\mathfrak{A} \leq \bigoplus_{i<n} \mathfrak{B}_i$.

Proof. The number $\delta(\mathfrak{C}/\mathfrak{A}) = |C - A| - \sum_{R\in L} \alpha_R(e_R(\mathfrak{C}) - e_R(\mathfrak{A}))$, while $\delta(\mathfrak{D}/\mathfrak{B}) = |C - A| - \sum_{R\in L} \alpha_R(e_R(\mathfrak{D}) - e_R(\mathfrak{B}))$. Since for any $R \in L$, any set $X \in R^{\mathfrak{C}} - R^{\mathfrak{A}}$ is necessarily in $R^{\mathfrak{D}} - R^{\mathfrak{B}}$,

$$e_R(\mathfrak{C}) - e_R(\mathfrak{A}) \leq e_R(\mathfrak{D}) - e_R(\mathfrak{B})$$

and the first inequality follows. If $\mathfrak{D}$ is a free join, then equality holds throughout, while if $R^{\mathfrak{D}} \neq R^{\mathfrak{B}} \cup R^{\mathfrak{C}}$ then $e_R(\mathfrak{C}) - e_R(\mathfrak{A}) + 1 \leq e_R(\mathfrak{D}) - e_R(\mathfrak{B})$, which proves (1). The verification of (2) is immediate by induction on n. ∎

3 Two finiteness results

This section is devoted to setting notation and obtaining two finiteness results which will be used throughout the paper. Both of these are achieved by combining the notion of granularity with the definition of $\overline{\mathbf{K}}_\alpha$. The first, Proposition 3.1, asserts that any sufficiently large collection of substructures of an element of $\overline{\mathbf{K}}_\alpha$ contains an arbitrarily large free join.

Proposition 3.1 *Fix $m \in \omega$ and $\mathfrak{D} \in \overline{\mathbf{K}}_\alpha$. For any infinite set $\{B_i : i \in \omega\}$ of m-element substructures of $\mathfrak{D}$ there is an infinite subset $Y \subseteq \omega$ and a finite $\mathfrak{A} \subseteq \mathfrak{D}$ such that*

1. *$\{\mathfrak{B}_i : i \in Y\}$ is a free join over $\mathfrak{A}$ and are pairwise isomorphic over $\mathfrak{A}$; and*
2. *$\mathfrak{A} \leq \mathfrak{B}_i$ for every $i \in Y$.*

Moreover, for any $m, s \in \omega$ there is an integer $N(m,s)$ large enough such that for any set $\{\mathfrak{B}_i : i < N(m,s)\}$ of substructures, each of size at most m, of any $\mathfrak{D} \in \overline{\mathbf{K}}_\alpha$, there is a subset $Y \subseteq N(m,s)$ and an $\mathfrak{A}$ such that $\{\mathfrak{B}_i : i \in Y\}$ is a free join over $\mathfrak{A}$ and $A \leq \mathfrak{B}_i$ for all $i \in Y$.

Proof. Fix a set $\{B_i : i \in \omega\}$ of m-element substructures of a fixed $\mathfrak{D} \in \overline{\mathbf{K}}_\alpha$. By replacing ω by an infinite subset of itself, it follows from the finite Δ-system lemma that we may assume that there is a fixed $\mathfrak{A}$ such that $B_i \cap B_j = A$ for all $i < j < \omega$. Fix an enumeration $\bar{a}$ of A and enumerations $\bar{b}_i$ of each B_i extending $\bar{a}$. Let r^* denote the maximum arity of the relations $R \in L$. Since L is finite and $|B_i| = m$ for all i, there are only finitely many possibilities for the quantifier-free type $qftp(\bar{b}_{i_1}, \ldots, \bar{b}_{i_{r^*}}/A)$ over A among all possible sequences $i_1 < \ldots < i_{r^*} < \omega$. Thus, by Ramsey's theorem there is an infinite $Y \subseteq \omega$ so that the quantifier-free type $qftp(\bar{b}_{i_1}, \ldots, \bar{b}_{i_{r^*}}/A)$ over A is constant among all sequences $i_1 < \ldots < i_{r^*}$ from Y.

Since $B_i \cap B_j = A$ for all distinct i, j from Y, $\{\mathfrak{B}_i : i \in Y\}$ is clearly a join over $\mathfrak{A}$. That they are pairwise isomorphic over $\mathfrak{A}$ is immediate since $qftp(\bar{b}_i/A)$ is constant. Assume by way of contradiction that it is not a free join. Then there are $R \in L$, $2 \leq t \leq r^*$, and $X_{(i_1, \ldots i_t)} \subseteq R^{\mathfrak{B}_{i_1} \cup \ldots \cup \mathfrak{B}_{i_t}} \setminus \bigcup\{R^{\mathfrak{B}_{i_\ell}} : 1 \leq \ell \leq t\}$ for every increasing sequence $i_1 < \ldots < i_t$ from Y. For every integer N, let Y_N be the first N elements of Y and let $\mathfrak{C}_N$ be the finite substructure of $\mathfrak{D}$ with universe $\bigcup\{B_i : i \in Y_N\}$. Now $|\mathfrak{C}_N|$ grows linearly in N, while (since $t \geq 2$) the number of subsets of $\mathfrak{C}_N$ satisfying R grows at least quadratically. So, if N is large enough, $\delta(\mathfrak{C}_N)$ would be negative, contradicting $\mathfrak{D} \in \overline{\mathbf{K}}_\alpha$. Thus $\{\mathfrak{B}_i : i \in Y\}$ is a free join over $\mathfrak{A}$. Arguing similarly, if $\mathfrak{A} \not\leq \mathfrak{B}_i$ for some (equivalently for every) $i \in Y$, then choose $\mathfrak{A}_i$ such that $\mathfrak{A} \subseteq \mathfrak{A}_i \subseteq \mathfrak{B}_i$ and $\delta(\mathfrak{A}_i/\mathfrak{A}) < 0$. Since $|A_i \setminus A| < m$, it follows from granularity that $\delta(\mathfrak{A}_i/\mathfrak{A}) \leq -Gr(m)$. So, for any integer N if we let $\mathfrak{C}_N$ be the substructure of $\mathfrak{D}$ with universe $\bigcup\{A_j : j \in Y_N\}$ (where A_j is the substructure of $\mathfrak{B}_j$ corresponding to A_i) then by Lemma 2.3(2), $\delta(\mathfrak{C}_N/\mathfrak{A}) \leq -NGr(m)$. Thus, $\delta(\mathfrak{C}_N) < 0$ whenever N is sufficiently large, which again contradicts $\mathfrak{D} \in \overline{\mathbf{K}}_\alpha$.

The 'Moreover' clause follows from the infinitary version by the standard König's Lemma argument. ∎

Our second finiteness result revolves around the idea of a *minimal pair,* which is a minimal instance of $\mathfrak{A} \not\leq \mathfrak{B}$. More precisely:

Definition 3.2 A pair $(\mathfrak{A}, \mathfrak{B})$ from $\mathbf{K}_\alpha$ is a *minimal pair* if $\mathfrak{A} \subseteq \mathfrak{B}$, $\delta(\mathfrak{B}/\mathfrak{A}) < 0$, but $\delta(\mathfrak{A}'/\mathfrak{A}) \geq 0$ for all proper $\mathfrak{A} \subseteq \mathfrak{A}' \subsetneq \mathfrak{B}$.

Lemma 3.3 *Suppose $\mathfrak{D} \in \overline{\mathbf{K}}_\alpha$ and $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are finite substructures of $\mathfrak{D}$ satisfying $(\mathfrak{A}, \mathfrak{B})$ is a minimal pair, $|B \setminus A| < m$, $\mathfrak{A} \subseteq \mathfrak{C}$, but $\mathfrak{B} \not\subseteq \mathfrak{C}$. Then $\delta(\mathfrak{D}'/\mathfrak{C}) \leq -Gr(m)$, where $\mathfrak{D}'$ is the substructure of $\mathfrak{D}$ with universe $B \cup C$.*

Proof. Let $\mathfrak{B}^*$ be the substructure of $\mathfrak{D}$ with universe $B \cap C$. Then $\mathfrak{A} \leq \mathfrak{B}^* \subseteq \mathfrak{B}$ and $\{\mathfrak{B}, \mathfrak{C}\}$ are disjoint over $\mathfrak{B}^*$, so $\mathfrak{D}'$, the substructure of $\mathfrak{D}$ with universe $B \cup C$, is a join of $\{B, \mathfrak{C}\}$. Then

$$\delta(\mathfrak{D}'/\mathfrak{C}) \leq \delta(\mathfrak{B}/\mathfrak{B}^*) \leq -Gr(m)$$

where the first inequality follows from Lemma 2.3 and the second follows from $(\mathfrak{A}, \mathfrak{B})$ being a minimal pair and granularity. ∎

Definition 3.4 Fix $m \in \omega$ and $\mathfrak{A} \in \mathbf{K}_\alpha$. An *$m$-minimal chain over* $\mathfrak{A}$ is a sequence $\langle \mathfrak{A}_i : i \leq j \rangle$ of structures from $\mathbf{K}_\alpha$ such that $\mathfrak{A}_0 = \mathfrak{A}$, $|A_{i+1} \setminus A_i| < m$, and $(\mathfrak{A}_i, \mathfrak{A}_{i+1})$ is a minimal pair for all $i < j$.

Our second finiteness result is almost immediate:

Lemma 3.5 *Fix $m \in \omega$ and $\mathfrak{A} \in \mathbf{K}_\alpha$. Every m-minimal chain $\langle \mathfrak{A}_i : i \leq j \rangle$ over $\mathfrak{A}$ has length $j \leq \delta(\mathfrak{A})/Gr(m)$.*

Proof. Since $|A_{i+1} \backslash A_i| < m$ and $\delta(\mathfrak{A}_{i+1}/\mathfrak{A}_i) < 0$, it follows immediately from the definition of $Gr(m)$ that $\delta(\mathfrak{A}_{i+1}/\mathfrak{A}_i) \leq -Gr(m)$. Thus, for each $i \leq j$, $0 \leq \delta(\mathfrak{A}_i) \leq \delta(\mathfrak{A}) - iGr(m)$, so $j \leq \delta(\mathfrak{A})/Gr(m)$. ∎

We conclude the section with another computation and an application of Lemma 3.5.

Lemma 3.6 *Let $\mathfrak{D} \in \overline{\mathbf{K}}_\alpha$, let $\langle \mathfrak{A}_i : i \leq j \rangle$ be an m-minimal chain over $\mathfrak{A}$ of substructures of $\mathfrak{D}$, and suppose that $\mathfrak{B} \subseteq \mathfrak{D}$ is finite, $\mathfrak{A} \subseteq \mathfrak{B}$, but $\mathfrak{A}_j \not\subseteq \mathfrak{B}$. Then $\delta(\mathfrak{D}_j/\mathfrak{B}) \leq -Gr(m)$, where $\mathfrak{D}_j$ is the substructure of $\mathfrak{D}$ with universe $A_j \cup B$.*

Proof. For each $i \leq j$, let $\mathfrak{D}_i$ denote the substructure of $\mathfrak{D}$ with universe $A_i \cup B$. Note that $\mathfrak{D}_0 = \mathfrak{B}$. By iterating Lemma 3.3 $\delta(\mathfrak{D}_{i+1}/\mathfrak{D}_i) \leq 0$ for all $i < j$, with equality holding when $\mathfrak{D}_{i+1} = \mathfrak{D}_i$ and $\delta(\mathfrak{D}_{i+1}/\mathfrak{D}_i) \leq -Gr(m)$ otherwise. Since $\mathfrak{A}_j \not\subseteq \mathfrak{B}$, $\mathfrak{D}_{i+1} \neq \mathfrak{D}_i$ for at least one i, so $\delta(\mathfrak{D}_j/\mathfrak{B}) = \delta(\mathfrak{D}_j/\mathfrak{D}_0) = \sum_{i=0}^{j-1} \delta(\mathfrak{D}_{i+1}/\mathfrak{D}_i) \leq -Gr(m)$. ∎

Definition 3.7 Fix $m \in \omega$ and $\mathfrak{D} \in \overline{\mathbf{K}}_\alpha$. A finite $\mathfrak{B} \subseteq \mathfrak{D}$ is *m-strong in $\mathfrak{D}$* if $\mathfrak{B} \leq \mathfrak{C}$ for all $\mathfrak{C}$ satisfying $|C \setminus B| < m$ and $\mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{D}$.

Lemma 3.8 *Fix $m \in \omega$, $\mathfrak{D} \in \overline{\mathbf{K}}_\alpha$, and a finite $\mathfrak{A} \subseteq \mathfrak{D}$. Let $\langle \mathfrak{A}_i : i \leq j \rangle$ be a* maximal *m-chain over $\mathfrak{A}$ in $\mathfrak{D}$. Then $\mathfrak{A}_j$ is m-strong and $\mathfrak{A}_j \subseteq \mathfrak{B}$ for any m-strong $\mathfrak{B}$ satisfying $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{D}$. In particular, $\mathfrak{A}_j = \mathfrak{A}'_k$ whenever $\langle \mathfrak{A}'_i : i \leq k \rangle$ is any maximal m-chain over $\mathfrak{A}$ in $\mathfrak{D}$.*

Proof. We first argue that $\mathfrak{A}_j$ is m-strong in $\mathfrak{D}$. By way of contradiction, assume there were $\mathfrak{B}$ satisfying $\mathfrak{A}_j \subseteq \mathfrak{B} \subseteq \mathfrak{D}$, $|B \setminus A_j| < m$, and $\delta(\mathfrak{B}/\mathfrak{A}_j) < 0$. Let $\mathfrak{C}$ be $\subseteq$-minimal such that $\mathfrak{A}_j \subseteq \mathfrak{C} \subseteq \mathfrak{B}$ and $\delta(\mathfrak{C}/\mathfrak{A}_j) < 0$. Then $(\mathfrak{A}_j, \mathfrak{C})$ is a minimal pair, contradicting the maximality of the m-chain. So $\mathfrak{A}_j$ is m-strong in $\mathfrak{D}$.

Now suppose that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{D}$ and that $\mathfrak{B}$ is m-strong in $\mathfrak{D}$. We argue that $\mathfrak{A}_j \subseteq \mathfrak{B}$. If this were not the case, then choose the largest $i < j$ such that $\mathfrak{A}_i \subseteq \mathfrak{B}$. Let $\mathfrak{C}$ be the substructure of $\mathfrak{D}$ with universe $A_{i+1} \cup B$. Then $\delta(\mathfrak{C}/\mathfrak{B}) < 0$ by Lemma 3.6, contradicting $\mathfrak{B}$ being m-strong in $\mathfrak{D}$. ∎

Remark 3.9 As a special case of Lemma 3.8, suppose that $\mathfrak{A} \subseteq \mathfrak{B}$ are from $\mathbf{K}_\alpha$. Let $m = |B|$ and let $\langle \mathfrak{A}_i : i \leq j \rangle$ be a maximal m-chain over $\mathfrak{A}$ of substructures of $\mathfrak{B}$. Then $\mathfrak{A}_j \leq \mathfrak{B}$. As well, it is easily checked that $\delta(\mathfrak{A}_j)$ is minimal among all $\mathfrak{C}$ satisfying $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{B}$.

4 Rational approximations and existence theorems

The goal of this section is to prove Lemma 4.1 and Proposition 4.2 which, together with our finiteness lemmas, form the basis of our understanding of models of S_α. The proof of Lemma 4.1 is a variant of a construction of Ikeda [5], which is based on the existence of a good rational approximation to an irrational α.

Fix an irrational α satisfying $0 < \alpha < 1$. A (soft) classical result of number theory (see e.g., Theorem 6.8 of [6]) states that there are infinitely many pairs of positive integers (a, b) satisfying $\left|\frac{a}{b} - \alpha\right| < 1/b$. It follows that the set $G = \{a - b\alpha : a, b \in \mathbb{N}^+\}$ is dense in the real numbers.

For each $n \in \mathbb{N}^+$, let q_n be the unique positive integer satisfying $0 < n - q_n\alpha < \alpha$. As notation, let $q_n^+ = q_n + 1$. Since $\alpha < 1$, $q_n < q_m$ whenever $n < m < \omega$.

Call a positive integer p *locally optimal* if

$$|p - q_p^+\alpha| < |n - q_n^+\alpha|$$

for all $1 \leq n < p$. Since G is dense in $\mathbb{R}$, infinitely many positive integers p are locally optimal.

Now fix a locally optimal $p > 1$. For $1 \leq n < p$ let $d_n = n - q_n\alpha$ and let $d_p = p - q_d^+\alpha$. Note that

a) $0 < d_n < \alpha$ for all $1 \leq n < p$;

b) $d_p < 0$; and

c) $d_n - d_m < \alpha$ whenever $1 \leq n < m \leq p$;

where the verification of c) when $m = p$ uses the local optimality of p.

Also, define a sequence $\langle s_n : 1 \leq n \leq p\rangle$ by: $s_1 = q_1$; $s_n = q_n - q_{n-1} - 1$ for $1 < n < p$; and $s_p = q_p^+ - q_{p-1} - 1$. Then

d) When $1 \leq n < p$, $\sum_{i=1}^n s_i$ is telescoping and equals $q_n - (n-1)$; hence

e) $\sum_{i=1}^p s_i = q_p^+ - (p-1)$.

Since $q_n < n/\alpha < q_n + 1$ for all $n \in \mathbb{N}^+$ it follows that

f) $0 \le s_n < 1/\alpha$ for all $1 \le n < p$ and $0 \le s_p < 1 + 1/\alpha$.

Lemma 4.1 *Fix any $\overline{R} \in L$ of arity $\bar{r} \ge 2$. Given any $\mathfrak{B} \in \mathbf{K}_\alpha$ with at least $1/\alpha_{\overline{R}} + \bar{r}$ elements and any $\epsilon > 0$, there is $\mathfrak{D} \in \mathbf{K}_\alpha$ extending $\mathfrak{B}$ such that*

1. *$-\epsilon < \delta(\mathfrak{D}/\mathfrak{B}) < 0$; and*

2. *For any proper substructure $\mathfrak{D}' \subsetneq \mathfrak{D}$, $\delta(\mathfrak{D}'/\mathfrak{D}' \cap \mathfrak{B}) \ge 0$.*

Moreover, $\mathfrak{D}$ can be chosen so that $R^{\mathfrak{D}} = R^{\mathfrak{B}}$ for all $R \ne \overline{R}$.

Proof. To ease notation, denote $\alpha_{\overline{R}}$ by α. Since $\mathfrak{B}$ is nonempty, $\delta(\mathfrak{B}) > 0$. We may assume that ϵ is less than both $\delta(\mathfrak{B})$ and α. Since G is dense in $\mathbb{R}$ we can find infinitely many locally optimal integers p such that $-\epsilon < p - q_p^+\alpha < 0$. Fix one such p that also satisfies $p((1/\alpha) - 1) > |B|$ and define the sequences $\langle d_n, s_n : 1 \le n \le p\rangle$ as above. Our lower bound on $|B|$ and f) ensure that $\binom{|B|}{\bar{r}-1} \ge 1 + 1/\alpha > \max\{s_i : 1 \le i \le p\}$. Since $q_p^+ > p/\alpha$ it follows from e) that $\sum_{i=1}^p s_i > |B|$.

Let $C = \{c_1, \ldots, c_p\}$ be disjoint to B. We let $\mathfrak{D}$ be any L-structure satisfying the following conditions:

i. The universe of $\mathfrak{D}$ is $B \cup C$;

ii. $\mathfrak{B} \subseteq \mathfrak{D}$;

iii. For each relation symbol $R \ne \overline{R}$, $R^{\mathfrak{D}} = R^{\mathfrak{B}}$;

iv. For each $1 \le i \le p$ there are exactly s_i subsets Q of B, each of size $\bar{r} - 1$ such that $Q \cup \{c_i\} \in \overline{R}^{\mathfrak{D}}$; (this is possible since $\binom{|B|}{\bar{r}-1} > s_i$)

v. Each $b \in B$ is in at least one of the sets Q from the previous clause; (this is possible since $\sum_{i=1}^p s_i > |B|$)

vi. There is exactly one (possibly empty) subset $Z \subseteq B$ of size $\bar{r} - 2$ such that $Z \cup \{c_i, c_{i+1}\} \in \overline{R}^{\mathfrak{D}}$ for each $1 \le i < p$; and

vii. $\overline{R}^{\mathfrak{D}}$ contains no other subsets of D.

Once we establish the inequalities in the conclusion of the Lemma it will be evident that $\mathfrak{D} \in \mathbf{K}_\alpha$. First, $D - B$ has p elements, Clause *iv*) contributes

$\sum_{i=1}^{p} s_i$ subsets on which $\overline{R}^{\mathfrak{D}}$ holds, and Clause *vi*) contributes an additional $p-1$ subsets. Thus

$$\delta(\mathfrak{D}/\mathfrak{B}) = p - \alpha\left(\sum_{i=1}^{p} s_i + (p-1)\right) = p - q_p^+\alpha$$

so we obtain the first inequality from the definition of q_p^+. Similarly, for any $1 \le n < p$, if $\mathfrak{D}_n \subseteq \mathfrak{D}$ has universe $B \cup \{c_1, \ldots, c_n\}$, then

$$\delta(\mathfrak{D}_n/\mathfrak{B}) = n - \alpha\left(\sum_{i=1}^{n} s_i + (n-1)\right) = n - q_n\alpha = d_n > 0$$

Furthermore, if $1 \le n < m \le p$ and $\mathfrak{D}_{n,m}$ is the substructure of $\mathfrak{D}$ with universe $B \cup \{c_{n+1}, \ldots, c_m\}$ then $\delta(\mathfrak{D}_{n,m}/\mathfrak{B}) = \delta(\mathfrak{D}_m/\mathfrak{D}_n) + \alpha$ since Clause *vi*) gives a single relationship between c_n and c_{n+1}. But $\delta(\mathfrak{D}_m/\mathfrak{D}_n) = \delta(\mathfrak{D}_m/\mathfrak{B}) - \delta(\mathfrak{D}_n/\mathfrak{B}) = d_m - d_n$, so $\delta(\mathfrak{D}_{n,m}/\mathfrak{B}) = d_m - d_n + \alpha > 0$. But now, if A is *any* nonempty proper subset of C and $\mathfrak{D}_A$ is the substructure of $\mathfrak{D}$ with universe $B \cup A$, then $\mathfrak{D}_A$ is the free join over $\mathfrak{B}$ of structures of the form $\mathfrak{D}_n$ and $\mathfrak{D}_{n,m}$. Thus $\delta(\mathfrak{D}_A/\mathfrak{B}) > 0$ by Lemma 2.3(2). So far we have verified (1) and, since $\delta(\mathfrak{B}/\mathfrak{B}) = 0$, we have verified (2) for all proper substructures $\mathfrak{D}' \subsetneq \mathfrak{D}$ that contain $\mathfrak{B}$.

To verify (2) in the general case, let $\mathfrak{D}'$ be any proper substructure of $\mathfrak{D}$. Let $\mathfrak{B}_0$ and $\mathfrak{D}^*$ be the substructures of $\mathfrak{D}$ with universes $D' \cap B$ and $D' \cup B$, respectively. Thus $\mathfrak{D}^*$ is a join of $\{\mathfrak{D}', \mathfrak{B}\}$ over $\mathfrak{B}_0$. There are now two cases.

First, if $\mathfrak{D}^*$ is a proper subset of $\mathfrak{D}$, then $\delta(\mathfrak{D}^*/\mathfrak{B}) \ge 0$ from our work above. Also, Lemma 2.3 gives that $\delta(\mathfrak{D}'/\mathfrak{B}_0) \ge \delta(\mathfrak{D}^*/\mathfrak{B})$, so $\delta(\mathfrak{D}'/\mathfrak{B}_0) \ge 0$.

On the other hand, if $\mathfrak{D}^* = \mathfrak{D}$, then $B - B_0$ must be nonempty since $\mathfrak{D}'$ is proper. Since $\mathfrak{D}^* = \mathfrak{D}$, Clause *v*) implies that $\overline{R}^{\mathfrak{D}^*} \ne \overline{R}^{\mathfrak{B}_0} \cup \overline{R}^{\mathfrak{D}'}$, so the second inequality of Lemma 2.3 implies that $\delta(\mathfrak{D}'/\mathfrak{B}_0) \ge \delta(\mathfrak{D}^*/\mathfrak{B}) + \alpha$. But $\delta(\mathfrak{D}^*/\mathfrak{B}) > -\epsilon$, so again $\delta(\mathfrak{D}'/\mathfrak{B}_0) > 0$.

We have completed the verification of (2) in the general case, which as noted above, additionally implies that $\mathfrak{D} \in \mathbf{K}_\alpha$. ∎

Proposition 4.2 *Suppose that $\mathfrak{A} \le \mathfrak{B}$ from $\mathbf{K}_\alpha$, $\mu > 0$, and a finite set $\Phi \subseteq \mathbf{K}_\alpha$ are given such that $\mathfrak{B} \subseteq \mathfrak{C}$ but $\mathfrak{B} \not\le \mathfrak{C}$ for all $\mathfrak{C} \in \Phi$. Then there is $\mathfrak{D}^* \supseteq \mathfrak{B}$, $\mathfrak{D}^* \in \mathbf{K}_\alpha$ such that*

1. *$\delta(\mathfrak{D}^*/\mathfrak{A}) < \mu$;*

2. $\mathfrak{A} \leq \mathfrak{D}^*$*; and*

3. No $\mathfrak{C} \in \Phi$ *isomorphically embeds into* $\mathfrak{D}^*$ *over* $\mathfrak{B}$*.*

Proof. Fix $\mathfrak{A}, \mathfrak{B}, \mu$ and Φ as above. If $\mathfrak{A} = \mathfrak{B}$, then we can take $\mathfrak{D}^* = \mathfrak{B}$, so assume that $\mathfrak{A} \neq \mathfrak{B}$, hence $\delta(\mathfrak{B}/\mathfrak{A}) > 0$ by our constraints on $\{\alpha_R : R \in L\}$. By replacing each $\mathfrak{C} \in \Phi$ by a minimal $\mathfrak{C}'$ such that $\mathfrak{B} \subseteq \mathfrak{C}' \subseteq \mathfrak{C}$ but $\mathfrak{B} \not\leq \mathfrak{C}'$ we may assume that $\delta(\mathfrak{C}/\mathfrak{B}) < 0$ for each $\mathfrak{C} \in \Phi$. Also, by adding a finite set X to $\mathfrak{B}$ with no extra relations and replacing each $\mathfrak{C} \in \Phi$ by the (finite) set of all $\mathfrak{C}^*$ with universe $C \cup X$ extending this enlarged $\mathfrak{B}$, we may assume that $|\mathfrak{B}| \geq 1/\alpha + \bar{r}$.

Choose an integer $s > |\mathfrak{C}|$ for all $\mathfrak{C} \in \Phi$ and choose $\epsilon > 0$ such that

i) $\epsilon < \mu$;

ii) $\epsilon < \delta(\mathfrak{B}/\mathfrak{A})$; and

iii) $s\epsilon < -\delta(\mathfrak{C}/\mathfrak{B})$ for all $\mathfrak{C} \in \Phi$.

Apply Lemma 4.1 to obtain $\mathfrak{D}$ for $\mathfrak{B}$ and ϵ. Let $\gamma = -\delta(\mathfrak{D}/\mathfrak{B})$. So $0 < \gamma < \epsilon$. Choose an integer $k > 0$ such that

$$k\gamma \leq \delta(\mathfrak{B}/\mathfrak{A}) < (k+1)\gamma$$

Let $\{\mathfrak{D}_i : i < k\}$ be k copies of $\mathfrak{D}$ with $D_i \cap D_j = B$ for all $i \neq j$ and let $\mathfrak{D}^* = \bigoplus_{i<k} \mathfrak{D}_i$ be the free join of $\{\mathfrak{D}_i : i < k\}$ over $\mathfrak{B}$.

By Equation 1.1 $\delta(\mathfrak{D}^*/\mathfrak{B}) = -k\gamma$, so

$$\delta(\mathfrak{D}^*/\mathfrak{A}) = \delta(\mathfrak{D}^*/\mathfrak{B}) + \delta(\mathfrak{B}/\mathfrak{A}) = \delta(\mathfrak{B}/\mathfrak{A}) - k\gamma < \gamma < \epsilon < \mu$$

To show that $\mathfrak{A} \leq \mathfrak{D}^*$, choose any $\mathfrak{C}$, $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{D}^*$. We must show that $\delta(\mathfrak{C}/\mathfrak{A}) \geq 0$. Let $\mathfrak{B}_0 = \mathfrak{C} \cap \mathfrak{B}$ and let $\mathfrak{C}_i = \mathfrak{C} \cap \mathfrak{D}_i$ for each $i < k$. There are now two cases.

Case 1. $\mathfrak{B}_0 = \mathfrak{B}$.

By Lemma 4.1 $\delta(\mathfrak{D}'/\mathfrak{B}) \geq -\gamma$ for all $\mathfrak{D}'$ satisfying $\mathfrak{B} \subseteq \mathfrak{D}' \subseteq \mathfrak{D}$, so $\delta(\mathfrak{C}_i/\mathfrak{B}) \geq -\gamma$ for each $i < k$. Thus

$$\delta(\mathfrak{C}) = \delta(\mathfrak{B}) + \sum_{i<k} \delta(\mathfrak{C}_i/\mathfrak{B}) \geq \delta(\mathfrak{B}) - k\gamma \geq \delta(\mathfrak{A})$$

Case 2. $\mathfrak{B}_0 \neq \mathfrak{B}$.

Since $\mathfrak{A} \le \mathfrak{B}$, $\delta(\mathfrak{B}_0/\mathfrak{A}) \ge 0$. Furthermore, each $\mathfrak{C}_i$ is a proper substructure of $\mathfrak{D}_i$, so $\delta(\mathfrak{C}_i/\mathfrak{B}_0) \ge 0$ for all $i < k$ by Clause (2) of Lemma 4.1. Thus $\delta(\mathfrak{C}/\mathfrak{A}) \ge 0$ as required. So we have established that $\mathfrak{A} \le \mathfrak{D}^*$.

Finally, we show that no $\mathfrak{C} \in \Phi$ can embed into $\mathfrak{D}^*$ over $\mathfrak{B}$. To see this, suppose that $\mathfrak{C}$ satisfies $|\mathfrak{C}| < s$ and $\mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{D}^*$. We argue that $\delta(\mathfrak{C}/\mathfrak{B}) \ge -s\epsilon$, which by Condition iii) on ϵ implies that $\mathfrak{C}$ is not isomorphic over $\mathfrak{B}$ to any element of Φ. To establish this inequality, let $\mathfrak{C}_i = \mathfrak{C} \cap \mathfrak{D}_i$ for each $i < k$. Since $|\mathfrak{C}| < s$ there are fewer than s i's for which $\mathfrak{C}_i \ne \mathfrak{B}$. Since $\mathfrak{C}$ is the free join of $\{\mathfrak{C}_i : i < k\}$ over $\mathfrak{B}$, Equation 1.1 and Lemma 4.1 imply

$$\delta(\mathfrak{C}) = \delta(\mathfrak{B}) + \sum_{i<k} \delta(\mathfrak{C}_i/\mathfrak{B}) \ge \delta(\mathfrak{B}) - s\epsilon$$

so $\delta(\mathfrak{C}/\mathfrak{B}) \ge -s\epsilon$. ∎

Definition 4.3 Let $\mathfrak{B} \in \mathbf{K}_\alpha$ and let Φ be a finite subset of $\mathbf{K}_\alpha$ such that each $\mathfrak{C} \in \Phi$ extends $\mathfrak{B}$. For any $\mathfrak{M} \models S_\alpha$, an embedding $g : \mathfrak{B} \to \mathfrak{M}$ *omits* Φ if there is no embedding $h : \mathfrak{C} \to \mathfrak{M}$ extending g for any $\mathfrak{C} \in \Phi$.

Proposition 4.4 *Suppose that $\mathfrak{A} \le \mathfrak{B}$ are from $\mathbf{K}_\alpha$ and Φ is a finite subset of $\mathbf{K}_\alpha$ such that for each $\mathfrak{C} \in \Phi$, $\mathfrak{A} \le \mathfrak{C}$, $\mathfrak{B} \subseteq \mathfrak{C}$, but $\mathfrak{B} \not\le \mathfrak{C}$. Then for any $\mathfrak{M} \models S_\alpha$, for any embedding $f : \mathfrak{A} \to \mathfrak{M}$ there are infinitely many embeddings $g_i : \mathfrak{B} \to \mathfrak{M}$ extending f such that each g_i omits Φ and $\{g_i(\mathfrak{B}) : i \in \omega\}$ is disjoint over $f(\mathfrak{A})$.*

Proof. To ease notation we may assume $f = id$, i.e., $\mathfrak{A} \subseteq \mathfrak{M}$. By replacing each $\mathfrak{C} \in \Phi$ by a $\subseteq$-minimal $\mathfrak{C}'$ satisfying $\mathfrak{B} \subseteq \mathfrak{C}' \subseteq \mathfrak{C}$ and $\delta(\mathfrak{C}'/\mathfrak{B}) < 0$, we may assume that $(\mathfrak{B}, \mathfrak{C})$ is a minimal pair for all $\mathfrak{C} \in \Phi$. Choose an integer m so that $|C \setminus A| < m$ for all $\mathfrak{C} \in \Phi$. Using Proposition 4.2, choose $\mathfrak{D} \in \mathbf{K}_\alpha$ such that $\mathfrak{A} \le \mathfrak{D}$, $\mathfrak{B} \subseteq \mathfrak{D}$, but $\delta(\mathfrak{D}/\mathfrak{A}) < Gr(m)$. Choose a disjoint family $\{\mathfrak{D}_i : i \in \omega\}$ over $\mathfrak{A}$ and isomorphisms $k_i : \mathfrak{D} \to \mathfrak{D}_i$ over $\mathfrak{A}$ for each i. Since $\bigoplus_{i<n} \mathfrak{D}_i \le \bigoplus_{i \le n} \mathfrak{D}_i$ for each n and since $\mathfrak{M} \models S_\alpha$, one can inductively construct an embedding $j : \bigoplus_{i \in \omega} \mathfrak{D}_i \to \mathfrak{M}$ extending f. As notation, for each $i \in \omega$ let $g_i = j \circ k_i$, let $\mathfrak{B}'_i = g_i(\mathfrak{B})$, and let $\mathfrak{D}'_i = g_i(\mathfrak{D})$. So $\mathfrak{A} \subseteq \mathfrak{B}'_i \subseteq \mathfrak{D}'_i \subseteq \mathfrak{M}$ for each i and $\{\mathfrak{D}'_i : i \in \omega\}$ is disjoint over $\mathfrak{A}$.

We complete the proof by showing that the set $Z = \{i \in \omega : g_i$ does not omit $\Phi\}$ is finite. Assume by way of contradiction that Z were infinite. For each $i \in Z$, choose $\mathfrak{C}_i \in \Phi$ and an embedding $h_i : \mathfrak{C}_i \to \mathfrak{M}$ extending $g_i|\mathfrak{B}$.

For each such i, let $\mathcal{H}_i$ be the substructure of $\mathfrak{M}$ with universe $D'_i \cup h_i(C_i)$. Note that $|\mathcal{H}_i| < |\mathfrak{D}| + m$ for each $i \in Z$. By Proposition 3.1 there is an $\mathcal{F}$ and an infinite $Y \subseteq Z$ such that $\{\mathcal{H}_i : i \in Y\}$ is disjoint over $\mathcal{F}$ and $\mathcal{F} \leq \mathcal{H}_i$ for each $i \in Y$. Fix any $i(*) \in Y$. Since $\{\mathfrak{D}'_i : i \in Y\}$ are disjoint over $\mathfrak{A}$, $\mathfrak{A} \subseteq \mathcal{F} \subseteq h_i(\mathfrak{C}_{i(*)})$. Since $\mathfrak{A} \leq \mathfrak{C}_{i(*)}$ by hypothesis, this implies $\mathfrak{A} \leq \mathcal{F}$, hence $\mathfrak{A} \leq \mathcal{H}_{i(*)}$ by transitivity. But this is impossible, since $\delta(\mathcal{H}_{i(*)}/\mathfrak{D}'_{i(*)}) < 0$ (hence $\leq -Gr(m)$), while $\delta(\mathfrak{D}'_{i(*)}/\mathfrak{A}) < Gr(m)$ and $\delta(\mathcal{H}_{i(*)}/\mathfrak{A}) = \delta(\mathcal{H}_{i(*)}/\mathfrak{D}'_{i(*)}) + \delta(\mathfrak{D}'_{i(*)}/\mathfrak{A})$. ∎

We close this section with a corollary showing that one of the properties of genericity holds for all $\aleph_0$-saturated models of S_α. Corollary 6.6 will demonstrate that the saturation assumption is necessary.

Definition 4.5 A *strong substructure* $\mathfrak{A} \subseteq \mathfrak{M}$ is a finite substructure of $\mathfrak{M}$ such that $\mathfrak{A} \leq \mathfrak{B}$ for all finite $\mathfrak{B}$ satisfying $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$. An embedding $f : \mathfrak{A} \to \mathfrak{M}$ is a *strong embedding* if $f(\mathfrak{A})$ is a strong substructure of $\mathfrak{M}$.

Corollary 4.6 *Suppose $\mathfrak{A} \leq \mathfrak{B}$ are from $\mathbf{K}_\alpha$ and $f : \mathfrak{A} \to \mathfrak{M}^*$ is strong, where $\mathfrak{M}^* \models S_\alpha$ is $\aleph_0$-saturated. Then there is a strong embedding $g : \mathfrak{B} \to \mathfrak{M}^*$ extending f. In particular, every $\mathfrak{B} \in \mathbf{K}_\alpha$ embeds strongly into $\mathfrak{M}^*$.*

Proof. First, note that if $\mathfrak{C} \in \mathbf{K}_\alpha$ extends $\mathfrak{B}$, but $\mathfrak{A} \not\leq \mathfrak{C}$, then since f is strong, *any* embedding $g : \mathfrak{B} \to \mathfrak{M}^*$ omits $\mathfrak{C}$. So let Φ be the (infinite) set of all isomorphism types (over $\mathfrak{B}$) of $\mathfrak{C} \in \mathbf{K}_\alpha$ such that $\mathfrak{B} \subseteq \mathfrak{C}$, $\mathfrak{A} \leq \mathfrak{C}$, but $\mathfrak{B} \not\leq \mathfrak{C}$. By Proposition 4.4, for every finite $\Phi_0 \subseteq \Phi$ there is an embedding $g : \mathfrak{B} \to \mathfrak{M}^*$ extending f. Since $\mathfrak{M}^*$ is $\aleph_0$-saturated there is $g : \mathfrak{B} \to \mathfrak{M}^*$ extending f that omits all of Φ. Combining this with the note above, g omits every extension $\mathfrak{C} \supseteq \mathfrak{B}$ such that $\mathfrak{B} \not\leq \mathfrak{C}$. Thus g is a strong embedding. The final sentence follows immediately since $\emptyset \leq \mathfrak{B}$ for any $\mathfrak{B} \in \mathbf{K}_\alpha$. ∎

5 Quantifier Elimination and completeness of S_α

In this section we prove that S_α admits quantifier elimination down to the level of Boolean combinations of chain-minimal extension formulas (see Definition 5.5). It follows easily from this that S_α is complete, is equivalent to the theory of the $(\mathbf{K}_\alpha, \leq)$-generic, and in the case of graphs is precisely the Shelah-Spencer almost sure theory. In keeping with the overall presentation, the proofs offered are combinatorial and highly syntactic. More model-theoretic proofs of these results are possible by passing to sufficiently saturated elementary extensions and using Corollary 4.6.

Definition 5.1 For each $\mathfrak{A} \in \mathbf{K}_\alpha$ and $m \in \omega$, we say $\mathfrak{B} \in \mathbf{K}_\alpha$ is *constructed by an m-chain over* $\mathfrak{A}$ if there is an m-chain $\langle \mathfrak{A}_i : i \leq j \rangle$ over $\mathfrak{A}$ and $\mathfrak{B} = \mathfrak{A}_j$. Let $X_m(\mathfrak{A})$ be a set of representatives of isomorphism types of $\mathbf{K}_\alpha$ that are constructed by m-chains over $\mathfrak{A}$.

Clearly, $\mathfrak{A} \in X_m(\mathfrak{A})$, every $\mathfrak{A}' \in X_m(\mathfrak{A})$ extends $\mathfrak{A}$, and by Lemma 3.5 $X_m(\mathfrak{A})$ is finite.

Definition 5.2 For $\mathfrak{A}', \mathfrak{A}'' \in X_m(\mathfrak{A})$, write $\mathfrak{A}' \sqsubset \mathfrak{A}''$ if there is an embedding $g : \mathfrak{A}' \to \mathfrak{A}''$ over $\mathfrak{A}$ such that $g(A') \neq A''$. If $\mathfrak{M} \models S_\alpha$ and $f : \mathfrak{A} \to \mathfrak{M}$ is an embedding, a structure $\mathfrak{A}^* \in X_m(\mathfrak{A})$ is *maximally embeddable in* $\mathfrak{M}$ *over* f if there is an embedding $f' : \mathfrak{A}^* \to \mathfrak{M}$ extending f, but for any $\mathfrak{A}'$ such that $\mathfrak{A}^* \sqsubset \mathfrak{A}'$, there is no embedding $g : \mathfrak{A}' \to \mathfrak{M}$ that extends f.

Remark 5.3 Fix $\mathfrak{A} \in \mathbf{K}_\alpha$, $m \in \omega$, $\mathfrak{M} \models S_\alpha$, and an embedding $f : \mathfrak{A} \to \mathfrak{M}$. Since $\mathfrak{A} \in X_m(\mathfrak{A})$ and $X_m(\mathfrak{A})$ is finite, a maximally embeddable $\mathfrak{A}^* \in X_m(\mathfrak{A})$ in $\mathfrak{M}$ over f exists. For any such $\mathfrak{A}^*$, if $f' : \mathfrak{A}^* \to \mathfrak{M}$ is an embedding extending f, then $f'(\mathfrak{A}^*)$ is m-strong. Conversely, if $\langle \mathfrak{A}_i : i \leq j \rangle$ is a maximal m-chain in $\mathfrak{M}$ over $f(\mathfrak{A})$, then by Lemma 3.8 $\mathfrak{A}_j$ is isomorphic (over f) to some $\mathfrak{A}^* \in X_m(\mathfrak{A})$ that is maximally embeddable in $\mathfrak{M}$ over f.

Fix $\mathfrak{A}, \mathfrak{B} \in \mathbf{K}_\alpha$, Φ a finite subset of $\mathbf{K}_\alpha$, and $m \in \omega$ such that $\mathfrak{A} \subseteq \mathfrak{B}$ and for each $\mathfrak{C} \in \Phi$, $\mathfrak{C} \supseteq \mathfrak{B}$ and $|C \setminus A| < m$. For each such quadruple, let $Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$ denote the (finite) set of all $\mathfrak{A}^* \in X_m(\mathfrak{A})$ such that there is $\mathfrak{D} \in \mathbf{K}_\alpha$ and an embedding $g : \mathfrak{B} \to \mathfrak{D}$ over $\mathfrak{A}$ such that $\mathfrak{A}^* \leq \mathfrak{D}$, $D = A^* \cup g(B)$, and it is NOT the case that there are $\mathcal{H} \in \mathbf{K}_\alpha$, $\mathfrak{C} \in \Phi$, and $h : \mathfrak{C} \to \mathcal{H}$ extending g such that $\mathfrak{D} \leq \mathcal{H}$.

The following Theorem forms the crux of our quantifier elimination. The significance is that the existence of an extension g omitting Φ is described in terms of extensions (and nonextensions) of f itself.

Theorem 5.4 *Fix any $\mathfrak{A}, \mathfrak{B} \in \mathbf{K}_\alpha$, Φ a finite subset of $\mathbf{K}_\alpha$, and $m \in \omega$ such that $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{B} \subseteq \mathfrak{C}$, and $|C \setminus A| < m$ for all $\mathfrak{C} \in \Phi$. As well, fix $\mathfrak{M} \models S_\alpha$ and an embedding $f : \mathfrak{A} \to \mathfrak{M}$.*

There is an embedding $g : \mathfrak{B} \to \mathfrak{M}$ extending f and omitting Φ if and only if there is $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$ that is maximally embeddable in $\mathfrak{M}$ over f.

Proof. First suppose that there is $g : \mathfrak{B} \to \mathfrak{M}$ extending f and omitting Φ. Let $\langle \mathfrak{A}'_i : i \leq j \rangle$ be a maximal m-chain of minimal pairs in $\mathfrak{M}$ over $f(\mathfrak{A})$. By Remark 5.3 there is $\mathfrak{A}^* \in X_m(\mathfrak{A})$ that is maximally embeddable in $\mathfrak{M}$ over f via an isomorphism $f' : \mathfrak{A}^* \to \mathfrak{A}'_j$ extending f. Also, by Lemma 3.8 $\mathfrak{A}'_j$ is m-strong in $\mathfrak{M}$.

It suffices to show that $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$. Let $\mathfrak{D}'$ be the substructure of $\mathfrak{M}$ with universe $A'_j \cup g(B)$. Let $\mathfrak{D} \supseteq \mathfrak{A}^*$ be isomorphic to $\mathfrak{D}'$ via an isomorphism $j : \mathfrak{D} \to \mathfrak{D}'$ that extends f'. Since $\mathfrak{A}'_j$ is m-strong in $\mathfrak{M}$, $\mathfrak{A}'_j \leq \mathfrak{D}'$, hence $\mathfrak{A}^* \leq \mathfrak{D}$. Put $g^* := j^{-1} \circ g$. Then $g^* : \mathfrak{B} \to \mathfrak{D}$ and $D = A^* \cup g^*(B)$. To finish this direction, assume by way of contradiction that there is $\mathcal{H} \geq \mathfrak{D}$, $\mathfrak{C} \in \Phi$ and $h : \mathfrak{C} \to \mathcal{H}$ extending g^*. Since $\mathfrak{M} \models S_\alpha$ and $\mathfrak{D} \leq \mathcal{H}$, the embedding $j : \mathfrak{D} \to \mathfrak{M}$ extends to an embedding $j^* : \mathcal{H} \to \mathfrak{M}$. But then $j^* \circ h : \mathfrak{C} \to \mathfrak{M}$ extends g, contradicting the fact that g omitted Φ.

Conversely, suppose that $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$ and that $\mathfrak{A}^*$ is maximally embeddable in $\mathfrak{M}$ over f. Choose an embedding $f' : \mathfrak{A}^* \to \mathfrak{M}$ extending f. By Remark 5.3 $f'(\mathfrak{A}^*)$ is m-strong in $\mathfrak{M}$.

Choose $\mathfrak{D} \in \mathbf{K}_\alpha$ and $g : \mathfrak{B} \to \mathfrak{D}$ over $\mathfrak{A}$ witnessing $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$. Fix Φ^*, a (finite) set of representatives of all isomorphism types over $\mathfrak{D}$ of all $\mathcal{H} \in \mathbf{K}_\alpha$ that satisfy $\mathfrak{A}^* \leq \mathcal{H}$, $|H \setminus A^*| < m$, $\mathfrak{D} \subseteq \mathcal{H}$, but $\mathfrak{D} \not\leq \mathcal{H}$. By Proposition 4.4 there is an embedding $j : \mathfrak{D} \to \mathfrak{M}$ extending f' that omits every $\mathcal{H} \in \Phi^*$. We argue that $g' : \mathfrak{B} \to \mathfrak{M}$ omits every $\mathfrak{C} \in \Phi$, where $g' := j \circ g$.

By way of contradiction, suppose that there were $\mathfrak{C} \in \Phi$ and $h : \mathfrak{C} \to \mathfrak{M}$ extending g'. Let $\mathcal{H}'$ be the substructure of $\mathfrak{M}$ with universe $j(D) \cup h(C)$. There are two cases. On one hand, if $j(\mathfrak{D}) \not\leq \mathcal{H}'$ then we would contradict j omitting Φ^*. On the other hand, if $j(\mathfrak{D}) \leq \mathcal{H}'$ then we would contradict $\mathfrak{D}$ being a witness to $\mathfrak{A}^* \in Y(\mathfrak{A}, \mathfrak{B}, \Phi, m)$. ∎

Definition 5.5 For $\mathfrak{A} \in \mathbf{K}_\alpha$ (and a fixed enumeration $\bar{a}$ of A) $\Delta_{\mathfrak{A}}(\bar{x})$ is atomic diagram of $\mathfrak{A}$ (i.e., the conjunction of all atomic and neg-atomic formulas true of $\bar{a}$). If $\mathfrak{A} \subseteq \mathfrak{B}$ (and the enumeration of A is an initial segment of the enumeration of B) let

$$\Psi_{\mathfrak{A},\mathfrak{B}}(\bar{x}) := \Delta_{\mathfrak{A}}(\bar{x}) \wedge \exists \bar{y} \Delta_{\mathfrak{B}}(\bar{x}, \bar{y})$$

Such formulas are collectively called *extension formulas* (over $\mathfrak{A}$). A *chain-minimal extension formula* is an extension formula $\Psi_{\mathfrak{A},\mathfrak{B}}$ where $\mathfrak{B}$ is the union of a minimal chain over $\mathfrak{A}$.

Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$ are from $\mathbf{K}_\alpha$. Let $\mathfrak{C}$ be the union of a maximal chain of minimal pairs over $\mathfrak{A}$. By Remark 3.9, $\mathfrak{C} \leq \mathfrak{B}$. Since the sentence $\forall \bar{x}[\Delta_{\mathfrak{C}}(\bar{x}) \rightarrow \Psi_{\mathfrak{C},\mathfrak{B}}(\bar{x})]$ is an axiom of S_α the extension formula $\Psi_{\mathfrak{A},\mathfrak{B}}$ is S_α-equivalent to the chain-minimal extension formula $\Psi_{\mathfrak{A},\mathfrak{C}}$. That is, every extension formula is S_α-equivalent to a chain-minimal extension formula.

Theorem 5.6 *Every L-formula is S_α-equivalent to a boolean combination of chain-minimal extension formulas.*

Proof. It suffices to show that every L-formula is S_α-equivalent to a boolean combination of extension formulas. By taking $\mathfrak{A} = \mathfrak{B}$, every Δ-formula describing the isomorphism type of any $\mathfrak{A}$ is equivalent to an extension formula. It is easily seen that every atomic formula $\varphi(\bar{x})$ is equivalent to a disjunction of $\Delta_{\mathfrak{A}}$-formulas for which φ holds. Thus, every quantifier-free formula is equivalent to a boolean combination of extension formulas.

It suffices to show that if $\theta(\bar{x}, \bar{y})$ is a boolean combination of extension formulas, then $\exists \bar{y}\theta(\bar{x}, \bar{y})$ is S_α-equivalent to a boolean combination of extension formulas. Since existential quantification commutes with disjunction we may assume that $\theta(\bar{x}, \bar{y}) \vdash \Delta_{\mathfrak{A}}(\bar{x}) \wedge \Delta_{\mathfrak{B}}(\bar{x}, \bar{y})$ for some $\mathfrak{A} \subseteq \mathfrak{B}$ and that θ is a conjunction of extension formulas and negations of extension formulas over $\mathfrak{B}$. We must show that $\exists \bar{y}\theta(\bar{x}, \bar{y})$ is S_α-equivalent to a boolean combination of extension formulas over $\mathfrak{A}$.

Fix such a θ, let Γ be the set of $\mathfrak{C}$ such that $\Psi_{\mathfrak{B},\mathfrak{C}}$ occurs positively in θ, and Φ be the set of $\mathfrak{C}$ for which $\neg\Psi_{\mathfrak{B},\mathfrak{C}}$ occurs as a conjunct of θ. Let $m = \sum_{\mathfrak{C} \in \Gamma \cup \Phi} |\mathfrak{C}|$ (more reasonable bounds are possible). Call a $\mathfrak{D} \in \mathbf{K}_\alpha$ a *candidate* if $\mathfrak{B} \subseteq \mathfrak{D}$, $|D| < m$, for every $\mathfrak{C} \in \Gamma$ there is $h : \mathfrak{C} \rightarrow \mathfrak{D}$, while for each $\mathfrak{C} \in \Phi$, there is NO $h : \mathfrak{C} \rightarrow \mathfrak{D}$. For each candidate $\mathfrak{D}$, let $\Phi^*_{\mathfrak{D}}$ consist of representatives of all isomorphism types of $\mathfrak{F} \in \mathbf{K}_\alpha$ such that $\mathfrak{D} \subseteq \mathfrak{F}$, $|F \setminus D| < \max\{|C| : \mathfrak{C} \in \Phi\}$, and there is an embedding $h : \mathfrak{C} \rightarrow \mathfrak{F}$ over $\mathfrak{B}$. Let Z consist of a representative of every isomorphism type over $\mathfrak{B}$ of candidates. We claim that $\exists \bar{y}\theta(\bar{x}, \bar{y})$ is S_α-equivalent to

$$\chi(\bar{x}) := \bigvee_{\mathfrak{D} \in Z} \bigvee_{\mathfrak{A}^* \in Y(\mathfrak{A},\mathfrak{D},\Phi^*_{\mathfrak{D}},m)} \left[\Psi_{\mathfrak{A},\mathfrak{A}^*}(\bar{x}) \wedge \bigwedge_{\mathfrak{A}' \in X_m(\mathfrak{A}), \mathfrak{A}' \sqsupset \mathfrak{A}^*} \neg\Psi_{\mathfrak{A},\mathfrak{A}'}(\bar{x}) \right]$$

To see this, fix $\mathfrak{M} \models S_\alpha$ and $\bar{a}$ from M. Let $\mathfrak{A}$ be the substructure of $\mathfrak{M}$ with universe $\bar{a}$. First assume that $\mathfrak{M} \models \exists \bar{y} \theta(\bar{a}, \bar{y})$. Fix a tuple $\overline{B}$ from M realizing $\theta(\bar{a}, \bar{y})$ and let $\mathfrak{B}$ be the substructure of $\mathfrak{M}$ with universe $\bar{a} \cup \bar{b}$. For each $\mathfrak{C} \in \Gamma$ choose an embedding $g_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{M}$ over $\mathfrak{B}$. Let $\mathfrak{D} = \bigcup \{g_{\mathfrak{C}}(C) : \mathfrak{C} \in \Gamma\} \subseteq \mathfrak{M}$. Since each $\mathfrak{C} \in \Phi$ is omitted over $\mathfrak{B}$, $\mathfrak{D}$ is a candidate. Moreover, the identity map $id : \mathfrak{D} \to \mathfrak{M}$ omits $\Phi^*_{\mathfrak{D}}$, so $\mathfrak{M} \models \chi(\bar{a})$ by Theorem 5.4.

Conversely, suppose that $\mathfrak{M} \models \chi(\bar{a})$. Choose a candidate $\mathfrak{D}$ witnessing this. By Theorem 5.4 again, there is an embedding $g : \mathfrak{D} \to \mathfrak{M}$ over $\mathfrak{A}$ omitting $\Phi^*_{\mathfrak{D}}$. Let $\bar{b}$ enumerate the image of the restriction $g|B$. It is easily checked that $\mathfrak{M} \models \theta(\bar{a}, \bar{b})$. ∎

Corollary 5.7 *The theory S_α is complete.*

Proof. Since the empty structure is an element of $\mathbf{K}_\alpha$ and since $\emptyset \leq \mathbf{K}_\alpha$, S_α decides every extension *sentence* (i.e., extension formula with no free variables). Thus, S_α decides every L-sentence by Theorem 5.6. ∎

We now show that this theory S_α is both the almost sure theory of random graphs of Shelah-Spencer, and the theory of the $(\mathbf{K}_\alpha, \leq)$-generic structure.

Corollary 5.8 *When L consists of a single binary relation R and $0 < \alpha < 1$ is irrational, then S_α is equal to the almost sure theory of the class $G(n, n^{-\alpha})$ of finite graphs with edge probability $n^{-\alpha}$.*

Proof. Since S_α is complete, one only needs to check that each axiom of S_α holds almost surely. The verification of this is straightforward and uses only the 'easy' lower bound of Theorem 3 of [7]. In particular, the only use of probabalistic methods is an application of Chebyshev's Inequality. ∎

Definition 5.9 An L-structure $\mathfrak{M}$ is $(\mathbf{K}_\alpha, \leq)$-*generic* if (1) $\mathfrak{M} = \bigcup \{\mathfrak{A}_n : n \in \omega\}$, where $\mathfrak{A}_n \in \mathbf{K}_\alpha$ and $\mathfrak{A}_n \leq \mathfrak{A}_{n+1}$ for each n; and (2) for all $\mathfrak{A} \leq \mathfrak{B}$ from $\mathbf{K}_\alpha$, every strong embedding $f : \mathfrak{A} \to \mathfrak{M}$ extends to a strong embedding $g : \mathfrak{B} \to \mathfrak{M}$.

In [3] it is shown that a $(\mathbf{K}_\alpha, \leq)$-generic structure exists and is unique up to isomorphism.

Corollary 5.10 *S_α is the theory of the $(\mathbf{K}_\alpha, \leq)$-generic $\mathfrak{M}$.*

Proof. Since S_α is complete, it suffices to show that $\mathfrak{M} \models S_\alpha$. Say $\mathfrak{M} = \bigcup\{\mathfrak{A}_n : n \in \omega\}$, where each $\mathfrak{A}_n \in \mathbf{K}_\alpha$, $\mathfrak{A}_n \leq \mathfrak{A}_{n+1}$, and A_n is a strong substructure of $\mathfrak{M}$. First, let $\mathfrak{B}$ be any finite substructure of $\mathfrak{M}$. Choose n such that $\mathfrak{B} \subseteq \mathfrak{A}_n$. Since membership in $\mathbf{K}_\alpha$ is hereditary, it follows that $\mathfrak{B} \in \mathbf{K}_\alpha$.

Second, suppose that $\mathfrak{B} \leq \mathfrak{C}$ and $f : \mathfrak{B} \to \mathfrak{M}$ is given. Choose n such that $f(\mathfrak{B}) \subseteq \mathfrak{A}_n$. Let $f' : \mathfrak{C} \to \mathfrak{C}'$ be any isomorphism extending f such that $\{\mathfrak{A}_n, \mathfrak{C}'\}$ are disjoint over $f(\mathfrak{B})$. (We do NOT require that $\mathfrak{C}' \subseteq \mathfrak{M}$.) Let $\mathfrak{D}'$ be the free join of $\{\mathfrak{A}_n, \mathfrak{C}'\}$ over $f(\mathfrak{B})$. Since $f(\mathfrak{B}) \leq \mathfrak{C}'$, Lemma 2.3 implies that $\mathfrak{A}_n \leq \mathfrak{D}'$. Since $\mathfrak{M}$ is $(\mathbf{K}_\alpha, \leq)$-generic, choose an embedding $g : \mathfrak{D}' \to \mathfrak{M}$ over $\mathfrak{A}_n$. Then $h = g \circ f'$ is an embedding of $\mathfrak{C}$ into $\mathfrak{M}$ extending f. ∎

6 Algebraic closure, existentially closed, and locally finite models of S_α

Our first goal is to characterize the algebraically closed sets in models of S_α. None of the results in 6.1–6.3 are new. Indeed, they appear as Lemmas 3.22 and 4.5 of [3] and Wagner [9] establishes these in an axiomatic setting. We include them here for completeness.

Recall that by Lemma 3.8, for every $\mathfrak{M} \models S_\alpha$, every finite $\mathfrak{A} \subseteq \mathfrak{M}$ and every $m \in \omega$, there is a unique smallest m-strong $\mathfrak{B}$ satisfying $\mathfrak{A} \leq \mathfrak{B} \leq \mathfrak{M}$. We denote this $\mathfrak{B}$ by $\mathrm{cl}_m^{\mathfrak{M}}(A)$. When $\mathfrak{M}$ is understood we simply write $\mathrm{cl}_m(A)$.

Proposition 6.1 *Fix $\mathfrak{M} \models S_\alpha$ and $A \subseteq M$. The following are equivalent:*

1. *A is algebraically closed (i.e., if $\varphi(x, \bar{a})$ is any algebraic $L(A)$-formula, then $\varphi(M, \bar{a}) \subseteq A$);*

2. *For any minimal pair $(\mathfrak{B}, \mathfrak{C})$ of (finite) substructures of $\mathfrak{M}$, if $B \subseteq A$, then $C \subseteq A$.*

3. *For any finite $\mathfrak{B} \subseteq \mathfrak{M}$, $\mathfrak{B} \cap A \leq \mathfrak{B}$.*

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$ Assume A is algebraically closed and fix $\mathfrak{B} \subseteq \mathfrak{A}$ and a minimal pair $(\mathfrak{B}, \mathfrak{C})$ with $\mathfrak{C} \subseteq \mathfrak{M}$. Then, letting $\bar{b}$ be an enumeration of $\mathfrak{B}$, $\Delta_{\mathfrak{C}}(\bar{x}, \bar{b})$ is an algebraic formula in $\mathfrak{M}$, hence $C \subseteq A$.

(2) $\Rightarrow$ **(3)** Choose any finite $\mathfrak{B} \subseteq \mathfrak{M}$. If $\mathfrak{B} \cap \mathfrak{A} \not\leq \mathfrak{B}$ then let $\mathfrak{C}$ be minimal such that $\mathfrak{B} \cap A \subseteq \mathfrak{C} \subseteq \mathfrak{B}$ and $\mathfrak{B} \cap A \not\leq \mathfrak{C}$. Then $C \subseteq A$, so $\mathfrak{B} \cap A = \mathfrak{C}$, contradiction.

(3) $\Rightarrow$ **(1)** Assume that (3) holds. Let $b \in M \setminus A$ and let $\varphi(x, \bar{a})$ be any $L(A)$-formula such that $\mathfrak{M} \models \varphi(b, \bar{a})$. We argue that $\varphi(x, \bar{a})$ is not algebraic. Let $\mathfrak{B}$ denote the substructure of $\mathfrak{M}$ with universe $\bar{a}b$. By Theorem 5.6 we may assume that φ is a boolean combination of chain-minimal extension formulas. By writing φ in Disjunctive Normal Form it suffices to assume that $\varphi(x, \bar{a})$ has the form

$$\bigwedge_{\mathfrak{C} \in \Gamma} \exists \bar{z} \Delta_{\mathfrak{C}}(x, \bar{a}, \bar{z}) \quad \wedge \quad \bigwedge_{\mathfrak{C} \in \Phi} \neg \exists \bar{z} \Delta_{\mathfrak{C}}(x, \bar{a}, \bar{z})$$

for finite sets Γ, Φ of chain-minimal extensions of $\mathfrak{B}$. Choose m large (at least $|\mathfrak{B}| + \sum_{\mathfrak{C} \in \Gamma \cup \Phi} |C|$), let $\mathfrak{B}^* = \mathrm{cl}_m(\mathfrak{B})$, let $\mathfrak{A}_0 = \mathfrak{B} \cap A$, and let Φ be a (finite) set of isomorphism types of all $\mathfrak{D} \supseteq \mathfrak{B}^*$ with $|D \setminus B^*| < m$. By (3) $\mathfrak{A}_0 \leq \mathfrak{B}^*$, so by Proposition 4.4 there are infinitely many embeddings $g_i : \mathfrak{B}^* \to \mathfrak{M}$, each omitting Φ, such that $\{g_i(\mathfrak{B}^*) : i \in \omega\}$ is disjoint over $\mathfrak{A}_0$. It is easily checked that $\mathfrak{M} \models \varphi(g_i(b), \bar{a})$ for each $i \in \omega$. ∎

Proposition 6.2 *For any $\mathfrak{M} \models S_\alpha$ and any finite $\mathfrak{A} \subseteq \mathfrak{M}$,* $\mathrm{acl}(A) = \bigcup_{m \in \omega} \mathrm{cl}_m(A)$*. In particular,* $\mathrm{acl}(A)$ *is the union of a (possibly countably infinite) chain $\langle \mathfrak{A}_i : i \leq j \leq \omega \rangle$ of minimal pairs.*

Proof. Since for every $m \in \omega$ $\mathrm{cl}_m(A)$ is the union of an m-chain of minimal pairs, $\bigcup_{m \in \omega} \mathrm{cl}_m(A) \subseteq \mathrm{acl}(A)$ follows from Proposition 6.1(2). For the converse it suffices to show that $\bigcup_{m \in \omega} \mathrm{cl}_m(A)$ is algebraically closed in $\mathfrak{M}$. To see this, choose $\mathfrak{B} \subseteq \mathfrak{M}$ finite, let $\mathfrak{A}_0$ be the substructure of $\mathfrak{M}$ with universe $B \cap \bigcup_{m \in \omega} \mathrm{cl}_m(A)$, and assume by way of contradiction that $\mathfrak{A}_0 \not\leq \mathfrak{B}$. By replacing $\mathfrak{B}$ by some $\mathfrak{B}'$ satisfying $\mathfrak{A}_0 \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$ we may assume that $(\mathfrak{A}_0, \mathfrak{B})$ is a minimal pair. Choose m such that $\mathfrak{A}_0 \subseteq \mathrm{cl}_m(A)$ and $|B \setminus B_0| < m$ and let $\mathfrak{D}$ be the substructure of $\mathfrak{M}$ with universe $\mathrm{cl}_m(A) \cup B$. By Lemma 3.6 $\delta(\mathfrak{D}/\mathrm{cl}_m(A)) \leq -Gr(m)$, which contradicts $\mathrm{cl}_m(A)$ being m-strong in $\mathfrak{M}$.

To establish the final sentence, recall that by Lemma 3.8 for any m, $\mathrm{cl}_m(A)$ is the union of *any* m-chain of minimal pairs over $\mathfrak{A}$. Moreover, if $m < m'$ then any m-chain of minimal pairs over $\mathfrak{A}$ is also an m'-chain of minimal pairs over $\mathfrak{A}$. It follows that we can form a single chain of minimal pairs over $\mathfrak{A}$ whose union is $\mathrm{acl}(A)$. ∎

Lemma 6.3 *Suppose* $\mathfrak{M} \models S_\alpha$ *and* $\bar{a}, \bar{a}'$ *are tuples from* M *of the same length. Then* $\mathrm{tp}_{\mathfrak{M}}(\bar{a}) = \mathrm{tp}_{\mathfrak{M}}(\bar{a}')$ *if and only if there is an isomorphism* $h : \mathrm{acl}(\bar{a}) \to \mathrm{acl}(\bar{a}')$ *such that* $h(\bar{a}) = \bar{a}'$.

Proof. Left to right is straightforward and holds for any structure $\mathfrak{M}$. Conversely, suppose that $\mathfrak{A}, \mathfrak{A}'$ are finite substructures of $\mathfrak{M}$ such that there is an isomorphism $h : \mathrm{acl}(A) \to \mathrm{acl}(A')$ such that $h(A) = A'$ (pointwise). Then A and A' have the same quantifier-free type. By Theorem 5.6 and symmetry, in order to establish that $\mathrm{tp}_{\mathfrak{M}}(A) = \mathrm{tp}_{\mathfrak{M}}(A')$ we need to show that if $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}$ $\mathfrak{B} \in \mathbf{K}_\alpha$, then there is an embedding $g : \mathfrak{B} \to \mathfrak{M}$ extending $h|A$. Fix such a $\mathfrak{B}$ and let $\langle \mathfrak{A}_i : i \leq j \rangle$ be a maximal chain of minimal pairs over $\mathfrak{A}$ *inside* $\mathfrak{B}$. Since $\mathfrak{B}$ is finite, the chain is finite. Thus $\mathfrak{A}_j \leq \mathfrak{B}$ and $A_j \subseteq \mathrm{acl}(A)$. Now $h|A_j : \mathfrak{A}_j \to \mathfrak{M}$ is an embedding extending $h|A$. Since $\mathfrak{A}_j \leq \mathfrak{B}$ and $\mathfrak{M} \models S_\alpha$ there is an embedding $g : \mathfrak{B} \to \mathfrak{M}$ extending $h|A_j$ and we finish. ∎

Since S_α is AE-axiomatizable, its class of models is closed under unions of increasing chains. It follows easily from this that *existentially closed* (e.c.) models of S_α exist. Let U_α be the subset of (universal) axioms of S_α characterizing membership in $\overline{\mathbf{K}}_\alpha$ i.e., asserting that any finite substructure of is in $\mathbf{K}_\alpha$. Using Lemma 2.3 the free join of models of U_α over an arbitrary model of U_α is again a model of U_α. Combining this with the fact that models of U_α are closed under unions of increasing chains, it is readily seen that any model of U_α extends to a model of S_α. Thus, U_α is the universal theory of S_α.

In order to characterize the e.c. models of S_α (equivalently of U_α) we make the following definition, which appears in many places, e.g. [3, 1, 2].

Definition 6.4 *If* $\mathfrak{M} \models S_\alpha$ *and* $\mathfrak{A} \subseteq \mathfrak{M}$ *is finite, then*

$$d_{\mathfrak{M}}(\mathfrak{A}) = \inf\{\delta(\mathfrak{B}) : \mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{M}, \mathfrak{B} \text{ finite}\}$$

Theorem 6.5 *An L-structure* $\mathfrak{M}$ *is an e.c. model of* S_α *if and only if* $\mathfrak{M} \models S_\alpha$ *and* $d_{\mathfrak{M}}(\mathfrak{A}) = 0$ *for every finite* $\mathfrak{A} \subseteq \mathfrak{M}$.

Proof. Fix $\mathfrak{M} \models S_\alpha$. By virtue of Theorem 5.6 $\mathfrak{M}$ is an e.c. model of S_α if and only if for every extension formula $\Psi_{\mathfrak{A},\mathfrak{B}}(\bar{x})$ and every $\bar{a}$ from $\mathfrak{M}$, IF $\mathfrak{N} \models \Psi_{\mathfrak{A},\mathfrak{B}}(\bar{a})$ for some $\mathfrak{N} \supseteq \mathfrak{M}$ modelling S_α, THEN $\mathfrak{M} \models \Psi_{\mathfrak{A},\mathfrak{B}}(\bar{a})$.

Now assume that every finite $\mathfrak{A} \subseteq \mathfrak{M}$ satisfies $d_{\mathfrak{M}}(\mathfrak{A}) = 0$. By way of contradiction, assume that $\mathfrak{M}$ is not an e.c. model of S_α. Then there are triples $(\mathfrak{A}, \mathfrak{B}, \mathfrak{N})$ such that $\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{N}$, $\mathfrak{N} \supseteq \mathfrak{M}$ is a model of S_α, $\mathfrak{A} \subseteq \mathfrak{M}$, but there is no embedding of $\mathfrak{B}$ into $\mathfrak{M}$ over $\mathfrak{A}$. Among all such triples, choose $(\mathfrak{A}, \mathfrak{B}, \mathfrak{N})$ such that $|B - A|$ is as small as possible. Note that this minimality implies that $B \cap M = A$.

We claim that $\mathfrak{A} \leq \mathfrak{B}$. To see this, assume by way of contradiction that $\delta(\mathfrak{B}'/\mathfrak{A}) < 0$ for some $\mathfrak{B}'$ satisfying $\mathfrak{A} \subseteq \mathfrak{B}' \subseteq \mathfrak{B}$. Since $d_{\mathfrak{M}}(\mathfrak{A}) = 0$ there is a substructure $\mathfrak{C}$ such that $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{M}$ with $\delta(\mathfrak{C}) < -\delta(\mathfrak{B}'/\mathfrak{A})$. It follows from our minimality condition that $B' \cap C = A$. Thus, taking $\mathfrak{D}$ to be the substructure of $\mathfrak{N}$ with universe $B' \cup C$, $\mathfrak{D}$ is a join of $\{\mathfrak{B}', \mathfrak{C}\}$ over $\mathfrak{A}$. Applying Lemma 2.3 yields $\delta(\mathfrak{D}/\mathfrak{C}) \leq \delta(\mathfrak{B}'/\mathfrak{A})$. But then

$$\delta(\mathfrak{D}) = \delta(\mathfrak{C}) + \delta(\mathfrak{D}/\mathfrak{C}) \leq \delta(\mathfrak{C}) + \delta(\mathfrak{B}'/\mathfrak{A}) < 0$$

which contradicts $\mathfrak{N} \models S_\alpha$.

But now, since $\mathfrak{A} \leq \mathfrak{B}$ and $\mathfrak{A} \subseteq \mathfrak{M}$, there is an embedding of $\mathfrak{B}$ into $\mathfrak{M}$ over $\mathfrak{A}$ since $\mathfrak{M} \models S_\alpha$.

For the converse, suppose that $\mathfrak{M}$ is an e.c. model of S_α, $\mathfrak{A}$ is a finite substructure of $\mathfrak{M}$, and $\epsilon > 0$. In order to show that $d_{\mathfrak{M}}(\mathfrak{A}) = 0$ it suffices to find a finite substructure $\mathfrak{D}'$ such that $\mathfrak{A} \subseteq \mathfrak{D}' \subseteq \mathfrak{M}$ and $\delta(\mathfrak{D}') < \epsilon$. Since $\emptyset \leq \mathfrak{A}$ we can apply Proposition 4.2 to get $\mathfrak{D} \in \mathbf{K}_\alpha$ such that $\mathfrak{A} \subseteq \mathfrak{D}$ and $\delta(\mathfrak{D}) < \epsilon$. By replacing $\mathfrak{D}$ by an isomorphic copy we may assume that $D \cap M = A$.

The free join $\mathcal{H} = \mathfrak{M} \oplus_{\mathfrak{A}} \mathfrak{D}$ is a model of U_α, so there is a model $\mathfrak{N}$ of S_α containing $\mathcal{H}$. Without loss, we may assume that $\mathfrak{N} \supseteq \mathfrak{M}$. Now $\mathfrak{A} \subseteq \mathfrak{D} \subseteq \mathfrak{N}$, $\mathfrak{A} \subseteq \mathfrak{M}$, and $\mathfrak{M}$ is an e.c. model of S_α, so there is an embedding $g : \mathfrak{D} \to \mathfrak{M}$ over $\mathfrak{A}$. Then $g(\mathfrak{D})$ is as desired. ∎

The following Corollary, when contrasted with Corollaries 4.6 and 5.10, indicates that e.c. models are very different than saturated models or the generic model.

Corollary 6.6 *If $\mathfrak{M}$ is an e.c. model of S_α then there is no strong embedding $g : \mathfrak{A} \to \mathfrak{M}$ for any nonempty $\mathfrak{A} \in \mathbf{K}_\alpha$.*

Proof. Let $\mathfrak{A} \neq \emptyset$ and let $g : \mathfrak{A} \to \mathfrak{M}$ be any embedding. Since $\mathfrak{A} \in \mathbf{K}_\alpha$ is nonempty, $\delta(\mathfrak{A}) > 0$, so $\delta(g(\mathfrak{A})) > 0 = d_{\mathfrak{M}}(g(\mathfrak{A}))$. Thus g is not strong. ∎

Since $\emptyset \leq \mathfrak{A}$ for all $\mathfrak{A} \in \mathbf{K}_\alpha$, it follows that $\mathrm{acl}(\emptyset) = \emptyset$ in any model $\mathfrak{M} \models S_\alpha$. By contrast, the algebraic closure of a singleton can be the whole model. To see this, we require a lemma that is of interest in its own right.

Lemma 6.7 *Suppose $\mathfrak{A} \leq \mathfrak{B}$ are from $\mathbf{K}_\alpha$. Then there is $\mathfrak{C} \in \mathbf{K}_\alpha$ such that $\mathfrak{B} \subseteq \mathfrak{C}$, but $(\mathfrak{A}, \mathfrak{C})$ is a minimal pair.*

Proof. The proof is just like the proof of Proposition 4.2, but one simply takes $\mathfrak{C}$ to be 'one more' copy of $\mathfrak{D}$ in the construction of $\mathfrak{D}^*$, i.e., $\mathfrak{C}$ is the free join of $(k+1)$ copies of $\mathfrak{D}$ over $\mathfrak{B}$. The verification that $(\mathfrak{A}, \mathfrak{C})$ is a minimal pair is similar to computations in the proof of Proposition 4.2. ∎

Lemma 6.8 *Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$ are from $\mathbf{K}_\alpha$. Then there is $\mathfrak{C} \in \mathbf{K}_\alpha$ such that $\mathfrak{B} \subseteq \mathfrak{C}$ and for any $\mathfrak{M} \models S_\alpha$ containing $\mathfrak{C}$, $\mathrm{acl}(A) \supseteq C$.*

Proof. From the characterization of algebraic closure, it suffices to find $\mathfrak{C} \supseteq \mathfrak{B}$ that is a (finite) union of a chain of minimal pairs over $\mathfrak{A}$. So let $\langle \mathfrak{A}_i : i \leq j \rangle$ be a maximal chain of minimal pairs over $\mathfrak{A}$ inside $\mathfrak{B}$. Then $\mathfrak{A}_j \leq \mathfrak{B}$, so we can find $\mathfrak{C} \supseteq \mathfrak{B}$ such that $(\mathfrak{A}_j, \mathfrak{C})$ is a minimal pair. Any such $\mathfrak{C}$ is a union of minimal pairs over $\mathfrak{A}$. ∎

Proposition 6.9 *There is an e.c. model $\mathfrak{M}$ of S_α such that $\mathrm{acl}(\{a\}) = M$ for every $a \in M$.*

Proof. We construct $\mathfrak{M} = \bigcup\{\mathfrak{A}_n : n \in \omega\}$ as an increasing union of structures from $\mathbf{K}_\alpha$. We dovetail constraints so that $d_{\mathfrak{M}}(\mathfrak{B}) = 0$ for any finite $\mathfrak{B} \subseteq \mathfrak{M}$ and so that for any $a \in M$, M can be written as the union of a chain of minimal pairs over $\{a\}$. For a constraint of the first kind, say $\mathfrak{A}_n$ has been constructed. Since $\emptyset \leq \mathfrak{A}_n$, Proposition 4.2 says that there is $\mathfrak{A}_{n+1} \supseteq \mathfrak{A}_n$ such that $\delta(\mathfrak{A}_{n+1}) < 1/n$ so the first group of constraints provides no difficulty. The second group of constraints are handled using Lemma 6.8 with $\mathfrak{A} = \{a\}$ and $\mathfrak{B} = \mathfrak{A}_n$. For each a, in order to ensure $\mathrm{acl}(\{a\}) = M$ it is necessary to employ the second group of constraints infinitely often. ∎

Call a model $\mathfrak{M} \models S_\alpha$ *locally finite* if $\mathrm{acl}(X)$ is finite for all $X \subseteq M$. Clearly, the $(\mathbf{K}_\alpha, \leq)$-generic model is locally finite, while Corollary 6.6 implies that no e.c. model is locally finite. However, local finiteness hardly characterizes the generic. Indeed, Proposition 6.10 below illustrates two of the types of freedom we have in constructing locally finite models of S_α. It is easily seen that if $\mathfrak{M} \models S_\alpha$ is countable, then $\mathfrak{M}$ is locally finite if and only if $\mathfrak{M} = \bigcup\{\mathfrak{D}_n : n \in \omega\}$, where $\mathfrak{D}_0 \leq \mathfrak{D}_1 \leq \ldots$ and each $\mathfrak{D}_n \in \mathbf{K}_\alpha$. As notation, let $\delta^*(\mathfrak{M}) = \sup\{\delta(\mathfrak{D}_n) : n \in \omega\} \in \mathbb{R} \cup \{\infty\}$ for some (any) representation of $\mathfrak{M}$ as $\bigcup \mathfrak{D}_n$ with $\mathfrak{D}_0 \leq \mathfrak{D}_1 \leq \ldots$ from $\mathbf{K}_\alpha$.

Proposition 6.10 *For any real number $r > 0$ and any $\mathcal{P} = \{\mathfrak{A}_i : i \in I\} \subseteq \mathbf{K}_\alpha$ such that $\sum_{i \in I} \delta(\mathfrak{A}_i) < r$, there is a countable, locally finite $\mathfrak{M}_{r,\mathcal{P}} \models S_\alpha$ such that $\delta^*(\mathfrak{M}_{r,\mathcal{P}}) = r$ and for each $i \in I$ there is a strong embedding $g_i : \mathfrak{A}_i \to \mathfrak{M}_{r,\mathcal{P}}$.*

Proof. Fix r and $\mathcal{P}$ as above. By adding or deleting copies of $\emptyset$ from $\mathcal{P}$ as needed we may assume that $\mathcal{P}$ is infinite and indexed by ω. Let $s = \sum_{n \in \omega} \delta(\mathfrak{A}_n)$.

We inductively construct a sequence $\mathfrak{D}_0 \leq \mathfrak{D}_1 \leq \dots$ from $\mathbf{K}_\alpha$ and a sequence $\epsilon_0 \geq \epsilon_1 \geq \dots$ of positive real numbers converging to 0 as follows:

- $\mathfrak{D}_0 = \emptyset$ and $\epsilon_0 = r - s$.
- At even stages, i.e., when $\mathfrak{D}_{2i}$ and ϵ_{2i} have been defined, let $\mathfrak{D}_{2i+1}$ be the free join of $\mathfrak{D}_{2i}$ and $\mathfrak{A}_i$ over $\emptyset$ (or an isomorphic copy of $\mathfrak{A}_i$ if it is not disjoint from $\mathfrak{D}_{2i}$) and let $\epsilon_{2i+1} = \epsilon_{2i}$.
- At odd stages, i.e., when $\mathfrak{D}_{2i-1}$ and ϵ_{2i-1} have been defined and we are looking at a specific pair $\mathfrak{A} \leq \mathfrak{B}$ from $\mathbf{K}_\alpha$ such that $B \cap D_{2i-1} = A$, choose $\mathfrak{C} \in \mathbf{K}_\alpha$ such that $\mathfrak{B} \subseteq \mathfrak{C}$, $C \cap D_{2i-1} = A$, $\mathfrak{A} \leq \mathfrak{C}$, and $\epsilon_{2i-1}/2 \leq \delta(\mathfrak{C}/\mathfrak{A}) < \epsilon_{2i-1}$. The existence of such a $\mathfrak{C}$ follows from Proposition 4.2. Specifically, using Proposition 4.2 choose $\mathfrak{B}^* \in \mathbf{K}_\alpha$ such that $\mathfrak{B} \subseteq \mathfrak{B}^*$, $\mathfrak{A} \leq \mathfrak{B}^*$, $B^* \cap D_{2i-1} = A$, and $\delta(\mathfrak{B}^*/\mathfrak{A}) < \epsilon_{2i-1}/2$. Then choose $\ell \in \omega$ least such that $\ell \cdot \delta(\mathfrak{B}^*/\mathfrak{A}) \geq \epsilon_{2i-1}/2$ and let $\mathfrak{C}$ be the free join of ℓ copies of $\mathfrak{B}^*$ over $\mathfrak{A}$. Take $\mathfrak{D}_{2i}$ to be the free join of $\mathfrak{D}_{2i-1}$ and $\mathfrak{C}$ over $\mathfrak{A}$ and let $\epsilon_{2i} = \epsilon_{2i-1} - \delta(\mathfrak{C}/\mathfrak{A})$. Note that $0 < \epsilon_{2i} \leq \epsilon_{2i-1}/2$.

Let $\mathfrak{M}_{r,\mathcal{P}} = \bigcup\{\mathfrak{D}_n : n \in \omega\}$. As $\mathfrak{D}_n \leq \mathfrak{D}_{n+1}$ for all $n \in \omega$, $\mathfrak{M}_{r,\mathcal{P}}$ is locally finite. So long as we organize the bookkeeping of the odd stages sufficiently well, we guarantee that $\mathfrak{M}_{r,\mathcal{P}} \models S_\alpha$. The even stages ensure that there is a strong embedding $g_i : \mathfrak{A}_i \to \mathfrak{D}_{2i+1}$, hence g_i is a strong embedding into $\mathfrak{M}_{r,\mathcal{P}}$. As for the computation of $\delta^*(\mathfrak{M}_{r,\mathcal{P}})$, at each even stage $\delta(\mathfrak{D}_{2i+1}/\mathfrak{D}_{2i}) = \delta(\mathfrak{A}_i)$, so $\sum_{i \in \omega} \delta(\mathfrak{D}_{2i+1}/\mathfrak{D}_{2i}) = s$, while $\epsilon_{2i} \leq \epsilon_{2i-1}/2$ for each i, hence $\sum_{i>0} \delta(\mathfrak{D}_{2i}/\mathfrak{D}_{2i-1}) = r - s$. Thus $\delta^*(\mathfrak{M}_{r,\mathcal{P}}) = r$. ∎

7 Stability, forking, nfcp, and DOP

In this final section we prove that S_α is stable, unsuperstable, has weak elimination of imaginaries, does not have the finite cover property, but does have the Dimensional Order Property (DOP). As well, we present a workable characterization of forking in models of S_α. Some of these results have appeared with different proofs in [1, 2, 3], at least in the special case of graphs.

For this section we adopt some of the standard conventions of stability theory. In particular, we work in a fixed, large saturated model $\mathfrak{M}^*$ of S_α. Whenever we compute $\mathrm{acl}(A)$ or $\mathrm{cl}_m(A)$ it will be with respect to $\mathfrak{M}^*$.

Proposition 7.1 *The theory S_α is stable.*

Proof. We argue that every formula is stable. Because of Theorem 5.6 and the fact that the set of stable formulas is closed under boolean combinations, it suffices to show that every chain-minimal extension formula is stable. So fix $\varphi(\bar{x}, \bar{y}) := \exists \bar{z} \Delta_{\mathfrak{C}}(\bar{x}, \bar{y}, \bar{z})$, where $\mathfrak{C}$ is a chain-minimal extension of $\mathfrak{B}$, and assume by way of contradiction that there are $\langle \bar{a}_i \bar{b}_i : i \in \omega \rangle$ from $\mathfrak{M}^*$ such that $\varphi(\bar{a}_i, \bar{b}_j)$ if and only if $i < j$. (The other versions of the order property, i.e., where $<$ is replaced by any of $\leq, >, \geq$ are handled similarly.)

Let $r = lg(\bar{z})$. For each $i < j < \omega$ choose an r-tuple $\bar{c}_{i,j}$ such that $\Delta_{\mathfrak{C}}(\bar{a}_i, \bar{b}_j, \bar{c}_{i,j})$ holds. By an application of Ramsey's theorem on pairs and replacing ω by an infinite subset, we may assume that for each $l < r$ one of the following four conditions hold of $(\bar{c}_{i,j})_l$, the l^{th}-coordinate of $\bar{c}_{i,j}$:

1. There is c^*_l such that $(\bar{c}_{i,j})_l = c^*_l$ for all $i < j < \omega$;
2. There is $c'_{i,l}$ such that $(\bar{c}_{i,j})_l = c'_{i,l}$ for all $i < j < \omega$;
3. There is $c''_{i,l}$ such that $(\bar{c}_{j,i})_l = c''_{i,l}$ for all $j < i < \omega$; or
4. $(\bar{c}_{i,j})_l \neq (\bar{c}_{i',j'})_l$ unless $i = i'$ and $j = j'$.

The proof splits into two cases.

Case 1. For all $l < r$, one of Clauses (1), (2), (3) hold.

In this case, let $\bar{e}_i$ be the r-tuple where $(\bar{e}_i)_l = c^*_l$ if (1) holds; $(\bar{e}_i)_l = c'_{i,l}$ if (2) holds, and $(\bar{e}_i)_l = c''_{i,l}$ if (3) holds. For each $i \in \omega$ let $\overline{d}_i = \bar{a}_i \bar{b}_i \bar{e}_i$ and let $\mathfrak{D}_i$ be the substructure of $\mathfrak{M}^*$ with universe $\overline{d}_i$. As well, for any $i, j \in \omega$ let $\mathfrak{D}_{i,j}$ be the substructure of $\mathfrak{M}^*$ with universe $\overline{d}_i \cup \overline{d}_j$. Note that the conditions of Case 1 imply that $\bar{c}_{i,j} \subseteq D_{i,j}$ whenever $i < j < \omega$. In particular, $\mathfrak{D}_{i,j}$ contains a witness to $\varphi(\bar{a}_i, \bar{b}_j)$ whenever $i < j < \omega$. By applying Proposition 3.1 to the set $\{D_i : i \in \omega\}$ one obtains an $\mathfrak{F}$ and an infinite $Y \subseteq \omega$ such that $\{D_i : i \in Y\}$ is a free join over $\mathfrak{F}$ of constant quantifier-free type over $\mathfrak{F}$.

As there are only finitely many enumerations of each $\mathfrak{D}_i$, there must be $i(0) < j(0)$ from Y such that

$$qftp(\overline{d}_{i(0)}\overline{d}_{j(0)}) = qftp(\overline{d}_{j(0)}\overline{d}_{i(0)})$$

Thus, $\mathfrak{D}_{i(0),j(0)} \cong \mathfrak{D}_{j(0),i(0)}$, hence $\varphi(\bar{a}_{j(0)}, \bar{b}_{i(0)})$, contradicting our assumptions about φ.

Case 2. For some $l < r$, Clause (4) holds.

Let $\mathfrak{A}_\omega$ denote the substructure of $\mathfrak{M}^*$ with universe $\bigcup\{\bar{a}_i\bar{b}_i : i \in \omega\}$. By a second application of Ramsey's theorem and a further shrinking of ω we may assume that $(\bar{c}_{i,j})_l \notin A_\omega$ for all $i < j < \omega$. To see this, first note that by Ramsey's theorem on pairs there is an infinite subset $X \subseteq \omega$ such that $(\bar{c}_{i,j})_l \notin \bar{a}_i\bar{b}_i$ for all $i < j$ from X. Similarly, by trimming X further we may additionally assume that $(\bar{c}_{i,j})_l \notin \bar{a}_j\bar{b}_j$ for all $i < j$ from X. Additionally, by Ramsey's theorem on triples we may replace X by an infinite subset of itself so that $(\bar{c}_{i,j})_l \notin \bar{a}_k\bar{b}_k$, $(\bar{c}_{i,k})_l \notin \bar{a}_j\bar{b}_j$, and $(\bar{c}_{j,k})_l \notin \bar{a}_i\bar{b}_i$ for all $i < j < k$ from X. Thus, after reindexing this set X by ω we obtain that $(\bar{c}_{i,j})_l$ is not an element of the trimmed version of A_ω for all $i < j < \omega$.

Similarly, by additional applications of Ramsey's theorem (this time on 4-tuples (i, j, i', j')) and a further shrinking of ω we may assume that $(\bar{c}_{i,j})_l \notin \bar{c}_{i',j'}$ for any $\{i', j'\} \neq \{i, j\}$ for all $i < j < \omega$.

Choose n large and fix an enumeration $\langle u_s : s < \binom{n}{2}\rangle$ of all pairs (i, j) satisfying $i < j < n$. For each $t < \binom{n}{2}$ let $\mathfrak{B}_{n,t}$ denote the substructure of $\mathfrak{M}^*$ with universe $\{\bar{a}_i\bar{b}_i : i < n\} \cup \{\bar{c}_s : s < t\}$. We will obtain a contradiction by showing that $\delta(\mathfrak{B}_{n,t}) < 0$ when n and t are sufficiently large. To see this, first note that $\delta(\mathfrak{B}_{n,0})$ grows linearly in n. In fact, $\delta(\mathfrak{B}_{n,0}) \leq n(lg(\bar{x}\bar{y}))$. But, our assumptions on $(\bar{c}_s)_l$ guarantee that $\bar{c}_s \not\subseteq B_{n,s}$ for any $s < \binom{n}{2}$. So, since $\mathfrak{C}$ is a chain-minimal extension of $\mathfrak{B}$, Lemma 3.6 yields $\delta(\mathfrak{B}_{n,s+1}) \leq \delta(\mathfrak{B}_{n,s}) - Gr(r)$, hence $\delta(\mathfrak{B}_{n,t}) \leq n(\lg(\bar{x}\bar{y}) - tGr(r)$. As $\binom{n}{2}$ grows quadratically, this implies $\delta(\mathfrak{B}_{n,t}) < 0$ whenever n and t are large enough. ∎

Lemma 7.2 *For any cardinal κ, every algebraically closed set B, and every $\bar{c}$, there is a set $\{C_i : i \in \kappa\}$ of algebraically closed extensions of B that are pairwise isomorphic over B, $C_0 = \mathrm{acl}(B\bar{c})$, and satisfy $\{C_i : i \in \kappa\}$ freely joined over B and $\bigcup\{C_i : i \in \kappa\}$ is algebraically closed. In particular, $\{C_i : i \in \kappa\}$ is fully indiscernible over B. Moreover, if A is any set disjoint from C_0, then we may additionally assume that $A \cap C_i = \emptyset$ for each i.*

Proof. By compactness it suffices to show that for any finite $B_0 \subseteq B$ and any $n, m \in \omega$ there are finite sets B_0' and $C^* = \bigcup\{C_i : i < n\}$ such that $B_0 \subseteq B_0' \subseteq B$, $C_0 = cl_m(B_0'\bar{c})$, C^* is the free join of n copies of C_0 over B_0', and C^* is m-closed.

So fix B_0, n, m as above. Let $C_0 = cl_m(B_0\bar{c})$ and let $B_0' = C_0 \cap B$. Since B is algebraically closed, $\mathfrak{B}_0' \leq \mathfrak{C}_0$. Let $\mathfrak{C}^* = \bigoplus_{i<n} \mathfrak{C}_i \in \mathbf{K}_\alpha$ be the free join of n copies of $\mathfrak{C}_0$ over $\mathfrak{B}_0'$. By Lemma 2.3(2) we have $\mathfrak{B}_0' \leq \mathfrak{C}^*$, so Corollary 4.6 gives a strong embedding of $\mathfrak{C}^*$ into $\mathfrak{M}^*$ over $\mathfrak{B}_0'$. It follows from Proposition 6.1 that the image of $\mathfrak{C}^*$ is algebraically closed, hence m-closed in $\mathfrak{M}^*$. ∎

Proposition 7.3 *Let A, B, C be algebraically closed sets such that $B \subseteq A \cap C$. Then $A \underset{B}{\perp} C$ if and only if $\{A, C\}$ are freely joined over B and $A \cup C$ is algebraically closed.*

Proof. Since S_α is stable, $\mathrm{tp}(A/B)$ has a nonforking extension to C. By Lemma 6.3 the two conditions $\{A, C\}$ freely joined over B and $A \cup C$ is algebraically closed describe a unique type $\mathrm{tp}(AC)$, hence there is a unique extension of $\mathrm{tp}(A/B)$ to $S(C)$. Thus, once we show that the failure of either of the two conditions implies dividing (hence forking) over B, it follows that the conjunction of the two conditions describe the (unique) nonforking extension.

We begin by showing that if $\{A, C\}$ is not a free join over B, then $\mathrm{tp}(A/C)$ contains a formula that divides over B. Clearly, if $c \in (A \cap C) \setminus B$, then '$x = c$'$\in \mathrm{tp}(A/C)$ divides over B (recall that B is algebraically closed), so we may assume that $A \cap C = B$.

Suppose that there are $\bar{a} \subseteq A \setminus B$, $\bar{b} \subseteq B$, $\bar{c} \subseteq C \setminus B$, and $R \in L$ such that $R(\bar{a}, \bar{b}, \bar{c})$ with $\bar{a}, \bar{c} \neq \emptyset$. By Lemma 7.2 choose $\{C_i : i \in \omega\}$ freely joined and indiscernible over B with $C_0 = acl(B\bar{c})$ and $\bigcup\{C_i : i \in \omega\}$ algebraically closed. For each $i \in \omega$ fix an isomorphism $f_i : C_0 \to C_i$ over B and let $\bar{c}_i = f_i(\bar{c})$. Fix $m > |\bar{a}|$ and let C_i^m denote the m-closure of $\bar{b}\bar{c}_i$ in C_i. We argue that $\{\varphi(\bar{x}, \bar{b}, \bar{c}_i) : i \in \omega\}$ is inconsistent, where $\varphi(\bar{x}, \bar{b}, \bar{c}_i) := R(\bar{x}, \bar{b}, \bar{c}_i) \wedge \bigwedge_j x_j \notin C_i^m$. By way of contradiction, assume that some $\bar{a}'$ from M^* realizes it. Choose n large enough such that $n\alpha > |\bar{a}'|$ and let $\overline{C}_n = \bigcup\{C_i^m : i < n\}$. Since $R(\bar{a}', \bar{b}, \bar{c}_i)$ holds for all $i < n$, $\delta(\overline{C}_n\bar{a}'/\overline{C}_n) \leq |\bar{a}'| - n\alpha < 0$, which contradicts $\overline{C}_n$ being m-closed.

Finally, assume that $\{A, C\}$ are freely joined over B, but $A \cup C$ is not algebraically closed. Choose $\bar{a} \subseteq A \setminus B$, $\bar{b} \subseteq B$, $\bar{c} \subseteq C \setminus B$ and $\overline{d}$ disjoint from

$A \cup C$, so that letting D be the substructure with universe $\bar{a}\bar{b}\bar{c}\bar{d}$, $(\bar{a}\bar{b}\bar{c}, D)$ is a minimal pair. Since D is not to be embeddable in A over $\bar{a}\bar{b}\bar{c}$ we may assume that for at least one $R \in L$, at least one element of R^D contains at least one element of $\bar{c}$.

Let $m > |D|$, choose n such that $nGr(M) > |\bar{a}|$, and let $m^* > n|D|$. By Lemma 7.2 choose $\{C_i : i \in \omega\}$ to be fully indiscernible and freely joined over $\mathfrak{B}$ with $\bigcup\{C_i : i \in \omega\}$ algebraically closed. Let $C_0^{m^*} = cl_{m^*}(\bar{b}\bar{c})$ and let $C_i^{m^*} = f_i(C_0^{m^*})$, where $f_i : C_0 \to C_i$ is an isomorphism over B. Let $\overline{C}_n = \bigcup\{C_i^{m^*} : i < n\}$. We argue that

$$\varphi(\bar{x}, \bar{b}, \bar{c}_i) := \exists \bar{z}[\Delta_{\mathfrak{D}}(\bar{x}, \bar{b}, \bar{c}_i, \bar{z}) \wedge \bigwedge_j x_j \notin C_i^{m^*}]$$

divides over B. If not, then there would be $\bar{a}'$ from M^* such that $\varphi(\bar{a}', \bar{b}, \bar{c}_i)$ holds for all $i \in \omega$. If this were the case, then for each i, choose $\overline{d}_i$ such that $\Delta_{\mathfrak{D}}(\bar{a}', \bar{b}, \bar{c}_i, \overline{d}_i)$ holds. Apply the Δ-system lemma to $\{\overline{d}_i : i \in \omega\}$. There are now two cases.

Case 1. For infinitely many i, $\overline{d}_i = \overline{d}^*$ for some fixed $\overline{d}^*$. In this case, arguing as above $\delta(\bar{a}'\overline{d}^*\overline{C}_n/\overline{C}_n) < 0$, contradicting the fact that $\overline{C}_n$ is m-strong in $\mathfrak{M}^*$.

Case 2. For infinitely many i, $\overline{d}_i = \overline{d}^* \hat{\ } \bar{e}_i$ for some $\overline{d}^*$ and some pairwise disjoint $\{\bar{e}_i : i \in \omega\}$.

For each $l \leq n$, let D_l denote the substructure with universe $\overline{C}_n \bar{a}' \cup \bigcup\{\overline{d}_i : i < l\}$. Since $(\bar{a}\bar{b}\bar{c}, D)$ is a minimal pair and the $\bar{e}_i$ are pairwise disjoint, it follows from Lemma 3.6 that $\delta(D_{l+1}/D_l) \leq -Gr(m)$ for each $l < n$, so $\delta(\mathfrak{D}_n/\overline{C}_n) \leq |\bar{a}'| - nGr(m) < 0$, contradicting the fact that $\overline{C}_n$ was m^*-strong in $\mathfrak{M}^*$. ∎

Corollary 7.4 *The theory S_α has weak elimination of imaginaries, i.e., every complete type over an algebraically closed set is stationary.*

Proof. Fix an algebraically closed B and a type $p \in S(B)$. To prove that p is stationary, it suffices to show that it has a unique nonforking extension to any algebraically closed $C \supseteq B$. Fix C and choose a, a' realizing p such that neither $\text{tp}(a/C)$ and $\text{tp}(a'/C)$ fork over B. Let $A = \text{acl}(Ba)$ and $A' = \text{acl}(Ba')$. Since $\text{tp}(a/B) = \text{tp}(a'/B)$, Lemma 6.3 gives an isomorphism $f : A \to A'$ over B such that $f(a) = a'$. Since neither $\text{tp}(A/C)$ and $\text{tp}(A'/C)$

fork over B, it follows from Proposition 7.3 that $\{A, C\}$ and $\{A', C\}$ are free joins over B and both $A \cup C$ and $A' \cup C$ are algebraically closed. The first statement implies that the extension $g : A \cup C \to A' \cup C$ of f formed by $g(c) = c$ for all $c \in C$ is an isomorphism. Since both the domain and range are algebraically closed Lemma 6.3 again implies that $\mathrm{tp}(A/C) = \mathrm{tp}(A'/C)$, hence $\mathrm{tp}(a/C) = \mathrm{tp}(a'/C)$. ∎

Proposition 7.5 *S_α does not have the finite cover property (i.e., has nfcp).*

Proof. Recall that in a stable theory T, not having the finite cover property is equivalent to: For any $E(\bar{x}, \bar{y}, \bar{z})$ there is a number N such that for any $\bar{c}$, if $E(\bar{x}, \bar{y}, \bar{c})$ is an equivalence relation with finitely many classes, then it has fewer than N classes. Fix such an $E(\bar{x}, \bar{y}, \bar{z})$ and let $t = |\bar{x}| + |\bar{y}| + |\bar{z}|$. Choose a number m such that for all tuples $\bar{e}, \bar{e}'$ of length t, if there is an isomorphism $g : \mathrm{cl}_m(\bar{e}) \to \mathrm{cl}_m(\bar{e}')$ satisfying $g(\bar{e}) = \bar{e}'$, then $E(\bar{e}) \leftrightarrow E(\bar{e}')$. The existence of such an m follows from Lemma 6.3 and compactness. Next, using Lemma 3.2 choose s such that $|cl_m(\bar{e})| \leq s$ for any tuple $\bar{e}$ of length t. Finally, arguing as in the proof of Proposition 3.1, choose N such that if $\{\overline{d}_i : i < N\}$ is any set of tuples, each of length at most s, then there are $i(0) < j(0) < N$ such that, letting $B = \overline{d}_{i(0)} \cap \overline{d}_{j(0)}$, $\{\overline{d}_{i(0)}, \overline{d}_{j(0)}\}$ is a free join over B and the natural map $g : \overline{d}_{i(0)} \mapsto \overline{d}_{j(0)}$ defined by taking the ℓ'th element of $\overline{d}_{i(0)}$ to the ℓ'th element of $\overline{d}_{j(0)}$ is an isomorphism over B. The existence of such an N follows from the finite Δ-system lemma and the finiteness of the language L. We argue that for any $\bar{c}$, if $E(\bar{x}, \bar{y}, \bar{c})$ has only finitely many classes, then it has fewer than N classes.

To see this, call a sequence of sets $(\bar{c}, \bar{a}, \bar{a}', A, A', B)$ *good* if

1. $\bar{c}$ is an initial segment of both $\bar{a}$ and $\bar{a}'$;
2. $A = \mathrm{cl}_m(\bar{a})$ and $A' = \mathrm{cl}_m(\bar{a}')$;
3. $A \cap A' = B$ and $\{A, A'\}$ is a free join over B; and
4. There is an isomorphism $g : A \to A'$ over B satisfying $g(\bar{a}) = \bar{a}'$.

Claim. For any good sequence $(\bar{c}, \bar{a}, \bar{a}', A, A', B)$, if $E(\bar{x}, \bar{y}, \bar{c})$ has only finitely many classes, then $E(\bar{a}, \bar{a}', \bar{c})$ holds.

Proof. Fix a good sequence $(\bar{c}, \bar{a}, \bar{a}', A, A', B)$ such that $E(\bar{x}, \bar{y}, \bar{c})$ has only finitely many classes. Let $B^* = \mathrm{acl}(B)$, and let $p = \mathrm{tp}(\bar{a}/B^*)$. By

Corollary 7.4 p is stationary. Let $\{\bar{a}_i : i \in \omega\}$ be a Morley sequence in p over B. For each $i \in \omega$ let $A_i = \mathrm{cl}_m(\bar{a}_i)$. Then for all $i \neq j$, there is an isomorphism $h_{i,j} : A \cup A' \to A_i \cup A_j$ such that $h_{i,j}(\bar{a}) = \bar{a}_i$ and $h_{i,j}(\bar{a}') = \bar{a}_j$. As the sets $A \cup A'$ and $A_i \cup A_j$ are m-closed, it follows that $E(\bar{a}, \bar{a}', \bar{c}) \leftrightarrow E(\bar{a}_i, \bar{a}_j, \bar{c})$ for all $i < j < \omega$. Since $E(\bar{x}, \bar{y}, \bar{c})$ has only finitely many classes, it follows that $E(\bar{a}, \bar{a}', \bar{c})$, which establishes the Claim.

Now fix $\bar{c}$ such that $E(\bar{x}, \bar{y}, \bar{c})$ has only finitely many classes. Choose tuples $\{\bar{a}_i : i < N\}$ such that $\bar{c}$ is an initial segment of each $\bar{a}_i$. From our choice of N there are $i(0) < j(0) < N$ such that, letting $A_{i(0)} = \mathrm{cl}_m(\bar{a}_{i(0)})$, $A_{j(0)} = \mathrm{cl}_m(\bar{a}_{j(0)})$, and $B = A_{i(0)} \cap A_{j(0)}$, $\{A_{i(0)}, A_{j(0)}\}$ is a free join over B and there is an isomorphism $g : A_{i(0)} \to A_{j(0)}$ over B satisfying $g(\bar{a}_{i(0)}) = \bar{a}_{j(0)}$. We argue that $E(\bar{a}_{i(0)}, \bar{a}_{j(0)}, \bar{c})$ holds, hence $E(\bar{x}, \bar{y}, \bar{c})$ has fewer than N classes. To see this, let $B^* = \mathrm{acl}(B)$ and let $p(\bar{x}) = \mathrm{tp}(\bar{a}_{i(0)}/B^*)$. Since S_α is stable, choose $\bar{a}^*$ realizing the (unique) nonforking extension of p to $A_{i(0)} \cup A_{j(0)} \cup B^*$. Let $A^* = \mathrm{cl}_m(B\bar{a}^*)$. It follows from Proposition 7.3 that the sequences $(\bar{c}, \bar{a}_{i(0)}, \bar{a}^*, A_{i(0)}, A^*, B)$ and $(\bar{c}, \bar{a}_{j(0)}, \bar{a}^*, A_{j(0)}, A^*, B)$ are both good. By the Claim both $E(\bar{a}_{i(0)}, \bar{a}^*, \bar{c})$ and $E(\bar{a}_{j(0)}, \bar{a}^*, \bar{c})$ hold, hence $E(\bar{a}_{i(0)}, \bar{a}_{j(0)}, \bar{c})$ holds as well. ∎

The proof of the following Proposition uses ideas from Ikeda [5].

Proposition 7.6 *The theory S_α is not superstable.*

Proof. Choose $a \in M^*$ such that $\mathrm{acl}(\{a\}) = \{a\}$, let $B_0 = \emptyset$ and let $D_0 = \{a\}$. We will produce a nested sequence $\langle B_n : n \in \omega\rangle$ of finite substructures of $\mathfrak{M}^*$ such that $a \not\downarrow_{B_n} B_{n+1}$ for all $n \in \omega$.

To accomplish this, we also construct an ancillary sequence $\langle \mathfrak{D}_n : n \in \omega\rangle$ of finite substructures of $\mathfrak{M}^*$ such that:

- $\{a\} \cup \bigcup\{B_n : n \in \omega\}$ is discrete (i.e., no R-relations hold among any subset);
- $\mathfrak{B}_n \leq \mathfrak{B}_{n+1}$ and $\mathfrak{D}_n \leq \mathfrak{D}_{n+1}$;
- $B_n = \mathrm{acl}(B_n)$, $D_n = \mathrm{acl}(aB_n)$, but $D_{n+1} \neq D_n \cup B_{n+1}$.

It follows from the characterization of forking given in Proposition 7.3 that these conditions imply $a \not\downarrow_{B_n} B_{n+1}$ for each $n \in \omega$, so it suffices to perform the construction.

Assume that B_n and D_n have been defined and satisfy the conditions. Choose $\mathfrak{B}_{n+1} \in \mathbf{K}_\alpha$ (not necessarily in $\mathfrak{M}^*$) such that $B_{n+1} = B_n \cup \{b_n\}$, $\mathfrak{B}_{n+1}$ is discrete, and $\{\mathfrak{B}_{n+1}, \mathfrak{D}_n\}$ are disjoint over $\mathfrak{B}_n$. Let $\mathcal{F}$ denote the free join of $\{\mathfrak{B}_{n+1}, \mathfrak{D}_n\}$ over $\mathfrak{B}_n$. Apply Lemma 4.1 to obtain $\mathfrak{D}_{n+1}$ for $\mathcal{F}$ and $\epsilon = 1$. That is, $\mathcal{F} \subseteq \mathfrak{D}_{n+1}$, $-1 < \delta(\mathfrak{D}_{n+1}/\mathcal{F}) < 0$, but $\delta(\mathcal{H}/\mathcal{H} \cap \mathcal{F}) \geq 0$ for every proper $\mathcal{H} \subseteq \mathfrak{D}_{n+1}$. It follows that $(\mathcal{F}, \mathfrak{D}_{n+1})$ is a minimal pair, but $\mathfrak{D}_n \leq \mathfrak{D}_{n+1}$. Now apply Corollary 4.6 to get a strong embedding $g : \mathfrak{D}_{n+1} \to \mathfrak{M}^*$ over $\mathfrak{D}_n$. ∎

Definition 7.7 For any finite set A and $\bar{e}$, define $d(\bar{e}/A) = d_{\mathfrak{M}^*}(A\bar{e}) - d_{\mathfrak{M}^*}(A)$. It is easily checked that if $\bar{e}$ and $\bar{e}'$ have the same type over A, then $d(\bar{e}/A) = d(\bar{e}'/A)$. Accordingly, for a complete type $p \in S(A)$, define $d(p/A) = d(\bar{e}/A)$ for any realization $\bar{e}$ of p.

The following Proposition and the subsequent Remark show that types of dimension 0 occur in abundance.

Proposition 7.8 *For any $\mathfrak{A} \leq \mathfrak{B}$ from $\mathbf{K}_\alpha$ with $\mathfrak{A} \neq \mathfrak{B}$ there is an isomorphic embedding f of $\mathfrak{B}$ into $\mathfrak{M}^*$ such that, taking $A' = f(A)$ and an enumeration $\bar{e}$ of $f(B) - f(A)$, $d(p/A') = 0$, where $p = \mathrm{tp}(\bar{e}/A')$.*

Proof. Since $\mathfrak{A} \leq \mathfrak{B}$ we can repeatedly apply Proposition 4.2 to obtain a sequence $\langle \mathfrak{D}_n : n \in \omega \rangle$ of elements of $\mathbf{K}_\alpha$ such that

1. $\mathfrak{D}_0 = \mathfrak{B}$;
2. $\mathfrak{D}_n \subseteq \mathfrak{D}_{n+1}$ for all $n \in \omega$;
3. $\mathfrak{A} \leq \mathfrak{D}_n$ for all $n \in \omega$;
4. $\delta(\mathfrak{D}_n/\mathfrak{A}) < 1/n$ for all $n \geq 1$.

Given such a sequence, let $X = \bigcup\{D_n : n \in \omega\}$. Since $\mathfrak{M}^*$ is large and saturated there is an embedding $f : X \to \mathfrak{M}^*$ such that $\mathrm{acl}(X) = X$.

Let $A' = f(A)$, let $\bar{e}$ enumerate $f(B) - f(A)$, and let $p = \mathrm{tp}(\bar{e}/A')$. We first argue that $d(p/A') \geq 0$. Let E be any finite set in $\mathfrak{M}^*$ containing $A'\bar{e}$. Since $f(X)$ is algebraically closed $E \cap f(X) \leq E$, hence $\delta(E \cap f(X)/A') \leq \delta(E/A')$. However, any finite F satisfying $A'\bar{e} \subseteq F \subseteq f(X)$, is a subset of $f(D_n)$ for some n. Since $A' \leq f(D_n)$ it follows that $\delta(F/A') \geq 0$. Taking $F = E \cap f(X)$ yields $\delta(E/A') \geq 0$.

For the opposite inequality note that $\delta(f(D_n)/A') < 1/n$, hence $d(p/A') < 1/n$ for all integers $n \geq 1$. It follows that $d(p/A') = 0$. ∎

Remark 7.9 The reader should note that by choosing appropriate finite sets Φ in our application of Proposition 4.2 in the proof above we can inductively construct a perfect tree of types of dimension zero. More precisely, for any $\mathfrak{A} \in \mathbf{K}_\alpha$ there is a family $\{f_\eta : \eta \in {}^\omega 2\}$ of isomorphisms taking $\mathfrak{A}$ into $\mathfrak{M}$ and a family $\{p_\eta \in S(f_\eta(A)) : \eta \in {}^\omega 2\}$ of complete types of dimension zero over their base, both indexed by a perfect tree, that are not conjugate, i.e., $f_\eta(f_\mu^{-1}(p_\mu)) \neq p_\eta$ for $\eta \neq \mu$.

Finally, let $\mathfrak{A}$ be the L-structure with two elements $A = \{a, b\}$ with no relations. In Section 2 of [2] Baldwin and Shelah show that the existence of a nonalgebraic type $p \in S(A)$ such that $d(p/A) = 0$, but $d(p/a)$ and $d(p/b) > 0$ is sufficient to demonstrate that the theory S_α has the Dimensional Order Property (DOP).

Corollary 7.10 *For any (symmetric, irreflexive) finite relational language L, the associated theory S_α has DOP.*

Proof. Since the notion of DOP is invariant under the addition of finitely many constants to the language, we may do so and reduce to the case where we have some distinguished relation symbol $\overline{R}$ of arity 2. To simplify notation let $\alpha = \alpha_{\overline{R}}$. Let $\mathfrak{B}$ be the L-structure whose universe has 4 points $\{a, b, x, y\}$, with the sets $\{a, x\}, \{b, y\} \in \overline{R}^{\mathfrak{B}}$ and no other relations, and let $\mathfrak{A}$ be the substructure of $\mathfrak{B}$ with universe $A = \{a, b\}$. A quick inspection of the cases confirms that $\mathfrak{A} \leq \mathfrak{B}$. Apply Proposition 7.8 and get an embedding f of $\mathfrak{A}$ into $\mathfrak{M}^*$ (as notation let $A' = \{a', b'\} = \{f(a), f(b)\}$) and a type $p(x, y) \in S(A')$ such that $\{e_1, e_2\} \cup A' \cong \mathfrak{B}$ over A' and $d(\bar{e}/A') = 0$ for any $\bar{e} = (e_1, e_2)$ realizing p. Since extensions of nonnegative dimension occur in abundance, p is nonalgebraic. Now fix any $\bar{e} = (e_1, e_2)$ realizing p. We will finish the proof by showing that $d(\bar{e}/a') \geq \alpha$ (the argument for showing that $d(\bar{e}/b') > 0$ is symmetric). Choose any $E \subseteq \mathfrak{M}^*$ such that $\bar{e}a' \subseteq E$. Since $\delta(\{a'\}) = 1$, it suffices to show that $\delta(E) \geq 1 + \alpha$. To accomplish this, consider the substructure with universe Eb'. On one hand, since $\{e_2, b'\} \in \overline{R}$, $\delta(Eb'/E) \leq 1 - \alpha$. On the other hand, since $Eb' \supseteq \bar{e}A'$ and $d(p/A') \geq 0$, $\delta(Eb'/A') \geq 0$. As $\delta(A') = 2$, this implies $\delta(Eb') \geq 2$. Since

$$\delta(Eb') = \delta(Eb'/E) + \delta(E)$$

$\delta(E) = \delta(Eb') - \delta(Eb'/E) \geq 2 - (1 - \alpha) = 1 + \alpha$ and we finish. ∎

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