# THE ESSENTIAL DIMENSION OF A g-DIMENSIONAL COMPLEX ABELIAN VARIETY IS 2g

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Abstract. We compute the Buhler-Reichstein essential dimension of a complex abelian variety using Kummer theory.

## 1. Introduction

Let $\mathcal{F} : \text{Fields}_k \to \text{Sets}$ be a covariant functor from the category of fields containing a fixed field $k$ to sets. Let $L$ be an object of $\text{Fields}_k$ and $a \in \mathcal{F}(L)$. The Buhler-Reichstein *essential dimension* $\text{ed}(a)$ of $a$ is the minimum taken over all fields $K$ such that $a \in \text{im}(\mathcal{F}(K) \to \mathcal{F}(L))$ of the transcendence degree $\text{trdeg}_k K$. The *essential dimension* $\text{ed}(\mathcal{F})$ of $\mathcal{F}$ is the maximum of $\text{ed}(a)$ taken over all $L$ and all $a \in \mathcal{F}(L)$. (This formulation is due to A. Merkurjev; see [1].)

Let $G$ be an algebraic group over $k$. It is then natural to consider the functor $\mathcal{F}_G : \text{Fields}_k \to \text{Sets}$ defined by sending $L$ to the set $H^1(L, G)$ of isomorphism classes of $G$-torsors (for the étale topology) over $L$.

We write $\text{ed}\, G$ for the essential dimension $\text{ed}(\mathcal{F}_G)$ of this functor.

Recently, a substantial body of literature has been built up concerning $\text{ed}\, G$ for various linear algebraic groups. Even for $G$ finite over $\mathbb{C}$, the computation of $\text{ed}\, G$ is an interesting and usually open question. However, the following result, a special case of [2, Theorem 6.1], is "classical."

**Theorem 1.1** (Buhler-Reichstein)**.** *Let $G$ be a finite abelian group viewed as an algebraic group over $\mathbb{C}$. Then $\text{ed}\, G$ is equal to the rank of $G$.*

In this note, I use Theorem 1.1 and the elementary theory of torsors over abelian varieties to prove the theorem stated in the title (generalized to include the case of arbitrary algebraically closed fields of characteristic 0).

**Theorem 1.2.** *Let $A$ be an abelian variety (viewed as an algebraic group) of dimension $g$ over an algebraically closed field $k$ of characteristic 0. Then $\text{ed}\, A = 2g$.*

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In fact, the main point of the proof is that $\operatorname{ed} A$ is the maximum of the $\operatorname{ed} H$ taken over all finite abelian subgroups $H < A$. From this, the result follows directly from an application of Theorem 1.1.

*Remark* 1.3. One interesting aspect of the computation is that it is done without the use of a versal torsor for the abelian variety $A$. (See [1, Definition 6.3] for an explanation of this concept.) It is not difficult to see that, in fact, no such versal torsor exists.

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## 2. Generalities

In this section, we collect some generalities that will be needed for the computation of the essential dimension of an abelian variety. As in the statement of Theorem 1.2, we work over an algebraically closed field $k$ of characteristic 0.

**Definition 2.1.** A field $K$ over $k$ is *unirational* if $K \subset k(t_1, \dots, t_r)$ for some $r \in \mathbb{N}$.

**Proposition 2.2.** *For $A$ an abelian variety over $k$ and $K$ a unirational field over $k$, $A(K)$ is divisible.*

*Proof.* This is a consequence of the fact that there are no non-constant maps from a rational variety over $k$ to $A$. □

**Proposition 2.3.** *For every non-negative integer $r$, there exists a unirational field $K$ of transcendence degree $r$ over $k$ and a $G := (\mathbb{Z}/2)^r$-torsor $T \in H^1(K, G)$ such that $\operatorname{ed} T = r$.*

*Proof.* In fact, we can take $K = k(t_1, \dots, t_r)$ and $T = \langle (t_1), \dots, (t_r) \rangle \in H^1(K, G) = H^1(K, \mathbb{Z}/2)^r$. It is easy to see that $\operatorname{ed} T = r$. (For example, by using the proof of Proposition 3.7 of [1]). □

*Remark* 2.4. Although we do not need it, the same result holds for $G$ any finite abelian group of rank $r$.

**Principle 2.5** (Berhuy-Favi)**.** *Let $\mathcal{F}, \mathcal{G} : \mathrm{Fields}_k \to \mathrm{Sets}$ be two functors equipped with a natural transformation $\psi : \mathcal{F} \to \mathcal{G}$, and assume that $\mathcal{F}(L) \twoheadrightarrow \mathcal{G}(L)$ for any field $L \in \mathrm{Fields}_k$. Then $\operatorname{ed}(\mathcal{F}) \geq \operatorname{ed}(\mathcal{G})$.*

*Proof.* This is Lemma 1.9 of [1]. □

**Corollary 2.6.** *Let $\mathcal{F}_i : \mathrm{Fields}_k \to \mathrm{Sets}$ be a family functors.*

1. $\operatorname{ed}(\coprod \mathcal{F}_i) = \sup \operatorname{ed}(\mathcal{F}_i)$.
2. *If $\mathcal{G} : \mathrm{Fields}_k \to \mathrm{Sets}$ is a functor and we are given maps $\psi_i : \mathcal{F}_i \to \mathcal{G}$ such that $\cup \psi_i(\mathcal{F}_i) = \mathcal{G}$, then $\operatorname{ed} \mathcal{G} \leq \sup \operatorname{ed}(\mathcal{F}_i)$.*

*Proof.* (1) is obvious. (2) follows from (1) and Principle 2.5 applied to the surjective map $\coprod \mathcal{F}_i \twoheadrightarrow \mathcal{G}$. □

**Lemma 2.7.** *Let $G$ be an algebraic group of $k$ and $E$ a $G$-torsor in $H^1(L, G)$ for some $L \in \mathrm{Fields}_k$. Suppose there is a variety $X$ with a free $G$-action defined over a subfield $M \in \mathrm{Fields}_k$ of $L$ and a $G$-equivariant morphism $E \to X_L$. Then $\operatorname{ed} E \leq \operatorname{trdeg}_k M + \dim X - \dim G$.*

*Proof.* Since $G$ acts freely on $X$, the quotient $B = X/G$ exists as an algebraic space [3, Proposition 22]. Since $E$ is a $G$-torsor over $\operatorname{Spec} L$, the map $\operatorname{Spec} L \to \operatorname{Spec} M$ factors through $B$. Let $b$ denote the image of $\operatorname{Spec} L$ in $B$ and let $X_b$ denote the pullback to $X$. Then the algebraic space $X_b$ is a $G$-torsor over $k(b)$. By descent, it is a scheme over $k(b)$. Moreover, the diagram

$$\begin{array}{ccc} E & \longrightarrow & X_b \\ \downarrow & & \downarrow \\ \operatorname{Spec} L & \longrightarrow & \operatorname{Spec} k(b) \end{array} \tag{2.8}$$

is a pullback.

Thus $E$ is in the image of the map $H^1(\operatorname{Spec} k(b), G) \to H^1(L, G)$. Since $\operatorname{trdeg}_k k(b) \leq \operatorname{trdeg}_k M + \dim X - \dim G$, the result follows. □

**Principle 2.9.** *Let $\psi : A \to B$ be an inclusion of algebraic groups over $k$ and let $E$ be a torsor in $H^1(L, A)$ for $L \in \mathrm{Fields}_k$. Then,* $\operatorname{ed} E + \dim A - \dim B \leq \operatorname{ed}(\psi_*(E)) \leq \operatorname{ed} E$.

*In particular,* $\operatorname{ed} E = \operatorname{ed}(\psi_* E)$ *if* $\dim A = \dim B$.

*Proof.* Clearly $\operatorname{ed}(\psi_* E) \leq \operatorname{ed} E$. Thus, we are reduced to proving the first inequality.

The pushforward of $E$ to a $B$-torsor is $\psi_*(E) = B \times^A E$. (Here $B \times^A E$ is the quotient of $B \times E$ identifying $(ba, e)$ with $(b, ae)$.) Let $f : E \to B \times^A E$ denote the $A$-equivariant morphism given by $e \mapsto (1, e)$.

Suppose now that there is an $M \subset L$ in $\mathrm{Fields}_k$ and a $B$-torsor $F$ such that $F_L \cong B \times^A E$ as a $B$-torsor. The induced $A$-equivariant map $f : E \to F_L$ then satisfies the hypotheses of Lemma 2.7. From this, we see that $\operatorname{ed} E \leq \operatorname{trdeg}_k M + \dim B \times^A E - \dim A = \operatorname{trdeg}_k M + \dim B - \dim A$. Thus $\operatorname{trdeg}_k M \geq \operatorname{ed} E + \dim A - \dim B$. The desired inequality follows. □

*Remark* 2.10. The first inequality of Principle 2.9 proves [1, Theorem 6.19]. Indeed, the proof of the principle is essentially the same as Berhuy and Favi's proof of their theorem.

## 3. Principal Homogeneous Spaces

The goal of this section is to prove Theorem 1.2. J. Silverman's book [4] is a good reference for the simple facts about torsors for abelian varieties used here. (They are stated there in the context of elliptic curves but the generalizations are obvious).

As in the statement of Theorem 1.2, let $A$ be an abelian variety of dimension $g$ over an algebraically closed field $k$ of characteristic 0. If $E$ is a torsor for $A$ over a field extension $K$ of $k$, then there is a finite extension $L/K$ such that $E_L$ is split (i.e., $E(L) \neq \emptyset$, hence, $E_L \cong A_L$).

Let us write $i : K \to L$ for the inclusion and $i_* : H^1(K, A) \to H^1(L, A)$ (resp. $i^* : H^1(L, A) \to H^1(K, A)$) for the corestriction (resp. restriction) map on the Weil-Châtelet group. It is well-known that $i_* i^*$ is multiplication by $[L : K]$. Hence, every $A$-torsor is in the subgroup $H^1(K, A)[n]$ of $n$-torsion elements of the Weil-Châtelet group, for some positive integer $n$. That is,

$$H^1(K, A) = \cup_{n=1}^{\infty} H^1(K, A)[n]. \tag{3.1}$$

The sequence

$$0 \to A(\overline{K})[n] \to A(\overline{K}) \stackrel{\times n}{\to} A(\overline{K}) \to 0$$

of Galois modules (for the absolute Galois group, $G_K$, of $K$) gives an exact sequence

$$A(K)/nA(K) \to H^1(K, A[n]) \to H^1(K, A)[n] \to 0 \tag{3.2}$$

of groups.

By (3.1), (3.2) and Corollary 2.6, we see that $\operatorname{ed} A \leq \sup_{n=1}^{\infty} \operatorname{ed}(A[n])$. Since $A[n] \cong (\mathbb{Z}/n)^{2g}$, $\operatorname{ed}(A[n]) = 2g$ by Theorem 1.1. Thus $\operatorname{ed} A \leq 2g$.

To see that $\operatorname{ed}(A) = 2g$, let $K$ be a unirational fields of transcendence degree $2g$ equipped with the $(\mathbb{Z}/2)^{2g}$-torsor $T$ with $\operatorname{ed} T = 2g$. (Such a field is provided by Proposition 2.3.) Note that,

$$H^1(M, A[n]) \stackrel{\cong}{\to} H^1(M, A)[n] \tag{3.3}$$

for any $M \subset K$ and any integer $n$. This is because $A(M)$ is divisible.

Let $E$ denote the image of $T$ under the composition

$$H^1(K, (\mathbb{Z}/2)^{2g}) = H^1(K, A[2]) = H^1(K, A)[2] \to H^1(K, A).$$

Suppose $\operatorname{ed} E < 2g$. Then there is a field $M \subset K$ of transcendence degree less than $2g$ and a torsor $E' \in H^1(M, A)$ such that $E$ is the image of $E'$ under the map $H^1(M, A) \to H^1(K, A)$. We clearly have $E' \in H^1(M, A)[2s]$ for some non-zero integer $s$. But then note that $H^1(M, A)[2s] = H^1(M, A[2s])$. It follows that the image $T'$ of $T$ in

$H^1(K, A[2s])$ is equal to the image of $E'$. This contradicts Principle 2.9. The proof of Theorem 1.2 is thus complete.

## References


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