ON MOTIVIC DECOMPOSITIONS ARISING FROM THE METHOD OF BIAŁYNICKI-BIRULA

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Abstract. Recently, V. Chernousov, S. Gille and A. Merkurjev have obtained a decomposition of the motive of an isotropic smooth projective homogeneous variety analogous to the Bruhat decomposition. Using the method of A. Bialynicki-Birula and a corollary, which is essentially due to S. del Baño, I generalize this decomposition to the case of a (possibly anisotropic) smooth projective variety homogeneous under the action of an isotropic reductive group. This answers a question of N. Karpenko.

1. Introduction

An important difference between the category of motives and the category of algebraic varieties over a field is the existence of interesting direct sum decompositions of motives. The simplest of these is the decomposition of the Chow motive $M(\mathbb{P}^n)$ of n-dimensional projective space over a field k:

$$M(\mathbb{P}^n) = \oplus_{i=0}^{n} \mathbb{Z}(n). \tag{1.1}$$

This is one of the elementary results in Grothendieck's theory of motives (which will be recalled in §2).

An example of a less elementary decomposition is the following theorem due to M. Rost, which is in an important ingredient in his construction of the "Rost motive" [23, Proposition 2].

Theorem 1.1 (Rost decomposition)**.** *Let Q be a smooth, projective, n-dimensional isotropic quadric over a field k of characteristic not equal to 2. Then*

$$M(Q) = \mathbb{Z} \oplus M(Q')(1) \oplus \mathbb{Z}(n) \tag{1.2}$$

where Q' is a smooth, projective sub-quadric of codimension 2 in Q.

Since both projective spaces and quadrics are examples of projective homogeneous varieties, it is natural to look for decompositions generalizing (1.1) and (1.2) in the motives of such varieties. In the case that G is a split reductive group (i.e., when the base field k is separably closed), a decomposition for the motive of G/P was found by B. Köck [19]. In this case $M(G/P)$ splits completely as a sum of Tate motives. A more general decomposition was later found by N. Karpenko [17] in the case of motives

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of flag varieties for classical groups, and recently Köck's decomposition was further generalized by V. Chernousov, S. Gille and A. Merkurjev [9, Theorem 7.4] to the case of motives for isotropic projective homogeneous varieties for adjoint semi-simple groups. Both generalizations explicitly describe the factors appearing in the decomposition (which are, in general, not Tate motives) in terms of smaller projective homogeneous spaces. Moreover, Rost's theorem appears as a special case of either generalization when the quadric Q is viewed as a homogeneous space for the group $\mathrm{PSO}(q)$ with q a quadratic form whose corresponding projective quadric is Q.

Theorem 3.5 of this paper applies results of Białynicki-Birula (as extended by W. Hesselink) to obtain a decomposition of the motive of any smooth, projective algebraic variety admitting an action of the multiplicative group. In the case of motives with rational coefficients (see Remark 2.1 for the distinction), the theorem is due to del Baño [10]. (I thank B. Köck for bringing this to my attention.) I include a sketch of the proof (which is similar to del Baño's) for completeness and convenience of the reader.

As the class of varieties admitting multiplicative group actions includes both the homogeneous spaces considered by Karpenko and the isotropic homogeneous spaces considered by Chernousov, Gille and Merkurjev, we obtain a generalization of the Chernousov-Gille-Merkurjev decomposition. In the end, we can give a rather explicit decomposition of the motive $M(X)$ of a projective homogeneous variety for a reductive group G as a sum of Tate twisted motives of certain "quasi-homogeneous schemes" X_i for the anisotropic kernel of G. Theorem 4.5 gives a rough form of this description which is refined in Theorem 7.4, the final theorem of the paper. In particular, the fact that the motive of a projective homogeneous variety admits a decomposition in terms of motives of quasi-homogeneous schemes for the anisotropic kernel answers the fundamental question posed by Karpenko in the introduction to [17].

1.1. **Notation.** All notions of Chow groups are taken from Fulton's book on intersection theory [11]. The official reference for reductive groups is SGA3 [1], but some notation is taken from Springer's book [25]. The main difference between these two references that will be important for this paper is that SGA3 demands that a reductive group is *connected* and a parabolic subgroup is *smooth*, while Springer does not make these assumptions. Since these are convenient assumptions for us, we will have to agree with SGA3. In several places, the symbol "k" is used to denote both the base field and an index. This does not seem to produce any confusion.

1.2. **Outline.** With one exception, the results in this paper build sequentially from general facts about motivic decompositions to specific information about the motivic decomposition of a projective homogeneous space in terms of double cosets of the Weyl group given in Theorem 7.4. Specifically, §2 reviews the theory of motives, §3 explains how Białynicki-Birula's theorem yields a motivic decomposition and §4 introduces the concept of projective quasi-homogeneous schemes and studies motivic decompositions of such schemes in the presence of a $\mathbb{G}_m$-action. The last two mathematical sections, §6 and §7, formulate the general theory in terms of root systems and reflection groups. The one exceptional section is §5 where I give a proof of a generalization of Rost's nilpotence theorem following Chernousov, Gille and Merkurjev. The results in this section are not needed anywhere else in the paper.

2. Motives and the category of correspondences

The category of motives can be defined by first defining the category of correspondences and then applying the functor of idempotent completion. In fact, the decomposition theorems of this paper such as Theorem 3.5 will hold before (or after) taking idempotent completion, but, to make the connection with Chow motives explicit, I will describe both categories.

The category of correspondences is the category Corr_k whose objects are pairs (X, n) with X a smooth projective scheme over the field k and n an integer. The morphisms are given by

$$Hom_{\mathrm{Corr}_k}((X,n),(Y,m)) = \oplus A_{d_i+n-m} X_i \times Y \tag{2.1}$$

where $X = \coprod X_i$ with X_i connected, $d_i = \dim X_i$ and A_k denotes the k-th Chow groups graded by dimension.

For a smooth, projective, variety X, $M(X)$ denotes the object $(X, 0)$. When $M = (X, n)$ is an object, $M(k)$ denotes the *Tate twisted* object $(X, n + k)$. Let $\mathbb{Z}$ denote the object $(\operatorname{Spec} k, 0)$. The "twists" $\mathbb{Z}(k)$ of $\mathbb{Z}$ are called *Tate objects.* Clearly, $\mathrm{Hom}_{\mathrm{Corr}_k}(Z(k), M(X)) = A_k X$ and $\mathrm{Hom}_{\mathrm{Corr}_k}(M(X), Z(k)) = A_{d-k}X$ for X irreducible of dimension d.

The objects of the category Chow_k of Chow motives over k are triples (X, n, p) with $p \in \mathrm{End}(X, n)$ a morphism such that $p^2 = p$. The morphisms in Chow_k are given by

$$Hom_{\mathrm{Chow}_k}((X,n,p),(Y,m,q)) = qHom_{\mathrm{Corr}_k}((X,n),(Y,m))p.$$

The category Chow_k is *idempotent complete* in the sense that every idempotent morphism has both a kernel and a cokernel. Clearly, there is a fully, faithful embedding $\mathrm{Corr}_k \hookrightarrow \mathrm{Chow}_k$ given by $(X, n) \rightsquigarrow (X, n, \mathrm{id})$. Moreover, this embedding is universal for functors from Corr_k with idempotent complete targets.

Both categories admit a tensor structure defined (on Chow_k) by

$$(X, n, p) \otimes (Y, m, q) = (X \times Y, n + m, p \times q).$$

The category Chow_k also admits direct sums defined as follows. Let r_k denote the idempotent on $M(\mathbb{P}^k)$ such that $\mathbb{Z}(k) = (\mathbb{P}^k, 0, r_k)$. Explicitly, it is given by the cycle $[\mathrm{pt} \times \mathbb{P}^k]$ where pt denotes an arbitrary degree 1 closed point in $\mathbb{P}^k$. Then, for two motives (X, n, p) and (Y, m, q) with $n \leq m$,

$$(X, n, p) \oplus (Y, m, q) = (X \coprod (Y \times \mathbb{P}^{m-n}), n, p + (q \times r_{m-n})).$$

The direct sum is the coproduct in the category of motives. In the category of correspondences, coproducts do not always exist. For example, it is not hard to see that the object $\mathbb{Z}\coprod\mathbb{Z}(2)$ does not exist in the category of correspondences over $\mathbb{C}$. However, when coproducts do exist in Corr_k, they coincide with those in the category of Chow motives.

Remark 2.1. The categories of motives occurring in this paper (and in those papers of Rost, Karpenko, and Chernousov-Gille-Merkurjev) have *integral coefficients*. If we were to tensor the morphism sets with $\mathbb{Q}$, replacing $\mathrm{Hom}_{\mathrm{Corr}_k}(M(X), M(Y))$ with $\mathrm{Hom}_{\mathrm{Corr}_k}(M(X), M(Y)) \otimes \mathbb{Q}$, we would obtain a category $\mathrm{Corr}_k \otimes \mathbb{Q}$ which is closer to the categories of motives Grothendieck originally considered [16, 21]. However, we would also loose information. For example, using the fact that any quadric is totally isotropic over a finite separable extension, it is easy to see that every quadric decomposes as a direct sum of the Tate objects $\mathbb{Q}(i) = \mathbb{Z}(i) \otimes \mathbb{Q}$ in $\mathrm{Corr}_k \otimes \mathbb{Q}$. On the other hand, Springer's theorem on quadrics isotropic over an odd degree extension [20, Theorem 2.3 p. 198] implies that the integral motive $M(Q) \in \mathrm{Corr}_k$ of a smooth quadric contains a factor of $\mathbb{Z}(0)$ if an only if $Q(k) \neq \emptyset$.

3. Motivic decomposition

The most general theorem in this paper on motivic decompositions is essentially a corollary of two results which I will now recall after giving one definition. As mentioned in the introduction, a version of the theorem (3.5) can also be found in Theorem 2.4 of S. del Baño's paper, [10].

Definition 3.1. A flat morphism $\phi : X \to Z$ is called an *affine fibration* (resp. an *affine quasi-fibration*) of relative dimension d if, for every point $z \in Z$, there is a Zariski open neighborhood $U \subset Z$ such that $X_U \cong Z \times \mathbb{A}^d$ with $\phi : X_U \to Z$ isomorphic to the projection on the first factor (resp. the fiber X_z of ϕ is isomorphic to $\mathbb{A}^d_{k(z)}$).

Clearly an affine fibration is an affine quasi-fibration. It is a well-known consequence of the homotopy invariance of Chow groups that an affine quasi-fibration between smooth varieties of relative dimension d induces an isomorphism $\phi^* : A_i Z \to A_{i+d} X$.

Theorem 3.2 (Karpenko)**.** *Let X be a smooth, projective variety over a field k with a filtration*

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

where the X_i are closed subvarieties. Assume that, for each integer $i \in [0, n]$, there is a smooth projective variety Z_i and an affine fibration $\phi_i : X_i - X_{i-1} \to Z_i$ of relative dimension a_i. Then, in the category of correspondences, $M(X) = \coprod_{i=0}^n M(Z_i)(a_i)$.

The theorem was stated by Karpenko [17] for the special case that the maps $\phi_i : X_i - X_{i-1} \to Z_i$ are vector bundle morphisms. However, in [9, Theorem 7.1], Chernousov, Gille and Merkurjev noticed that Karpenko's proof actually applies to any affine quasi-fibration. (Del Baño gives a slightly different proof of the result in the proof of his Theorem 2.4).

The second result is the method of Białynicki-Birula which gives a natural situation where Karpenko's theorem applies.

Theorem 3.3 (Białynicki-Birula, Hesselink, Iversen). *Let X be a smooth, projective variety over a field k equipped with an action of the multiplicative group $\mathbb{G}_m$. Then*

(1) The fixed point locus $X^{\mathbb{G}_m}$ is a smooth, closed subscheme of X.

(2) There is a numbering $X^{\mathbb{G}_m} = \coprod_{i=1}^n Z_i$ of the connected components of the fixed point locus, a filtration

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$$

and affine fibrations $\phi_i : X_i - X_{i-1} \to Z_i$.

(3) The relative dimension a_i of the affine fibration ϕ_i is the dimension of the positive eigenspace of the action of $\mathbb{G}_m$ on the tangent space of X at an arbitrary point $z \in Z_i$. The dimension of Z_i is the dimension of $TX_z^{\mathbb{G}_m}$.

As the theorem stated is the product of several results of different authors, I will give a short history of the result (in lieu of a proof).

Iversen [15] showed that $X^{\mathbb{G}_m}$ is smooth. Białynicki-Birula [2] showed, under the assumption that k is algebraically closed, that X is a union of locally closed subschemes X_i^+ with affine fibrations $\phi_i : X_i^+ \to Z_i$ where the Z_i are the connected components of $X^{\mathbb{G}_m}$. In fact, this was shown with the assumption that X is projective replaced with the assumption that X is complete. Shortly thereafter, Białynicki-Birula showed that, when X is projective, there is a filtration of X and an ordering of the connected components as in the theorem such that $X_i^+ = X_i - X_{i-1}$ [3]. Thus, in the case that k is algebraically closed, the theorem as stated was proved by Białynicki-Birula and Iversen. Note that the existence of a filtration is deduced by embedding X equivariantly in a projective space with a diagonalized $\mathbb{G}_m$-action. It is then easy to construct a filtration on the projective space and see that that it restricts to one on X. (However, for X smooth and complete but not projective, there are examples where no filtration satisfying $X_i^+ = X_i - X_{i-1}$ exists [8, Example 2, p. 30].)

Białynicki-Birula's proofs make use of the assumption that k is algebraically closed. Hesselink removed this restriction and was able to show that X is a union of locally closed X_i^+ for X a smooth, proper scheme over an arbitrary base [13] provided that there is a covering of X by $\mathbb{G}_m$-stable Zariski open affine subsets. To construct a filtration in the case X is projective, it then suffices to find a $\mathbb{G}_m$-equivariant embedding of X into a projective space with a diagonal action of $\mathbb{G}_m$ or, equivalently, a very ample $\mathbb{G}_m$-linearized line bundle over X. The fact that such bundles exist can be found in Mumford's GIT [22]. Since a diagonal action of $\mathbb{G}_m$ on $\mathbb{P}^r$ preserves the coordinate hyperplanes (and thus their complements), any X embedded by a $\mathbb{G}_m$-linearized very ample line bundle automatically has a covering by $\mathbb{G}_m$-invariant open affines. Thus Hesselink's hypotheses are verified for X smooth and projective.

The Białynicki-Birula decomposition is explicit in the sense that the locally closed subscheme X_i^+ is the set of all points $x \in X$ such that $\lim_{t\to 0} tx \in Z_i$ where $(t, x) \mapsto tx$ is the $\mathbb{G}_m$ action. Moreover, the map $\phi_i : X_i^+ \to Z_i$ is then given by $x \mapsto \lim_{t\to 0} tx$.

Remark 3.4. Since X separated, $\lim_{t\to 0} tx$ has at most one meaning, since X is proper it has exactly one.

Theorem 3.5. *Let X be a smooth, projective scheme over a field k equipped with an action of the multiplicative group $\mathbb{G}_m$. Then, in the category* Corr_k*,*

$$M(X) = \coprod M(Z_i)(a_i) \tag{3.1}$$

where the Z_i are the connected components of $X^{\mathbb{G}_m}$ and the a_i are determined as in Theorem 3.3 (3).

Proof. This is a corollary of the two previous theorems. □

We will refer to the decomposition of (3.1) as the *motivic Białynicki-Birula decomposition.* The rest of this paper will focus on the application of the theorem to the special case where X is a projective homogeneous variety. Before proceeding to the general theory, I use the theorem directly to derive Rost's decomposition theorem.

Example 3.6. Let $q : V \to k$ be a non-degenerate quadratic form of dimension $n + 2$ over a field k with char $k \neq 2$, and let Q be the associated n-dimensional smooth projective quadric. Suppose that q is isotropic, that is, there exists a nonzero vector $v \in V$ such that $q(v) = 0$. Then there is a subspace $W \subset V$ and two linearly independent vectors v_1 and v_2 such that $V = kv_1 \oplus kv_2 \oplus W$ and

$$q(xv_1 + yv_2 + w) = xy + q'(w)$$

for a non-degenerate quadratic form q' on W. (This is an easy exercise which can also be found in almost any book on quadratic forms over a field, e.g., [18, Proposition 3.7.1].) In this case, the multiplicative group $\mathbb{G}_m$ acts on Q by

$$[xv_1 + yv_2 + w] \mapsto [txv_1 + t^{-1}yv_2 + w].$$

Assume that $\dim Q \neq 2$. The fixed point set $Q^{\mathbb{G}_m}$ then has three components: the points $[v_1]$ and $[v_2]$ and the quadric $Q' = \{[w] \in \mathbb{P}(W) \mid q'(w) = 0\}$ which we denote by Z_1, Z_2 and Z_3 respectively. Let T_i denote the tangent space at an (arbitrary) point of Z_i. The action of $\mathbb{G}_m$ on T_1 has only negative weights. Therefore, in the decomposition of Theorem 3.5, $a_1 = 0$. The weights of T_2 are all positive, therefore, $a_2 = n$. Finally, T_3 has weights -1

and 1 each occurring once and 0 occurring $n-2$ times. Therefore $a_3 = 1$. Thus we have

$$M(Q) = \mathbb{Z} \coprod \mathbb{Z}(n) \coprod M(Q')(1). \tag{3.2}$$

It is easy to see that the above decomposition also holds when $\dim Q = 2$, however there is a possibility that the quadric Q' can split as a disjoint union of 2 copies of $\operatorname{Spec} k$.

4. Projective homogeneous schemes

Let G denote a *reductive* group over a field k in the terminology of SGA3. That is, G is smooth and connected with trivial unipotent radical. Recall that a *parabolic* subgroup of G is a subgroup P such that G/P is projective and P is smooth over k. Let $\mathrm{G{-}Sch}_k$ denote the category of G-schemes over k. The objects of this category are schemes X over k equipped with a G-action $G \times X \to X$. The morphisms are the G-equivariant scheme-theoretic morphisms. Base change induces an obvious functor $\mathrm{G{-}Sch}_k \rightsquigarrow \mathrm{G{-}Sch}_L$ for L an extension of k.

Let $\overline{k}$ denote an algebraic closure of k. A G-scheme X is a *projective homogeneous variety* for G if $X_{\overline{k}}$ is isomorphic as a $G_{\overline{k}}$-scheme to $G_{\overline{k}}/P$ for $P \subset G_{\overline{k}}$ a parabolic subgroup. It is well-known that such a projective homogeneous variety is projective over k. We will call a G-scheme X a *projective quasi-homogeneous scheme* if X is smooth and projective over k and the morphism $\psi = (a, \mathrm{pr}_2) : G \times X \to X \times X$ given by $(g, x) \mapsto (gx, x)$ is smooth.

Proposition 4.1. *Let X be a G-scheme over k. Then the following are equivalent.*

(1) X is a projective quasi-homogeneous G-scheme.
(2) $X_{\overline{k}}$ is a disjoint union of projective homogeneous varieties.
(3) X is smooth, projective, and, for every geometric point $x \in X_{\overline{k}}$, the orbit map $m_x : G_{\overline{k}} \to X_{\overline{k}}$ (given by $g \mapsto gx$) induces a surjection $dm_x : L(G) \to TX_x$.

Proof. Since all of the properties listed are invariant under base change of k, we can assume that k is algebraically closed.

(1) $\Rightarrow$ (2): A scheme is projective quasi-homogeneous if and only if all of its connected components are projective quasi-homogeneous. (This is easy.) Therefore we can assume that X is connected. Since $G \times X \to X \times X$ is smooth, all orbits are open. Thus, since X is smooth and connected (hence irreducible), all orbits must intersect. It follows that there is only one orbit, namely, X itself. Thus $X = G/P$ for some subgroup P. The smoothness of ψ then implies that P is smooth.

(2) $\Rightarrow$ (3): Here we can assume $X = G/P$ with P parabolic. The claim then follows from the assumption that P is smooth.

(3)⇒ (1): From (3), it follows that $\psi : G\times X \to X\times X$ induces surjections on the tangent spaces. The claim then follows from [12, p. 270, Proposition 10.4 (iii)]. □

Remark 4.2. The motivation for considering projective quasi-homogeneous schemes in addition to projective homogeneous varieties is already apparent in the Rost decomposition of quadrics. Let q be an n-dimensional non-degenerate quadratic form, let Q be the associated $n-2$ dimensional smooth projective quadric and let $\mathrm{PSO}(q)$ denote the special orthogonal group. Then Q is a projective homogeneous space for $\mathrm{PSO}(q)$ if and only if $\dim Q > 0$. When $\dim Q = 0$, Q can either be irreducible or a disjoint union of two copies of $\operatorname{Spec} k$. In either case, it is not a projective homogeneous variety for $\mathrm{PSO}(q)$. However, regardless of the dimension, Q is projective quasi-homogeneous for $\mathrm{PSO}(q)$.

Now let X denote a projective quasi-homogeneous scheme for G, and let $\mathbb{G}_m \overset{\lambda}{\tilde{\to}} L \subset G$ denote the inclusion of a k-split torus in G. (The group G is called *isotropic* if such a split torus exists.) In this case, L acts on X and Theorem 3.5 applies to give a decomposition of $M(X)$. The main result of this section is that, in fact, the summands appearing are themselves projective quasi-homogeneous schemes for the centralizer $H = Z(\lambda)$ of L in G. A more detailed description will be obtained in §6 and §7.

Theorem 4.3. *(1) H is connected, reductive and defined over k.*
(2) H acts on the fixed point set X^λ.
(3) The action map $\psi_H : H \times X^\lambda \to X^\lambda \times X^\lambda$ is smooth. Thus X^λ is projective quasi-homogeneous.

Proof. (1) The fact that H is connected is [Springer, 13.4.2 (i)]. The fact that it is reductive is [Springer, 7.6.4 (i)]. It is defined over k by [Springer, 13.3.1 (ii)].

(2) To see that H acts on X^λ, let T be a scheme over k and consider T-valued points $x \in X^\lambda(T)$, $h \in H(T)$ and $t \in L(T)$. Then $thx = htx = hx$, thus, hx is in $X^\lambda(T)$.

(3) Since the smoothness of ψ_H is invariant under field extension of k, we may assume that $k = \bar{k}$. Pick a closed point $z \in Z$. The orbit map $l : G \to X$ given by $g \mapsto gz$, induces a surjection $dl : L(G) \to TX_z$ because, by assumption, X is a projective quasi-homogeneous scheme for G. The multiplicative group $\mathbb{G}_m$ acts on G via conjugation by λ, i.e., $g \mapsto \lambda(t)g\lambda(t^{-1})$. The group $\mathbb{G}_m$ also acts on X via right multiplication, i.e., $x \mapsto \lambda(t)x$. Since z is a fixed point of λ, the orbit map l is equivariant for the $\mathbb{G}_m$-actions. Moreover, we obtain a $\mathbb{G}_m$-action on TX_z compatible with the *Ad*-action of $\mathbb{G}_m$ on $L(G)$.

Now $L(G) \cong L(G)_+ \oplus L(H)$ where $L(G)_+$ consists of the non-zero weight space of $L(G)$ and $L(H)$ is the Lie algebra of H. Analogously, $TX_z \cong TX_{z+} \oplus TX_z^\lambda$ where TX_{z+} is the non-zero weight space of TX_z. Since dl

respects the weight decomposition, $dl(L(H)) = TX^\lambda$. Thus Proposition 4.1 (3) is satisfied. □

Corollary 4.4. *Let X be a projective quasi-homogeneous scheme for an isotropic reductive group G, and let $\lambda : \mathbb{G}_m \to G$ be the embedding of a split torus. Then, in the motivic Białynicki-Birula decomposition*

$$M(X) = \coprod M(Z_i)(a_i)$$

of Theorem 3.5, the Z_i are all projective quasi-homogeneous schemes for the centralizer H of λ.

Proof. The corollary holds because the Z_i appearing in Theorem 3.5 are components of X^λ. □

4.1. **Adjoint groups.** For a reductive group G, let $\mathcal{QH}_G$ denote the full subcategory of $G-\mathrm{Sch}_k$ consisting of projective quasi-homogeneous schemes. If Z_G is the center of G, then $G_{\mathrm{ad}} = G/Z_G$ is the *adjoint* group of G, an adjoint semi-simple group [1, Proposition 22.4.3.5]. The restriction functor $G_{\mathrm{ad}}-\mathrm{Sch}_k \rightsquigarrow G-\mathrm{Sch}_k$ induces an equivalence of categories $\mathcal{QH}_{G_{\mathrm{ad}}} \rightsquigarrow \mathcal{QH}_G$. This is because Z_G is smooth and acts trivially on all quasi-homogeneous schemes over G.

4.2. **Anisotropic Kernels.** [26] Let S denote a maximal k-split torus of G, and let $Z(S)$ denote its centralizer. The derived subgroup $DZ(S)$ is the *semi-simple anisotropic kernel*. Since $Z(S)$ is reductive, there is an almost direct product decomposition $Z(S) = DZ(S) \cdot Z$ where Z is the center of $Z(S)$[5, Proposition 2.2]. It follows that the adjoint group of $Z(S)$ is isomorphic to the adjoint group of the semi-simple anisotropic kernel. Thus the categories $\mathcal{QH}_{Z(S)}$, $\mathcal{QH}_{DZ(S)}$ and $QH_{Z(S)_{\mathrm{ad}}}$ are all equivalent via the restriction of group functors. (Moreover, the objects are identical.)

Applying Corollary 4.4 inductively, we obtain an answer to a question of N. Karpenko [17].

Theorem 4.5. *Let X be a projective quasi-homogeneous scheme for a reductive group G. Then*

$$M(X) = \coprod M(Y_i)(a_i) \tag{4.1}$$

where the Y_i are irreducible projective quasi-homogeneous schemes for the anisotropic kernel of G (resp. for $Z(S)$, for $Z(S)_{\mathrm{ad}}$).

Remark 4.6. A projective homogeneous variety X is said to be *isotropic* if $X = G/P$ for a parabolic subgroup P defined over k. Otherwise it is said to be *anisotropic*. X is anisotropic if and only if $X(k)$ is empty. If X is an isotropic projective homogeneous space for a reductive group G, then there exists at least one k-split torus L in G. (See [25] or [1].) In other words, if X is isotropic then G is as well. It follows that the schemes Y_i appearing in (4.1) are all either anisotropic or isomorphic to $\mathrm{Spec}\, k$.

Exercise 4.7. It is interesting to see an example of Corollary 4.4 at work on an anisotropic projective homogeneous variety for an isotropic reductive group. One such is given by the variety X of two dimensional isotropic subspaces for the quadratic form $q = x_1^2 + \cdots + x_{2n}^2 + yz$ over the reals with $n \geq 2$. Using the methods of Example 3.6, the decomposition can be computed explicitly in terms of smaller quadrics and varieties of two dimensional isotropic subspaces. In Example 7.6, we will return to this matter, computing the decomposition using the Lie theory of $\mathrm{PSO}(q)$.

5. The nilpotence theorem of Chernousov, Gille and Merkurjev

As a corollary of the results of the previous section, we obtain the following theorem of Chernousov, Gille and Merkurjev [9, Theorem 8.2].

Theorem 5.1. *Let X be a projective homogeneous variety for a reductive group G over a field k. Then the kernel of the map*

$$\mathrm{End}(M(X)) \to \mathrm{End}(M(X \otimes \overline{k}))$$

consists of nilpotent endomorphisms.

The proof follows that of [9]. I include it here for the convenience of the reader and to make the point that the theorem can be obtained without the full description of the motivic decomposition obtained in [9, Theorem 7.4].

For a field extension L/k, let n_L denote the number of terms appearing in the decomposition

$$M(X_L) = \coprod_{i=1}^{n_L} M(Z_i)(a_i) \tag{5.1}$$

of (4.1) for the projective homogeneous G_L-variety X_L. (Here the Z_i will depend on L.) Clearly, $M \supset L \Rightarrow n_M \geq n_L$, and the maximal number of terms in the coproduct occurs precisely when each Z_i is $\mathrm{Spec}\, L$. In particular, this happens when $L = \mathrm{Spec}\, k_{\mathrm{sep}}$.

Claim 5.2. Set $N(d, n) = (d + 1)^{\overline{n}-n}$ with $\overline{n} = n_{k_{\mathrm{sep}}}$. Then, for any morphism $f \in \mathrm{End}(M(X))$ with $f \otimes \overline{k} = 0$, $f^{N(d,n_k)} = 0$.

Evidently the claim implies the theorem.

Now in the case that the maximal number of terms appears in the decomposition (i.e., $n_k = \overline{n}$), the claim is trivial because each of the objects appearing is Tate. In fact, the only morphism in $\mathrm{End}(M(X))$ which vanishes in $\mathrm{End}(M(X) \otimes \overline{k})$ is the 0 morphism. Thus the claim is valid for $n_k = \overline{n}$.

Now reason by descending induction on $n = n_k$. (Properly speaking, we reason by ascending induction on $\overline{n} - n_k$ starting with the case $\overline{n} - n_k = 0$.) Let $f \in \mathrm{End}(M(X))$ be an endomorphism in the kernel of the map to $\mathrm{End}(M(X \otimes \overline{k}))$ and pick a point z in one of the anisotropic components Z_i appearing in (5.1). (If all components are isotropic, n is maximal and the claim is already proved.) Set $L = k(z)$. Over L, Z_i is isotropic. Therefore the number $n_i = n_L$ of terms appearing in the motivic decomposition of X_L is greater than n. Thus the claim holds for X_L and $f_L^{N(d,n_i)} = 0$. Since $N(d, n_i) \le N(d, n+1)$, it follows that $f_L^N = 0$ where $N = N(d, n+1)$.

I now use [7, Theorem 3.1 and Remark 3.2] in the form in which it was used by Gille, Chernousov and Merkurjev [9, Proposition 8.1].

Lemma 5.3. *Let X be a smooth, projective variety over a field k and Z an r-dimensional scheme of finite type over k. Let $f \in \operatorname{End}(M(X))$ be an endomorphism such that, for every point $z \in Z$, the morphism $f_{z*} : A_*(X \otimes k(z)) \to A_*(X \otimes k(z))$ vanishes. Then $f_*^{r+1} : A_*(X \times Z) \to A_*(X \times Z)$ vanishes.*

From the lemma and the fact that $\dim Z_i \le d$, it follows that the composition

$$M(Z_i)(a_i) \xrightarrow{j_i} M(X) \overset{f^{(d+1)N(d,n+1)}}{\rightarrow} M(X) \tag{5.2}$$

vanishes where the first arrow, j_i, is the canonical one coming from the coproduct decomposition. Thus, for each anisotropic Z_i in the coproduct, $f^{(d+1)N} \circ j_i = 0$. On the other hand, if $Z_i = \operatorname{Spec} k$, then it is easy to see that the composition in (5.2) vanishes even with $f^{(d+1)N(d,n+1)}$ replaced by f.

Since $N(d, n+1) = (d+1)^{\overline{n} - n_k - 1}$, the claim is proved.

Remark 5.4. Clearly the exponent $(d+1)^{\overline{n} - n_k}$ is not optimal.

6. The Weyl group and its double cosets

In this section and the next, I give an explicit description of the components Z_i appearing in the motivic decomposition (3.1) of a projective homogeneous variety X for an isotropic reductive group G. Roughly speaking, the geometric components are in one-to-one correspondence with certain double cosets of the Weyl group. The algebraic components correspond to equivalences classes of these double cosets under the so-called "*-action" of the absolute Galois group of the base field k (6.1).

While the language of schemes was used in the previous sections, in this section I abuse notation slightly (e.g. in the proof of lemma 6.2) and confuse points with k-valued points. This facilitates comparison with the reference [25] which is written in the language of varieties.

For $T \subset G$ a maximal (but not necessarily split) torus defined over k, set $W = W(G, T)$, the corresponding Weyl group. For a subtorus $C \subset T$, let $W_C = W(Z_G(C), T)$ where $Z_G(C)$ denotes the centralizer of C. It is a subgroup of W. Likewise, for a character $\phi : \mathbb{G}_m \to T$, let $W_\phi = W(Z_G(\phi), T)$, also a subgroup of W.

Now, in the situation of the §4, G has a cocharacter $\lambda : \mathbb{G}_m \xrightarrow{\cong} L \subset G$. We can assume that $L \subset S \subset T$ with S a maximal k-split torus and T a maximal torus with T defined over k [25, Theorem 13.3.6 and Remark 13.3.7].

If X is isotropic, there is a parabolic subgroup P such that $X = G/P$. Since G is reductive, we may assume that $P = P(\mu)$ for a cocharacter $\mu : \mathbb{G}_m \to S$. (Roughly, $P(\mu)$ is defined as the set of $g \in G$ such that $\lim_{t\to\infty} \mu(t)g\mu(t)^{-1}$ exits. See [25, §13.4.1].) Set

$$\mathcal{X} = W_\lambda \backslash W/W_\mu. \tag{6.1}$$

Theorem 6.1. *If k is separably closed, the connected components Z_i appearing in the motivic Białynicki-Birula decomposition of X are in one-one correspondence with the elements of $\mathcal{X}$.*

To begin the proof of the theorem, first note that, since k is separably closed, X is isotropic and $X = G/P$ with $P = P(\mu)$ as above. It follows that the maximal torus T acts on G/P with fixed points corresponding to cosets in

$$\mathcal{Y} = W/W_\mu. \tag{6.2}$$

To see this, suppose that $y \in G(k)$ is such that the coset yP is fixed by T. Then $y^{-1}Ty$ is a maximal torus contained in P. On the other hand, since $P = P(\mu)$ with $\mu : \mathbb{G}_m \to S \subset T$, T is also contained in P. Since all maximal tori are conjugate within P [1, Corollary 5.7, p. 496], there is a $p \in P(k)$ such that

$$p^{-1}Tp = y^{-1}Ty. \tag{6.3}$$

Thus yp^{-1} is in the normalizer $N_G(T)$ of T and, thus, represents an element $w \in W = W(G,T)$. If p' is another element of P satisfying (6.3), then $p'p^{-1}$ normalizes T. This implies that $p'p^{-1} \in Z_G(\mu)$ by the following.

Lemma 6.2. *For G reductive with maximal torus T and $\mu \in X_*(T)$, $P(\mu) \cap N_G(T) \subset Z_G(\mu)$.*

Proof. Take $p \in P(\mu) \cap N_G(T)$ and set $\beta(t) = p\mu(t)p^{-1}$. Since p normalizes T, $\beta \in X_*(T)$. Set $a(t) = \mu(t)p\mu(t)^{-1}$. Since $p \in P(\mu)$, $a := \lim_{t\to 0} a(t)$ exists. Therefore

$$\lim_{t\to 0} \beta(t)\mu(t)^{-1} = pa. \tag{6.4}$$

In particular, the limit in (6.4) exists. But, since $\beta(t)\mu(t)^{-1}$ is a cocharacter, this is only possible if $\beta(t) = \lambda(t)$. □

It follows, therefore, that y and equation (6.3) determine the class $\pi(y)$ of yp^{-1} in W/W_μ. Moreover, if $y' = yp'$ is another element of G representing the coset yP, then it is easy to check that $\pi(y) = \pi(y')$.

Thus, there is a map $\pi : (G/P)^T \to \mathcal{Y}$. It is not hard to see that the map $\tilde{s} : W \to (G/P)^T$ given by $w \mapsto wP$ induces a map $s : \mathcal{Y} \to (G/P)^T$ inverse to π. Thus $(G/P)^T \cong \mathcal{Y}$.

Now let yP and $y'P$ be two points in the same component Z of $(G/P)^\lambda$ which are both fixed by the T action. Without loss of generality, we can then assume that y and y' normalize T. By Proposition 4.3, $yP = hy'P$ for some $h \in Z_G(\lambda)$. It follows then that h also normalizes T and is, thus, in W_λ. Thus π induces a map

$$q : \pi_0((G/P)^\lambda) \to \mathcal{X}. \tag{6.5}$$

To see that q is an isomorphism, it suffices to check that $s : \mathcal{Y} \to (G/P)^T$ induces an inverse map $r : \mathcal{X} \to \pi_0((G/P)^\lambda$. I leave this verification, which completes the proof of Theorem 6.1, to the reader.

6.1. **The Galois action.** Let $\Gamma = \mathrm{Gal}(k_{\mathrm{sep}}/k)$ denote the absolute Galois group. If $X = G/P(\mu)$ is an isotropic projective homogeneous variety, then Γ acts on $G(k_{\mathrm{sep}})$, $T(k_{\mathrm{sep}})$ and $W(k_{\mathrm{sep}})$ stabilizing $P = P(\mu)$ and, thus, W_μ. It follows that Γ acts on the double coset space $\mathcal{X}$. Clearly Γ also acts on $(G/P)^\lambda(k_{\mathrm{sep}})$, and it is easy to see that the map $r : \mathcal{X} \to \pi_0(X^\lambda)$ is an isomorphism of Γ-sets.

Computing the Galois action on $\pi_0(X^\lambda)$ can be reduced to the case of isotropic X using the fact that every reductive k-group has an quasi-split inner form G_{inn} [25, Proposition 16.4.9] given by a class $\sigma \in H^1(\Gamma, G_{\mathrm{ad}})$ where G_{ad} denotes the adjoint group of G. Let $p : G \to G_{\mathrm{ad}}$ be the canonical quotient map. It is easy to see that we can arrange that T is stabilized and λ is fixed by σ. Then, in fact, σ is in the image of the map

$$H^1(\Gamma, Z_{G_{\mathrm{ad}}}(p \circ \lambda) \cap N_{G_{\mathrm{ad}}}(p(T))) \to H^1(\Gamma, G_{\mathrm{ad}}).$$

We have an action of G_{ad} on X and, thus, a twist X_σ of X with an action of λ. Under the twist, the action of the Galois group on $W_{\mathrm{inn}} = W(G_{\mathrm{inn}}, T_\sigma)$ is given by the $*$-action [26]. Since G_{inn} is quasi-split, X_σ is isotropic and thus corresponds to a parabolic $P(\mu)$ in G_{inn}. From the previous section, we then have

$$\pi_0(X_\sigma^\lambda) = W_\lambda \backslash W_{\mathrm{inn}} / W_\mu.$$

Proposition 6.3. *There is an isomorphism*

$$\pi_0(X^\lambda) \cong \pi_0(X_\sigma^\lambda) = W_\lambda \backslash W_{\mathrm{inn}} / W_\mu$$

of étale schemes over $\operatorname{Spec} k$. *(In other words, the above sets are isomorphic as* Γ*-sets.)*

Proof. It is easy to see that $\pi_0(X_\sigma^\lambda)$ viewed as an étale scheme over $\operatorname{Spec} k$ is the twist of $\pi_0(X^\lambda)$ by σ. Since $Z_{G_{\mathrm{ad}}}(p \circ \lambda)$ is geometrically connected, it acts trivially on the geometric points of $\pi_0(X^\lambda)$. Thus the two schemes are isomorphic. □

7. Explicit Description and Examples

With a little extra work, we can give an explicit description of the twists and the spaces Z_i appearing in Corollary 4.4 and Theorem 4.5 in terms of the relevant reflection groups, Dynkin diagrams and root systems. This is a generalization of the description appearing in [9].

From the previous section, we know that the quasi-homogeneous schemes Z_i are in correspondence with the orbits of the $*$-action on the double cosets in $W_\lambda \backslash W_{\text{inn}}/W_\mu$. Over k_{sep}, each such Z_i decomposes as a disjoint union Z_w over the elements of the $*$-orbit. Our goal is then to describe the Tate twist associated to Z_w and also the projective homogeneous space Z_w in terms of the Dynkin diagram of $Z(\lambda)$.

In obtaining our description, it will be convenient to consider the case where G is split first. Therefore, assume G is split with maximal torus T. Let R denote the set of roots of G. Choose a Borel subgroup B or, equivalently, a set R_+ of positive roots. Then $R = R_+ \cup R_-$ with R_- the negative roots. Let Σ denote the corresponding set of simple positive roots. The Weyl group $W = W(G,T)$ is then generated by the reflections s_α in the hyperplanes defined by the $\alpha \in \Sigma$. We let $\ell(w)$ denote the corresponding length function on W: $\ell(w)$ is the length l of a minimal expression $w = s_1 s_2 \cdots s_l$ of w in terms of the simple roots.

Now let X be a projective homogeneous variety. We have $X = G/P(\mu)$ for some cocharacter $\mu : \mathbb{G}_m \to T$ which is non-negative on R_+. (Any cocharacter can be conjugated to a non-negative one.) Let $J = \{\alpha \in \Sigma \mid \langle \alpha, \mu \rangle = 0\}$. Then W_μ is the subgroup of W generated by the s_α with $\alpha \in J$. Accordingly, we will also write W_J for this subgroup. Now, if G has a non-central cocharacter $\lambda : \mathbb{G}_m \to T$, there is one which is non-negative on Σ. Thus, setting $I = \{\alpha \in \Sigma \mid \langle \alpha, \lambda \rangle = 0\}$, we have $W_\lambda \backslash W / W_\mu = W_I \backslash W / W_J$. Note that the correspondence $X \rightsquigarrow J$ between isomorphism classes of projective homogeneous varieties for G and subsets $J \subset \Sigma$ is one-to-one and onto. We will call X the homogeneous variety associated to J, and we will call J the set of roots of X.

We now use the result of an exercises in Humphrey's book on reflection groups [14, Ex. 1 on p. 20].

Exercise 7.1. Any double coset in $W_I \backslash W / W_J$ has a unique element b of minimal length. The element b satisfies the following equivalent properties:

(1) $\ell(b s_\alpha) = \ell(b) + 1$ for $\alpha \in J$ and $\ell(s_\alpha b) = \ell(b) + 1$ for $\alpha \in I$.
(2) $b\alpha > 0$ for $\alpha \in J$ and $b^{-1}\alpha > 0$ for $\alpha \in I$.

Moreover, any element $w \in W$ may be written as

$$w = abc \tag{7.1}$$

with $a \in W_I$, $c \in W_J$ and $\ell(w) = \ell(a) + \ell(b) + \ell(c)$.

Solution. The equivalence of the two properties is Lemma 1.6 on p. 12 of [14]. Let b be an element of minimal length in the double coset. Then $\ell(b s_\alpha) \geq \ell(b)$ for all $\alpha \in J$, and, since lengths either go up by one or down by one upon multiplying by a reflection, this implies that $\ell(b s_\alpha) = \ell(b) + 1$ for all $\alpha \in J$. Similarly, $\ell(s_\alpha b) = \ell(b) + 1$ for $\alpha \in I$. Thus b satisfies both properties (1) and (2).

Now suppose $w \in W$. Write $w = abc$ with $\ell(a)$ and $\ell(c)$ minimal. We can write a, b and c out as reduced words in the simple reflections as follows:

$$a = r_1 r_2 \cdots r_l,$$
$$b = s_1 s_2 \cdots s_m,$$
$$c = t_1 t_2 \cdots t_n.$$

Since $a \in W_I$ (resp. $c \in W_J$) and the length function on W restricts to that on W_I (resp. W_J), we can assume that the $r_i's$ are in I and the $t_i's$ are in J. So $\ell(w) \leq l + m + n$ and, if $\ell(w) < l + m + n$, there must be a pair of reflections in the word for w that can be deleted [14, Theorem 1.7]. This pair cannot involve one of the $s_i's$ because otherwise b would not be minimal in the double coset. On the other hand, it cannot simply involve two of the $r_i's$ because then our word for a would not be reduced. Likewise it cannot simply involve two of the $t_i's$. Finally, the word cannot involve an r_i and a t_j because we assumed that $\ell(a)$ and $\ell(c)$ were minimal.

Now suppose b' were another coset representative for $W_I b W_J$ of minimal length. Then $b' = abc$ for a and c as above. But, since $\ell(b') = \ell(a) + \ell(b) + \ell(c)$, we must have $a = c = 1$. □

Remarks 7.2. (1) In contrast with the single coset case [14, Proposition 10.10], the elements a and c are not unique. This is clear, for example, if $W = \mathbb{Z}/2 \times \mathbb{Z}/2$ and $W_I = W_J$ is the first factor of $\mathbb{Z}/2$. (2) In the special case where $I = J$, this exercise is used in [9] (see Proposition 3.4). (3) What I have stated as the "exercise" is actually the solution to Humphrey's question which is whether or not a minimal element exists. A more complete version of the exercise is in Bourbaki [6, Exercise 1.3, p. 37].

For a subset $K \subset \Sigma$, let R_K denote the set of roots generated by K. Let $R_K^+ = R_K \cap R^+$ (resp. $R_K^- = R_K \cap R^-$). We can now give our explicit decomposition in the split case.

Proposition 7.3. *Let G be a split reductive k-group with maximal torus T and simple roots Σ. Let $X = G/P$ be a projective homogeneous variety for G with J the corresponding set of simple roots, let λ be a non-central cocharacter of G which is non-negative on Σ and vanishing precisely on $I \subset \Sigma$ and let E be the set of minimal length coset representatives of $W_I \backslash W / W_J$ with W_I and W_J as above. Then, under the Białynicki-Birula decomposition for λ,*

$$M(X) = \coprod_{w \in E} M(Z_w)(\ell(w))$$

with Z_w the orbit of wP under $Z(\lambda)$. The set $I \subset \Sigma$ is the set of simple roots of $Z(\lambda)$. Moreover, the roots of Z_w are

$$J_w = \{\alpha \in I \mid w^{-1}\alpha \in R_J\}.$$

Proof. First note that, if $w \in E$ and $\alpha \in I$, $w^{-1}\alpha \in R^+$. This is condition (2) of the exercise. Now the twist a_w associated to the motive $M(Z(\lambda)wP)$ in the motivic decomposition of Corollary 4.4 is the rank of the positive weight space of λ on $T(G/P)_{wP}$. This is the same as the rank r of the positive weight space of $\mathrm{Ad}(w^{-1})\lambda$ on $T(G/P)_P$. Now $T(G/P)_P$ is naturally identified with $L(G)/(kR_J^- \oplus L(T) + kR^+)$. (Here I write kR for the free vector space on the set R and view it as a subspace of $L(G)$ in the natural way.) It follows that r is the number of negative roots α not in R_J^- such that $\langle w^{-1}\lambda, \alpha\rangle = \langle \lambda, w\alpha\rangle > 0$. Thus

$$\begin{aligned} r &= \#\{\alpha \in R^- - R_J \,|\, w\alpha \in R^+ - R_I\} \\ &= \#(R^+ - R_I^+) \cap w(R^- - R_J^-) \\ &= \#(R^+ \cap wR^- - R^+ \cap wR_J^- - R_I^+ \cap wR^-). \end{aligned}$$

Now $R^+ \cap wR_J^-$ and $R_I^+ \cap wR^-$ are both empty by part (2) of the exercise. So

$$\begin{aligned} r &= \#(R^+ \cap wR^-) \\ &= \ell(w). \end{aligned}$$

Determining the roots of Z_w is now easy. We have

$$Z_w = Z(\lambda)/(Z(\lambda) \cap wPw^{-1}).$$

Set $P_w = Z(\lambda) \cap wPw^{-1}$. By definition, P_w is a parabolic subgroup of $Z(\lambda)$. Note that P_w is actually a *standard parabolic subgroup* in that it contains the Borel subgroup $B_\lambda = B \cap Z(\lambda)$ where B is the standard Borel subgroup such that $L(B) = kR^+$. This follows from the fact that $w^{-1}R_I^+ \subset R^+$. Now, $L(P) = kR_+ \oplus L(T) \oplus kR_J^-$ and $L(Z(\lambda)) = kR_I^+ \oplus L(T) \oplus kR_I^-$. Thus, using the exercise, we have

$$\begin{aligned} L(P_w) &= (kR_I^+ \oplus L(T) \oplus kR_I^-) \cap w(kR_+ \oplus L(T) \oplus kR_J^-) \\ &= kR_I^+ \oplus L(T) \oplus (kR_I^- \cap wkR_J^-) \\ &= kR_I^+ \oplus L(T) \oplus (kR_I^- \cap wkR_J). \end{aligned}$$

Now it follows that

$$J_w = \{\alpha \in I \,|\, w^{-1}\alpha \in R_J\}.$$

□

7.1. **The Tits Index.** To every reductive k-group G, J. Tits has associated a part-algebraic, part-combinatorial object known as the *Tits index* of G. It consists of the Dynkin diagram for G, a graph with vertices Σ corresponding to the simple roots of a given maximal k-torus, together with a Galois action on the vertices preserving a set Σ_0 of distinguished vertices. The subset Σ_0 consists of the simple roots orthogonal to a maximal k-split torus. Thus Σ_0 is the set of roots of the semi-simple anisotropic kernel. The group $\mathrm{Gal}(k_{\mathrm{sep}}/k)$ acts on Σ via the $*$-action stabilizing Σ_0. When drawing the diagram, the

Galois orbits of the vertices not in Σ_0 are circled and vertices in the same $*$-orbit are supposed to be drawn "close together."

Now, in this picture, projective homogeneous varieties (i.e., k-defined conjugacy classes of parabolics) are in one-to-one correspondence with $*$-invariant subset J of Σ [26, (2.5.4)]. Isotropic projective homogeneous varieties (i.e., conjugacy classes containing a k-defined parabolic) are in one-to-one correspondence with $*$-invariant subsets J of Σ containing Σ_0. Moreover, it is easy to see that for every $*$-invariant subset I of Σ containing Σ_0, there is a k-defined cocharacter λ of T which vanishes on I but is positive on $\Sigma - I$. In this case, $Z(\lambda)$ is the Levi component L_I of the parabolic subgroup P_I associated to I. (The necessary argument is given in the proof of [25, Lemma 15.1.2]).

We now have the following result which follows from the previous proposition by standard methods of descent.

Theorem 7.4. *Suppose I and J are $*$-invariant subsets of Σ with $I \supset \Sigma_0$. Let E be the set of minimal length coset representatives for $W_I \backslash W_{\text{inn}} / W_J$, and let $\overline{E}$ be the set of orbits of E under the $*$-action. Let X be the projective homogeneous variety associated to J. We have*

$$M(X) = \coprod_{\overline{w} \in \overline{E}} M(Z_{\overline{w}})(\ell(w))$$

where $Z_{\overline{w}}$ is a projective quasi-homogeneous scheme for the reductive group L_I $(= Z(\lambda))$. Moreover, the base change of $Z_{\overline{w}}$ to k_{sep} is a disjoint union

$$Z_{\overline{w}} \otimes k_{\text{sep}} = \coprod_{w \in \overline{w}} Z_w$$

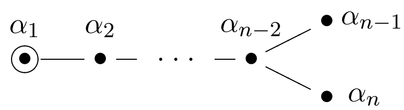
where Z_w is the projective homogeneous variety for $L_I \otimes k_{\text{sep}}$ corresponding to the subset

$$J_w = \{\alpha \in I \,|\, w^{-1}\alpha \in R_J\}.$$

Remark 7.5. If the $*$-orbit $\overline{w}$ consists of one element w, then $Z_{\overline{w}}$ is the projective homogeneous variety corresponding to J_w. In particular, its structure as a k-variety is determined by the combinatorics. At any rate, since $Z_{\overline{w}}$ is a projective quasi-homogeneous variety, it is a subvariety of the variety of parabolics of the reductive group L_I.

Example 7.6. We now return to the example of (4.7). Here we have $G = PSO(q)$ with $q = x_1^2 + \cdots + x_{2n-2}^2 + yz$ (with $k = \mathbb{R}$) and X the projective quasi-homogeneous scheme of two-dimensional subspaces. As long as $n \geq 3$, X is a projective homogeneous variety for G. (When $n = 2$, X has two geometric components.)

Now the Dynkin diagram of G (decorated as in the Tits index) is the following picture.



The $*$-action exchanges α_{n-1} with α_n and leaves all other roots fixed. When $n \geq 4$, the set $J \subset \Sigma$ corresponding to X is $\Sigma - \{\alpha_2\}$. For $n = 3$, $J = \Sigma - \{\alpha_2, \alpha_3\}$. We assume $n \geq 4$ at first and sketch the case where $n = 3$ (where the $*$-action plays a significant role) at the end of this example.

If we set $I = \Sigma_0 = \Sigma - \{\alpha_1\}$, then we are in the setting of Theorem 7.4. Write $s_i = s_{\alpha_i}$ for the generators of the Weyl group. From the theory of Coxeter complexes ([14, § 1.15]), we see that $W_I \backslash W$ is identified with the set of vectors of the form $\pm e_i$ in the real vector space $\mathbb{R}^n = \mathbb{R}e_1 + \cdots + \mathbb{R}e_n$. The action of W on the right is given by $e_i s_j = e_{(j,j+1)i}$ for $1 \leq j < n$ where $(j, j+1)$ denotes the transposition in the symmetric group exchanging j and $j+1$. For $j = n$, we have $e_i s_n = -e_{(n-1,n)i}$. Now it is fairly easy to see that the cosets in $W_I \backslash W$ containing elements of E are $e_1 = W_I 1$, $e_3 = W_I s_1 s_2$, and $-e_2 = W_I s_1 s_2 \cdots s_{n-2} s_{n-1} s_n s_{n-2} \cdots s_3 s_2$. It is also easy to see that the representatives listed are in fact the elements of E. That is,

$$E = \{1, s_1 s_2, s_1 s_2 \cdots s_{n-2} s_{n-1} s_n s_{n-2} \cdots s_2\}.$$

Writing w_1, w_2 and w_3 for the elements listed in order, we have

$$\ell(w_1) = 0, \ell(w_2) = 2, \ell(w_3) = 2n - 3.$$

Clearly $J_{w_1} = I \cap J = \{\alpha_3, \ldots, \alpha_n\}$. To compute J_{w_2}, note that

$$\begin{aligned} w_2^{-1}\alpha_2 &= s_2 s_1 \alpha_2 \\ &= s_2(\alpha_1 + \alpha_2) \\ &= (\alpha_1 + \alpha_2) - \alpha_2 \\ &= \alpha_1 \in J. \end{aligned}$$

Thus $\alpha_2 \in J_{w_2}$. A similar computation shows that, for $i > 2$, $w_2^{-1}\alpha_i \in J_{w_2}$ if and only if α_i is not connected to α_2 in the Dynkin diagram. Thus, for $n \geq 5$, $J_{w_2} = \{\alpha_2, \alpha_4, \ldots, \alpha_n\}$. However, for $n = 4$, $J_{w_2} = \{\alpha_2\}$.

Finally, to compute J_{w_3}, note that $w_3^{-1}\alpha_i = \alpha_i$ for $2 < i \leq n-2$, $w_3^{-1}\alpha_2 = \alpha_1 + \alpha_2$, $w_3^{-1}\alpha_{n-1} = \alpha_n$ and $w_3^{-1}\alpha_n = \alpha_{n-1}$. It follows that $J_{w_3} = J_{w_1} = I - \{\alpha_2\}$.

Putting all of this together, we have the decomposition

$$M(X) = M(Q) \oplus M(Y)(2) \oplus M(Q)(2n-3) \tag{7.2}$$

where Q is the motive of a quadric of isotropic lines for the quadratic form $q' = x_1^2 + \cdots + x_{2n}^2$, and Y is isomorphic to the space of isotropic planes for q'.

When $n = 3$, $W_I \backslash W$ (viewed in terms of the Coxeter complex) has four W_J orbits containing the minimal W_I-cosets

$$\begin{aligned} e_1 &= W_I w_1, \quad w_1 = 1; \\ e_3 &= W_I w_2, \quad w_2 = s_1 s_2; \\ -e_3 &= W_I w_3, \quad w_3 = s_1 s_3; \\ -e_2 &= W_I w_r, \quad w_4 = s_1 s_2 s_3. \end{aligned}$$

The elements w_i are written in reduced form so we have $\ell(w_1) = 1, \ell(w_2) = \ell(w_3) = 2$ and $\ell(w_4) = 3$. Note that w_2 and w_3 are conjugate under the $*$-action. Clearly $J_{w_1} = \emptyset$ and some computation shows that $J_{w_4} = \emptyset$ as well. On the other hand, $J_{w_2} = \alpha_2$ while $J_{w_3} = \alpha_3$.

It turns out then that $Z_{\overline{w_1}} = Z_{\overline{w_4}} = Q$ and $Z_{\overline{w_3}} = Y$ with Q and Y as in 7.2. In other words, we obtain the same decomposition as in the case $n = 4$. However, we learn that $Y \otimes \mathbb{C} = Z_{w_2} \coprod Z_{w_3}$ with $Z_{w_2} = Z_{w_3} = \mathbb{P}^1$.

It is also possible (and perhaps easier) to work out (7.2) directly using the geometry of the $\mathbb{G}_m$-action on X and the weight decomposition of TX at the various fixed loci as suggested in Exercise 4.7.

8. Acknowledgments

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