# FIXED POINTS, LOCAL MONODROMY, AND INCOMPRESSIBILITY OF CONGRUENCE COVERS

PATRICK BROSNAN AND NAJMUDDIN FAKHRUDDIN

Abstract. We prove a fixed point theorem for the action of certain local monodromy groups on étale covers and use it to deduce lower bounds in essential dimension. In particular, we give more geometric proofs of many (but not all) of the results of the preprint of Farb, Kisin and Wolfson, which uses arithmetic methods to prove incompressibility results for Shimura varieties and moduli spaces of curves. Our method allows us to prove results for exceptional groups, and also for the reduction modulo good primes of Shimura varieties and moduli spaces of curves.

Contents

## 1. Introduction

The purpose of this paper is to prove incompressibility results and lower bounds on essential dimension using a new method for producing fixed points, which we formulate in Proposition 16 and Theorem 18 below. This recovers many, but not all, of the incompressibility results obtained using arithmetic methods in the preprint [FKW19] of Farb, Kisin and Wolfson. For example, we get geometric proofs of the incompressibility of congruence covers related to the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties (Corollary 41) and the moduli space $\mathcal{M}_g$ of smooth genus $g$ curves (Theorem 23). Our method also allows us to prove new results including an extension of the theorems of [FKW19] on incompressibility of congruence covers of locally symmetric varieties associated to certain groups of type $E_7$. In contrast to the method of [FKW19], our approach also allows us to prove incompressibility of congruence covers of both $\mathcal{A}_g$ and $\mathcal{M}_g$ over fields of positive characteristic.

1.1. **Essential dimension.** Essential dimension is a numerical measure of the complexity of algebraic objects which first appeared in a paper by J. Buhler and Z. Reichstein [BR97] from 1997. Since then, many different, but equivalent, ways of looking at it have arisen. To explain our results in more detail and to fix terminology, we give a quick review of the concepts of incompressibility and essential dimension here in the introduction. However, for more details, we refer the reader to §2 and to [BR97, FKW19, Mer13].

Let $k$ be any field. By a variety over $k$ we shall mean a reduced, separated scheme of finite type. If $f : X \to Y$ is a generically étale morphism of varieties with $Y$ integral, then the *essential dimension* of $f$, $\operatorname{ed} f$, is defined to be the minimum of the dimensions of irreducible varieties $Y'$ such that the following conditions hold:

(1) There is a dominant rational map $Y \dashrightarrow Y'$ and a finite étale morphism of schemes $X' \to Y'$.
(2) Over the generic point of $Y$, the morphism $X \to Y$ is the pullback of the morphism $X' \to Y'$.

We will often abuse notation slightly and write $\operatorname{ed}(X \to Y)$, or even $\operatorname{ed} X$, instead of $\operatorname{ed} f$. For any prime number $p$, we define $\operatorname{ed}(f; p)$ (or $\operatorname{ed}(X; p)$), the *essential dimension at $p$* of $f$, to be the minimum of $\operatorname{ed} g$, where $g$ ranges over all morphisms of the form

$g : X \times_Y Z \to Z$, with $Z$ integral and $Z \to Y$ a generically finite dominant morphism of degree prime to $p$ such that $X \times_Y Z$ is reduced. (The above definitions can be extended to allow reducible $Y$ by setting $\operatorname{ed} f = \max_i \operatorname{ed}(X \times_{Y_i} Y \to Y_i)$, where the $Y_i$ run over the irreducible components of $Y$, and similarly for $\operatorname{ed}(f; p)$).

If $G$ is a finite group, $Y$ is an integral variety and $f : X \to Y$ is a $G$-torsor, then $f$ is finite étale and the above definitions apply to $X$. A $G$-torsor is said to be *incompressible* if $\operatorname{ed} X = \dim X$ and *$p$-incompressible* if $\operatorname{ed}(X; p) = \dim X$. The essential dimension $\operatorname{ed} G$ of $G$ is defined to be the maximum of $\operatorname{ed} X$ over all $G$-torsors $X$ as above and we define $\operatorname{ed}(G; p)$ analogously.

1.2. **Incompressibility results.** The recent paper of Farb, Kisin and Wolfson [FKW19] mentioned above proves the incompressibility of a large class of congruence covers of Shimura varieties. For example, let $\mathcal{A}_{g,N}$ denote the moduli space of principally polarized abelian varieties of dimension $g$ with (symplectic) level $N$-structure [FC90, Definition I.4.4, p. 19]. If $N \geqslant 3$ and if $p$ is a prime number not dividing $N$, then the natural morphisms $\mathcal{A}_{g,pN} \to \mathcal{A}_{g,N}$ is an $\mathbf{Sp}_{2g}(\mathbb{F}_p)$-torsor. One of the key theorems of [FKW19] is the $p$-incompressibility of $\mathcal{A}_{g,pN}$ or, equivalently, of the cover $\mathcal{A}_{g,pN} \to \mathcal{A}_{g,N}$.

The results of [FKW19] were proved using arithmetic methods involving reduction modulo $p$ and a subtle argument involving Serre–Tate coordinates. Their results go far beyond the case of $\mathcal{A}_{g,pN}$, proving incompressibility for congruence covers of many interesting varieties related to $\mathcal{A}_g$ such as the moduli space of curves $\mathcal{M}_g$ and a large class of Hodge type Shimura varieties (and even certain subvarieties thereof).

Following a suggestion of Z. Reichstein, our initial goal in this paper was to recover the $p$-incompressibility of $\mathcal{A}_{g,pN}$ using an elementary (and, by now, standard) geometric criterion for incompressibility, which we refer to as *the fixed point method*. As we explain below, the fixed point method allows us not only to recover some of the results of [FKW19] for $\mathcal{A}_g$ but also to prove several new results.

We will give a stronger statement (Proposition 10) of the fixed point method in §2. However, for the purposes of this introduction it boils down to the following fact: Suppose $G$ is a finite group, $X$ is a $G$-torsor over an irreducible variety $Y$, and $\overline{X}$ is a $G$-equivariant compactification of $X$. If a finite abelian $p$-subgroup $H \leqslant G$ of rank $\dim X$ has a smooth fixed point on $\overline{X}$, then $X$ is $p$-incompressible.

In §3, we prove a criterion, Proposition 16, for the existence of fixed points in a compactification of a variety $X$, which is presented as a finite étale Galois cover $f : X \to Y$ with Galois group $G$. This criterion, which depends on a partial compactification $\overline{Y}$ of $Y$ and the local monodromy of the cover $f$ on a partial toroidal resolution $S$ of $\overline{Y}$, leads to Theorem 18, which combines Proposition 16 with the fixed point method to give lower bounds on essential dimension based on local monodromy. Applying Theorem 18 to various situations, e.g., where $Y$ is the moduli space $\mathcal{M}_g[N]$ of genus $g$ curves with (symplectic) level $N$ structure or where $Y$ is $\mathcal{A}_{g,N}$ or, more generally, a locally symmetric variety, allows us to prove lower bounds on essential dimension and, in many cases, even incompressibility results.

1.3. **Contents of the paper.** We now explain the incompressibility results in more detail along with the topics of the sections in the paper. Section 2 briefly reviews essential dimension and the notion of versal and $p$-versal torsors. It also proves the version of the fixed point method, Proposition 10, mentioned above. Most of this section is due to Z. Reichstein, and we are grateful to him for allowing us to use it.

Section 3 proves our main general result on essential dimension and local monodromy, Theorem 18. The rest of the paper consists essentially of examples. Section 4 proves the incompressibility of two types of covers related to the moduli space $\mathcal{M}_g[N]$ of genus $g$ curves with (symplectic) level $N$ structure. The first result, Theorem 23, recovers the $p$-incompressibility of the cover $\mathcal{M}_g[pN] \to \mathcal{M}_g[N]$, which was proved by arithmetic means in [FKW19, Theorem 3.3.2]. Our proof is characteristic free and works over all fields of characteristic not dividing $pN$. The second result, Theorem 30, proves the incompressibility of certain "quantum covers" of $\mathcal{M}_g[N]$ arising from the TQFTs constructed in [BHMV95].

Section 5 proves our main general results on the essential dimension of congruence covers of locally symmetric varieties. We begin by recalling the notion of a locally symmetric variety in § 5.1 essentially following the terminology of Ash, Mumford, Rapoport and Tai [AMRT10]. These are quotients $\Gamma\backslash D$ of Hermitian symmetric domains by arithmetic subgroups $\Gamma$. (See Remark 36 for an explanation of why we prefer to use the language of locally symmetric varieties rather than the closely related language of Shimura varieties in the context of this paper.)

The main theorems of Section 5 are Theorem 33, which proves a general lower bound on the essential dimension of congruence covers of locally symmetric varieties in terms of boundary components, and Theorem 34, which deals with the special case where $D$ is a tube domain with a zero dimensional rational boundary component. In this special tube domain case, our results often imply $p$-incompressibility. Both theorems are proved by applying our main theorem on essential dimension and local monodromy, Theorem 18, to the case where $X \to Y$ is a congruence cover of locally symmetric varieties and $\overline{Y}$ is the Baily–Borel compactification of $Y$. In Corollary 41 of §5.2, we apply Theorem 34 to recover the result from [FKW19] on the incompressibility of the congruence cover $\mathcal{A}_{g,pN} \to \mathcal{A}_{g,N}$. We also extend this result to fields of positive characteristic in §5.6 using the integral toroidal compactifications of Faltings and Chai [FC90].

Section 6 considers the problem of constructing incompressible congruence covers of locally symmetric varieties analogous to the congruence covers of modular curves with full level structure. These are the covers that are called *principal $p$-coverings* in [FKW19], and much of what we do in Section 6 is motivated by similar considerations in [FKW19]. However, owing to our methods, which depend on boundary components, we have to deal with issues not present in [FKW19]. Our main result is Theorem 67. As an application, we can produce $p$-incompressible congruence covers of certain locally symmetric varieties of type $E_7$ and with Galois group $\mathbb{G}(\mathbb{F}_{p^n})$, where $\mathbb{G}$ is the simply connected form of $E_7$ over $\mathbb{F}_p$. We also recover most of the results of [FKW19] on existence of $p$-incompressible principal $p$-covers for classical groups. The notable exception is when the group is of type $A_n$ for $n$ even. See Remark 68.

There are two appendices in which we reproduce (with permission) results shown to us by other people. Appendix A gives an argument by M. Nori, which proves a weak version of Conjecture 1 below on essential dimension of variations of Hodge structure. Appendix B gives a proof, due to Dave Benson, that the essential dimension at $p$ of $\mathbf{Sp}_{2g}(\mathbb{F}_p)$ is $p^{g-1}$.

This shows that there is a large (in fact, exponential) difference between the essential dimension of congruence covers $\mathcal{A}_{g,pN} \to \mathcal{A}_{g,N}$, which are very particular $\mathbf{Sp}_{2g}(\mathbb{F}_p)$-torsors, and the essential dimension of the versal torsor, which is $p^{g-1}$. (See also Corollary 7 for a general lower bound on essential dimension of groups with non-abelian $p$-Sylow subgroups.)

Jesse Wolfson has informed us that his student, Hannah Knight, has also, independently, computed $\mathrm{ed}(\mathbf{Sp}_{2g}(\mathbb{F}_p); p)$. Moreover, she has made significant progress towards the goal of computing $\mathrm{ed}(G; p)$ for much more general finite quasisimple groups of Lie type. In particular, she has computed it for groups $G = \mathbf{Sp}_{2g}(\mathbb{F}_{p^r})$ for $r > 1$ as well as for analogous orthogonal groups.

1.4. **Comparison with work of Farb, Kisin and Wolfson.** The methods of this paper and of [FKW19], although very different, have a key point in common, i.e., the use of elementary abelian $p$-groups contained in the Galois group of the congruence covers. We use these subgroups in a very direct way via the fixed point method, whereas their appearance in [FKW19] is slightly indirect, as wild ramification groups of degenerations of congruence covers. These different ways of exploiting such subgroups accounts, to a large extent, for the fact that although the results of both papers have a large intersection, there are results provable by each method not accessible to the other.

For example, the methods of [FKW19] apply to many congruence covers of Shimura varieties whose connected components are compact. As these congruence covers are étale (at least when the congruence subgroups involved are neat, see §5.3), the fixed point method cannot possibly apply (as there are no fixed points). On the other hand, the methods of [FKW19] rely on embedding Shimura varieties in $\mathcal{A}_g$ and the fact that $\mathcal{A}_g$ has a good integral model over which the congruence covers degenerate (at primes dividing the level). Therefore, they apply essentially only to Hodge type Shimura varieties, and, for example, not to those of type $E_7$ where our methods yield new incompressibility results. Furthermore, they do not apply to $\mathcal{A}_g$ and $\mathcal{M}_g$ over fields of positive characteristic, since the congruence covers over such a field cannot have unequal characteristic degenerations.

It would be interesting to prove $p$-incompressibility for locally symmetric varieties associated to groups of type $E_6$. Since such varieties are not of Hodge type, the methods of [FKW19] do not apply, but, since they are not quotients of tube domains, our methods also do not suffice to prove incompressibility (Remark 68(3)).

1.5. **A conjecture.** Recall from [Del79, §1] that Hermitian symmetric domains are special examples of the period domains studied in Hodge theory. Given any integral variation of Hodge structure $\mathbb{H}$ on a smooth variety $B$, we get an associated period map $\varphi : B^{\mathrm{an}} \to \Gamma\backslash D$ where $D$ is a period domain, or, more generally, a Mumford–Tate domain, with generic Mumford–Tate group $\mathbf{G}$, an algebraic group over $\mathbb{Q}$, and $\Gamma \leqslant \mathbf{G}(\mathbb{Q})$ is an arithmetic lattice depending on the monodromy. See the paper [BBT18] of Bakker, Brunebarbe and Tsimerman for this notation and for their main theorem, which states that the image $Y$ of $\varphi$ has the structure of a quasiprojective complex variety. We call $Y$ the *image of the period map.*

Combining the results obtained by our methods with those of [FKW19] along with some wishful thinking leads us to guess that the dimension of the image of the period map should bound the essential dimension of congruence covers from below. To make this explicit, suppose $\mathbb{H}$ is a torsion-free variation of Hodge structure on a smooth complex variety $B$. Write $\mathbb{H}_{\mathbb{Z}}$ for the local system on $B$ corresponding to $\mathbb{H}$. Then, for each prime $p$, we get a family of local systems $\mathbb{H}_{\mathbb{Z}}/p^n$ of free $\mathbb{Z}/p^n$-modules over $B$. Moreover, the étalé spaces of the sheaves $\mathbb{H}_{\mathbb{Z}}/p^n$ are algebraic. So it makes sense to formulate:

**Conjecture 1.** Suppose $\mathbb{H}$ is a torsion-free integral variation of Hodge structure on a smooth irreducible complex variety $B$. Let $Y$ be the image of the period map and set $d = \dim Y$. There exists an integer $N$ such that, if $p$ is a prime number and $n$ is a nonnegative integer with $p^n \geqslant N$, then $\operatorname{ed}(\mathbb{H} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \to B; p) \geqslant d$.

We could modify the conjecture by replacing $\operatorname{ed}(\mathbb{H} \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \to B; p)$ with $\operatorname{ed}(\mathbb{H}_{\mathbb{Z}} \to B)$ suitably defined. In Appendix A, we give a precise formulation and a proof, due to M. Nori, of this modified statement. This is our main evidence for the validity of the above conjecture beyond the case where the period domain is Hermitian symmetric.

1.6. **General notation and notational conventions.** We try to maintain the convention of writing algebraic groups $\mathbf{G}$ in boldface and abstract groups $G$, as well as Lie groups, in non-bold. For an algebraic group $\mathbf{G}$ over a subring of $\mathbb{R}$, we write $\mathbf{G}(\mathbb{R})_+$ for the connected component of the identity of the Lie group $\mathbf{G}(\mathbb{R})$. If $\mathbf{G}$ is defined over $\mathbb{Q}$, then we write $\mathbf{G}(\mathbb{Q})_+ := \mathbf{G}(\mathbb{R})_+ \cap \mathbf{G}(\mathbb{Q})$. In §5, $\mathbf{G}$ is usually an adjoint group over $\mathbb{Q}$, but, in §6, $\mathbf{G}$ is usually taken to be simply connected with adjoint group $\mathbf{G}^{\mathrm{ad}}$.

We warn the reader that we have reversed what seems to be the usual convention of writing stacks such as the moduli stacks of curves or principally polarized abelian varieties in calligraphic script and the associated course moduli spaces in roman font. Fortunately, there are very few stacks in the paper and, with regard to the above moduli stacks, we are usually taking a large enough level structure so that the stack and the space coincide. Still, we apologize in advance if this causes confusion.

1.7. **Acknowledgements.** As we already mentioned, this work owes its existence to a suggestion from Zinovy Reichstein. We are very grateful to him for this suggestion and for many other smaller, but still very significant, contributions he generously made to this paper.

Brosnan would also like to thank Michael Rapoport for several useful discussions about Shimura varieties and toroidal compactifications and Dave Benson for showing us how to compute the essential dimension of the group $\mathbf{Sp}_{2g}(\mathbb{F}_p)$. He would like to thank Jesse Wolfson for several useful suggestions and encouraging emails as well as Burt Totaro for typo corrections. Moreover, he thanks the Isaac Newton Institute for hosting the workshop where the conversations with Benson took place in January of 2020 and the Simons Foundation for a Collaboration Grant, which helped make it possible to travel before the lockdown of March 2020. Fakhruddin would also like to thank Gregor Masbaum for useful correspondence on TQFTs, Arvind Nair for useful conversations on Hermitian symmetric domains, and Madhav Nori for useful discussions related to Conjecture 1. He was supported by the DAE, Government of India, under project no. RTI4001.

## 2. Essential dimension, versality and the fixed point method

2.1. **Essential dimension of $G$-varieties.** Let $G$ be a finite group and $k$ be a base field. In §1, we have defined the essential dimension $\operatorname{ed} X$ of a $G$-torsor $X \to Y$ and also its essential dimension at $p$, $\operatorname{ed}(X;p)$. We have also defined $\operatorname{ed} G$, the essential dimension of $G$, and $\operatorname{ed}(G;p)$, the essential dimension at $p$ of $G$. In general, these numbers depend on the field $k$, so we sometimes write $\operatorname{ed}_k(G)$ and $\operatorname{ed}_k(G;p)$ to emphasize this.

If $X \to Y$ is an irreducible $G$-torsor, then $X$ is also an $H$-torsor for any subgroup $H$ of $G$ (with base $H\backslash X$), so we may also consider the essential dimension of $X$ as an $H$-torsor. In this case, we shall write $\operatorname{ed}_G(X)$ and $\operatorname{ed}_H(X)$ if there is any risk of confusion.

The following lemma is the analogue of [BR97, Lemma 2.2] for $G$-torsors and is very similar to [FKW19, Lemma 2.1.4].

**Lemma 2.** *Let $X \to Y$ be a $G$-torsor (with $Y$ integral), $Y'$ an integral variety with a dominant rational map $\phi : Y \to Y'$, and $X' \to Y'$ a finite étale morphism such that over the generic point of $Y$, $X'$ is equal to $Y \times_{Y'} X'$. Then there exists a finite étale cover $Y''$ of $Y$ such that the rational map $Y \dashrightarrow Y'$ lifts to $Y''$ and there is a structure of $G$-torsor on $X'' = Y'' \times_{Y'} X' \to Y''$ such that the induced rational map $X \to X''$ is $G$-equivariant.*

We omit the proof since it is essentially the same as the proof of [FKW19, Lemma 2.1.4].

**Remark 3.** Suppose $X$ is an irreducible $G$-torsor and $H \leqslant G$. Then $\operatorname{ed}_H(X) \leqslant \operatorname{ed}_G(X)$ and $\operatorname{ed}_H(X;p) \leqslant \operatorname{ed}_G(X;p)$ for every prime $p$. This follows from Lemma 2 since if we have $Y', \phi$ as in the lemma and $X'$ is a $G$-torsor then we may set $Z$ to be $X/H$, $Z'$ to be $X'/H$, and then the rational map $Y \dashrightarrow Y'$ induces a rational map $Z \to Z'$ such that $X$ is equal to $Z \times_{Z'} X'$ over the generic point of $Z$.

**Remark 4.** Let $k$ be a field of characteristic $\neq p$ containing a primitive $p$th root of unity, and suppose $G$ is a finite $p$-group. By a theorem of Karpenko and Merkurjev, $\operatorname{ed} G = \operatorname{ed}(G;p)$ is the smallest dimension of a faithful linear representation of $G$ defined over $k$; see [KM08, Theorem 4.1].

Of particular interest to us will be the case where $G = (\mathbb{Z}/p)^r$ is an elementary abelian group of rank $r$. Here $\operatorname{ed} G = \operatorname{ed}(G;p) = r$. This special case predates the Karpenko–Merkurjev theorem and is considerably easier to prove; see, [Rei10, Example 2.6] or [Mer13, Example 3.5].

Now suppose $k$ is an arbitrary field of characteristic $\neq p$ and let $k'$ be the field obtained from $k$ by adjoining a primitive $p$th root of unity. Since $[k' : k]$ is prime to $p$, we have $\operatorname{ed}_k(G;p) = \operatorname{ed}_{k'}(G;p)$, see [KM08, Remark 4.8]. In particular, $\operatorname{ed}(\mathbb{Z}/p)^r \geqslant \operatorname{ed}((\mathbb{Z}/p)^r;p) = r$ over any field $k$ of characteristic $\neq p$. ♠

2.2. **Versality.** In this section it will be convenient for us to also consider irreducible varieties $X$ with faithful $G$-actions which are not free. Since $G$ is finite, $X$ always has a dense affine open $G$-invariant subset $U$ on which $G$ acts freely. Then $U \to U/G$ is a $G$-torsor and we set $\operatorname{ed} X := \operatorname{ed} U$ and $\operatorname{ed}(X;p) := \operatorname{ed}(U;p)$. It is easy to see that this is independent of the choice of $U$.

We say that an irreducible $G$-variety $V$ is *weakly versal* (respectively, *weakly $p$-versal*) if, for every $G$-torsor $X \to Y$, with $Y$ integral, there exists a $G$-equivariant rational map $X \dashrightarrow V$ (respectively a $G$-equivariant correspondence $X \rightsquigarrow V$ of degree prime to $p$). Here $p$ is a fixed prime number and by a $G$-equivariant correspondence $X \rightsquigarrow V$ of degree prime to $p$ we mean a dominant morphism $Y' \dashrightarrow Y$ of degree prime to $p$ together with a $G$-equivariant rational map $Y' \times_Y X$ to $V$. We say that $V$ is *versal* (respectively, *$p$-versal*) if every dense open $G$-invariant subvariety $V_0 \subset V$ is weakly versal (respectively, weakly $p$-versal). Note that versality and $p$-versality are birational properties of $V$, whereas weak versality and weak $p$-versality are not.

**Theorem 5** (Duncan–Reichstein)**.** *Let $G$ be a finite $p$-group and $X$ an irreducible $G$-variety over a base field $k$. If $X$ has a smooth $G$-fixed $k$-point, then $X$ is $p$-versal.*

*Proof.* See [DR15, Corollary 8.6]. ♠ ♠

**Lemma 6.** *Let $G$ be a finite group, $X$ an irreducible $G$-variety and $p$ a prime integer.*

*(a) If $X$ is versal, then $\operatorname{ed} X = \operatorname{ed} G$.*

*(b) If $X$ is $p$-versal, then $\operatorname{ed}(X; p) = \operatorname{ed}(G; p)$.*

*Proof.* (a) We need to show that $\operatorname{ed} X' \leqslant \operatorname{ed} X$ for every $G$-torsor $X' \to Y'$. Let $U$ be a $G$-invariant affine open subset of $X$ on which $G$ acts freely. By shrinking $U$ if necessary, we may find a morphism $\alpha : U/G \to Z$ and a $G$-torsor $W \to Z$ such that $U = (U/G) \times_Z W$. Since $U$ is a $G$-invariant open subset of $X$, $U$ is weakly versal. Thus there exists a $G$-equivariant rational map $X' \dashrightarrow U$, equivalently a map $\beta : Y' \to U/G$ inducing an isomorphism $X' = Y' \times_{U/G} U$ . Composing $\beta$ and $\alpha$ we obtain a rational map $\gamma\colon Y' \dashrightarrow Z$ such that $X'$ is (generically) equal to $Y' \times_Z W$. We conclude that $\operatorname{ed} Y \leqslant \dim Z = \operatorname{ed} X$, as claimed.

Part (b) is proved by the same argument, with rational maps replaced by correspondences of degree prime to $p$. ♠

**Corollary 7.** *Let $G$ be a finite group, $p$ a prime number, and $G_p$ a Sylow $p$-subgroup of $G$. If $G_p$ is non-abelian, and $X$ is a $p$-versal $G$-variety, then $\dim X \geqslant p$.*

*Proof.* We have $\dim X \overset{(i)}{\geqslant} \operatorname{ed}_G(X) \overset{(ii)}{\geqslant} \operatorname{ed}_{G_p}(X; p) \overset{(iii)}{=} \operatorname{ed}(G_p; p) \overset{iv)}{\geqslant} p$.

Here (i) follows from the definition of essential dimension, (ii) from Remark 3, (iii) from Lemma 6(b), and (iv) from [MR10, Theorem 1.3]. ♠

Since $\dim \mathcal{A}_g = \dfrac{g(g+1)}{2}$, Corollary 7 allows us to see easily that the congruence cover $\mathcal{A}_{g,pN} \to \mathcal{A}_{g,N}$ of the moduli space of principally polarized abelian varieties from the introduction is (usually) not $p$-versal.

**Corollary 8.** *Fix $g \geqslant 2$. Then the $\mathbf{Sp}_{2g}(\mathbb{F}_p)$-torsor $\mathcal{A}_{pN} \to \mathcal{A}_N$ from §1.2 is not $p$-versal for any prime $p > \dfrac{g(g+1)}{2}$.*

*Proof.* By Corollary 7 it suffices to show that $G = \mathbf{Sp}_{2g}(\mathbb{F}_p)$ contains a non-abelian $p$-subgroup $H$. Since $G$ contains $\mathbf{Sp}_4(\mathbb{F}_p)$, we may assume without loss of generality that $g = 2$. We may also assume that $G$ is the automorphism group of the symplectic form $x_1 \wedge x_4 + x_2 \wedge x_3$. Let $H$ be the subgroup of lower-triangular matrices in $\mathbf{Sp}_4(\mathbb{F}_p)$. Clearly, $H$ is a $p$-group. An easy computation shows that the lower-triangular matrices

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

do not commute. Thus $H$ is a non-abelian subgroup of $G$, as claimed. ♠

**Remark 9.** As mentioned in the introduction, D. Benson showed us a proof that, for $p > 2$, $\mathrm{ed}_{\mathbb{C}}(\mathbf{Sp}_{2g}(\mathbb{F}_p); p) = p^{g-1}$. With his permission, we give it below in Theorem 79 of Appendix B.

2.3. **The fixed point method.** Throughout this paper we will refer to the following result (and in particular to Proposition 10(b)) as "the fixed point method."

**Proposition 10.** *Let $G$ be a finite group, $X$ an irreducible generically free $G$-variety over a base field $k$ of characteristic $\neq p$, with $p$ a prime number. Suppose $G$ has a subgroup $H$ such that $H$ is a $p$-group and $H$ has a smooth fixed point $X$. Then*

*(a)* $\mathrm{ed}_G(X_0; p) \geqslant \mathrm{ed}_H(X_0; p) = \mathrm{ed}(H; p)$ *for any dense open $G$-invariant subvariety $X_0 \subset X$.*

*(b) If $H = (\mathbb{Z}/p)^r$, then* $\mathrm{ed}_G(X_0; p) \geqslant \mathrm{ed}_H(X_0; p) = r$.

In the case, where $\mathrm{char}(k) = 0$, Proposition 10(b) follows from [RY00, Theorem 7.7], whose proof relies on equivariant resolution of singularities. The proof we present here does not use resolution of singularities; in particular, it remains valid in prime characteristic, as long as $\mathrm{char}(k) \neq p$.

*Proof.* (a) The inequality $\mathrm{ed}_G(X_0; p) \geqslant \mathrm{ed}_H(X_0; p)$ follows from Remark 3. Since essential dimension at $p$ is a birational invariant of $H$-varieties, $\mathrm{ed}_H(X_0; p) = \mathrm{ed}_H(X; p)$. Thus we may assume without loss of generality that $X_0 = X$. By Theorem 5, $X$ is a $p$-versal $H$-variety. Now Lemma 6 tells us that $\mathrm{ed}_H(X; p) = \mathrm{ed}(H; p)$.

(b) By Remark 4, $\mathrm{ed}(H; p) = r$. The rest follows from part (a). ♠

## 3. A criterion for the existence of fixed points

In this section $k$ is an arbitrary algebraically closed field.

**Definition 11.**

(1) A *toroidal singularity* is a scheme $S$ over $k$ together with an isomorphism of $S$ (which we suppress from the notation unless there is a possibility of confusion) with the spectrum of the completion of the local ring of a (normal) affine toric variety over $k$ at a torus fixed point. We say that a toroidal singularity is *simplicial* if the corresponding affine toric variety is simplicial.

(2) An action of a finite abelian group on a toroidal singularity is said to be toroidal if the action is induced via completion by the action of a finite subgroup of the torus on the corresponding toric variety.

(3) A *toroidal map* of toroidal singularities is a morphism of schemes which is induced by completion from a toric morphism, (i.e., a morphism induced by a map of the corresponding semigroups) of the corresponding affine toric varieties.

**Remark 12.** The definitions (2) and (3) do depend on the choice of the toroidal structure, i.e., the isomorphism in (1).

**Lemma 13.** *Let $\pi : T \to S$ be a finite surjective toroidal map of toroidal singularities. Then any normal scheme $S'$ with a finite map $\pi' : S' \to S$ through which $\pi$ factors is toroidal and the map $\pi'$ is also toroidal. Furthermore, if $S$ is simplicial then so is $S'$.*

*Proof.* By definition, there exist affine toric varieties $U_T$ and $U_S$ and a toric morphism $p : U_T \to U_S$ inducing $\pi$ by completion at the torus fixed points. The morphism $p$ corresponds to a map of lattices $p_M : M_T \to M_S$ together with rational polyhedral cones $C_T \subset M_T \otimes \mathbb{R}$, $C_S \subset M_S \otimes \mathbb{R}$, such that $p_M(C_T) \subset C_S$. Since $\pi$ is finite and surjective, $M_T$ and $M_S$ must have the same rank and $p_M$ must be injective. We claim that we must also have $p_M(C_T) = C_S$: if this does not hold, then some positive dimensional face $F$ of $C_T$ must map to the interior of $C_S$. This implies that the closure in $U_T$ of the torus orbit corresponding to $F$, $Z_F$, maps to the torus fixed point of $U_S$. Since $Z_F$ contains the torus fixed point of $U_T$, it follows that $\pi$ cannot be finite, which gives a contradiction.

From $p_M(C_T) = C_S$ it follows that $p$ is finite, and then by the Galois correspondence we see that $S'$ is the completion of the toric variety corresponding to the cone induced by $C_T$ in a sublattice $M_{S'}$ of $M_S$ containing $M_T$. The assertions of the lemma follow immediately from this.

♠

**Lemma 14.** *Let $S$ be a simplicial toroidal singularity and $S^o \subset S$ the open set which is the complement of the completion of the boundary divisor. Let $\pi^o : T^o \to S^o$ be a connected finite étale Galois cover of degree not divisible by* $\mathrm{char}(k)$ *and let $T$ be the normalisation of $S$ in $T^o$. Then $T$ is a simplicial toroidal singularity and the induced map $\pi : T \to S$ is also toroidal.*

*Proof.* Since $S$ is simplicial, there exists a smooth toroidal singularity $S_1$ together with a finite toroidal map $\pi_1 : S_1 \to S$ satisfying $S_1 \times_S S^o = S_1^o$, where $S_1^o$ is the open complement of the boundary. Let $S_2^o$ be a connected component of $S_1^o \times_{S^o} T^o$. The projection $\pi_2^o$ to $S_1^o$ makes $S_2^o$ a finite étale Galois cover of $S_1$ of degree not dividing $\mathrm{char}(k)$. Let $S_2$ be the normalisation of $S_1$ in $S_2^o$ and let $\pi_2 : S_2 \to S_1$ be the induced map.

By Abhyankar's lemma [SGA71, Expose XIII, §5], which is applicable since $S_1$ is complete and regular and $S_1 \backslash S_1^o$ is a normal crossings divisor, there exists a formally smooth toroidal singularity $S_3$, a toroidal map $\pi_3 : S_3 \to S_1$ and a map $\pi_{3,2} : S_3 \to S_2$ such that $\pi_3 = \pi_2 \circ \pi_{3,2}$. Lemma 13 now implies that $S_2$ and $\pi_2$ are both toroidal.

The composition of two finite toroidal maps of toroidal singularities is again toroidal, so it follows that $\pi_1 \circ \pi_3$ is toroidal. By construction, there is a map $\pi_T : S_3 \to T$ such that $\pi_1 \circ \pi_3 = \pi \circ \pi_T$, so by Lemma 13 once again, we see that $T$ is a simplicial toroidal singularity and $\pi$ is also toroidal.

♠

The lemma below is a version of [RY00, Proposition A.2], which we also use in the proof.

**Lemma 15.** *Let $A$ be a finite abelian group acting on a variety $U$ and $p : U \to V$ a proper $A$-equivariant map, with $A$ acting trivially on $V$. If there exists a toroidal singularity $T$ with a toroidal $A$-action and an $A$-equivariant rational map $h : T \dashrightarrow U$ such that the map $p \circ h : T \dashrightarrow V$ is a morphism, then $A$ has a fixed point in $U$.*

*Proof.* Let $C$ be an affine toric variety with a torus fixed point $c_0$ such that $T \cong \widehat{\mathscr{O}}_{c_0,C}$. Let $C^\sharp$ be the toric variety corresponding to the star subdivision of the cone corresponding to $C$ with respect to a ray in its interior. We have a birational proper morphism $C^\sharp \to C$ with the property that the fibre over $c_0$ is an irreducible divisor $E$. Let $T^\sharp = T \times_C C^\sharp$; $T^\sharp$

is normal because the local rings at its closed points, which are the closed points of $E$, are the completions of the corresponding local rings on $C^\sharp$ which is normal (and excellent). The map $C^\sharp \to C$ induces a proper birational morphism $\phi : T^\sharp \to T$ whose fibre over $t_0$, the closed point of $T$, is $E$. Since the action of $A$ on $T$ is induced from the torus action on $C$, $A$ acts on $T^\sharp$ and $E$ and the action on $E$ is toric.

Let $\mathscr{O}_{E,T^\sharp}$ be the local ring of $E$ on $T^\sharp$. Since $E$ is a divisor and $T^\sharp$ is normal, $\mathscr{O}_{E,T^\sharp}$ is a dvr. The map $h$ may be viewed as an $A$-equivariant rational map from $T^\sharp$ to $U$, so it gives an $A$-equivariant map $\mathrm{Spec}(\mathrm{Frac}(\mathscr{O}_{E,T^\sharp})) \to U$. It follows from the assumptions that the composite of the maps

$$\mathrm{Spec}(\mathscr{O}_{E,T^\sharp}) \to T \overset{h}{\dashrightarrow} U \xrightarrow{p} V.$$

is a morphism. By the valuative criterion of properness, the composed morphism $\mathrm{Spec}(\mathscr{O}_{E,T^\sharp}) \to V$ lifts to a morphism $\mathrm{Spec}(\mathscr{O}_{E,T^\sharp}) \to U$ which is $A$-equivariant, so we get an $A$-equivariant rational map from $E$ to $U$. Since $E$ lies over the closed point $t_0$ of $T$, this rational map factors through $Z = p^{-1}(\bar{f} \circ h(t_0)) \subset U$, which is a proper scheme. We now replace $E$ by any toric resolution $\tilde{E}$ of $E$, so we have an $A$-equivariant rational map from $\tilde{E}$ to $Z$. Since the $A$-action on $E$ comes from the torus and $\tilde{E}$ is proper, $E$ has a (smooth) $A$-fixed point. By [RY00, Proposition A.2] it follows, that $Z$, hence also $U$, has an $A$-fixed point. ♠

**Proposition 16.** *Let $Y$ be an irreducible variety over $k$ and $f : X \to Y$ an irreducible finite étale Galois cover with Galois group $G$. Let $S$ be a simplicial toroidal singularity and $S^o \subset S$ the complement of the boundary divisor. Suppose there exists a morphism $g : S^o \to Y$ such that the image of the composite of $\pi_1^{\mathrm{et}}(S^o, s) \xrightarrow{g_*} \pi_1^{\mathrm{et}}(Y, g(s)) \twoheadrightarrow G$ [1] is a finite (abelian) group $A$ of order not divisible by $\mathrm{char}(k)$ (for $s$ any geometric point of $S^o$). Assume that $g$ extends to a morphism $\bar{g} : S \to \overline{Y}$, where $\overline{Y} \supset Y$ is a partial compactification of $Y$. Then*

1. *Any $G$-equivariant partial compactification $\overline{X} \supset X$ admitting a proper morphism $\bar{f} : \overline{X} \to \overline{Y}$ extending $f$ has an $A$-fixed point.*
2. *Any smooth proper variety $X'$ with a $G$-action which is equivariantly birational to $X$ has an $A$-fixed point.*

The statement and proof of this proposition have several elements in common with [CGR06, §6].

*Proof.* Consider the scheme $S^o \times_Y X$. It is finite étale over $S^o$ and has an action of $G$ induced by the action on $X$. The assumption on the map on fundamental groups implies that the connected components of $S \times_Y X$ are Galois covers of $S$ with Galois group $A$. Let $T^o$ be any one of these components and let $\pi^o : T^o \to S^o$ be the covering map.

Let $T$ be the normalisation of $S$ in the function field of $T^o$, so $T \times_S S^o = T^o$. It follows from Lemma 14 that $T$ and the induced map $\pi : T \to S$ are both toroidal. Since $A$ is the Galois group of the covering $\pi^o : T^o \to S^o$, the action of $A$ on $T$ is also toroidal.

Let $h : T \dashrightarrow \overline{X}$ be the $A$-equivariant rational map corresponding to the natural morphism from $T$ to $X$. The composed map $\bar{f} \circ h$ is equal to $\bar{g} \circ \pi$, so it is a morphism.

[1] The second map is well defined only up to conjugation, but the claim does not depend on this choice.

We now apply Lemma 15, with $U = \overline{X}$, $V = \overline{Y}$, $p = \bar{f}$, to complete the proof of (1). Part (2) follows from (1) and "going down" [RY00, Proposition A.2]. ♠

**Remark 17.** If $k = \mathbb{C}$ we may take $S$ to be the analytic germ of a toric variety at a torus fixed point, $S^o$ the complement of the toric boundary in $S$, and $g$, $\bar{g}$ to be complex analytic maps. This follows from Proposition 16 by completing the local ring corresponding to $S$.

**Theorem 18.** *Let $X$ be a smooth variety over an algebraically closed field $k$ with a free action of a finite group $G$. Let $A \subset G$ be an elementary abelian $p$-group of rank $r$ and let $Y = X/G$. Let $S$ be a simplicial toroidal singularity and $S^o \subset S$ the complement of the boundary divisor. Suppose there exists a morphism $g : S^o \to Y$ such that the image of the composite of $\pi_1^{\mathrm{et}}(S^o, s) \xrightarrow{g_*} \pi_1^{\mathrm{et}}(Y, g(s))$ is $A$. Furthermore, assume that $g$ extends to a morphism $\bar{g} : S \to \overline{Y}$, where $\overline{Y} \supset Y$ is a partial compactification of $Y$. If there exists a smooth partial compactification $\overline{X} \supset X$ together with a proper morphism $\bar{f} : \overline{X} \to \overline{Y}$ extending $f$ then*

$$\mathrm{ed}_G(X;p) \geqslant \mathrm{ed}_A(X;p) = r.$$

Note that the assumption on the existence of $\overline{X}$ is always satisfied if $\mathrm{char}(k) = 0$.

*Proof.* Since the $G$-action on $X$ is free, the quotient map $f : X \to Y$ is finite étale. By Proposition 16, $\overline{X}$ has an $A$-fixed point, so the theorem follows by applying Proposition 10 (taking $H$ there to be $A$). ♠

## 4. The moduli space of curves

In this section we prove incompressibility results for two types of covers of $\mathcal{M}_g$, first for covers that are pullbacks of congruence covers of $\mathcal{A}_g$ and then for certain covers arising from TQFTs. For congruence covers, our proof is characteristic free and so extends the incompressibility over fields of characteristic zero already proved in [FKW19] to positive characteristics; for the covers of the second type, the methods of *op. cit.* do not apply. Note that in *op. cit.*, incompressibility for $\mathcal{M}_g$ is deduced using the Torelli map to $\mathcal{A}_g$, but our proof does not use this.

Both results are applications of Theorem 18. In §4.1 we make some monodromy computations needed for both proofs, incompressibility for congruence covers is then proved in §4.2 and for the "quantum" covers in §4.3.

For the basics of mapping class groups needed for this section the reader may consult [FM12].

### 4.1. **A monodromy computation.**

4.1.1. Let $g \geqslant 2$ and let $\Sigma$ be a closed oriented surface of genus $g$. Let $\mathrm{Mod}(\Sigma)$ be the mapping class group of $\Sigma$. Corresponding to any pants decomposition of $\Sigma$, equivalently a collection $P$ of $3g-3$ mutually non-isotopic and non-intersecting loops $\gamma_i$ in $\Sigma$, there is a free abelian group $F_P(\Sigma) \subset \mathrm{Mod}(\Sigma)$ of rank $3g-3$ generated by the Dehn twists around these loops. Let $\Gamma_P$ be the dual graph of the pants decomposition given by $P$: the vertices of $\Gamma_P$ are the connected components of $\Sigma \setminus \cup_i \gamma_i$ and for each $\gamma \in P$ there is an edge $e_\gamma$ joining the vertices corresponding to the two components of which $\gamma$ is in the boundary. This is a trivalent graph, possibly with loops and multiple edges.

There is a canonical map $h_P : F_P(\Sigma) \to \mathrm{Aut}(H_1(\Sigma,\mathbb{Z}))$ sending a diffeomorphism to its action on homology. This map is not always injective; for example, the Dehn twist around a separating loop acts trivially on homology. However, we have the following:

**Lemma 19.** *For suitable choices of $P$ the map $h_P$ is injective; in fact, for any integer $N > 1$, the reduction of $h_P(F_P(\Sigma))$ in $\mathrm{Aut}(H_1(\Sigma,\mathbb{Z}/N\mathbb{Z}))$ is a free $\mathbb{Z}/N\mathbb{Z}$-module of rank $3g-3$.*

*Proof.* We explain the construction of one such $P$ by giving the dual graph as a graph embedded in the plane: Start with a convex $g$-gon $R$ with edges labelled (consecutively) $e_1, e_2, \dots, e_g$. Inside $R$ we choose $g-2$ distinct collinear points and connect them with edges $e_{g+1}, e_{g+2}, \dots, e_{2g-3}$. We then connect each of the vertices of $R$ with one of the chosen points in the interior by an edge in such a way that no edges intersect (except at vertices) and such that the resulting graph is trivalent. It is easy to see that this is always possible; the cases $g = 2$ require some minor modifications which we leave to the reader. We call these edges $e_{2g-2}, e_{2g-1}, \dots, e_{3g-3}$ and the resulting graph $\Gamma$. The edges of $\Gamma$ decompose the interior of $R$ into $g$ polygons with disjoint interiors; we label these $R_1, R_2, \dots, R_g$ with the edge $e_j$ lying in $R_j$, $j = 1, 2, \dots, g$.

We construct a genus $g$ surface $\Sigma$ and a pants decomposition $P$ of $\Sigma$ such that $\Gamma \cong \Gamma_P$ by "fattening" all the edges of $\Gamma$ with the loops $\gamma_i$ being circles (in $\mathbb{R}^3$) centered at a point on the interior of each edge $e_i$. A basis of $H_1(\Sigma,\mathbb{Z})$ is given as follows: let $A_j$, $j = 1, 2, \dots, g$ be the homology class of the circle $\gamma_j$. Each polygon $R_j$ also gives rise to a simple loop on $\Sigma$ whose homology class we call $B_j$. Note that we have not specified orientations of the various loops: for our purposes it suffices to arbitrarily fix one choice for each loop.

We now compute the effect on homology of the Dehn twist $t_{\gamma_i}$ around the loop $\gamma_i$, starting with $i = 1, 2 \dots, g$. By construction, we have $A_j \cdot B_k = \pm\delta_{j,k}$ it follows that we have:

$$t_{\gamma_i}(A_j) = A_j \ \forall j$$
$$t_{\gamma_i}(B_j) = B_j \ \forall j \neq i$$
$$t_{\gamma_i}(B_i) = B_i \pm A_i$$

We now consider the Dehn twists around the loops $\gamma_i$ for $i = g+1, \dots, 3g-3$. For such $i$ it is clear that

$$t_{\gamma_i}(A_j) = A_j \ \forall j.$$

The edge $e_i$ corresponding to the loop $\gamma_i$ lies on the boundary of exactly two polygons $R_k$ and $R_l$, so

$$t_{\gamma_i}(B_j) = B_j \ \forall j \neq k, l.$$

Finally, we have

$$t_{\gamma_i}(B_k) = B_k \pm A_k \pm A_l; \ t_{\gamma_i}(B_l) = B_l \pm A_l \pm A_k.$$

The condition that the graph $\Gamma$ is trivalent implies that distinct polygons $R_k$ and $R_l$ share at most one edge. From this and the formulae above it follows that the images of the elements $\{h_P(t_{\gamma_i})\}_{i=1}^{3g-3}$ in $\mathrm{Aut}(H_1(\Sigma,\mathbb{Z}/N\mathbb{Z}))$ are linearly independent for all $N > 1$ ♠

4.1.2. Let $C_o$ be a totally degenerate stable curve of genus $g$ over $\mathbb{C}$, and let $\nu : C \to B$ be the (analytic) versal deformation of $C_o$, so $B$ is a ball of dimension $3g-3$ and $C_o$ is the fibre over a point $o \in B$. Let $D \subset B$ be the divisor over which $\nu$ is not smooth; this is a normal crossings divisor with $3g-3$ irreducible components $D_i$. Let $B^o = B \backslash D$, and let $C_{B^o} = C \times_B B^o$, so $\nu : C_{B^o} \to B^o$ is a family of smooth projective curves of genus $g$. Fixing a point $u \in B^o$, we get a monodromy representation $\rho : \pi_1(B^o, u) \to \mathrm{Mod}(C_u)$. The image of this homomorphism is a free abelian group of rank $3g-3$; in fact, it is a subgroup of the form $F_P(\Sigma)$ that we have considered above for a suitable $P$. This can be seen as follows:

Choose a disc $B'$ in $B$ centered at the point $b_0$ corresponding to $C_o$ and meeting each component $D_i$ of $D$ transversally and let $C_{B'} = C \times_B B'$. Since $C_o$ is totally degenerate it has $3g-3$ singular points $p_i$ each of which is an ordinary double point. The divisors $D_i$ are in a natural bijection with the singular points, $D_i$ being the locus of all points in $B$ over which the point $p_i$ "remains singular". For each $p_i$ choose a small sphere $S_i$ in $C_{B^o}$ (in some fixed metric) centered at $p_i$. Then if $b \in B' - \{b_0\}$ is sufficiently close to $b_0$, $S_i \cap C_b$ is a simple loop $\gamma_i$ and these loops are pairwise disjoint (see, e.g., [SGA73, Expose XIV]). These $3g-3$ loops give rise to a pants decomposition $P$ of $C_b$ (viewed as a differential manifold) and as a topological space $C_o$ is obtained from $C_b$ by contracting each $\gamma_i$ to a point. The monodromy representation then maps a loop $\delta_i$ in $B^o$ based at $b$ and going round the divisor $D_i$ once to the Dehn twist (up to a sign depending on orientations) corresponding to the loop $\gamma_i$ [Don06, §2].

4.1.3. Any trivalent graph $\Gamma$ with $2g-2$ vertices gives rise to a unique totally degenerate stable curve $C_\Gamma$ of genus $g$ over any field $k$. We choose a copy of $\mathbb{P}^1_k$ with three marked rational points for each vertex of $\Gamma$ and label the marked points with the edges incident on the vertex, a loop being counted twice. We glue these copies of $\mathbb{P}^1_k$ along the marked points by identifying pairs of points marked by the same edge of $\Gamma$. The pants decomposition $P$ that one gets by taking $C_o$ to be $C_\Gamma$ has $\Gamma_P \cong \Gamma$.

For an arbitrary field $k$ and a totally degenerate stable curve $C_o$ over $k$, we have a versal deformation $\nu : C \to B$ of $C_o$, where now $B = \operatorname{Spec} k[[x_1, x_2, \dots, x_{3g-3}]]$. As above, $\nu$ is smooth outside a normal crossings divisor $D \subset B$ (which we may take to be the zero set of $x_1 x_2, \dots x_{3g-3}$) and we let $B^o = B \backslash D$. Let $u$ be a geometric point of $B^o$. Although the mapping class group is no longer meaningful over $k$, we still have a monodromy representation $\rho : \pi_1^{\mathrm{et}}(B^o, u) \to \operatorname{Aut} H^1_{\mathrm{et}}(C_u, \mathbb{Z}/N\mathbb{Z})$, where $(N, \mathrm{char}(k)) = 1$.

**Lemma 20.** *Let $k$ be an algebraically closed field with $(N, \mathrm{char}(k)) = 1$. If the dual graph of $C_o$ is as in Lemma 19, then the image of the monodromy representation $\rho$ is a free $\mathbb{Z}/N\mathbb{Z}$-module of rank $3g-3$.*

*Proof.* If $k = \mathbb{C}$, then the comparison theorem for étale fundamental groups and étale cohomology shows that in this case the Lemma is implies by Lemma 19 and the discussion in §4.1.2. This implies the lemma for $k$ of characteristic zero by Abhyankar's lemma.

If $\mathrm{char}(k) > 0$ we let $W(k)$ be the ring of Witt vectors of $k$ and let $\mathcal{B}$ be the versal deformation of $C_o$ over $W(k)$. So $\mathcal{B} = \operatorname{Spec} W(k)[[x_2, x_2, \dots, x_{3g-3}]]$ and the universal family $\nu : \mathcal{C} \to \mathcal{B}$ is smooth over the complement $\mathcal{B}^o$ of a relative normal crossings divisor

$\mathcal{D} \subset \mathcal{B}$. The lemma then follows from the case $\mathrm{char}(k) = 0$ using the structure of the tame fundamental group of $\mathcal{B}^o$ ([SGA71, Expose XIII, Corollaire 5.3]). ♠

4.2. **Incompressibility for congruence covers of $\mathcal{M}_g$.** In this section $k$ is an arbitrary algebraically closed field.

4.2.1. Let $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus $g \geqslant 2$ over $k$. Let $\overline{\mathcal{M}}_g$ be the moduli space of stable curves; this is an irreducible normal projective variety. For an integer $N > 2$ not divisible by $\mathrm{char}(k)$, let $\mathcal{M}_g[N]$ be the moduli space of smooth curves with (symplectic) level $N$-structure. Although $\mathcal{M}_g$ is not smooth, $\mathcal{M}_g[N]$ is an irreducible and smooth variety. Let $\overline{\mathcal{M}}_g[N]$ be the normalisation of $\overline{\mathcal{M}}_g$ in $\mathcal{M}_g[N]$; it is a normal projective variety but it is not smooth.

**Lemma 21.** *Let $C_o$ be a totally degenerate curve of genus $g$ over $k$ whose dual graph $\Gamma$ is of the form described in Lemma 19. If $N > 2$ then $\overline{\mathcal{M}}_g[N]$ is smooth at all points lying above the point corresponding to $C_o$ in $\overline{\mathcal{M}}_g$.*

*Proof.* This follows immediately from [Mos83, Satz II], at least if $\mathrm{char}(k) = 0$. We give a self-contained proof below that also works if $\mathrm{char}(k) > 0$.

Let $B$, $C$, etc., be as in §4.1.3. Let $B_0[N]$ be the $\mathbf{Sp}(2g, \mathbb{Z}/N\mathbb{Z})$-torsor over $B_0$ given by adding (symplectic) level $N$ structure and let $B[N]$ be the normalisation of $B$ in $B_0[N]$. Let $B'$ be any connected component of $B[N]$. It follows from Lemma 20 that the map $B'$ is given by extracting $N$-th roots of all the $x_i$, so $B' \cong \mathrm{Spec}\, k[[y_1, y_2, \ldots, y_{3g-3}]]$ with $y_i^N = x_i$. In particular, $B'$ is regular.

The finite group $\mathrm{Aut}(C_o)$ acts on all the objects above. It acts faithfully on $H^1_{\mathrm{et}}(C_o, \mathbb{Z}/N\mathbb{Z})$ which is identified with the invariants of the monodromy on $H^1_{\mathrm{et}}(C_u, \mathbb{Z}/N\mathbb{Z})$. It follows that $B'$ has trivial stabilizer in $\mathrm{Aut}(C_o)$.

We have a natural map $\iota : B_0[N] \to \mathcal{M}_g[N]$ since $\mathcal{M}_g[N]$ is a fine moduli space and this extends (by normality and the finiteness of the morphism $B[N] \to B$) to a morphism $\bar{\iota} : B[N] \to \overline{\mathcal{M}}_g[N]$. Since the map $B \to \overline{\mathcal{M}}_g$ induces an isomorphism of $B/\mathrm{Aut}(C_o)$ with the formal neighbourhood of $[C_0] \in \overline{\mathcal{M}}_g$, the fact that $B'$ has trivial stabilizer in $\mathrm{Aut}(C_o)$ implies that $\bar{\iota}$ identifies $B'$ with the formal neighbourhood of $\bar{\iota}(o') \in \overline{\mathcal{M}}_g[N]$, where $o'$ is the closed point of $B'$. The lemma follows from this since $B'$ is regular.

♠

**Remark 22.** We only need Lemma 21 for the $g$ for which there is no graph $\Gamma$ as in Lemma 19 such that $\mathrm{Aut}(\Gamma)$ is trivial. It can be easily checked that such graphs exist for all $g \geqslant 7$.

As above, we continue to assume that $\mathrm{char}(k) \nmid N$. Suppose $p$ is a prime such that $p \nmid \mathrm{char}(k)N$. Then forgetting the level $p$ structure induces a finite étale morphism $f : \mathcal{M}_g[pN] \to \mathcal{M}_g[N]$ making $\mathcal{M}_g[pN]$ into an $\mathbf{Sp}_{2g}(\mathbb{F}_p)$-torsor over $\mathcal{M}_g[N]$.

The following theorem gives a new proof of [FKW19, Corollary 4] and also extends it to fields of positive characteristic.

**Theorem 23.** *$\mathcal{M}_g[pN]$ is $p$-incompressible as an $\mathbf{Sp}_{2g}(\mathbb{F}_p)$-torsor. Furthermore, for $g > 2$, and any $n > 1$ such that $p \mid n$ and $\mathrm{char}(k) \nmid n$, the map $\mathcal{M}_g[n] \to \mathcal{M}_g$ is $p$-incompressible.*

*Proof.* Let $C_o$ be a totally degenerate stable curve of genus $g$ with dual graph as in Lemma 19. Let $B$ be the versal deformation space of $C_o$ as in §4.1.3. Let $S^0$ be a connected component of the étale cover of $B^o$ given by trivialising the finite local system $R^1\nu_*(\mathbb{Z}/N\mathbb{Z})$ and let $S$ be the normalisation of $B$ in $S^0$. Then $S$ is a simplicial toroidal singularity since $\mathrm{char}(k) \nmid N$; it is in fact smooth, but we will not need this. The pullback of $C$ to $S^0$ induces a morphism $g : S^0 \to \mathcal{M}_g[N]$ and since $p \nmid N$, by Lemma 20 for any geometric point $s$ of $S^0$ the image of the composite $\pi_1^{\mathrm{et}}(S^0, s) \xrightarrow{g_*} \pi_1^{\mathrm{et}}(\mathcal{M}_g[N], g(s)) \twoheadrightarrow \mathbf{Sp}_{2g}(\mathbb{F}_p)$ is an elementary abelian $p$-group of rank $3g - 3 = \dim(\mathcal{M}_g[N])$.

By Lemma 21 there exists a nonsingular Zariski open subset $U$ (resp. $V$) of $\overline{\mathcal{M}}_g[pN]$ (resp. $\overline{\mathcal{M}}_g[N]$) containing $\mathcal{M}_g[pN]$ (resp. $\mathcal{M}_g[N]$) and all the points parametrising curves isomorphic to $C_o$ such that the map $f : \mathcal{M}_g[pN] \to \mathcal{M}_g[N]$ extends to a finite morphism $\bar{f} : U \to V$. We now set $X = \mathcal{M}_g[pN]$, $Y = \mathcal{M}_g[N]$, $\overline{X} = U$, $\overline{Y} = V$ and $f : X \to Y$, $\bar{f} : \overline{X} \to \overline{Y}$ as above. The first part of the theorem will follow from Theorem 18 if we can show that the map $g : S^0 \to Y$ extends to a morphism $\bar{g} : S \to \overline{Y}$

To see this, we note that the universal family of stable curves over $B$ induces a morphism $h : B \to \overline{\mathcal{M}}_g$. Let $B' = B \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_g[N]$. $B'$ is finite over $B$ and the map $g : S^0 \to Y$ clearly factors through $B'$. Since $S$ is normal and finite over $B$, the map $S^0 \to B'$ extends to a morphism $S \to B'$ which gives an extension of $g$ as a map $g' : S \to \overline{\mathcal{M}}_g[N]$. By construction, the closed point of $S$ maps to a point in $\overline{X}$, so it follows that $g$ extends to a morphism $\bar{g} : S \to \overline{Y}$ as desired.

To prove the second statement, we may assume that $n = p$. Choose any integer $N > 2$ such that $(N, p\,\mathrm{char}(k)) = 1$. Since $g > 2$, the generic curve of genus $g$ has no nontrivial automorphisms, so $\mathcal{M}_g[p] \times_{\mathcal{M}_g} \mathcal{M}_g[N]$ is equal to $\mathcal{M}_g[pN]$ over the generic point of $\mathcal{M}_g[N]$. We complete the proof by applying the first part of the theorem. ♠

**Remark 24.** The second part of Theorem 23 for $g = 2$ follows from the corresponding statement for $\mathcal{A}_g$ proved in Corollary 41 and Theorem 50.

4.3. **Incompressibility for some "quantum" covers of $\mathcal{M}_g$.** We now apply Proposition 16 to prove incompressibility for certain covers of moduli spaces of curves arising from certain (3-dimensional) Topological Quantum Field Theories. Although our method is quite general, for the sake of concreteness we restrict ourselves to the TQFTs constructed explicitly in [BHMV95].

Throughout this section the base field is $\mathbb{C}$.

4.3.1. We first recall some of the properties of the TQFTs that we shall need. Let $g \geqslant 2$ and let $\Sigma$ be a closed oriented surface of genus $g$. The three dimensional TQFTs constructed in [BHMV95], which are indexed by integers $p \geqslant 1$, give rise to a projective representation $\rho_p$ of the mapping class group $\mathrm{Mod}(\Sigma)$ in a vector space $V_p(\Sigma)$ (over the cyclotomic field $\mathbb{Q}(\exp(\frac{2\pi i}{4p})))$.

We now assume that $p$ is odd for simplicity: this is not a significant restriction since the $V_p(\Sigma)$ for even $p$ can be expressed as a tensor product of $V_p(\Sigma)$ for odd $p$ and another simple representation (depending only on $g$). For each $P$ as in §4.1.1 there is a natural basis $\{u_\sigma\}$ of $V_p(\Sigma)$ in which the action of $F_P(\Sigma)$ is diagonalised [BHMV95, Theorem 4.11]. The basis is parametrised by *admissible colorings* of $\Gamma_P$ [BHMV95, Definition 4.5] which are maps

$$\sigma : E(\Gamma_P) \to \{0, 2, \dots, p-2\}$$

satisfying the following admissibility condition: for all vertices $v$ of $\Gamma_P$, if $e_1$, $e_2$ and $e_3$ are the three edges incident on $v$ (for a loop the same edge is repeated twice) then

$$|\sigma(e_1) - \sigma(e_2)| \leqslant \sigma(e_3) \leqslant \sigma(e_1) + \sigma(e_2). \tag{25}$$

The Dehn twist corresponding to an edge $\gamma$ acts on $u_\sigma$ by multiplication by a $p$-th root of unity depending only on the colour $\sigma(e_\gamma)$: if $A = \exp(\frac{2\pi i}{4p})$ then for any colour $i$ this is $(-1)^i A^{i^2+2i}$ [BHMV95, §5.8, Remark 7.6 (ii)]. One sees from the formula that if $p$ is an odd prime, then these roots of unity are distinct $p$-th roots of unity for distinct even colours.

**Lemma 26.** *The groups $\rho_p(F_P(\Sigma))$ are elementary abelian $p$-groups of rank $3g-3$ if $p > 5$ is a prime.*

*Proof.* Let $\gamma \in P$. If $\gamma$ is a loop, let $\sigma_\gamma$ be the coloring of $\Gamma_P$ given by

$$\sigma_\gamma(e_{\gamma'}) = \begin{cases} 2 \text{ if } \gamma' = \gamma \\ 0 \text{ if } \gamma' \neq \gamma. \end{cases} \tag{27}$$

If $\gamma$ is not a loop let $\sigma_\gamma$ be the coloring of $\Gamma_P$ given by

$$\sigma_\gamma(e_{\gamma'}) = \begin{cases} 4 \text{ if } \gamma' = \gamma \\ 2 \text{ if } \gamma' \neq \gamma. \end{cases} \tag{28}$$

One easily sees that $\sigma_\gamma$ is an admissible coloring in both cases. Moreover, the Dehn twist $t_\gamma$ around $\gamma$ acts by a non-trivial $p$-th root of unity on $u_{\sigma_\gamma}$ in both cases, whereas the other Dehn twists $t_{\gamma'}$ all act by a different root of unity independent if $\gamma \neq \gamma'$.

Let $\sigma_0$ (resp. $\sigma_2$) be the coloring which is the constant function 0 (resp. 2). Suppose $\prod_{\gamma \in P} t_\gamma^{a_\gamma}$ is in $\ker(\rho_p(F_P(\Sigma)))$. If $\gamma$ is a loop then by considering the restriction of the representation to the subspace spanned by $u_{\sigma_0}$ and $u_{\sigma_\gamma}$ we see immediately that $a_\gamma$ must be a multiple of $p$. If $\gamma$ is not a loop we get the same conclusion by using the subspace spanned by $u_{\sigma_2}$ and $u_{\sigma_\gamma}$. Thus, $\rho_p(F_P(\Sigma))$ is is an elementary abelian $p$-group of rank equal to $\#P = 3g-3$. Since $F_P(\Sigma)$ acts trivially on $u_{\sigma_0}$ we conclude that the same holds for the projective image in $PGL(V_p(\Sigma))$. ♠

4.3.2. Since $\mathrm{Mod}(\Sigma)$ is finitely generated, the representations $\rho_p$ (for any $p \geqslant 1$) can be reduced modulo all but finitely many primes $q$ of the subring of $\mathbb{C}$ generated by the matrix coefficients, giving rise to an infinite sequence of finite quotients $G^{p,q}$ of $\mathrm{Mod}(\Sigma)$. In general, these quotients might depend on the choice of lattice used to define the reduction of the representations, but for $p$ an odd prime the explicit lattices in $V_p(\Sigma)$ invariant under the action of $\mathrm{Mod}(\Sigma)$ that were constructed in [GM07] can be used to uniquely specify the $G^{p,q}$ (for all $q$).

The group $\mathrm{Mod}(\Sigma)$ can be identified (after choosing a base point) with the fundamental group of the moduli *stack* $\mathbf{M}_g$ of smooth projective curves of genus $g$ over $\mathbb{C}$. Therefore, the finite quotients $G^{p,q}$ of $\mathrm{Mod}(\Sigma)$ give rise to étale covers of the stack $\mathbf{M}_g$, which we denote by $\mathbf{M}_g^{p,q}$.

**Lemma 29.** *Suppose $p > 3$ is a prime and $n \neq p$ is also a prime. Let $\Gamma^{p,q}$ be the kernel of the surjection $\mathrm{Mod}(\Sigma) \to G^{p,q}$ and let $\Gamma_n$ be the kernel of the surjection $\mathrm{Mod}(\Sigma) \to \mathrm{Aut}(H_1(\Sigma, \mathbb{Z}/n\mathbb{Z}))$. Then $\Gamma^{p,q} \cdot \Gamma_n = \mathrm{Mod}(\Sigma)$.*

*Proof.* From the proof of Lemma 26 we see that the Dehn twists $\delta$ around any of the loops $\gamma$ maps to an element of order $p$ under $\rho_p$, so $\delta^p \in \Gamma^{p,q}$. Since $p \neq n$, if $\gamma$ is a non-separating loop then $\delta^p$ is an element of exact order $n$ in $\mathrm{Aut}(H_1(\Sigma, \mathbb{Z}/n\mathbb{Z}))$. Since the only normal subgroup of this group is its centre, it follows that $\Gamma^{p,q}$ surjects onto $\mathrm{Aut}(H_1(\Sigma, \mathbb{Z}/n\mathbb{Z}))$. ♠

For any integer $N > 2$, let $\mathbf{M}_g[N]$ be the moduli stack of smooth projective curves of genus $g$ with level $n$ structure. It is in fact equal to its coarse moduli space, the variety $\mathcal{M}_g[N]$ from §4.2. We set $\mathcal{M}_g^{p,q}[N] := \mathbf{M}_g^{p,q} \times_{\mathbf{M}_g} \mathcal{M}_g[N]$. By Lemma 29, $\mathcal{M}_g^{p,q}[N]$ is irreducible and the projection map $\pi_g^{p,q} : \mathcal{M}_g^{p,q}[N] \to \mathcal{M}_g[N]$ is a finite etale $G^{p,q}$-cover.

**Theorem 30.** *The $G^{p,q}$-covers $\pi_g^{p,q} : \mathcal{M}_g^{p,q}[N] \to \mathcal{M}_g[N]$ are $p$-incompressible for $p > 5$, $N > 2$ and $p \nmid N$ (and all but finitely many primes $q$ of the relevant cyclotomic field).*

**Remark 31.**

(1) If $g > 2$ then $\mathcal{M}_g$ is generically a variety, so we have a finite $G^{p,q}$-cover $\pi_g^{p,q}$ : $\mathcal{M}_g^{p,q} \to \mathcal{M}_g$ (which is only generically étale). Theorem 30 implies that this cover is generically incompressible.
(2) If $g > 3$ there exists a totally degenerate stable curve with trivial automorphism group. In this case the proof below may be carried out without the level $N$-structure.

*Proof.* Let $C_o$ be a totally degenerate stable curve of genus $g$, and let $\nu : C \to B$, $D$, $B$, etc. be as in §4.1.2. By the discussion there and Lemma 26, the image in $G^{p,q}$ of $F_P(\Sigma)$ (identified with $\pi_1(B^o, b)$) is an elementary abelian $p$-group of rank $3g - 3$. On the other hand, the image of the monodromy representation in $\mathrm{Aut}(H_1(C_b, \mathbb{Z}/N\mathbb{Z}))$ is a finite abelian group of exponent $N$ (since the mondromy around each irreducible component of $D$ is unipotent).

Let $S^o$ be a connected component of $B^o \times_{\mathbf{M}_g} \mathcal{M}_g[N]$, where the map to $\mathbf{M}_g$ is the classifying map. The map $S^o \to B^o$ is an abelian cover of exponent $N$. Let $S$ be the normalisation of $B$ in $S^o$, so this is a (simplicial) toroidal singularity and $S^o \subset S$ is the complement of the boundary divisor. Since $p \nmid N$, for $s \in S^o$ the image of the monodromy representation $\pi_1(S^o, s) \to \pi_1(\mathcal{M}_g[N]) \to G^{p,q}$ is still an elementary abelian $p$-group of rank $3g - 3$.

We conclude by applying Theorem 18 with $Y = \mathcal{M}_g[N]$, $X = \mathcal{M}_g^{p,q}[N]$, $f = \pi_q^{p,q}$, $\overline{Y} = \overline{\mathcal{M}}_g[N]$, $\overline{X}$ any equivariant smooth compactification of $\mathcal{M}_g^{p,q}[N]$ which has a map $\bar{f}$ to $\overline{\mathcal{M}}_g[N]$ extending $f$ (which exists because $k = \mathbb{C}$), $S^o$ and $S$ as above and $g$ and $\bar{g}$ the natural classifying maps.

♠

**Remark 32.** By a theorem of Masbaum and Reid [MR12], for each $g \geqslant 2$ there are infinitely many integers $N$, and for each such $N$ an infinite set of (rational) primes $P_N$, such that the group $PSL(N, r)$, for $r \in P_N$, occurs as one of the groups $G^{p,q}$. In contrast to this, it follows from the congruence subgroup property of $\mathbf{Sp}_{2g}(\mathbb{Z})$, $g \geqslant 2$, that there are no such covers of $\mathcal{A}_g$ for $g \geqslant 2$ if $N > 2g$.

## 5. Locally symmetric varieties

In this section, we first (in §5.1) fix some notation involving algebraic groups and Hermitian symmetric domains and state our main theorems regarding locally symmetric varieties: Theorems 33 and Theorem 34. Theorem 33 is then proved in §5.5 and Theorem 34 is proved in §5.4.

Our basic references for Hermitian symmetric domains are [Hel78] (where they are called Hermitian symmetric spaces of noncompact type) and [AMRT10].

### 5.1. **Locally symmetric varieties, boundary components and incompressibility.**

We say that a semisimple algebraic group $\mathbf{G}$ over $\mathbb{R}$ is of *Hermitian type* if the Lie group $G := \mathbf{G}^{\mathrm{ad}}(\mathbb{R})_+$ is isomorphic to $\mathrm{Aut}(D)^0$, the identity component of the group of biholomorphisms of a Hermitian symmetric domain $D$. We note that $G$ and $D$ determine each other, $D$ being biholomorphic to $G/K$, where $K$ is a maximal compact subgroup of $G$ and $G/K$ having its natural $G$-invariant complex structure. The choice of isomorphism does not matter for our purposes, we fix one and then identify $D$ with $G/K$. A semisimple algebraic group over $\mathbb{Q}$ is of *Hermitian type* if $\mathbf{G}_{\mathbb{R}}$ is of Hermitian type.

Suppose $\mathbf{G} = \mathbf{G}^{\mathrm{ad}}$ and $\Gamma$ is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ contained in $G$, then, by Baily–Borel [BB66], the complex-analytic space $\Gamma\backslash D$ has the structure of a quasi-projective complex variety. Moreover, if $\Gamma$ is neat, then this structure is unique by Borel's Extension Theorem [Bor72]. (See §5.3 for a review of the notions of arithmetic and congruence subgroups.) In other words, there is a uniquely defined quasi-projective variety $M_\Gamma$ whose analytification $M_\Gamma^{\mathrm{an}}$ is $\Gamma\backslash D$. The varieties $M_\Gamma$ will be called *locally symmetric varieties.* We often abuse notation and write $\Gamma\backslash D$ for $M_\Gamma$.

Now suppose that $\Delta$ is a finite index subgroup of $\Gamma$. We then have a morphism $\Delta\backslash D \to \Gamma\backslash D$ of complex analytic spaces and, again using [Bor72], we get a corresponding morphism $\pi : M_\Delta \to M_\Gamma$ of quasi-projective varieties. Moreover, if we assume that $\Gamma$ is neat and $\Delta \trianglelefteq \Gamma$, then $M_\Gamma$ is smooth and $\pi$ is a $\Gamma/\Delta$-torsor.

We have $D = G/K$ where $K$ is a maximal compact subgroup of $G$. Moreover, the Cartan involution $\sigma$ corresponding to $K$ gives us a splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of $G$ and $K$ respectively. Here $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and $-1$ eigenspaces of $\sigma$ respectively. The space $\mathfrak{p}$, which can be identified with the tangent space of $D$ at the point corresponding to $K$, has a complex structure $J$. This gives rise to a decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$, where $\mathfrak{p}_\pm$ is the $\pm i$-eigenspace of $J$. The Harish-Chandra embedding theorem [AMRT10, Theorem 2.1], then gives a holomorphic embedding of $D$ into $\mathfrak{p}_+$.

Write $\overline{D}$ for the closure of $D$ in $\mathfrak{p}_+$. Then the action of $G$ on $D$ extends to $\overline{D}$, which is itself a union of Hermitian symmetric domains called *boundary components* [AMRT10, Definition 3.2]. The *normalizer* of a boundary component $F$ is then $N(F) := \{g \in G : gF = F\}$. (Note that the space $D$ itself is a boundary component with $N(F) = G$.) If we write $G$ as a product $G_1 \times \cdots \times G_k$ with $G_i$ simple, then the association $F \mapsto N(F)$ sets up a one-one correspondence between boundary components and parabolic subgroups $P \leqslant G$ of the form $P_1 \times \cdots \times P_k$ with each $P_i$ containing a maximal parabolic subgroup of $G_i$. (So each $P_i$ is either maximal parabolic in $G_i$ or equal to $G_i$ itself.) Let $W(F)$ be the unipotent radical of $N(F)$ and $U(F)$ the centre of $W(F)$.

A boundary component is said to be *rational* if $N(F) = \mathbf{P}(\mathbb{R}) \cap G$ where $\mathbf{P}$ is a rational parabolic subgroup of $\mathbf{G}$. We will call such a parabolic subgroup *$d$-cuspidal* if the corresponding boundary component is $d$-dimensional. The Baily–Borel compactfication of $\Gamma\backslash D$ involves adding strata corresponding to rational boundary components [BB66], the stratum associated to a $d$-cuspidal parabolic being of dimension $d$. (When $D$ has no rational boundary components the space $\Gamma\backslash D$ is compact for any arithmetic subgroup $\Gamma \leqslant \mathbf{G}(\mathbb{Q}) \cap G(\mathbb{R})$.)

With the above notation, we can state a theorem.

**Theorem 33.** *Let $\mathbf{G}$ be an algebraic group over $\mathbb{Q}$ of Hermitian type with corresponding Hermitian symmetric domain $D$, let $F$ be a rational boundary component of $D$ and set $U := U(F)$. Let $\Delta$ and $\Gamma$ be neat arithmetic subgroups of $G \cap \mathbf{G}(\mathbb{Q})$ with $\Delta \trianglelefteq \Gamma$, and set $\mathbb{U} := (U \cap \Gamma)/(U \cap \Delta)$. Then any smooth $\mathbb{U}$-equivariant compactification $\overline{M}_\Delta$ of $M_\Delta$ has a $\mathbb{U}$-fixed point.*

Coupled with the fixed point method, Theorem 33 can be used to give lower bounds for the essential dimension of congruence covers. Tube domains, see §5.4, are a special class of Hermitian symmetric domains, and if $D$ is such and $\mathbf{P}$ is 0-cuspidal then we can even prove incompressibility for certain covers. Since the proof in this case is very simple and does not use the general theory of Hermitian symmetric domains, we state and prove this special case as Theorem 34 below.

(See §5.3 for the notion of a subgroup defined by congruence conditions, which we use in the statement.)

**Theorem 34.** *Let $U$ be a finite dimensional real vector space and let $D = C + iU$ be a tube domain in $U_{\mathbb{C}}$ associated to a self-adjoint homogenous open cone $C \subset U$. Let $\mathbf{G}$ be an adjoint algebraic group over $\mathbb{Q}$ such that $G := \mathbf{G}(\mathbb{R})_+ \cong \mathrm{Aut}(D)^0$. Let $P$ be the normalizer of $U$ in $G$ ($U$ acts on $D$ by translations) and assume that $P = \mathbf{P}(\mathbb{R}) \cap G$, where $\mathbf{P} \subset \mathbf{G}$ is a parabolic subgroup. Let $\Delta$ and $\Gamma$ be neat arithmetic subgroups of $G \cap \mathbf{G}(\mathbb{Q})$ with $\Delta \trianglelefteq \Gamma$, and set $\mathbb{U} := (U \cap \Gamma)/(U \cap \Delta)$. Then:*

*(1) Any smooth $\mathbb{U}$-equivariant compactification $\overline{M}_\Delta$ of $M_\Delta$ has a $\mathbb{U}$-fixed point.*
*(2) If the rank of $\mathbb{U} \otimes \mathbb{F}_p$ is equal to $\dim(D)$ then the cover $\pi : M_\Delta \to M_\Gamma$ is $p$-incompressible.*

*Furthermore, given $\Gamma$ as above, for any prime $p$ there exists $\Delta \trianglelefteq \Gamma$ such that the rank of $\mathbb{U} \otimes \mathbb{F}_p$ is equal to $\dim(D)$.*

A proof of Theorem 34 will be given in Section 5.4.

**Remark 35.** One can sometimes get essentially the same conclusions as in Theorems 33 and 34 even when $\Gamma$ and $\Delta$ are not neat by using a simple base change trick. See the proof of the last part of Corollary 41 for an illustration.

Section 6 is devoted to constructing examples, see in particular, Theorem 67 and Remark 68.

**Remark 36.** Our definition of groups of Hermitian type forces all the simple factors of $G$ to be noncompact. For the purpose of constructing the locally symmetric varieties $M_\Gamma$, for $\Gamma$ an arithmetic subgroup of $G$, this is not necessary, and such varieties are commonly studied, e.g., in the theory of Shimura varieties. However, if $\mathbf{G}$ is simple and $G$ has a compact factor, then $M_\Gamma$ is compact [BB66, Lemma 3.2 on page 469] and in this case our methods do not apply.

As we mentioned earlier, Farb, Kisin and Wolfson have been able to prove incompressibility of congruence covers in many instances where $D$ has no nontrivial rational boundary components; i.e, where $M_\Gamma$ is compact. So, we feel that, taken together, Theorem 34 and [FKW19] provide enough evidence to motivate the following conjecture. It is a specialization of Conjecture 1 from the introduction, but we feel that it is worthwhile here to make it separately in the context of locally symmetric varieties.

**Conjecture 37.** Let $p$ be a prime, $D$ a Hermitian symmetric domain, $G = \mathrm{Aut}^0(D)$ and $\Gamma$ an arithmetic subgroup of $G$ as above. Then there exists a normal subgroup $\Delta \trianglelefteq \Gamma$ of finite index such that the Galois cover $M_\Delta \to M_\Gamma$ is $p$-incompressible. That is, viewing $M_\Delta$ as a $\Gamma/\Delta$-variety, we have $\mathrm{ed}(M_\Delta; p) = \dim(D)$.

**Remark 38.** Using Corollary 7 and the method of Corollary 8, it is not hard to see that the covers $M_\Delta \to M_\Gamma$ studied in Theorem 34 are rarely $p$-versal.

5.2. **The example of $\mathcal{A}_g$.** One typical example where we can apply Theorem 34 occurs in the case of the moduli space of principally polarized abelian varieties considered above when $\mathbf{G} = \mathbf{PSp}_{2g,\mathbb{Q}}$ and $D$ is the Siegel upper half-space (which is a tube domain, see Example 44). In fact, this is the example which motivated this paper. It also concretely illustrates several of the issues that we will have to deal with later in a more abstract setting, so we explain it here.

For each positive integer $N \geqslant 3$, set

$$\Gamma_N := \ker(\mathbf{Sp}_{2g}(\mathbb{Z}) \longrightarrow \mathbf{Sp}_{2g}(\mathbb{Z}/N)). \tag{39}$$

(Here we view $\mathbf{Sp}_{2g}$ as the group of symplectomorphisms of the lattice $\mathbb{Z}^{2g}$ with a fixed symplectic form—a reductive group scheme over $\operatorname{Spec}\mathbb{Z}$.) Since $N \geqslant 3$, the restriction of the central isogeny $\mathbf{Sp}_{2g,\mathbb{Q}} \to \mathbf{G}$ to $\Gamma_N$ is injective. So $\Gamma_N$ is naturally isomorphic to its image $\overline{\Gamma}_N$ in $\mathbf{G}(\mathbb{Q})$. Moreover, this image is contained in $G = \mathbf{G}(\mathbb{R})_+$. (This follows from the fact that $\mathbf{Sp}_{2g}(\mathbb{R})$ is connected.)

The variety $M_{\overline{\Gamma}_N}$ is the coarse moduli space $\mathcal{A}_{g,N}$ of principally polarized $g$-dimensional abelian varieties with level $N$-structure, and, if $p$ is a prime not dividing $N$, the map $\pi : \mathcal{A}_{g,pN} \to \mathcal{A}_{g,N}$ is an $\mathbf{Sp}_{2g}(\mathbb{F}_p)$-torsor. In other words, if we set $\Gamma := \overline{\Gamma}_N$ and $\Delta := \overline{\Gamma}_{pN}$, then the map $\mathcal{A}_{g,pN} \to \mathcal{A}_{g,N}$ can be identified with the map $M_\Delta \to M_\Gamma$.

Let $\widetilde{\mathbf{P}}$ denote a parabolic in $\mathbf{Sp}_{2g,\mathbb{Q}}$ corresponding to a maximal isotropic subspace for the symplectic form on $\mathbb{Q}^{2g}$. For definiteness, fix the symplectic form $\phi$ (on $\mathbb{Z}^{2g}$) given by the matrix

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where $I$ denote the $g \times g$ identity matrix. It is not hard to see that the unipotent radical of $\widetilde{\mathbf{P}}$ consists of matrices of the form

$$\begin{pmatrix} I & T \\ 0 & I \end{pmatrix} \tag{40}$$

where $T$ is a symmetric $g \times g$ matrix. So the unipotent radical of $\widetilde{\mathbf{P}}$ is abelian, and is, therefore, its own center.

Let $\mathbf{P}$ denote the image of $\widetilde{\mathbf{P}}$ in $\mathbf{G}$, and let $P := G \cap \mathbf{P}$. Then $P$ is a maximal parabolic and corresponds to a unique boundary component $F$ (which is actually a single point in $\overline{D}$). In fact, in the tube domain description of $D$ (see Example 44), this $P$ is exactly the $P$ in Theorem 34. In particular, $U(F)$ is equal to the abelian algebraic group defined by the matrices in (40). The group $\mathbb{U} = U(F) \cap \Gamma / U(F) \cap \Delta$ from Theorem 33 is just the group $\mathrm{Sym}^2 \mathbb{F}_p^g$, the vector space of symmetric $2 \times 2$-matrices over $\mathbb{F}_p$ viewed as a group. So we get the following result recovering [FKW19, Theorem 2]:

**Corollary 41.** *Any smooth $\mathbb{U}$-equivariant compactification $\overline{\mathcal{A}}_{g,pN}$ of $\mathcal{A}_{g,pN}$ has a $\mathbb{U}$-fixed point. Consequently, the $\mathbf{Sp}_{2g}(\mathbb{F}_p)$-torsor $\mathcal{A}_{g,pN} \to \mathcal{A}_{g,N}$ described above is $p$-incompressible. Furthermore, for any $n > 1$ and $p \mid n$, the map $\mathcal{A}_{g,n} \to \mathcal{A}_g$ is $p$-incompressible.*

*Proof of Corollary 41 assuming Theorem 34.* The Siegel upper half-space is a tube domain so, since $\mathbb{U} \subset \mathbf{Sp}_{2g}(\mathbb{F}_p)$ and $\mathrm{rank}(\mathbb{U}) = \dim \mathcal{A}_{g,pN}$, the first and second parts of the corollary follow from Theorem 34.

To prove the second part, we may assume that $p = n$. Let $N > 2$ be any integer such that $p \nmid N$. The natural finite surjective morphism from $\mathcal{A}_{g,pN}$ to the normalisation $\mathcal{A}'_{g,pN}$ of the component of the fibre product $\mathcal{A}_{g,p} \times_{\mathcal{A}_g} \mathcal{A}_{g,N}$ dominating $\mathcal{A}_{g,N}$ is $\mathbb{U}$-equivariant, so it follows from the first part that any $\mathbb{U}$-equivariant compactification of $\mathcal{A}'_{g,pN}$ has a $\mathbb{U}$-fixed point. By Proposition 10, the map $\mathcal{A}'_{g,pN} \to \mathcal{A}_{g,N}$ is $p$-incompressible, from which we deduce that the same holds for the map $\mathcal{A}_{g,n} \to \mathcal{A}_g$. ♠

5.3. **Arithmetic and Congruence Subgroups.** Here we recall some standard facts and terminology about congruence subgroups. Our main references are [Ser95, Rag04].

Let $\mathbf{G}$ be a linear algebraic group defined over $\mathbb{Q}$ and let $\mathbb{A}^f$ denote the ring of finite rational adeles. The embedding of $\mathbb{Q}$ in $\mathbb{A}^f$ induces an embedding of $\mathbf{G}(\mathbb{Q})$ in $\mathbf{G}(\mathbb{A}^f)$, and we use this embedding to regard the first group (of rational points) as a subgroup of the second group (of adelic points).

A subgroup $\Gamma \leq \mathbf{G}(\mathbb{Q})$ is said to be a *congruence subgroup* if $\Gamma = K \cap \mathbf{G}(\mathbb{Q})$ for a compact open subgroup $K$ of $\mathbf{G}(\mathbb{A}^f)$. A subgroup $\Gamma \leq \mathbf{G}(\mathbb{Q})$ is said to be *arithmetic* if it is commensurable with a congruence subgroup. Here two subgroups $A$ and $B$ of an abstract group $G$ are *commensurable* if $[A : A \cap B]$ and $[B : A \cap B]$ are both finite.

Let $T_c$ denote the subspace topology induced on $\mathbf{G}(\mathbb{Q})$ by its inclusion in $\mathbf{G}(\mathbb{A}^f)$. So $T_c$ has a neighborhood basis of the identity consisting of congruence subgroups. Let $T_a$ denote the topology on $\mathbf{G}(\mathbb{Q})$ obtained by taking the arithmetic subgroups as a neighborhood basis of the identity. Then the topology $T_a$ is a priori at least as fine as the topology $T_c$. The congruence subgroup problem asks whether they are the same. In other words, the congruence subgroup problem for $\mathbf{G}$ asks whether every arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ is congruence. For $\mathbf{G}$ solvable this is known to be the case [Rag76, p. 108]. On the other hand, it is not the case for $\mathbf{G} = \mathbf{SL}_2$ [Ser95], a fact which was apparently already known to F. Klein. Moreover, for $G$ semisimple, it is never the case unless $\mathbf{G}$ is simply connected [Ser95, 1.2c]. See [Rag04] for a more complete account of what is known.

If $\Gamma \leq G(\mathbb{Q})$, we say that a subgroup $\Delta$ of $\Gamma$ is *defined by congruence conditions* if $\Delta = \Gamma \cap K$ for some compact open subgroup of $G(\mathbb{A}^f)$. Equivalently, $\Delta$ is open in $\Gamma$ for the subspace topology induced by $T_c$.

The group $\mathbf{GL}_n(\mathbb{A}^f)$ is a locally compact Hausdorff space, and, if $\rho : \mathbf{G} \to \mathbf{GL}_{n,\mathbb{Q}}$ is a faithful linear representation, then the induced map $\rho(\mathbb{A}^f) : \mathbf{G}(\mathbb{A}^f) \to \mathbf{GL}_n(\mathbb{A}^f)$ is a homeomorphism of $\mathbf{G}(\mathbb{A}^f)$ onto its image, which is necessarily closed. It follows that the topology $T_c$ on $\mathbf{G}(\mathbb{Q})$ has a basis of open neighborhoods of 1 of the form $\rho^{-1}(\Phi_{n,d})$, where $\Phi_{n,d}$ denotes the set of $n \times n$ matrices in $\mathbf{GL}_n(\mathbb{Z})$ which are congruent to the identity modulo $d$.

5.4. **Tube domains.** All the facts about tube domains that we use in this section can be found in [FK94, Chapter X].

**Definition 42.** A *tube domain* is a set of the form

$$D = U + iC \subset U_{\mathbb{C}} \tag{43}$$

where $U$ is a finite dimensional real vector space and $C$ is an open homogeneous self-adjoint cone in $U$. Then $D$ is an open subset of the complexification $U_{\mathbb{C}}$ of $U$ so it has a natural structure of complex manifold; in fact, it is always a Hermitian symmetric domain [FK94, Theorem X.1.1].

A tube domain is said to be *irreducible* if the cone $C$ cannot be written as a product of two cones in a nontrivial way. Any tube domain can be written as a product of irreducible tube domains in an essentially unique way [FK94, Proposition III.4.5].

The group $U$ acts holomorphically on $D$ by translations, so it is a subgroup of $\mathrm{Aut}(D)^0$. In fact, it is the unipotent radical of a parabolic subgroup $P$ of $\mathrm{Aut}(D)^0$. The tube domain is irreducible iff $P$ is a maximal parabolic.

We say that a pair $(\mathbf{G}, \mathbf{P})$ with $\mathbf{G}$ a semisimple algebraic group over $\mathbb{Q}$ and $\mathbf{P}$ a parabolic subgroup is of *tube type* if there is an isomorphism of $G := \mathbf{G}^{\mathrm{ad}}(\mathbb{R})_+$ with $\mathrm{Aut}(D)^0$, for $D$ a tube domain as above, such that $\mathbf{P}(\mathbb{R}) \cap G$ corresponds to $P$.

**Example 44.** The basic example of a tube domain is the Siegel upper half-space $\mathfrak{H}_g$. In this case, $U$ is the space of symmetric real $g \times g$ matrices, $C$ is the cone of positive definite matrices and the group $\mathbf{PSp}_{2g}(\mathbb{R})$ acts on $\mathfrak{H}_g$ by

$$\gamma \cdot \Omega = (A\Omega + B) \cdot (C\Omega + D)^{-1}$$

for $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{Sp}_{2g}(\mathbb{R})$ (with $A, B, C, D$ real $g \times g$ matrices) and $\Omega \in \mathfrak{H}_g$. The parabolic subgroup corresponding to this presentation is the subgroup $P$ consisting of elements $\gamma$ as above with $C = 0$. The group $U$, acting by translation on $\mathfrak{H}_g$, is naturally identified with the unipotent radical of $P$.

Although not needed for the proof of Theorem 34, we give a characterisation of Hermitian symmetric domains which are of tube type.

**Lemma 45.** *Let $D = G/K$ be a Hermitian symmetric domain with boundary component $F$ and let $N(F)$, $W(F)$ and $U(F)$ be as in §5.1. Then the following are equivalent:*

*(1)* $\dim U(F) = \dim D$;

*(2) $D$ is biholomorphic to a tube domain (as in Definition 42) in such a way that $N(F)$ corresponds to $P$ (equivalently, $W(F)$ corresponds to $U$, the unipotent radical of $P$).*

*Furthermore, if (1) or equivalently (2) holds, then $F$ is a 0-dimensional boundary component.*

**Note:** In Lemma 45, $\dim D$ denotes the dimension of $D$ as a complex manifold, and $\dim U(F)$ denotes the dimension of $U(F)$ as a real Lie group.

*Proof.* The reduction to the case that $D$ is irreducible is easy. So we assume $D$ is irreducible and leave the reduction to the reader.

For any boundary component $F$, by [AMRT10, III, (4.1)] there is a real analytic isomorphism $D \cong F \times C(F) \times W(F)$, where $C$ is a self-adjoint homogenous open cone in $U(F)$. It follows that $\dim U(F) = \dim D$ iff $F$ is 0-dimensional and $W(F) = U(F)$. Assuming this is the case, it follows from [AMRT10, III, Lemma 4.7] that $D$ is biholomorphic to the tube domain corresponding to the cone $C(F)$ (and $U(F) = W(F)$ corresponds to $U$).

Conversely, if $D$ is a tube domain as defined above and if we take $F$ to be the boundary component corresponding to the maximal parabolic subgroup $P$, then $U(F) = U$ so $\dim U(F) = \dim D$. ♠

**Remark 46.** For the explicit classification of tube domains, the reader may consult [FK94, X.5 ]. The irreducible ones correspond to simple Lie groups $G$ of Hermitian type with (real) root system of type $C_r$, where $r = \mathrm{rank}(G)$.

**Lemma 47.** *Let $(\mathbf{G}, \mathbf{P})$ be a pair of tube type with $\mathbf{G}$ adjoint. Suppose that $\Gamma \leqslant G \cap \mathbf{G}(\mathbb{Q})$ is an arithmetic subgroup. Let $p$ be a prime number. Then there is a normal subgroup $\Delta$ of $\Gamma$ defined by congruence conditions such that the $p$-torsion subgroup of $H := (\Gamma \cap U)/(\Delta \cap U)$ has rank $\dim U$.*

*Proof.* Set $U_\Gamma := \Gamma \cap U$. The group $pU_\Gamma$ is of finite index in $U_\Gamma$, so it is itself an arithmetic subgroup of $U$. In fact, since $U$ is unipotent, any arithmetic subgroup of $U$ is congruence. Therefore, $pU_\Gamma$ is a congruence subgroup of $U$.

Now, by [PR94, Proposition 4.2], we can find a positive integer $n$ and a faithful linear linear representation $\rho : \mathbf{G} \to \mathbf{GL}_{n,\mathbb{Q}}$ such that $\Gamma$ is a finite index subgroup of $\rho^{-1}\mathbf{GL}_n(\mathbb{Z})$. Since $pU_\Gamma$ is a congruence subgroup of $U$, there exists a positive integer $d$ such that $\rho^{-1}(\Phi_{n,d}) \cap U \leq pU_\Gamma$. So set $\Delta = \Gamma \cap \rho^{-1}\Phi_{n,d}$. Since $\Phi_{n,d} \trianglelefteq \mathbf{GL}_n(\mathbb{Z})$, $\Delta \trianglelefteq \Gamma$. So the lemma follows. ♠

**Lemma 48.** *Let $U$ be a real vector space and $C \subset U$ an open convex cone. Let $L \subset U$ be a lattice. Let $p : U_\mathbb{C} \to T := U_\mathbb{C}/L$ be the quotient map and $\overline{T} \subset T$ the partial compactification of $T$ defined by a maximal dimensional smooth rational polyhedral cone $C' \subset C$. For any $c \in U$, let $D_c \subset U_\mathbb{C}$ be the open set*

$$\{u \in U_\mathbb{C} \mid \mathrm{im}(u) - c \in C\} .$$

*For any sufficiently small polydisc $\overline{S}$ centred at the torus fixed point $t_0$ of $\overline{T}$, $S = \overline{S} \cap p(D_c)$ is a product of punctured polydiscs.*

*Proof.* We may choose coordinates such that $L = \mathbb{Z}^n \subset \mathbb{R}^n = U$ for some $n > 0$, $U_{\mathbb{C}} = \mathbb{C}^n$, and $T = (\mathbb{C}^\times)^n$. We may also assume that

$$C' = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0 \ \forall\, i\}$$

and then $\overline{T} = \mathbb{C}^n$, with $p : \mathbb{C}^n \to (\mathbb{C}^\times)^n$ given by

$$(z_1, z_2, \dots, z_n) \mapsto \left(e^{2\pi i z_1}, e^{2\pi i z_2}, \dots, e^{2\pi i z_n}\right) .$$

Since $C' \subset C$, and $C$ is open and convex, $\{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid \mathrm{im}(z_i) \gg 0 \ \forall\, i\} \subset D_c$. It follows that $p(D_c)$ contains

$$\{(w_1, w_2, \dots, w_n) \in (\mathbb{C}^\times)^n \mid |w_i| < \varepsilon \ \forall\, i\}$$

if $\varepsilon$ is sufficiently small. ♠

*Proof of Theorem 34.* Let $X = M_\Delta$, $Y = M_\Gamma$, $K := \Gamma/\Delta$, and $f : X \to Y$ the natural map making $X$ into a $K$-torsor over $Y$. Let $\overline{Y}$ be the Baily–Borel [BB66] compactification of $Y$ and $\overline{X}$ a smooth $\mathbb{U}$-equivariant compactification of $X$ such that there is a morphism $\bar{f} : \overline{X} \to \overline{Y}$ extending $f$.

The group $U \cap \Gamma$, which is a lattice in $U$, gives rise to an algebraic torus $T = (U \cap \Gamma)\backslash U_{\mathbb{C}}$ containing $(U \cap \Gamma)\backslash D$. Let $C' \subset C$ be a smooth rational polyhedral cone of maximal dimension, where the integral structure defining smoothness is given by $U \cap \Gamma$. The cone $C'$ gives rise to a torus embedding $T \subset \overline{T}$ with $\overline{T}$ an affine space. Since $C' \subset C$, the intersection of a polydisc $\overline{S}$ around $t_0$, the torus fixed point of $\overline{T}$, with $\overline{S}$ is a product $S$ of punctured polydiscs (apply Lemma 48 with $c = 0$) and the map $\pi_1(S) \to \pi_1(T) = (U \cap \Delta)$ is an isomorphism.

The natural map $(U \cap \Gamma)\backslash D \to \Gamma\backslash D$ induces a holomorphic map $g : S \to Y$ with the property that the induced map on $\pi_1$ corresponds to the inclusion $U \cap \Gamma \to \Gamma$. Therefore, the image of the map $g_* : \pi_1(S) \to K = \Gamma/\Delta$ is the finite abelian group $\mathbb{U}$. By the Borel extension theorem [Bor72], the map $g$ extends to a holomorphic map $\bar{g} : \overline{S} \to \overline{Y}$. By Proposition 16, $\mathbb{U}$ has a fixed point in $\overline{X}$, so this proves part (1) of the theorem.

Part (2) of the theorem follows from this by Proposition 10 and part (3) of the theorem is the content of Lemma 47. ♠

5.5. **General Hermitian symmetric domains.** General Hermitian symmetric domains do not have as simple a description as do tube domains, but the theory of Siegel domains of the third kind (see, e.g., [AMRT10, III §4]) allows us to give a proof of Theorem 33 which is very similar to the proof of Theorem 34.

Let $D$, $G$, $\mathbf{G}$ be as in §5.1 and let $F$ be a rational boundary component. We let $N(F)$ be the normalizer of $F$, $W(F)$ its unipotent radical and $U(F)$ the centre of $W(F)$.

We have already mentioned the Harish-Chandra embedding, which is a holomorphic embedding of $D$ in $\mathfrak{p}_+$. Recall that $D = G/K$ where $K$ is a maximal compact subgroup of $G$ (the stabilizer of a given element of $D$). Let $G_{\mathbb{C}}$ (resp. $K_{\mathbb{C}}$) denote the complexification of $G$ (resp. $K$) [Bou98, III §6, Proposition 20]. The subspaces $\mathfrak{p}_\pm$ of $\mathfrak{p}_{\mathbb{C}}$ are abelian subalgebras of $\mathfrak{g}$ corresponding to subgroups $P_\pm = \exp \mathfrak{p}_\pm$ of $G_{\mathbb{C}}$. Moreover, $K_{\mathbb{C}}$ normalizes $P_-$ and the resulting subgroup $K_{\mathbb{C}}P_-$ is parabolic. Set $\check{D} := G_{\mathbb{C}}/K_{\mathbb{C}}P_-$; this space, a complex generalized flag variety, is called the *compact dual* of $D$. The exponential map $\exp : \mathfrak{p}_+ \to G_{\mathbb{C}}$ followed by the quotient map to $\check{D}$ then gives rise to an open immersion of $\mathfrak{p}_+$

in $\check{D}$ [AMRT10, Theorem III.2.1]. The composition of the two embeddings $D \hookrightarrow \mathfrak{p}_+ \hookrightarrow \check{D}$ is known as the *Borel embedding*.

5.5.1. All the results stated in this subsection can be found in [AMRT10, III §4.3] (esp. pp 152-153) and the references therein.

Let $D(F) = U(F)_{\mathbb{C}} \cdot D \subset \check{D}$. This is a submanifold of $\check{D}$ on which there is an action of $U(F)_{\mathbb{C}}$ such that $D$ is preserved by $U(F)$. The group $U(F)_{\mathbb{C}}$ acts freely (and holomorphically) on $D(F)$ and the quotient by this action is a complex manifold $D(F)'$, so we have a map $\pi'_F : D(F) \to D(F)'$ whose fibres are principal homogenous spaces over $U(F)_{\mathbb{C}}$. Furthermore, $D(F)'$ is contractible.

For any $w \in D(F)'$, we may identify $(\pi'_F)^{-1}(w)$ with $U(F)$ by choosing any point as the origin. Then

$$D_w := (\pi'_F)^{-1}(w) \cap D = \{u \in U(F)_{\mathbb{C}} |\, \mathrm{im}(u) \in C(F) - c(w)\} \tag{49}$$

where $C(F)$ is an open self-adjoint homogenous cone in $U(F)$ and $c(w) \in U(F)$.

5.5.2.

*Proof of Theorem 33.* The proof is very similar to the case of tube domains.

Set $U := U(F)$. The inclusion of $D$ in $D(F)$ induces an inclusion $(\Gamma \cap U)\backslash D \to (\Gamma \cap U)\backslash D(F)$ which induces an isomorphism on fundamental groups. Since the map $\pi'_F$ is equivariant for the action of $U(F)_{\mathbb{C}}$, it is also equivariant for the action of $U \cap \Gamma$. For any $w \in D(F)'$, the inclusion $(U \cap \Gamma)\backslash(\pi'_F)^{-1}(w) \to (U \cap \Gamma\backslash D(F))$ also induces an isomorphism on fundamental groups. We now apply Lemma 48 to get $S \subset (U \cap \Gamma)\backslash D_w$, a product of punctured discs, and a sequence of maps

$$S \to (U \cap \Gamma)\backslash D_w \to (U \cap \Gamma)\backslash D \to \Gamma\backslash D.$$

The discussion above shows that the image of the map on fundamental groups induced by the composite of these maps is $U \cap \Gamma$. We may now complete the proof as in the case of tube domains by using the Baily–Borel compactification of $D/\Gamma$ and the Borel extension theorem. ♠

5.6. **Incompressibility in positive characteristic.** In many cases the locally symmetric varieties $M_\Gamma$ have a modular interpretation which leads to a natural model defined over a well-defined number field $L$. In fact, they often have natural smooth models $\mathcal{M}_\Gamma$ over a localisation $R$ of the ring of integers of $L$, so we can reduce $\mathcal{M}_\Gamma$ modulo maximal ideals $P$ of $R$. When $M_\Delta$ also has such a model, we get a finite étale covering $\mathcal{M}_{\Delta,k} \to \mathcal{M}_{\Gamma,k}$, where $k = R/P$. It is then natural to ask when these covers are incompressible or $p$-incompressible (over $\bar{k}$).

Our proof of incompressibility in the case of the moduli space of curves was characteristic free, but for locally symmetric varieties the proof was complex analytic and does not immediately extend to fields of positive characteristic: for example, in this case there is no analogue of Borel's extension theorem, even for $\mathcal{A}_g$. However, the theory of toroidal compactifications of integral models allows us to bypass this difficulty by using the existence of fixed points in characteristic zero to get fixed points (on suitable compactifications) over $k$ as well, from which we can deduce incompressibility using the fixed-point method. All this is best done using the language of Shimura varieties, however, rather than explaining this in detail we give a proof of the analogue of Corollary 41 and then point out the references which can be used to generalize this result.

Let $k$ be an algebraically closed field of characteristic $l > 0$. The varieties $\mathcal{A}_{g,N}$ are moduli spaces of $g$-dimensional principally abelian varieties with level $N$ structure, so can be defined over $k$ as long as $l \nmid N$ [MFK94]. If $p$ is a prime not dividing $N$ and $l \neq p$, then the cover $\mathcal{A}_{g,pN}/k \to \mathcal{A}_{g,N}/k$ is defined and is an $\mathbf{Sp}(2g, \mathbb{F}_p)$-torsor if $N \geqslant 3$. Here we use $/k$ to emphasize that the varieties are over the field $k$. The theorem below extends [FKW19, Theorem 2] to fields of positive characteristic.

**Theorem 50.** *The $\mathbf{Sp}_{2g}(\mathbb{F}_p)$-torsor $\mathcal{A}_{g,pN}/k \to \mathcal{A}_{g,N}/k$ is $p$-incompressible if $p \nmid N$, $N \geqslant 3$ and $l \nmid pN$. Furthermore, for any $n > 1$ and prime $p$ such that $p \mid n$ and $(n, p\,\mathrm{char}\,k) = 1$, the map $\mathcal{A}_{g,n} \to \mathcal{A}_g$ is $p$-incompressible.*

*Proof.* Let $R$ be the ring $\mathbb{Z}[1/n, \zeta_n]$, where $\zeta_n$ is a primitive $n$-th root of unity in $\mathbb{C}$, and assume $n \geqslant 3$. By results of Mumford [MFK94], there exists a smooth scheme $\mathcal{A}_{g,n}/R$ whose fibre at any prime $P$ of $R$ is the variety $\mathcal{A}_{g,n}/k_P$, where $k_P$ denotes the residue field at $P$. By [FC90, IV, Theorem 6.7] there exist smooth proper algebraic spaces $\overline{\mathcal{A}}_{g,n}/R$ (depending on some auxiliary data which we suppress) which contain $\mathcal{A}_{g,n}/R$ as a fibrewise dense open subspace and such that the natural action of $\mathbf{Sp}_{2g}(\mathbb{Z}/n\mathbb{Z})$ on $\mathcal{A}_{g,n}/R$ extends to $\overline{\mathcal{A}}_{g,n}/R$. Furthermore, by [FC90, V, Theorem 5.8], the auxiliary data may be chosen so that $\overline{\mathcal{A}}_{g,n}/R$ is projective, so in particular, it is a scheme.

We now take $n = pN$. Since $\overline{\mathcal{A}}_{g,pN}/\mathbb{C}$ is a smooth compactification of $\mathcal{A}_{g,pN}/\mathbb{C}$, it follows from the discussion in §5.2, that the group $\mathbb{U} \subset \mathbf{Sp}_{2g}(\mathbb{F}_p) \subset \mathbf{Sp}_{2g}(\mathbb{Z}/pN\mathbb{Z})$ has a fixed point in $\mathcal{A}_{g,pN}(\mathbb{C})$. Since $\overline{\mathcal{A}}_{g,pN}/R$ is proper, by specialisation it follows that $\mathbb{U}$ has a fixed point in $\mathcal{A}_{g,pN}(k)$. Since $\mathbb{U}$ is contained in $\mathbf{Sp}_{2g}(\mathbb{F}_p)$, the Galois group of the cover under consideration, and $\mathrm{rank}(\mathbb{U}) = \dim \mathcal{A}_{g,pN}/k$, the theorem follows from the fixed point method (Proposition 10).

To prove the second part of the theorem, we may assume that $p = n$. If $p > 2$, then $\overline{\mathcal{A}}_{g,p}/k$ is smooth and the above shows that it has a $\mathbb{U}$-fixed point so the statement follows from the fixed point method. Now suppose $p = 2$, let $N > 2$ be any integer such that $(N, p\,\mathrm{char}\,k) = 1$, and consider the morphism $\mathcal{A}_{g,pN}/k \to \mathcal{A}'_{g,pN}/k$ as in the proof of Corollary 41. Since $p = 2$ this is an isomorphism, so the statement follows from the second part of the theorem. ♠

**Remark 51.** As mentioned earlier, versions of Theorems 33 and 34 can be proved for certain more general locally symmetric varieties, by essentially the same argument as above, using integral toroidal compactifications of Shimura varieties of Hodge type; see [Lan13] for the PEL case and [MP19] for the general Hodge type setting. Here we need to assume that $\Gamma$ is a congruence subgroup which is hyperspecial at the prime $l$ in order to get a model which is smooth: this corresponds to the condition $l \nmid N$ above. We leave this as an exercise for the interested reader. However, we note that the theory of integral toroidal compactifications is not needed to prove these results for the reductions modulo primes of large (undetermined) residue characteristic: this follows immediately from the results in characteristic zero by "spreading out" any smooth proper equivariant compactification and applying the fixed point method to the reduction modulo such a prime.

## 6. Incompressible hyperspecial congruence covers

Pick a prime $p$, which we will keep fixed for this section. Our goal is to prove generalizations of some of the results of [FKW19, §4] producing congruence covers with group $\mathbb{G}(\mathbb{F}_q)$ where the $\mathbb{G}$ are certain semisimple algebraic groups over $\mathbb{F}_q$ with $q = p^r$ for some positive integer $r$. The main theorem here is Theorem 67. As explained in Remark 68, this allows us to produce congruence covers for most, but not all, of the classical groups $\mathbb{G}$ considered in [FKW19]. The main new result is the existence of $p$-incompressible congruence covers for locally symmetric varieties of type $E_7$ with congruence group also of type $E_7$.

The main technical tools and references are as follows:

(1) Results from SGA3 [SGA70] used to control the reduction modulo $p$ of the subgroup scheme $\mathbf{U}(F) \leqslant \mathbf{G}$ associated to the group $U(F)$ of Theorem 33.
(2) A well-known approximation result for number fields, Proposition 69 below.
(3) A theorem of Prasad and Rapinchuk on producing isotropic groups with specific behavior at a set of primes [PR06].

6.1. **General notation.** Suppose $\mathbf{H}$ is an algebraic group over $\mathbb{Q}$, and let $\mathbb{A}^f := \mathbb{A}_{\mathbb{Q}}$ denote the finite rational adeles. If $p$ is a prime number, we write $\mathbb{A}^{f,p}$ for the prime to $p$ adeles. So $\mathbb{A}^f = \mathbb{A}^{f,p} \times \mathbb{Q}_p$.

If $K$ is a compact open subgroup of $\mathbf{H}(\mathbb{A}^f)$ we set $\Gamma_K := K \cap \mathbf{H}(\mathbb{Q})$ and $\Gamma_K^+ = K \cap \mathbf{H}(\mathbb{Q})_+$. We say that $K$ is neat if it is neat in the sense of Pink [Pin90]. For $K$ neat, $\Gamma_K$ is neat as well.

We say that $\mathbf{H}$ has *strong approximation* if $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{A}^f)$. This obviously implies that $\mathbf{H}(\mathbb{Q})K = \mathbf{H}(\mathbb{A}^f)$ for any compact open subgroup $K \leqslant \mathbf{H}(\mathbb{A}^f)$.

**Proposition 52.** *Suppose that* $\mathbf{H}$ *is either*

*(a) a simply connected semisimple algebraic group over* $\mathbb{Q}$ *without compact* $\mathbb{Q}$*-simple factors, or*
*(b) a Cartesian power* $\mathbb{G}_a^n$ *of the additive group.*

*Then*

*(1) the Lie group* $\mathbf{H}(\mathbb{R})$ *is connected;*
*(2)* $\mathbf{H}$ *has strong approximation.*

*Proof.* Assertion (1) is obvious in case (b) and assertion (2) follows from the case $n = 1$, which is the usual strong approximation for the adeles. In case (a), strong approximation is proven in [PR94, Theorem 7.12] and assertion (1) is due to E. Cartan. See [BT72, Corollaire 4.7] or [PR94, Proposition 7.6]. ♠

6.2. **Smooth $\mathbb{Z}_p$-models and principal $p$-pairs.** Suppose $\mathbf{H}$ is an algebraic group of $\mathbb{Q}$. A smooth $\mathbb{Z}_p$-model of $\mathbf{H}$ is a smooth scheme $\mathbf{H}_x$ over $\mathbb{Z}_p$ together with an isomorphism $\mathbf{H}_{\mathbb{Q}_p} \cong \mathbf{H}_x \times_{\mathbb{Z}_p} \mathbb{Q}_p$. If $\mathbf{H}$ is reductive (resp. semisimple), then a reductive (resp. semisimple) $\mathbb{Z}_p$-model is a smooth model which is also a reductive (resp. semisimple) group scheme over $\mathbb{Z}_p$.

Given a smooth $\mathbb{Z}_p$-model $\mathbf{H}_x$, suppose $K^p \leqslant \mathbf{H}(\mathbb{A}^{f,p})$ is a compact open subgroup. Define compact open subgroups $K_p$ and $L_p$ of $\mathbf{H}(\mathbb{Q}_p)$ by setting

$$\begin{aligned} K_p &= \mathbf{H}_x(\mathbb{Z}_p), \\ L_p &= \ker(\mathbf{H}_x(\mathbb{Z}_p) \to \mathbf{H}_x(\mathbb{F}_p)). \end{aligned}$$

Then set $K = K^p \times K_p$ and $L = K^p \times L_p$. Clearly, $L \trianglelefteq K$. Moreover, since $\mathbf{H}_x$ is smooth, reduction mod $p$ induces an isomorphism

$$K/L \cong \mathbf{H}_x(\mathbb{F}_p). \tag{53}$$

For future reference, we call $(K, L)$ the *principal $p$-pair* arising from the smooth $\mathbb{Z}_p$-model $\mathbf{H}_x$ and the compact open subgroup $K^p \leqslant \mathbf{H}(\mathbb{A}^{f,p})$.

**Remark 54.** Given $\mathbf{H}_x$, it is easy to see that there exists an open neighborhood $V$ of the identity in $\mathbf{H}(\mathbb{A}^{f,p})$ such that $K$ is neat as long as $K^p \subseteq V$. In particular, as long as $\mathbf{H}$ has a smooth $\mathbb{Z}_p$-model, there exists principal $p$-pairs $(K, L)$ with $K$ neat.

**Proposition 55.** *Suppose $\mathbf{H}$ is an algebraic group over $\mathbb{Q}$ with strong approximation and with $\mathbf{H}(\mathbb{R})$ connected. Let $(K, L)$ be a principal $p$-pair arising from $\mathbf{H}_x$ and $K^p$ as above. Then inclusion induces an isomorphism of groups $\Gamma_K/\Gamma_L \cong K/L$.*

*Proof.* The homomorphism $\Gamma_K/\Gamma_L \to K/L$ is obviously one-one by the definition of $\Gamma_L$. Since $\mathbf{H}$ has strong approximation, $\mathbf{H}(\mathbb{Q})L = \mathbf{H}(\mathbb{A}^f)$. So, for any $k \in K$, we can find $\ell \in L$ and $h \in \mathbf{H}(\mathbb{Q})$ such that $k = h\ell$. Then $h = \ell k^{-1} \in H(\mathbb{Q}) \cap K = \Gamma_K$. It follows that $\Gamma_K/\Gamma_L \to K/L$ is onto. ♠

If $\mathbf{H}$ is a semisimple group, then $\mathbf{H}$ has a semisimple $\mathbb{Z}_p$-model if and only if the $\mathbb{Q}_p$-group $\mathbf{H}_{\mathbb{Q}_p}$ is quasisplit and split over an unramified extension of $\mathbb{Q}_p$ [Tit79]. If these two conditions hold, then $\mathbf{H}_{\mathbb{Q}_p}$ is called an *unramified group* and we say that $\mathbf{H}$ is *unramified at $p$*. Isomorphism classes of semisimple $\mathbb{Z}_p$-models of $\mathbf{H}$ are then in one-one correspondence hyperspecial points $x$ in the Bruhat–Tits building of $\mathbf{H}_{\mathbb{Q}_p}$. This motivates our notation for smooth models.

6.3. **Congruence subgroups for simply connected groups.** For the rest of this section we fix a simply connected group $\mathbf{G}$ over $\mathbb{Q}$ of Hermitian type (as in §5.1). We write $\rho : \mathbf{G} \to \mathbf{G}^{\mathrm{ad}}$ for the canonical homomorphism to the adjoint group, which we call the *adjoint homomorphism*.

For $K \leqslant \mathbf{G}(\mathbb{A}^{f,p})$ a compact open subgroup, we set $\Gamma_K^{\mathrm{ad}} = \rho(\Gamma_K)$. By Proposition 52, $\mathbf{G}(\mathbb{R})$ is connected. So $\Gamma_K^{\mathrm{ad}}$ is an arithmetic subgroup of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q})_+$. Analogously to §5.1, we set $M_K = \Gamma_K^{\mathrm{ad}} \backslash D$.

**Lemma 56.** *Suppose $L$ and $K$ are compact open subgroups of $\mathbf{G}(\mathbb{A}^{f,p})$ with $L \trianglelefteq K$ and with $K$ neat. Then*

(1) *The adjoint homomorphism $\rho$ induces an isomorphism $\Gamma_K \to \Gamma_K^{\mathrm{ad}}$.*
(2) *We have*
$$\Gamma_K^{\mathrm{ad}}/\Gamma_L^{\mathrm{ad}} \cong \Gamma_K/\Gamma_L \cong K/L.$$
(3) *The natural morphism $M_L \to M_K$ is a finite étale Galois cover with Galois group $K/L$.*

*Proof.* (1) Since $K$ is neat, $K$ does not meet the center of $\mathbf{G}$. It follows that $\rho_{|\Gamma_K} : \Gamma_K \to \Gamma_K^{\mathrm{ad}}$ is one-one. But this homomorphism is onto by the definition of $\Gamma_K^{\mathrm{ad}}$.

(2) This follows from (1) and Proposition 55.

(3) Since $K$ is neat, $\Gamma_K$ is torsion free. So, by (1), $\Gamma_K^{\mathrm{ad}}$ is torsion free as well. It follows that $M_L \to M_K$ is étale and Galois with Galois group $\Gamma_K^{\mathrm{ad}}/\Gamma_L^{\mathrm{ad}} = K/L$. ♠

**Corollary 57.** *Suppose $\mathbf{G}$ is unramified at a prime $p$, $x$ is a hyperspecial point of the Bruhat–Tits building of $\mathbf{G}_{\mathbb{Q}_p}$, $K^p$ is a compact open subgroup of $\mathbf{G}(\mathbb{A}^f)$ and $(K, L)$ is the principal $p$-pair arising from this data. Assume that $K$ is neat. Then $M_L \to M_K$ is an étale Galois cover with Galois group $\mathbf{G}_x(\mathbb{F}_p)$.*

*Proof.* By Lemma 56, $M_L \to M_K$ is étale and Galois with Galois group $K/L$. But, under the hypotheses, $K/L \cong \mathbf{G}_x(\mathbb{F}_p)$. ♠

6.4. **Boundary components and reduction modulo $p$.** Now fix a rational boundary component $F$ of $D$ and let $N(F)$, $W(F)$, $U(F)$ be as before. Write $\mathbf{N}^{\mathrm{ad}}(F)$ (resp. $\mathbf{W}^{\mathrm{ad}}(F), \mathbf{U}^{\mathrm{ad}}(F)$) for the corresponding algebraic subgroups of $\mathbf{G}^{\mathrm{ad}}$, and write $\mathbf{N}(F)$ for the inverse image of $\mathbf{N}^{\mathrm{ad}}(F)$ in $\mathbf{G}$, a parabolic subgroup. Write $\mathbf{W}(F)$ for the unipotent radical of $\mathbf{N}(F)$ and $\mathbf{U}(F)$ for the center of $\mathbf{W}(F)$. Then $\rho$ induces a isomorphisms $\mathbf{W}(F) \cong \mathbf{W}^{\mathrm{ad}}(F)$ (resp. $\mathbf{U}(F) \cong \mathbf{U}^{\mathrm{ad}}(F)$). Moreover, the group of real points of $\mathbf{W}(F)$ (resp. $\mathbf{U}(F)$) is isomorphic to the Lie group $W(F)$ (resp. $U(F)$).

Since $F$ will be fixed in this section, we allow ourselves to drop it from the notation writing, for example, $\mathbf{N}$ instead of $\mathbf{N}(F)$.

Suppose further that $\mathbf{G}$ is unramified and fix a hyperspecial point $x$ giving us a group scheme $\mathbf{G}_x$ as above in §6.2. As in [SGA70, Exposé XXVI], write $\operatorname{Par} \mathbf{G}_x$ for the $\mathbb{Z}_p$-scheme representing the functor of parabolic subgroup schemes of $\mathbf{G}_x$. Then $\operatorname{Par} \mathbf{G}_x$ is proper over $\mathbb{Z}_p$. So it follows that the parabolic subgroup $\mathbf{N}$ extends uniquely to a parabolic subgroup scheme $\mathbf{N}_x$ of $\mathbf{G}_x$.

Write $\mathbf{W}_x$ for the unipotent radical of $\mathbf{N}_x$ [SGA70, XXII.5.11.4]. This is a closed subgroup scheme of $\mathbf{N}_x$ whose geometric fibers are connected and unipotent with the property that $\mathbf{N}_x/\mathbf{W}_x$ is reductive. In particular, it is an extension to $\mathbb{Z}_p$ of $\mathbf{W}$.

**Proposition 58.** *The subgroup $\mathbf{U}$ of $\mathbf{W}$ extends to a smooth central closed subgroup scheme $\mathbf{U}_x$ of $\mathbf{W}_x$. Moreover, if we set $\mathbf{V}_x = \mathbf{W}_x/\mathbf{U}_x$ we have*

$$\mathbf{U}_x \cong \mathbb{G}_a^{r_U};$$
$$\mathbf{V}_x \cong \mathbb{G}_a^{r_V}$$

*with $r_U = \dim \mathbf{U}$ and $r_V = \dim \mathbf{W}/\mathbf{U}$.*

*Proof.* First note that, we have an exact sequence of unipotent algebraic groups over $\mathbb{Q}$,

$$1 \to \mathbf{U} \to \mathbf{W} \to \mathbf{V} \to 1 \tag{59}$$

where both $\mathbf{U}$ and $\mathbf{V}$ are abelian. Moreover, the Lie algebras of $\mathbf{U}$ and $\mathbf{W}$ are defined in terms of root spaces relative to a suitable maximal $\mathbb{R}$-split torus of $\mathbf{G}$. (See [AMRT10, p. 143].)

Now, from [SGA70, Exposé 26, Proposition 2.1], it follows that $\mathbf{W}_x$ admits a finite filtration by closed subgroup schemes

$$\mathbf{W}_0 = \mathbf{W}_x \supseteq \mathbf{W}_1 \supseteq \mathbf{W}_2 \supseteq \cdots \supseteq \mathbf{W}_n = \{1\} \tag{60}$$

where the quotients $\mathbf{W}_i/\mathbf{W}_{i+1}$ are group schemes associated to vector bundles on $\mathbb{Z}_p$. So, since any vector bundle on $\mathbb{Z}_p$ is trivial, for each $i$, we have $\mathbf{W}_i/\mathbf{W}_{i+1} \cong \mathbb{G}_a^{r_i}$ for some nonnegative integer $r_i$.

In fact, when $\mathbf{G}$ is pinned the vector bundles are root spaces of $\mathbf{G}$ and, in the general case, the result is deduced by descent. Examining the proof, one sees that, given $\mathbf{W}_x$ as above, there are at most two nontrivial vector bundles involved and that $\mathbf{U}_x$ is a central subgroup scheme of $\mathbf{W}_x$ with $\mathbf{U}_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbf{U}$. The proposition then follows easily. ♠

**Theorem 61.** *Suppose $\mathbf{G}$ is unramified at $p$ and $(K, L)$ is the principal $p$ pair associated to a smooth $\mathbb{Z}_p$-model $\mathbf{G}_x$ and a compact open subgroup $K^p \leqslant \mathbf{G}(\mathbb{A}^{f,p})$. Set*

$$\mathbb{U} := \frac{\Gamma_K \cap \mathbf{U}(\mathbb{Q})}{\Gamma_L \cap \mathbf{U}(\mathbb{Q})} \leqslant \frac{\Gamma_K}{\Gamma_L}.$$

*Then $\mathbb{U}$ is an $\mathbb{F}_p$-vector space with*

$$\dim_{\mathbb{F}_p} \mathbb{U} = r_U.$$

*Proof.* Since $\mathbf{U}_x$ is a closed subgroup scheme of $\mathbf{G}_x$ with generic fiber $\mathbf{U}$, Proposition 58 implies that $\mathbf{U}(\mathbb{Q}_p) \cap K_p = \mathbf{U}(\mathbb{Q}_p) \cap \mathbf{G}_x(\mathbb{Z}_p) = \mathbf{U}_x(\mathbb{Z}_p) \cong \mathbb{Z}_p^{r_U}$.

Let $R_p := \ker[\mathbf{U}_x(\mathbb{Z}_p) \to \mathbf{U}_x(\mathbb{F}_p)]$. Then, we have a commutative diagram of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & R_p \arrow[r] \arrow[d, hook] & \mathbf{U}_x(\mathbb{Z}_p) \arrow[r] \arrow[d, hook] & \mathbf{U}_x(\mathbb{F}_p) \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & L_p \arrow[r] & \mathbf{G}_x(\mathbb{Z}_p) \arrow[r] & \mathbf{G}_x(\mathbb{F}_p) \arrow[r] & 1 \end{tikzcd}$$

where the vertical arrows are monomorphisms and the nontrivial arrows on the right are reduction module $p$. From this, it follows that $R_p = L_p \cap \mathbf{U}_x(\mathbb{Z}_p)$. But then, since $L_p \leqslant K_p$ and $K_p \cap \mathbf{U}(\mathbb{Q}_p) = \mathbf{U}_x(\mathbb{Z}_p)$, we have $R_p = L_p \cap \mathbf{U}(\mathbb{Q}_p)$ as well.

Set $K_U^p := K^p \cap \mathbf{U}(\mathbb{A}^{f,p})$, $K_U = K \cap \mathbf{U}(\mathbb{A}^f)$ and $L_U = L \cap \mathbf{U}(\mathbb{A}^f)$. Then $K_U^p$ (resp. $K_U, L_U$) is a compact open subgroup of $\mathbf{U}(\mathbb{A}^{f,p})$ (resp. $\mathbf{U}(\mathbb{A}^f)$).

Moreover,

$$\begin{aligned} K_U &= (K^p \times K_p) \cap \mathbf{U}(\mathbb{A}^f) = (K^p \times K_p) \cap (\mathbf{U}(\mathbb{A}^{f,p}) \times \mathbf{U}(\mathbb{Q}_p)) \\ &= (K^p \cap \mathbf{U}(\mathbb{A}^{f,p})) \times (K_p \cap \mathbf{U}(\mathbb{Q}_p)) = (K^p \cap \mathbf{U}(\mathbb{A}^{f,p}) \times \mathbf{U}_x(\mathbb{Z}_p). \end{aligned}$$

Similarly,

$$\begin{aligned} L_U &= (K^p \times L_p) \cap \mathbf{U}(\mathbb{A}^f) = (K^p \cap \mathbf{U}(\mathbb{A}^{f,p})) \times (L_p \cap \mathbf{U}(\mathbb{Q}_p)) \\ &= (K^p \cap \mathbf{U}(\mathbb{A}^{f,p})) \times R_p. \end{aligned}$$

It follows that $K_U/L_U \cong \mathbf{U}_x(\mathbb{Z}_p)/R_p \cong \mathbb{F}_p^{r_U}$.

We have $\Gamma_{K_U} = \mathbf{U}(\mathbb{Q}) \cap K \cap \mathbf{U}(\mathbb{A}^f) = \mathbf{G}(\mathbb{Q}) \cap K \cap \mathbf{U}(\mathbb{A}^f) = \Gamma_K \cap \mathbf{U}(\mathbb{Q})$. And, similarly, $\Gamma_{L_U} = \Gamma_L \cap \mathbf{U}(\mathbb{Q})$.

Now apply Proposition 55 to deduce that $\mathbb{U} = \Gamma_{K_U}/\Gamma_{L_U} \cong K_U/L_U$. ♠

6.5. **Hermitian pairs adapted to semisimple groups over $\mathbb{F}_p$.** Suppose $p$ is a prime number and $\mathbb{G}$ is a simply connected semisimple group over $\mathbb{F}_p$ for some prime $p$. Our goal is to find a simply connected group $\mathbf{G}$ of Hermitian type with a 0-cuspidal parabolic subgroup $\mathbf{P}$ such that

(1) $\mathbf{G}$ is unramified at $p$.
(2) For some, hence any, reductive model $\mathbf{G}_x$ of $\mathbf{G}$ over $\mathbb{Z}_p$, we have $\mathbf{G}_x \otimes_{\mathbb{Z}_p} \mathbb{F}_p \cong \mathbb{G}$.

We will say that such a pair $(\mathbf{G}, \mathbf{P})$ is *adapted* to $\mathbb{G}$.

**Proposition 62.** *Suppose $(\mathbf{G}, \mathbf{P})$ is adapted to $\mathbb{G}$. Then there exists a neat principal $p$-pair $(K, L)$ in $\mathbf{G}(\mathbb{A}^f)$ with $\Gamma_K/\Gamma_L \cong \mathbb{G}$. Consequently, the cover*

$$M_L \to M_K \tag{63}$$

*is finite étale with Galois group $\mathbb{G}(\mathbb{F}_p)$ and with*

$$\mathrm{ed}_{\mathbb{G}(\mathbb{F}_p)} M_L \geqslant \dim U(F) \tag{64}$$

*where $F$ is any 0-dimensional boundary component. If $(\mathbf{G}, \mathbf{P})$ is of tube type then the cover (63) is $p$-incompressible.*

*Proof.* Using Remark 54, we can find a neat principal $p$-pair $(K, L)$. As in §6.2, this gives rise to the étale Galois cover (63). Then, by Theorem 61, we have $\dim_{\mathbb{F}_p} \mathbb{U} = \dim U(F)$. Then (64) follows from Theorem 33, and the incompressibility for tube domains follows from Lemma 45, or directly from Theorem 34. ♠

6.6. **Reduction and Statement of Theorem.** The following lemma reduces the problem of finding an adapted pair $(\mathbf{G}, \mathbf{P})$ to the case where $\mathbb{G}$ is almost simple over $\mathbb{F}_p$.

**Lemma 65.** *Suppose $(\mathbf{G}_i, \mathbf{P}_i)$ are adapted to $\mathbb{G}_i$ for $i = 1, 2$. Then $(\mathbf{G}_1 \times \mathbf{G}_2, \mathbf{P}_1 \times \mathbf{P}_2)$ is adapted to $\mathbb{G}_1 \times \mathbb{G}_2$.*

*Proof.* This follows by noting that if $\mathbf{P}_1$ and $\mathbf{P}_2$ are 0-cuspidal, then so is $\mathbf{P}_1 \times \mathbf{P}_2$. ♠

We can write any simply connected, semisimple, algebraic group $H$ over a field $L$ as

$$H = \prod_{i=1}^{k} \mathrm{Res}_L^{K_i} H_i \tag{66}$$

where the $K_i$ are finite separable extensions of $L$, the $H_i$ are absolutely almost simple, simply connected groups and Res denotes Weil restriction. Moreover, the description of $H$ in (66) is essentially unique in that the list of $K_i$ and $H_i$ appearing is unique up to reordering and isomorphisms. Let us say that a semisimple group $H$ over a finite field is of *potentially Hermitian type* if the following two conditions are satisfied:

(1) None of the $H_i$ in (66) are of type $E_8$, $F_4$ or $G_2$.
(2) None of the $H_i$ are triality forms of $D_4$.

**Theorem 67.** *Suppose that $\mathbb{G}$ is a simply connected group of potentially Hermitian type over $\mathbb{F}_p$. Then there exists a 0-cuspidal Hermitian pair $(\mathbf{G}, \mathbf{P})$ adapted to $\mathbb{G}$.*

**Remark 68.**

(1) It will follow from the proof of the theorem and the classification of tube domains [FK94, p. 213] that we can choose the pair $(\mathbf{G}, \mathbf{P})$ to be of tube type in the cases that $\mathbb{G}$ has no factors of triality type $D_4$, no factors over $\overline{\mathbb{F}}_p$ of type $A_r$, with $r$ even, and no factors of type $E_6$. In particular, we may take $(\mathbf{G}, \mathbf{P})$ to be of tube type when $\mathbb{G}$ is of type $E_7$.

When $(\mathbf{G}, \mathbf{P})$ is of tube type, by combining Theorem 34, Theorem 61 and Theorem 67 we get $p$-incompressible congruence covers with Galois group $\mathbb{G}(\mathbb{F}_p)$.

(2) In contrast to (1), in [FKW19] split factors of type $A_r$ with $r$ even are also allowed, but factors of type $E_7$ are not. However, we do get weaker results for groups of type $A_r$ with $r$ even.

(3) If $\mathbb{G}$ is a form of $E_6$, the dimension of the Hermitian symmetric domain corresponding to $\mathbf{G}$ is 16, but we only get a lower bound of 8—the dimension of the centre of the unipotent radical of $\mathbf{P}$—for the $p$-essential dimension of congruence covers with Galois group $\mathbb{G}(\mathbb{F}_p)$.

6.7. **Satake–Tits Index.** Associated to any reductive group $\mathbf{G}$ over a field $L$ we have the Satake–Tits index [Tit66, p. 38]. This consists of the following data:

(1) The Dynkin diagram $D = \operatorname{Dyn}\mathbf{G}$.
(2) An action $\tau : \operatorname{Gal}(L_{\text{sep}}/L) \to \operatorname{Aut} D$ of the absolute Galois group of $L$ on the Dynkin diagram, sometimes called the $*$-action.
(3) A collection of $D_0$ of vertices called the *uncircled vertices*.

The vertices of $D$, which are defined in terms of simple roots, can be identified with conjugacy classes of maximal parabolic subgroups in $\mathbf{G}_{L_{\text{sep}}}$. Conjugation then gives the $*$-action. The uncircled vertices correspond to simple roots in the anisotropic kernel of $\mathbf{G}$. Then conjugacy classes of maximal parabolic subgroups defined over $L$ are in one-one correspondence with $\tau$ orbits in $D \setminus D_0$. These are the orbits that Tits calls *distinguished*, and, in the Satake–Tits index these orbits are drawn with circles around them.

If $K/L$ is a field extension, then we can identify $D$ with the Dynkin diagram $D_K$ of $\mathbf{G}_K$. So the first ingredient in the Satake–Tits index is insensitive to field extension. However, the $*$-action obviously changes: for example, if $K \subset L_{\text{sep}}$, then the $*$-action $\tau_K : \operatorname{Gal}(L_{\text{sep}}/K) \to \operatorname{Aut} D$ is obtained by restriction. If we write $D_{K,0}$ for the set of uncircled vertices in the Satake–Tits index of $\mathbf{G}_K$, then we have $D_{K,0} \subseteq D_0$.

In the case that $L = \mathbb{R}$ and $(\mathbf{G}, \mathbf{P})$ is a 0-cuspidal pair, the $*$-action of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ on $D = D_{\mathbb{R}}$ is the opposition on the Dynkin diagram. It preserves connected components of $D$ and, for $\mathbf{G}$ almost simple, it is

(1) the unique nontrivial action in types $A$, $D_n$ for $n$ odd, and in type $E_6$,
(2) the trivial action in all other types.

If $\mathbf{G}$ is almost simple, then Deligne's special vertex $s$ (see [Del79, p. 258]) lies in $D_{\mathbb{R},0}$. Its orbit under the $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ $*$-action corresponds to conjugacy class of real parabolics associated to the 0-dimensional boundary components. For general $\mathbf{G}$, there is a special vertex in each component of $D$, whose orbit is circled in the real Satake–Tits index. Write $Z_\infty$ for the union of the orbits of these special vertices under the real $*$-action. Then there exists a 0-dimensional rational boundary component if and only if $Z_\infty$ is contained in the circled vertices of the Satake–Tits index of $\mathbf{G}$.

6.8. **Lemmas.** We believe the following result on approximation for number fields is probably well-known, but for the convenience of the reader and lack of a published reference we give a proof following ideas we learned from [Sal82].

**Proposition 69.** *Suppose $n$ is a positive integer, $F$ is a number field and $S$ is a finite set of places of $F$. Then the map*

$$\mathrm{H}^1(F, S_n) \to \prod_{\nu \in S} \mathrm{H}^1(F_\nu, S_n)$$

*is surjective.*

*Proof.* An $S_n$-torsor or, equivalently, a degree $n$ étale algebra, over a field $K$ is determined by a monic, separable polynomial of degree $n$ over the field. So write $U_n(K)$ for the set of such polynomials. We get a surjective map $U_n(K) \twoheadrightarrow \mathrm{H}^1(K, S_n)$ taking a polynomial $p$ to the isomorphism class of the étale algebra $K[x]/p$.

Now $U_n(F_\nu)$ is an open subset of the space $F_\nu^{n-1}$ of all monic degree $n$ polynomials. This gives rise to a commutative diagram

$$\begin{tikzcd} U_n(F) \arrow[r] \arrow[d, two heads] & \prod_{\nu \in S} U_n(F_\nu) \arrow[d, two heads] \\ \mathrm{H}^1(F, S_n) \arrow[r] & \prod_{\nu \in S} \mathrm{H}^1(F_\nu, S_n) \end{tikzcd} \tag{70}$$

where the top horizontal arrow is just (the diagonal) inclusion and the bottom horizontal arrow is restriction.

Now, it is well-known and easy to see that the fibers of the right vertical arrow are open. (In fact, the bottom right set is finite and the right vertical arrow is continuous.) And it follows from weak approximation that the image of the top arrow is dense. From these two facts along with the surjectivity of the downward arrows, the proposition follows directly. ♠

**Remark 71.** In [Sal82, Theorem 5.8], Saltman gives variants of this argument which are applicable to finite groups $G$ admitting a generic Galois extension. In particular, this holds for any group $G$ admitting a faithful linear representation $V$ such that $V /\!\!/ G$ is rational.

**Lemma 72.** *Suppose $p$ is a prime number and $q = p^n$ with $n \in \mathbb{Z}_+$. Then there exists a totally real extension $L/\mathbb{Q}$ of degree $n$ in which the prime $p$ is inert.*

*Proof.* This follows immediately from Proposition 69. ♠

The next lemma applies an approximation theorem of Prasad and Rapinchuk [PR06] to produce isotropic algebraic groups $\mathbf{H}$ with specific behavior at places of $L$.

**Lemma 73.** *Suppose $\mathbb{H}$ is an absolutely almost simple, simply connected group over $\mathbb{F}_q$ which is of potentially Hermitian type. Let $L/\mathbb{Q}$ be a field extension as in Lemma 72. Write $S_\infty$ for the set of real places of $L$ and $\{\nu\}$ for the place lying above $p$. Then there exists a simply connected group $\mathbf{H}$ over $L$ such that*

*(1) $\mathbf{H}_\omega$ is of Hermitian symmetric type for each place $\omega \in S_\infty$.*

*(2)* $\mathbf{H}$ *is unramified at* $\nu$*, and, for some, hence any, reductive model* $\mathbf{H}_x$ *of* $\mathbf{H}$ *over the ring* $R_\nu$ *of integers in* $L_\nu$*, we have* $\mathbf{H}_x \otimes_{R_\nu} \mathbb{F}_q \cong \mathbb{H}$*.*

*(3) The special vertices of* $\mathbf{H}_\omega$ *are all the same in the Dynkin diagram* $\operatorname{Dyn}\mathbf{H}$ *of* $\mathbf{H}$*. Moreover, the orbit* $Z_\infty$ *of this vertex under the opposition is stable under the* $\operatorname{Gal}(L)$ *action and circled in the Satake–Tits index of* $\mathbf{H}$ *over* $L$*.*

*Proof.* Let $D$ denote the Dynkin diagram of $\mathbb{H}$ with vertices $\alpha_1, \ldots, \alpha_r$ corresponding to the simple roots as in the diagrams in [Hel78, p. 532]. Notice that, if $\mathbb{H}$ is of type $D_4$, then, owing to our assumption that $\mathbb{H}$ is not triality, we can assume that $\alpha_1$ is fixed by action of $\operatorname{Gal}(\mathbb{F}_q)$ on $D$.

Now we want to choose a simply connected real group $\mathbf{H}_\infty$ for each possible type of $\mathbb{H}$ as follows. (For most of the types there is no real choice, but we want to choose carefully in type $A$ and $D$.)

(a) If $\mathbb{H}$ is type $A_r$, then $\mathbf{H}_\infty$ if $\mathbf{SU}_{\left(\frac{r+1}{2}, \frac{r+1}{2}\right)}$ is $r$ is odd and $\mathbf{SU}_{\left(\frac{r}{2}, \frac{r}{2}+1\right)}$ if $r$ is even.

(b) If $\mathbb{H}$ is of type $B_r$, then we take $\mathbf{H}_\infty$ to be the unique form of Hermitian symmetric type: $\mathbf{Spin}_{2r-1,2}$.

(c) If $\mathbb{H}$ is of type $C_r$ we take $\mathbf{H}_\infty$ to be the symplectic group $\mathbf{Sp}_{2r}$.

(d) If $\mathbb{H}$ is of type $D_r$, then we take $\mathbf{H}_\infty$ to be the group $\mathbf{Spin}_{2r-2,2}$.

(e) In types $E_6$ and $E_7$, we take $\mathbf{H}_\infty$ to be the unique form of Hermitian type.

In each of these choices, we also choose an identification of $\operatorname{Dyn}\mathbf{H}_\infty$ with $D$. We can do this so that the labels on the vertices in [Hel78, p. 532] match up with the labels $\alpha_1, \ldots, \alpha_r$ for $D$. This is especially important for $\mathbb{H}$ of type $D_4$, where it implies that Deligne's special vertex, which is in that case $\alpha_1$, is fixed by the action of $\operatorname{Gal}(\mathbb{F}_p)$ on $D$.

Now, note that we have

$$\operatorname{Aut} D = \begin{cases} \mathbb{Z}/2, & \text{for } \mathbb{H} \text{ of type } A,\ E_6 \text{ or } D_r \text{ with } r \neq 4; \\ S_3, & \text{for } \mathbb{H} \text{ of type } D_4; \\ \{1\}, & \text{otherwise.} \end{cases} \tag{74}$$

Let $V$ denote the subgroup of $\operatorname{Aut} D$ stabilizing the $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$-orbit of the special vertex in $\mathbf{H}_\infty$. It is easy to see that $V = \operatorname{Aut} D$ itself except in the case where $\mathbb{H}$ is of type $D_4$. In that case, $V$ is the subset of $\operatorname{Aut} D = S_3$ stabilizing $\alpha_1$. It follows from our nontriality assumption that, in any case, the *-action of $\operatorname{Gal}(\mathbb{F}_q)$ on $D$ factors through $V$. Using Proposition 69 for the group $V$, we can find a quasisplit group $\mathbf{H}^{\mathrm{qs}}$ over $L$, which is split by a quadratic extension $M/L$ with $\operatorname{Gal}(M/L)$ acting on $D$ through $V$, such that $\mathbf{H}^{\mathrm{qs}}$ is an inner form of $\mathbf{H}_\infty$ at each infinite place of $L$, and, at the prime $\nu$, $\mathbf{H}^{\mathrm{qs}}$ is an unramified group over $R_\nu$ with reduction modulo $\nu$ equal to $\mathbb{H}$. For example, if $\mathbb{H}$ is of type $D_r$ for $r$ odd, then $\mathbf{H}_\infty$ is of outer type over $\mathbb{R}$. If $\mathbb{H}$ is split, then producing $\mathbf{H}^{\mathrm{qs}}$ is equivalent to finding a totally imaginary quadratic extension $M/L$ in which $\nu$ splits. On the other hand, if $\mathbb{H}$ is the unique nonsplit simply connected group over $\mathbb{F}_q$ of type $D_r$, then we need to find a totally imaginary quadratic extension $M/L$ in which $\nu$ is inert. As Proposition 69 allows us to produce quadratic extensions of $L$ with arbitrary behaviour at the $\omega$ and at $\nu$, we can produce such groups $\mathbf{H}^{\mathrm{qs}}$.

Since the $\operatorname{Gal}(L)$ action on $\mathbf{H}^{\mathrm{qs}}$ factors through $V$, the orbit $Z_\infty$ of Deligne's special vertex in $D = \operatorname{Dyn}\mathbf{H}_\infty$ is fixed by $\operatorname{Gal}(L)$.

Now, by [PR06, Theorem 1], we can find an inner form $\mathbf{H}$ of $\mathbf{H}^{\mathrm{qs}}$ such that $\mathbf{H}\otimes L_\omega = \mathbf{H}_\infty$ at all real primes $\omega$ and $\mathbf{H}\otimes R_\nu$ is unramified with reduction equal to $\mathbb{H}$. Moreover, by [PR06, Theorem 1 (iii)], we can do this in such a way that $Z_\infty$ is contained in the set of circled vertices for the Satake–Tits index of $D_L$. This completes the proof. ♠

6.9. **Proof of Theorem 67.** By Lemma 65, we can assume that $\mathbb{G}$ is almost simple. So $\mathbb{G} = \mathrm{Res}_{\mathbb{F}_p}^{\mathbb{F}_q}\mathbb{H}$ for some $q = p^n$ and some absolutely almost simple group $\mathbb{H}$ of potentially Hermitian type over $\mathbb{F}_q$. Let $L/\mathbb{Q}$ be a field extension as in Lemma 72 and let $\mathbf{H}$ be a group as in Lemma 73. Set $\mathbf{G} = \mathrm{Res}_{\mathbb{Q}}^{L}\mathbf{H}$.

Since Weil restriction commutes with base change, $\mathbf{G}$ is of Hermitian type and the base-change of $\mathbf{G}$ to $\mathbb{Q}_p$ is unramified with reduction modulo $p$ isomorphic to $\mathbb{G}$.

Let $\Sigma$ be the orbit under the opposition of the set of special vertices in the Dynkin diagram $\mathrm{Dyn}\,\mathbf{G}$. This is the subset of the Dynkin diagram corresponding to the 0-dimensional boundary components. Since $Z_\infty \subset \mathrm{Dyn}\,\mathbf{H}$ is stable under $\mathrm{Gal}(L)$ and contained in the circled vertices, it follows that $\Sigma$ is stable under $\mathrm{Gal}(\mathbb{Q})$ and it also follows that $\Sigma$ is contained in the circled vertices of $\mathrm{Dyn}\,\mathbf{G}$. Therefore, $\mathbf{G}$ contains a 0-cuspidal parabolic $\mathbf{P}$.

6.10. **The group of type $E_7$.** There is a unique simply connected form of $E_7$ over $\mathbb{R}$ of Hermitian type, and this group has $\mathbb{R}$-rank 3 ([Hel78, p. 518]). As a special case of Theorem 67, or more directly from [PR94], we see that there exists a form $\mathbf{G}$ of $E_7$ over $\mathbb{Q}$ of Hermitian type which has a 0-cuspidal parabolic $\mathbf{P}$ or, equivalently, has $\mathbb{Q}$-rank 3.

**Lemma 75.** *Any (simply connected) form $\mathbf{G}$ of $E_7$ over $\mathbb{Q}$ of Hermitian type with $\mathbb{Q}$-rank 3 is split at all finite primes $p$. Furthermore, such a group is unique up to isomorphism and extends to a reductive group scheme over $\mathbb{Z}$ which is also unique up to isomorphism.*

Such a group over $\mathbb{Z}$ was constructed explicitly by Baily in [Bai70] using integral octonions and by Gross in [Gro96] using Galois cohomology.

*Proof.* Only the first part of the lemma is perhaps not explicitly in the literature. To prove this, we note that for each prime $p$, the $\mathbb{Q}_p$-rank of $\mathbf{G}$ is at least 3 and the special vertex must be circled in the Satake–Tits diagram of $\mathbf{G}_{\mathbb{Q}_p}$, since both statements hold over $\mathbb{Q}$ (the second because it holds over $\mathbb{R}$ and the ranks are the same over $\mathbb{Q}$ and $\mathbb{R}$). By consulting the table of forms of $E_7$ in [Tit66, p. 59], we see that the only such form of $E_7$ over any $p$-adic field is split.

The uniqueness of $\mathbf{G}$ over $\mathbb{Q}$ up to isomorphism follows from the Hasse principal for simply connected groups (see [Gro96, p. 266]) and the existence of an integral model follows, for example, from [Gro96, Proposition 1.2]. The uniqueness of integral models follows from [Gro96, Proposition 2.1, 1)] and the fact that hyperspecial subgroups of $\mathbf{G}(\mathbb{Q}_p)$ are conjugate under $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}_p)$ for all primes $p$: these facts, together with the central exact sequence

$$1 \to \mu_2 \to \mathbf{G} \to \mathbf{G}^{\mathrm{ad}} \to 1$$

of group schemes over $\mathbb{Q}$ imply that given any two extensions of $\mathbf{G}$ to reductive group schemes over $\mathbb{Z}$, there exists an element $g$ of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q})$ such that conjugation by $g$ on $\mathbf{G}$ extends to an isomorphism of the two integral models. ♠

For the rest of this section we shall use $\mathbf{G}$ to also denote a reductive integral model of the $\mathbb{Q}$-group as in Lemma 75. The group $\Gamma := \mathbf{G}(\mathbb{Z})$ is then an arithmetic subgroup of $\mathbf{G}(\mathbb{R})$ (and is in fact a maximal arithmetic subgroup [Bai70, Theorem 5.2]). For any integer $n > 1$, let $\Gamma(n)$ be the kernel of the (surjective) reduction map $\Gamma \to \mathbf{G}(\mathbb{Z}/n\mathbb{Z})$. We set $M_\Gamma = \Gamma\backslash D$ and $M_{\Gamma(n)} = \Gamma(n)\backslash D$, where $D$ is the Hermitian symmetric domain corresponding to $\mathbf{G}$. Note that the actions on $D$ are not always faithful, but the quotients have a natural algebraic structure such that the maps $M_{\Gamma(n)} \to M_\Gamma$ are algebraic [BB66].

**Corollary 76.** *For any integer $n > 1$ and any prime $p \mid n$, the cover $M_{\Gamma(n)} \to M_\Gamma$ is $p$-incompressible.*

*Proof.* We may clearly assume that $n = p$. Since the group $\Gamma$ is not neat we cannot immediately apply Theorem 34. However, by embedding $\mathbf{G}$ in $\mathbf{GL}_{r,\mathbb{Z}}$ for some $r$, we see that if $N > 2$ is any integer then $\Gamma(N)$ is neat. Using this, and the fact that $D$ is a tube domain, the corollary follows in exactly the same way as Corollary 41. ♠

## Appendix A. Essential dimension of variations of Hodge structure

Let $B$ be an irreducible variety over $\mathbb{C}$ and let $L$ be a locally constant sheaf on $B^{\mathrm{an}}$. The purpose of this appendix is to define the essential dimension $\mathrm{ed}(L)$ of $L$ and give a lower bound for this when $L$ is the local system $\mathbb{H}_\mathbb{Z}$ underlying a variation of Hodge structure (or VHS, for short) $\mathbb{H}$.

Analogously to the definition of $\mathrm{ed}(f)$ for a generically étale morphism in the introduction, we define the essential dimension $\mathrm{ed}(L)$ of $L$ to be the minimum of the dimensions of irreducible varieties $B'$ such that the following condition holds:

- There is a locally constant sheaf $L'$ on $B'^{\mathrm{an}}$, a dense open subvariety $U$ of $B$, and a morphism $f : U \to B'$, such that $L|_{U^{\mathrm{an}}} \cong (f^{\mathrm{an}})^{-1}(L')$.

We then have the following result, which provides some weak evidence for Conjecture 1 beyond the case of variations whose associated period domain is Hermitian symmetric.

**Proposition 77.** *Let $B$ be a smooth irreducible variety over $\mathbb{C}$ and let $\mathbb{H}$ be an integral (polarised) variation of Hodge structure on $B$ with underlying local system $\mathbb{H}_\mathbb{Z}$. Then*

$$\mathrm{ed}(\mathbb{H}_\mathbb{Z}) \geqslant d,$$

*where $d$ is the dimension of the image of the period map associated to the VHS $\mathbb{H}$ (as defined in §1.5).*

The simple proof below was explained to us by Madhav Nori.

*Proof.* Suppose $\mathrm{ed}(\mathbb{H}_\mathbb{Z}) = n$. It follows from the definitions that there exists a nonempty open subvariety $U \hookrightarrow B$, a variety $B'$ of dimension $n$, a morphism $f : U \to B'$, and a local system $L'$ on $B'$ such that $\mathbb{H}_\mathbb{Z}|_{U^{\mathrm{an}}} \cong (f^{\mathrm{an}})^{-1}(L')$. Replacing $B$ by $U$, we may assume that $f$ is defined on all of $B$. Clearly $(f^{\mathrm{an}})^{-1}(L)$ is constant on (the analytification of) all fibres of $f$, so the local system $\mathbb{H}_\mathbb{Z}$ is also constant on these fibres.

Let $\phi : B^{an} \to \Gamma\backslash D$ be the period map as in §1.5. Then, since $\mathbb{H}_\mathbb{Z}$ is constant on the fibers of $f$, it follows from the theorem of the fixed part that $\phi$ is constant on the connected components of the fibers of $f$ [Del71, Corollaire 4.1.2], [Sch73, Theorem 7.22]. This implies that $\dim V \geqslant \dim Y$, where $Y$ is the image of the period map, i.e., $n \geqslant d$. ♠

**Remark 78.**

(1) For the purposes of Proposition 77, the algebraicity of the image of the period map proved in [BBT18] is irrelevant. Without knowing anything about the image, we may define the "dimension of the image of the period map" simply to be the rank of $d\phi$ at a general point of $B^{\mathrm{an}}$.
(2) The inequality in Proposition 77 can be strict: $\mathbb{H}_{\mathbb{Z}}$ can be a nonconstant local system even if $\mathbb{H}_x$ is a constant Hodge structure. However, one can see that there always exists a finite étale cover $p : \tilde{B} \to B$ such that $\mathrm{ed}(p^*(\mathbb{H}))$ is equal to the dimension of the image of the period map.

## Appendix B. Essential dimension of $\mathbf{Sp}_{2g}(\mathbb{F}_p)$

The purpose of this appendix it to give the computation, due to Dave Benson, of the essential dimension at $p$ of the group $\mathbf{Sp}_{2g}(\mathbb{F}_p)$.

**Theorem 79** (Benson)**.** *Let $p$ be an odd prime and $g$ a positive integer. Then*

$$\mathrm{ed}_{\mathbb{C}}(\mathbf{Sp}_{2g}(\mathbb{F}_p); p) = p^{g-1}.$$

*Proof.* Let $V := \mathbb{F}_p^{2g}$ with basis $e_1, \dots, e_{2g}$ and let $\phi$ be the symplectic form on $V$ given by

$$\phi = \sum_{i=1}^{g} e^*_{2g+1-i} \wedge e^*_i. \tag{80}$$

Then the group $G$ of linear transformations of $V$ preserving $\phi$ is an explicit presentation of $\mathbf{Sp}_{2g}(\mathbb{F}_p)$.

Now, $G$ has a faithful irreducible representation $M$ of degree $(p^g-1)/2$ [War72, Proposition 2.7]. In fact, although we do not need this fact, $(p^g-1)/2$ is the minimum dimension of a nontrivial irreducible representation of $G$ [LS74]. So $M$ is a faithful linear representation of $G$ of minimum possible dimension.

Write $P$ for the subgroup of $G$ consisting of all upper-triangular matrices in $G$ with 1s on the diagonal. Then $P$ is a $p$-Sylow subgroup of $G$. So, by Karpenko–Merkurjev [KM08], $\mathrm{ed}(G; p) = \mathrm{ed}\, P = \dim W$, where $W$ is a faithful representation of $P$ of minimal dimension.

Let $H$ denote the set of matrices in $G$ with 1s on the diagonal but with 0s everywhere else except the top row and rightmost column. Then it is not hard to see that $H$ is an extraspecial $p$-group: its center is the subgroup $Z$ consisting of matrices with 1s on the diagonal and 0s everywhere else except the top-right entry. And $H/Z \cong C_p^{2g-2}$. Moreover, it is easy to see that the center of $P$ is also $Z$.

Since $P$ is a $p$-group with cyclic center, it follows that any faithful representation of $P$ has a faithful irreducible constituent (and similarly for $H$). So, letting $W$ denote (as above) a faithful representation of $P$ of minimal dimension, it follows that $W$ is irreducible. Since the degree of any complex character divides the order of the group [Ser77, Corollary 6.5.2, p. 52], it follows that $\dim W$ is a power of $p$. From this, and the fact that $G$ has a faithful representation of degree $(p^g-1)/2$, it follows that $\dim W = p^k$ with $k \leqslant g-1$.

On the other hand, since $H$ is an extraspecial $p$-group of order $p^{2g-1}$, any faithful representation of $H$ has dimension at least $p^{g-1}$ [Gor80, Theorem 5.5, p. 208]. It follows that $\dim W = p^{g-1}$, and this completes the proof. ♠

**Remark 81.** As mentioned in the introduction, in work in progress, Jesse Wolfson's PhD student Hannah Knight has computed $\mathrm{ed}_{\mathbb{C}}(G;p)$ for groups such as $G = \mathbf{Sp}_{2g}(\mathbb{F}_{p^r})$ for $r > 1$ as well as for analogous orthogonal groups. See also [BMKS16] for results which can be used to compute the essential dimension at $p$ of groups such as $\mathbf{GL}_n(\mathbb{F}_p)$.

## References


[AMRT10] Avner Ash, David Mumford, Michael Rapoport, and Yung-Sheng Tai, *Smooth compactifications of locally symmetric varieties*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, With the collaboration of Peter Scholze. MR 2590897 4, 19, 24, 25, 26, 30

[Bai70] Walter L. Baily, Jr., *An exceptional arithmetic group and its Eisenstein series*, Ann. of Math. (2) **91** (1970), 512–549. MR 269779 36, 37

[BB66] W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528. MR 216035 19, 20, 21, 25, 37

[BBT18] Benjamin Bakker, Yohan Brunebarbe, and Jacob Tsimerman, *o-minimal GAGA and a conjecture of Griffiths*, 2018. 5, 38

[BHMV95] C. Blanchet, N. Habegger, G. Masbaum, and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology **34** (1995), no. 4, 883–927. MR 1362791 4, 16, 17

[BMKS16] Mohammad Bardestani, Keivan Mallahi-Karai, and Hadi Salmasian, *Minimal dimension of faithful representations for p-groups*, J. Group Theory **19** (2016), no. 4, 589–608. MR 3518394 39

[Bor72] Armand Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Differential Geometry **6** (1972), 543–560, Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays. MR 0338456 19, 25

[Bou98] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation. MR 1728312 25

[BR97] J. Buhler and Z. Reichstein, *On the essential dimension of a finite group*, Compositio Math. **106** (1997), no. 2, 159–179. 2, 7

[BT72] Armand Borel and Jacques Tits, *Compléments à l'article: "Groupes réductifs"*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 253–276. MR 315007 28

[CGR06] V. Chernousov, Ph. Gille, and Z. Reichstein, *Resolving G-torsors by abelian base extensions*, J. Algebra **296** (2006), no. 2, 561–581. 11

[Del71] Pierre Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR 498551 37

[Del79] ———, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and $L$-functions, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289. MR 546620 5, 33

[Don06] S. K. Donaldson, *Lefschetz pencils and mapping class groups*, Problems on mapping class groups and related topics, Proc. Sympos. Pure Math., vol. 74, Amer. Math. Soc., Providence, RI, 2006, pp. 151–163. MR 2264539 14

[DR15] Alexander Duncan and Zinovy Reichstein, *Versality of algebraic group actions and rational points on twisted varieties*, J. Algebraic Geom. **24** (2015), no. 3, 499–530, With an appendix containing a letter from J.-P. Serre. MR 3344763 8

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford. MR 1083353 3, 4, 27

[FK94] Jacques Faraut and Adam Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1994, Oxford Science Publications. MR 1446489 23, 24, 33

[FKW19] Benson Farb, Mark Kisin, and Jesse Wolfson, *The Essential Dimension of Congruence Covers*, arXiv e-prints (2019), arXiv:1901.09013v2. 2, 3, 4, 5, 7, 12, 15, 21, 22, 27, 28, 33

[FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR 2850125 12

[GM07] Patrick M. Gilmer and Gregor Masbaum, *Integral lattices in TQFT*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 5, 815–844. MR 2382862 17

[Gor80] Daniel Gorenstein, *Finite groups*, second ed., Chelsea Publishing Co., New York, 1980. MR 569209 38

[Gro96] Benedict H. Gross, *Groups over* $\mathbf{Z}$, Invent. Math. **124** (1996), no. 1-3, 263–279. MR 1369418 36

[Hel78] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press, Inc., New York–London, 1978. MR 514561 19, 35, 36

[KM08] Nikita A. Karpenko and Alexander S. Merkurjev, *Essential dimension of finite p-groups*, Invent. Math. **172** (2008), no. 3, 491–508. MR MR2393078 (2009b:12009) 7, 38

[Lan13] Kai-Wen Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, London Mathematical Society Monographs Series, vol. 36, Princeton University Press, Princeton, NJ, 2013. MR 3186092 27

[LS74] Vicente Landazuri and Gary M. Seitz, *On the minimal degrees of projective representations of the finite Chevalley groups*, J. Algebra **32** (1974), 418–443. MR 360852 38

[Mer13] Alexander S. Merkurjev, *Essential dimension: a survey*, Transform. Groups **18** (2013), no. 2, 415–481. MR 3055773 2, 7

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. 27

[Mos83] Samy M. Mostafa, *Die Singularitäten der Modulmannigfaltigkeit* $\overline{M}_g(n)$ *der stabilen Kurven vom Geschlecht* $g \geqslant 2$ *mit* $n$*-Teilungspunktstruktur*, J. Reine Angew. Math. **343** (1983), 81–98. MR 705878 15

[MP19] Keerthi Madapusi Pera, *Toroidal compactifications of integral models of Shimura varieties of Hodge type*, Ann. Sci. Éc. Norm. Supér. (4) **52** (2019), no. 2, 393–514. MR 3948111 27

[MR10] Aurel Meyer and Zinovy Reichstein, *Some consequences of the Karpenko-Merkurjev theorem*, Doc. Math. (2010), no. Extra vol.: Andrei A. Suslin sixtieth birthday, 445–457. MR 2804261 8

[MR12] Gregor Masbaum and Alan W. Reid, *All finite groups are involved in the mapping class group*, Geom. Topol. **16** (2012), no. 3, 1393–1411. MR 2967055 18

[Pin90] Richard Pink, *Arithmetical compactification of mixed Shimura varieties*, Bonner Mathematische Schriften, vol. 209, Universität Bonn, 1990. MR 1128753 28

[PR94] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen. MR 1278263 24, 28, 36

[PR06] Gopal Prasad and Andrei S. Rapinchuk, *On the existence of isotropic forms of semi-simple algebraic groups over number fields with prescribed local behavior*, Adv. Math. **207** (2006), no. 2, 646–660. MR 2271021 28, 34, 36

[Rag76] M. S. Raghunathan, *On the congruence subgroup problem*, Inst. Hautes Études Sci. Publ. Math. (1976), no. 46, 107–161. MR 0507030 22

[Rag04] ———, *The congruence subgroup problem*, Proc. Indian Acad. Sci. Math. Sci. **114** (2004), no. 4, 299–308. MR 2067695 22

[Rei10] Zinovy Reichstein, *Essential dimension*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 162–188. MR 2827790 7

[RY00] Zinovy Reichstein and Boris Youssin, *Essential dimensions of algebraic groups and a resolution theorem for G-varieties*, Canad. J. Math. **52** (2000), no. 5, 1018–1056, With an appendix by János Kollár and Endre Szabó. 9, 10, 11, 12

[Sal82] David J. Saltman, *Generic Galois extensions and problems in field theory*, Adv. in Math. **43** (1982), no. 3, 250–283. MR 648801 34

[Sch73] Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319. MR 382272 37

[Ser77] Jean-Pierre Serre, *Linear representations of finite groups*, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. 38

[Ser95] ———, *Groupes de congruence (d'après H. Bass, H. Matsumoto, J. Mennicke, J. Milnor, C. Moore)*, Séminaire Bourbaki, Vol. 10, Soc. Math. France, Paris, 1995, pp. Exp. No. 330, 275–291. MR 1610421 22

[SGA70] *Schémas en groupes. III: Structure des schémas en groupes réductifs*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 153, Springer-Verlag, Berlin-New York, 1970. MR 0274460 28, 30, 31

[SGA71] *Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics, Vol. 224, Springer-Verlag, Berlin-New York, 1971, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud. MR 0354651 10, 15

[SGA73] *Groupes de monodromie en géométrie algébrique. II*, Lecture Notes in Mathematics, Vol. 340, Springer-Verlag, Berlin, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II), Dirigé par P. Deligne et N. Katz. MR MR0354657 (50 #7135) 14

[Tit66] J. Tits, *Classification of algebraic semisimple groups*, Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, 1966, pp. 33–62. MR 37 #309 33, 36

[Tit79] ———, *Reductive groups over local fields*, Automorphic forms, representations and $L$-functions, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69. MR 546588 29

[War72] Harold N. Ward, *Representations of symplectic groups*, J. Algebra **20** (1972), 182–195. MR 286909 38

(Brosnan) Department of Mathematics, 1301 Mathematics Building, University of Maryland, College Park, MD 20742-4015, USA
*Email address*: `pbrosnan@umd.edu`

(Fakhruddin) School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, INDIA
*Email address*: `naf@math.tifr.res.in`