SINGULARITIES OF ADMISSIBLE NORMAL FUNCTIONS (WITH AN APPENDIX BY NAJMUDDIN FAKHRUDDIN)*

PATRICK BROSNAN[1], HAO FANG[2], ZHAOHU NIE, AND GREGORY PEARLSTEIN[3]

Abstract. In a recent paper, M. Green and P. Griffiths used R. Thomas' work on nodal hypersurfaces to sketch a proof of the equivalence of the Hodge conjecture and the existence of certain singular admissible normal functions. Inspired by their work, we study normal functions using Morihiko Saito's mixed Hodge modules and prove that the existence of singularities of the type considered by Griffiths and Green is equivalent to the Hodge conjecture. Several of the intermediate results, including a relative version of the weak Lefschetz theorem for perverse sheaves, are of independent interest.

1. Introduction

Let S be a complex manifold. Let $\mathcal{H} = (\mathcal{H}_{\mathbb{Z}}, F^{\bullet}\mathcal{H}_{\mathbb{C}})$ be a variation of Hodge structure of weight -1 with $\mathcal{H}_{\mathbb{Z}}$ torsion free on S. Then $\mathcal{H}$ induces a holomorphic family of compact complex tori $\pi : J(\mathcal{H}) \to S$. Let $\mathcal{J}(\mathcal{H})$ denote the sheaf of holomorphic sections of π. The exact sequence

$$0 \to \mathcal{H}_{\mathbb{Z}} \to \mathcal{H}/F^0\mathcal{H} \to \mathcal{J}(\mathcal{H}) \to 0$$

of sheaves of abelian groups on S induces a long exact sequence in cohomology. Writing $\mathrm{cl}_{\mathbb{Z}} : \mathrm{H}^0(S, \mathcal{J}(\mathcal{H})) \to \mathrm{H}^1(S, \mathcal{H}_{\mathbb{Z}})$ for the first connecting homomorphism, we find that, to each holomorphic section ν of π, we can associate a cohomology class $\mathrm{cl}_{\mathbb{Z}}(\nu) \in \mathrm{H}^1(S, \mathcal{H}_{\mathbb{Z}})$.

Assume now that $j : S \to \bar{S}$ is an embedding of S as a Zariski open subset of a complex manifold $\bar{S}$ [24, Definition 1.4]. If U is an (analytic) open neighborhood of a point $s \in \bar{S}$, we can restrict $\mathrm{cl}_{\mathbb{Z}}(\nu)$ to $U \cap S$ to obtain a class in $\mathrm{H}^1(U \cap S, \mathcal{H}_{\mathbb{Z}})$. Taking the direct limit over all open neighborhoods U of s, we obtain a class

$$\sigma_{\mathbb{Z},s}(\nu) \in (R^1 j_*\mathcal{H}_{\mathbb{Z}})_s = \varinjlim_{s \in U} \mathrm{H}^1(U \cap S, \mathcal{H}_{\mathbb{Z}}).$$

We call this class the *singularity* of ν at s, and we say that ν *is singular on* $\bar{S}$ if there exists a point $s \in \bar{S}$ with a non-torsion singularity $\sigma_{\mathbb{Z},s}(\nu)$.

In this paper, we will study $\sigma_{\mathbb{Z},s}(\nu)$ when ν is an *admissible normal function*; that is, a horizontal holomorphic section of π satisfying a very restrictive (but, from the point of view of algebraic geometry, very natural) constraint on its local monodromy. These normal functions were systematically studied by Morihiko Saito in [24]. Following Saito, we write $\mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}}$ for the group of admissible normal functions.

Our interest in singularities of admissible normal functions arises naturally from the study of primitive Hodge classes using the family of all hyperplane sections, in relation with the Hodge conjecture. To define these, we fix some notation.

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1.1. Let X be a smooth projective complex variety of dimension $2n$ with n an integer and let $\mathcal{L}$ be a very ample line bundle on X. Set $\bar{P} := \bar{P}_{\mathcal{L}} := |\mathcal{L}|$ and let

$$\mathcal{X} := \mathcal{X}_{\mathcal{L}} := \{(x, f) \in X \times \bar{P} \,|\, f(x) = 0\}.$$

We call $\mathcal{X}$ the *incidence variety associated to the pair* $(X, \mathcal{L})$. Let $\mathrm{pr} : \mathcal{X} \to X$ denote the first projection and $\pi : \mathcal{X} \to \bar{P}$ denote the second projection. Let $d := \dim \bar{P}$. Then $\mathcal{X}$ is smooth of dimension $r := 2n + d - 1$ because pr is a Zariski local fibration with fiber $\mathbb{P}^{d-1}$. Let $X^\vee \subset \bar{P}$ denote the dual variety and let $P := \bar{P} \setminus X^\vee$. Then the restriction $\pi^{sm} : \pi^{-1}P \to P$ is smooth and, hence, determines a variation of Hodge structure $\mathcal{H}$ of weight -1 over $P = \bar{P} - X^\vee$ with integral structure $\mathcal{H}_{\mathbb{Z}} = R^{2n-1}\pi^{sm}_*\mathbb{Z}(n)/\{\text{torsion}\}$.

1.2. Let H be a pure Hodge structure of weight -1. Then,

$$J(H) = \mathrm{Ext}^1(\mathbb{Z}(0), H)$$

in the category or polarizable mixed Hodge structures is the intermediate Jacobian of H. If V is a smooth projective variety define $J^p(V) = J(H^{2p-1}(V, \mathbb{Z}(n))$. Recall that for such a variety V, the Deligne cohomology groups $\mathrm{H}^{2p}_{\mathcal{D}}(V, \mathbb{Z}(p))$ fit into a short exact sequence

$$0 \to J^p(V) \to \mathrm{H}^{2p}_{\mathcal{D}}(V, \mathbb{Z}(p)) \to \mathrm{H}^{p,p}(V, \mathbb{Z}(p)) \to 0$$

where $\mathrm{H}^{p,p}(V, \mathbb{Z}(p)) := \mathrm{H}^{p,p}(V) \cap \mathrm{H}^{2p}(V, \mathbb{Z}(p))$ is the group of Hodge classes. Moreover, the sequence is functorial with respect to morphisms between smooth, projective schemes.

In particular, given a pair $(X, \mathcal{L})$ as in (1.1), let $\mathrm{H}^{n,n}(X, \mathbb{Z})^{\mathrm{prim}}$ denote the subgroup of primitive classes in $\mathrm{H}^{n,n}(X, \mathbb{C}) \cap \mathrm{H}^{2n}(X, \mathbb{Z})$ (i.e., classes killed by cupping with $c_1(\mathcal{L})$). Let Y be a smooth hyperplane section of X. Then, by the functoriality of Deligne cohomology, a class $\zeta \in \mathrm{H}^{n,n}(X, \mathbb{Z})^{\mathrm{prim}}$ defines a point

$$\mathrm{AJ}_Y(\zeta) \in J^n(Y)/J^n(X)$$

and hence a section $\mathrm{AJ}(\zeta) : P \to J(\mathcal{H})/J^n(X)$. By [24, Remark 1.7 (iii)], the resulting normal function $\mathrm{AJ}(\zeta)$ admissible. (In the above construction, the necessity of passage to the quotient by $J^n(X)$ can be removed by making a choice of lifting $\tilde{\zeta}$ of ζ to $\mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$.)

Note that, for any $p \in \bar{P}$, $\sigma_p(J^n(X)) = 0$. Therefore, we obtain a map $\tau_p : \mathrm{NF}(P, \mathcal{H})^{\mathrm{ad}}/J^n(X) \to (R^1 j_*\mathcal{H}_{\mathbb{Q}})_p$ where $j : P \to \bar{P}$ is the open embedding.

The following Theorem is one of the main results of this paper.

Theorem 1.3. *Let X and $\mathcal{L}$ be as in* (1.1)*. Pick a positive integer k and set $\mathcal{X} = \mathcal{X}_{\mathcal{L}^k}$ and $\bar{P} = |\mathcal{L}^k|$. Let $p \in \bar{P}$. Then*

(i) *We have a commutative diagram*

$$\begin{tikzcd} \mathrm{H}^{n,n}(X, \mathbb{Z})^{\mathrm{prim}} \arrow[r, "\mathrm{AJ}"] \arrow[d, "\alpha_p"'] & \mathrm{NF}(P, \mathcal{H})^{\mathrm{ad}}/J^n(X) \arrow[d, "\tau_p"] \\ \mathrm{H}^{2n}(X_p, \mathbb{Q}(n)) \arrow[r, "\beta_p"] & (R^1 j_*\mathcal{H}_{\mathbb{Q}})_p \end{tikzcd}$$

where α_p is induced by restriction and tensoring with $\mathbb{Q}$, and β_p is a map induced from the decomposition theorem of Beilinson, Bernstein and Deligne [3].

(ii) *There is an integer N, depending only on X, such that, for $k \geq N$, the restriction of β_p to the image of α_p is injective.*

Remark 1.4. In fact, as the proof of Theorem 1.3 will show, the conclusion of part (ii) of the theorem holds as long as N is large enough that the the vanishing cycles of a Lefschetz pencil of hyperplane sections of $\mathcal{L}^k$ are non-zero for $k \geq N$. In Proposition 5.9 we quote a result from [1] to show that there is such an N. However, recently, Dimca and Saito have shown that $N = 3$ suffices [9].

Motivated by work of Green and Griffiths [10], we apply the theorem to show that the Hodge conjecture is equivalent to the following conjecture concerning normal functions.

Conjecture 1.5. *Let X and $\mathcal{L}$ be as in* (1.1). *For every non-torsion primitive Hodge class ζ, there is an integer k such that* $\mathrm{AJ}(\zeta)$ *is singular on* $|\mathcal{L}^k|$.

By an argument of B. Totaro stated in a paper of Thomas [26] and recast in the language of the present paper in Theorem 6.5 below, the Hodge conjecture is equivalent to the statement that, for every pair $X, \mathcal{L}$ as in (1.1), and every $\zeta \in \mathrm{H}^{n,n}(X, \mathbb{Z})^{\mathrm{prim}}$, there is an integer k and a $p \in \bar{P} = |\mathcal{L}^k|$ such that $\alpha_p(\zeta)$ is non-torsion. From this we obtain the following result.

Theorem 1.6. *Conjecture 1.5 holds (for every even dimensional X and every non-torsion primitive middle dimensional Hodge class ζ) if and only if the Hodge conjecture holds (for all smooth projective algebraic varieties).*

Proof. Suppose first that Conjecture 1.5 holds. Then, for every $\zeta \in \mathrm{H}^{n,n}(X, \mathbb{Z})^{\mathrm{prim}}$ with $\dim X = 2n$, there exists $k \in \mathbb{Z}$ and $p \in |\mathcal{L}^k|$ such that $\tau_p(\mathrm{AJ}(\zeta))$ is non-zero. Since the diagram in Theorem 1.3 is commutative, this implies that $\alpha_p(\zeta)$ is non-zero. Thus, by Theorem 6.5, the Hodge conjecture holds.

Suppose conversely that the Hodge conjecture holds. Then, again by Theorem 6.5, for every $\zeta \in \mathrm{H}^{n,n}(X, \mathbb{Z})^{\mathrm{prim}}$ with $\dim X$ even, every ample line bundle $\mathcal{L}$ on X, and every $k \gg 0$, there is an $p \in |\mathcal{L}^k|$ such that $\alpha_p(\zeta)$ is non-zero. By part (ii) of the theorem, this implies that $\beta \circ \alpha_p(\zeta)$ is non-zero. Hence, by the commutativity, $\mathrm{AJ}(\zeta)$ is singular at p. □

In the paper of Green and Griffiths [10], a result analogous to Theorem 1.6 is stated, where $\bar{P} = |\mathcal{L}^k|$ is replaced by a modification of $\bar{S} \to \bar{P}$ such that the inverse image of $X^{\vee}$ is a normal crossing divisor. This result, which we recover by our methods in Corollary 7.22, seems weaker than Theorem 1.6 in the direction of proving the Hodge conjecture but stronger in the converse direction.

We have two intermediate results which may be particularly interesting in their own right. The first is Lemma 2.18 which gives a criterion for the intermediate extension functor $j_{!*}$ of [3] to preserve the exactness of a sequence of mixed Hodge modules. The second is Theorem 5.1 which we call the "perverse weak Lefschetz." It is a relative weak Lefschetz for families of hypersurfaces.

The organization of this paper is as follows. In §2, we study the general properties of admissible normal functions and their singularities. In particular, we show that the singularity is always a Tate class which lies in the local intersection cohomology, a subgroup of the local cohomology. In §3, we generalize the notion of absolute Hodge cohomology slightly. In §4, we introduce some notation concerning the decomposition theorem of Beilinson-Bernstein-Deligne and Saito. In §5, we prove the perverse weak Lefschetz theorem alluded to above. In §6, we prove Theorem 1.3 and Theorem 6.5 which together complete the proof of Theorem 1.6.

As mentioned above, the last section, §7, links our work directly to that of Green and Griffiths [10]. Doing this involves showing that certain types of singularities of admissible normal functions do not disappear after modification of the base. This answers a question of Green and Griffiths (see note at bottom of [10, p. 225]).

The appendix by N. Fakhruddin improves the perverse weak Lefschetz and one of its consequences (Theorem 5.11) by adding certain hypotheses.

Notation. A complex variety will mean an integral separated scheme X of finite type over $\mathbb{C}$. Following Saito, we write d_X for dim X to shorten some of the expressions. If $\mathcal{E}$ is a locally free sheaf on X and $s \in \Gamma(X, \mathcal{E})$, we write $V(s)$ for the zero locus of s [11].

By a perverse sheaf we mean a perverse sheaf for the middle perversity. If $f : X \to Y$ is a morphism between complex varieties, we write $f_*, f_!, f^*, f^!$ for the derived functors between the bounded derived categories of constructable sheaves following the convention of [3, 1.4.2.3]. However, we deviate from this convention is §7 where we write $f_*\mathcal{F}$ (instead of $H^0 f_*\mathcal{F}$) for the usual push-forward of a constructible sheaf $\mathcal{F}$.

We write MHS for the category of mixed Hodge structures which are graded-polarizable in the sense that each graded piece is polarizable. When necessary for clarity, we write MHS_R for the category of mixed Hodge structures with coefficients in a ring R. Similarly, we write $\mathrm{VMHS}(S)$ or $\mathrm{VMHS}_R(S)$ for the category of (graded-polarizable) variations of mixed Hodge structures with R coefficients over a separated analytic space S.

Remarks 1.7. The reader might guess that analogues of the results in this paper can be obtained in characteristic p by replacing mixed Hodge modules by mixed perverse sheaves. Indeed this is the case. For example, we expect that analogues of our main results hold when the Griffiths intermediate Jacobian is replaced with the ℓ-adic intermediate Jacobian of [4].

The paper [6], which appeared on the ArXiv shortly after the present paper, has a parallel investigation of singularities of primitive Hodge classes. While we use Theorem 5.1, [6] uses an argument base on the perverse hard Lefschetz theorem.

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2. Admissible normal functions and Intersection Cohomology

Let $j : S \to \bar{S}$ be an open immersion of smooth complex manifolds. If E is a local system of $\mathbb{Q}$-vector spaces on S and $s \in \bar{S}$ is a closed point, we set

$$\mathrm{H}^i(E)_s := \varinjlim_{s \in U} \mathrm{H}^i(S \cap U, E)$$

where the limit is taken over all open neighborhoods U of s. If $i : \{s\} \to \bar{S}$ denotes the inclusion morphism, then $\mathrm{H}^i(E)_s = \mathrm{H}^i(\{s\}, i^* j_* E)$. (Note our convention for j_*. We also ask the reader to distinguish between the integer i and the morphism i based on the context.)

2.1. Now suppose that $\bar{S}$ is equidimensional of dimension d and $j : S \to \bar{S}$ of (2.1) is an open immersion of S as a Zariski open subset of $\bar{S}$ [24, Definition 1.4]. The local system E defines a perverse sheaf $E[d]$ on S (since S is smooth). Moreover, by intermediate extension, it defines a perverse sheaf $j_{!*}E[d]$ on $\bar{S}$. Adopting the standard notation, we set

$$\mathrm{IH}^i(\bar{S}, E) = \mathrm{H}^{i-d}(\bar{S}, j_{!*}E[d])$$
$$\mathrm{IH}^i(E)_s = \mathrm{H}^{i-d}(\{s\}, i^* j_{!*}E[d]).$$

Note that, $j_{!*}E[d]$ maps to $j_*E[d]$: it is defined as a subobject of ${}^p j_*E[d] := {}^pH^0(j_*E[d])$ in the category of perverse sheaves and j_* is left t-exact. Therefore we have natural maps

$$\mathrm{IH}^i(\bar{S}, E) \to \mathrm{H}^i(S, E); \quad \mathrm{IH}^i(E)_s \to \mathrm{H}^i(E)_s. \tag{2.2}$$

Lemma 2.3. *The maps in* (2.2) *are isomorphisms for $i = 0$ and monomorphisms for $i = 1$.*

Proof. Since j_* is left t-exact, we have a distinguished triangle

$$ {}^p j_*E[d] \to j_*E[d] \to {}^p\tau_{\geq 1} j_*E[d] \to {}^p j_*E[d+1]. \tag{2.4}$$

By [3, 2.1.2.1], $H^i({}^p\tau_{\geq 1} j_*E[d]) = 0$ for $i \leq -d$. Therefore, the map ${}^p j_*E[d] \to j_*E[d]$ induces isomorphisms

$$\mathrm{H}^i(\bar{S}, {}^p j_*E[d]) \to \mathrm{H}^i(S, E[d]),$$
$$\mathrm{H}^i({}^p j_*E[d])_s \to \mathrm{H}^i(j_*E[d])_s$$

for $i \leq -d$. Moreover, we have injections $\mathrm{H}^{-d+1}(\bar{S}, {}^p j_*E[d]) \to \mathrm{H}^{-d+1}(S, E[d])$ and $\mathrm{H}^{-d+1}({}^p j_*E[d])_s \to \mathrm{H}^{-d+1}(j_*E[d])_s$.

Similarly, there is an exact sequence

$$0 \to j_{!*}E[d] \to {}^p j_*E[d] \to F \to 0 \tag{2.5}$$

in $\mathrm{Perv}(\bar{S})$ where F is a perverse sheaf supported on $\bar{S} \setminus S$. It follows that $H^i(F) = 0$ for $i \leq -d$. The result now follows immediately from the long exact sequence in cohomology (resp. local cohomology at s) induced by (2.5). □

2.6. Suppose that $\mathcal{H}$ is a variation of Hodge structure of weight -1 on S. We write $\mathrm{NF}(S, \mathcal{H})$ for the group of horizontal normal functions from S into $J(\mathcal{H})$. By [24], there is a canonical isomorphism $\mathrm{NF}(S, \mathcal{H}) = \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbb{Z}, \mathcal{H})$. Moreover, if we let $\mathrm{VMHS}\,(S)^{\mathrm{ad}}_{\bar{S}}$ denote the subcategory of variations of mixed Hodge structure on S which are admissible with respect to the open immersion $j : S \to \bar{S}$, then the group $\mathrm{Ext}^1_{\mathrm{VMHS}\,(S)^{\mathrm{ad}}_{\bar{S}}}(\mathbb{Z}, \mathcal{H})$ is a subgroup of $\mathrm{NF}(S, \mathcal{H})$. Following [24, Definition 1.4], we call these the *admissible normal functions with respect to* $\bar{S}$ and write $\mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$ for this group.

Remark 2.7. Let $\nu \in \mathrm{NF}(S, \mathcal{H})$ be a normal function on S. Let $\mathrm{Shv}(S)$ denote the category of sheaves of abelian groups on S and write $r : \mathrm{VMHS}(S) \to \mathrm{Shv}(S)$ for the forgetful functor taking a variation of mixed Hodge structure $\mathcal{H}$ on S to its underlying sheaf of abelian groups $\mathcal{H}_{\mathbb{Z}}$. Then $\mathrm{cl}_{\mathbb{Z}}(\nu)$ is the image of ν under the composition

$$\mathrm{NF}(S, \mathcal{H}) = \mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbb{Z}, \mathcal{H}) \xrightarrow{r} \mathrm{Ext}^1_{\mathrm{Shv}(S)}(\mathbb{Z}, \mathcal{H}_{\mathbb{Z}}) = \mathrm{H}^1(S, \mathcal{H}_{\mathbb{Z}}).$$

Similarly, suppose $j : S \to \bar{S}$ is as in (2.1) and $i : \{p\} \to \bar{S}$ is the inclusion of a point. Then the map $\sigma_{\mathbb{Z},p} : \mathrm{NF}(S, \mathcal{H}) \to H^1(\mathcal{H}_{\mathbb{Z}})_p$ is given by the composition of the above displayed equation with

$$\mathrm{H}^1(S, \mathcal{H}_{\mathbb{Z}}) \to \mathrm{H}^1(\bar{S}, Rj_*\mathcal{H}_{\mathbb{Z}}) \to \mathrm{H}^1(\{p\}, i^*Rj_*\mathcal{H}_{\mathbb{Z}}).$$

We leave the verification of these compatibilities to the reader. In fact, we will always work with the above formulation of $\mathrm{cl}_{\mathbb{Z}}$ and $\sigma_{\mathbb{Z},p}$, and the reader who is willing to take the above composition as the definition of $\sigma_{\mathbb{Z},p}$ can dispense with this verification.

The following is a type of "universal coefficient theorem" for variations of mixed Hodge structure and normal functions.

Lemma 2.8. *Let S be as in 2.1.*

(i) *Let $\mathcal{V}$ and $\mathcal{W}$ be variations of mixed Hodge structure on S. If $\pi_0(S)$ is finite, then the natural map*

$$\mathrm{Hom}_{\mathrm{VMHS}_{\mathbb{Z}}(S)}(\mathcal{V},\mathcal{W})\otimes\mathbb{Q}\to\mathrm{Hom}_{\mathrm{VMHS}_{\mathbb{Q}}(S)}(\mathcal{V}_{\mathbb{Q}},\mathcal{W}_{\mathbb{Q}})$$

is an isomorphism.

(ii) *If $\pi_0(S)$ is finite and $\pi_1(S,s)$ is finitely generated for each $s\in S$, then the natural map*

$$\mathrm{Ext}^1_{\mathrm{VMHS}_{\mathbb{Z}}(S)}(\mathbb{Z},\mathcal{W})\otimes\mathbb{Q}\to\mathrm{Ext}^1_{\mathrm{VMHS}_{\mathbb{Q}}(S)}(\mathbb{Q},\mathcal{W}_{\mathbb{Q}})$$

is an isomorphism.

(iii) *If the conditions of (ii) are satisfied, then, for any variation of pure Hodge structure $\mathcal{H}$ of weight -1 on S, the natural map*

$$\mathrm{NF}(S,\mathcal{H})\otimes\mathbb{Q}=\mathrm{Ext}^1_{\mathrm{VMHS}_{\mathbb{Z}}(S)}(\mathbb{Z},\mathcal{H})\otimes\mathbb{Q}\to\mathrm{Ext}^1_{\mathrm{VMHS}_{\mathbb{Q}}(S)}(\mathbb{Q},\mathcal{H}_{\mathbb{Q}})$$

is an isomorphism.

Proof. (i) is obvious, and (iii) follows directly from (ii). We leave to the reader the fact that the map in (ii) is injective. To see that it is surjective, suppose

$$0\to\mathcal{W}_{\mathbb{Q}}\to\mathcal{V}\xrightarrow{p}\mathbb{Q}\to 0$$

is an exact sequence of rational variations of mixed Hodge structure on S. Assume first that S is connected. Then, using the fact that $\pi_1(S,s)$ is finitely generated, we can find a lattice $\mathcal{V}_{\mathbb{Z}}\subset\mathcal{V}$ such that $\mathcal{V}_{\mathbb{Z}}\cap\mathcal{W}_{\mathbb{Q}}=\mathcal{W}$. We then have $p(\mathcal{V}_{\mathbb{Z}})=\alpha\mathbb{Z}$ for some $\alpha\in\mathbb{Q}^*$. Scaling by α we obtain the desired result.

We leave the case where S has finitely many connected components (where we may have to scale by more than one α and add up the results) to the reader. □

Corollary 2.9. *Under the assumptions of Lemma 2.8 and the notation of* (2.6)*, we have*

$$\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar S}\otimes\mathbb{Q}=\mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar S}}(\mathbb{Q},\mathcal{H}_{\mathbb{Q}}).$$

Proof. This follows directly from the Lemma 2.8 because admissibility of variations of a mixed Hodge structure $\mathcal{V}$ depends only on $\mathcal{V}_{\mathbb{Q}}$. □

Definition 2.10. If $\mathcal{H}$ is a $\mathbb{Q}$-VMHS, we call $\nu\in\mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar S}}(\mathbb{Q},\mathcal{H})$ an *admissible* $\mathbb{Q}$-*normal function*. We write $\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}$ for the group of such functions (and $\mathrm{NF}(S,\mathcal{H})$ for $\mathrm{Ext}^1_{\mathrm{VMHS}(S)_{\bar S}}(\mathbb{Q},\mathcal{H})$)). We write $\mathrm{cl}:\mathrm{NF}(S,\mathcal{H})\to\mathrm{H}^1(S,\mathcal{H})$ and $\sigma_p:\mathrm{NF}(S,\mathcal{H})\to(Rj_*\mathcal{H})_p$ for the obvious analogues of $\mathrm{cl}_{\mathbb{Z}}$ and $\sigma_{\mathbb{Z},p}$.

The main result of this section is the following.

Theorem 2.11. *Let $j:S\to\bar S$ be an open immersion of smooth manifolds as in* (2.1) *and let $\mathcal{H}$ be a $\mathbb{Q}$ variation of pure Hodge structure of weight -1 on S. The group homomorphism* $\mathrm{cl}:\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar S}\to\mathrm{H}^1(S,\mathcal{H})$ *factors through* $\mathrm{IH}^1(\bar S,\mathcal{H})$*. Similarly, for each $s\in\bar S$, the map $\sigma_s:\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar S}\to\mathrm{H}^1(\mathcal{H})_s$ factors through* $\mathrm{IH}^1(\mathcal{H})_s$.

We will use a few lemmas concerning the intermediate extensions of perverse sheaves and mixed Hodge modules on S. The first concerns the fact that $j_{!*}$ is "End-exact" when applied to perverse sheaves on S; that is, it preserves injections and surjections. In N. Katz's book [15, p. 87], this fact is stated and a proof is sketched. For completeness and the convenience of the reader, we give a proof here.

Lemma 2.12. *Let $j : S \to \bar{S}$ be an open immersion as in 2.11. Suppose that the sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact in $\mathrm{Perv}(S)$. *Then $j_{!*}(f)$ is an injection and $j_{!*}(g)$ is a surjection in* $\mathrm{Perv}(\bar{S})$.

Proof. By [3, Prop 1.4.16], ${}^p j_!$ is right-exact and ${}^p j_*$ is left-exact. From the definition of the intermediate extension functor [3, 2.1.7], we have the following commutative diagram with exact top and bottom rows.

$$\begin{tikzcd}
 & {}^p j_! A \arrow[r] \arrow[d, two heads] & {}^p j_! B \arrow[r] \arrow[d, two heads] & {}^p j_! C \arrow[r] \arrow[d, two heads] & 0 \\
 & j_{!*} A \arrow[r] \arrow[d, tail] & j_{!*} B \arrow[r] \arrow[d, tail] & j_{!*} C \arrow[d, tail] & \\
0 \arrow[r] & {}^p j_* A \arrow[r] & {}^p j_* B \arrow[r] & {}^p j_* C &
\end{tikzcd}$$

The proposition now follows from chasing the diagram. □

2.13. For "_" a separated reduced analytic space, we write $\mathrm{MHM}(\_)$ for the category of mixed Hodge modules on "_" and $\mathrm{MHM}(\_)^p$ for the category of polarizable mixed Hodge modules [21, 2.17.8]. (It is understood that a left upper p stands for "perversity", while a right upper p stands for "polarization" in this paper.) If $j : S \to \bar{S}$ is an open immersion as in (2.1), then we write $\mathrm{MHM}(S)^p_{\bar{S}}$ for the category of polarizable mixed Hodge modules on S which are extendable to $\bar{S}$. Recall that a mixed Hodge module M in $\mathrm{MHM}(S)$ is said to be *smooth* if $\mathrm{rat}\, M$ is isomorphic to $E[d_S]$ where E is a local system on S where $\mathrm{rat} : \mathrm{MHM}(S) \to \mathrm{Perv}(S)$ denotes the functor of [21, Theorem 0.1]. By [21, Theorem 3.27] we have an equivalence of categories

$$\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}} \cong \mathrm{MHM}(S)^{ps}_{\bar{S}}$$

where the right hand denotes the full subcategory of $\mathrm{MHM}(S)^p_{\bar{S}}$ consisting of smooth mixed Hodge modules.

Definition 2.14. If $a, c \in \mathbb{Z}$, then we say that an object M in $\mathrm{MHM}(\_)$ has *weights in the interval* $[a, c]$ if $\mathrm{Gr}^W_i M = 0$ for $i \notin [a, c]$.

We write $j_{!*} : \mathrm{MHM}(S)_{\bar{S}} \to \mathrm{MHM}(\bar{S})$ for the functor given by

$$j_{!*} M = \mathrm{im}(H^0 j_! M \to H^0 j_* M).$$

By [21, 2.18.1], both $j_!$ and j_* preserve polarizability. Therefore, for M in $\mathrm{MHM}(S)^p_{\bar{S}}$, $j_{!*} M$ is in $\mathrm{MHM}(\bar{S})^p$.

Lemma 2.15. *If M is an object in* $\mathrm{MHM}(S)^p_{\bar{S}}$ *with weights in the interval* $[a, c]$, *then $j_{!*} M$ also has weights in* $[a, c]$.

Proof. By [21, Proposition 2.26], $H^0 j_! M$ has weights $\leq c$ and $H^0 j_* M$ has weights $\geq a$. Since maps between polarizable mixed Hodge modules are strict with respect to the weight filtration, the functor $\mathrm{Gr}^W_i : \mathrm{MHM}(\bar{S})^p \to \mathrm{MHM}(\bar{S})^p$ is exact [7, Proposition 1.1.11] for each $i \in \mathbb{Z}$. It follows that $j_{!*} M = \mathrm{im}(H^0 j_! M \to H^0 j_* M)$ has weights in $[a, c]$. □

2.16. The functor $j_{!*}$ is not in general exact. However, for C, A pure of respective weights c and a in $\mathrm{MHM}(S)^p$,

$$\mathrm{Ext}^j(C, A) = 0 \text{ if } c < a + j.$$

This is stated explicitly in the algebraic case in [21, Eq. 4.5.3]; however, the proof given there clearly applies to the polarizable analytic case.

From this and the fact that $j_{!*}$ commutes with finite direct sums, we see that $j_{!*}$ preserves the exactness of the sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{2.17}$$

provided A is pure of weight a and C is pure of weight c with $c < a + 1$.

Lemma 2.18. *Suppose that the entries in* (2.17) *consist of objects in* $\mathrm{MHM}(S)^p_{\bar{S}}$ *where A is pure of weight a and C is pure of weight c with $c \le a + 1$. Then the sequence*

$$0 \to j_{!*}A \xrightarrow{j_{!*}(f)} j_{!*}B \xrightarrow{j_{!*}(g)} j_{!*}C \to 0 \tag{2.19}$$

is exact in $\mathrm{MHM}(\bar{S})^p$.

Proof. Write $i : Z \to \bar{S}$ for the complement of S in $\bar{S}$. The lemma will follow mainly from [3, Corollary 1.4.25] which gives the following description of the intermediate extension in our context.

(*) $j_{!*}B$ is the unique prolongement of B in $\mathrm{MHM}(\bar{S})$ with no non-trivial sub-object or quotient object in the essential image of the functor $i_* : \mathrm{MHM}(Z) \to \mathrm{MHM}(\bar{S})$.

Here we have used the fact that $\mathrm{rat} : \mathrm{MHM}(\_) \to \mathrm{Perv}(\_)$ is faithful and exact to deduce (*) from the corresponding statement in [3].

By (2.16), we already know that the theorem holds for $c \le a$; thus, it suffices to consider the case $c = a + 1$.

By Lemma 2.15, we know that $j_{!*}B$ has weights in the interval $[a, c]$. By Lemma 2.12 and the exactness of Gr^W, we know that $\mathrm{Gr}^W_c\, j_{!*}B = j_{!*}C \oplus D$ for some object D in $\mathrm{MHM}(\bar{S})^p$ which is pure of weight c. By the definition of $j_{!*}B$, we know that D is supported on Z. But, since $j_{!*}B$ surjects onto D via the composition

$$j_{!*}B \twoheadrightarrow \mathrm{Gr}^W_c\, j_{!*}B \twoheadrightarrow D$$

this contradicts (*) unless $D = 0$.

Thus $\mathrm{Gr}^W_c\, j_{!*}B = j_{!*}C$. By similar reasoning, we see that $\mathrm{Gr}^W_a\, j_{!*}B = j_{!*}A$. □

Lemma 2.20. *Let S be as in Theorem 2.11. Then the functor* $\mathrm{VMHS}_{\mathbb{Q}}(S)^{\mathrm{ad}}_{\bar{S}} \rightsquigarrow \mathrm{MHM}(S)^p_{\bar{S}}$ *sending a variation $\mathcal{V}$ to $\mathcal{V}[d]$ induces isomorphisms*

$$\mathrm{Ext}^i_{\mathrm{VMHS}_{\mathbb{Q}}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathcal{V}, \mathcal{W}) \xrightarrow{\cong} \mathrm{Ext}^i_{\mathrm{MHM}(S)^p_{\bar{S}}}(\mathcal{V}[d], \mathcal{W}[d])$$

for $i = 0, 1$.

Proof. For $i = 0$ this follows from [21, Theorem 3.27]. For $i = 1$, this follows from the (easy) fact that an extension of smooth perverse sheaves is smooth. □

Corollary 2.21. *Suppose $j : S \to \bar{S}$ and $\mathcal{H}$ are as in Theorem 2.11. Then the restriction map*

$$\mathrm{Ext}^1_{\mathrm{MHM}(\bar{S})^p}(\mathbb{Q}[d], j_{!*}\mathcal{H}[d]) \xrightarrow{j^*} \mathrm{Ext}^1_{\mathrm{MHM}(S)^p_{\bar{S}}}(\mathbb{Q}[d], \mathcal{H}[d]) = \mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$$

is an isomorphism.

Proof. Lemma 2.18 shows that j^* is surjective. On the other hand, suppose $\overline{\nu} \in \mathrm{Ext}^1_{\mathrm{MHM}(\bar{S})^p}(\mathbb{Q}[d], j_{!*}\mathcal{H}[d])$ given by the sequence

$$0 \to j_{!*}\mathcal{H}[d] \to B \to \mathbb{Q}[d] \to 0$$

is in the kernel of j^*. Then there is a splitting $s : \mathbb{Q}[d] \to j^*B$. Applying $j_{!*}$ to s, we obtain a splitting $\mathbb{Q}[d] \to j_{!*}j^*B$. But it is easy to see from Lemma 2.18 that $B = j_{!*}j^*B$ (as both are extensions of $\mathbb{Q}[d]$ by $j_{!*}\mathcal{H}[d]$). Therefore $\overline{\nu} = 0$. It follows that j^* is injective. □

Proof of Theorem 2.11. The diagram

$$\begin{tikzcd} \mathrm{Ext}^1_{\mathrm{MHM}(\bar{S})^p}(\mathbb{Q}[d], j_{!*}\mathcal{H}[d]) \arrow[r, "j^*"] \arrow[d, "\mathrm{rat}"'] & \mathrm{Ext}^1_{\mathrm{MHM}(S)^p_{\bar{S}}}(\mathbb{Q}[d], \mathcal{H}[d]) \arrow[d, "\mathrm{rat}"] \\ \mathrm{IH}^1(\bar{S}, \mathcal{H}) \arrow[r, "j^*"] & \mathrm{H}^1(S, \mathcal{H}) \end{tikzcd} \tag{2.22}$$

commutes. The assertions in Theorem 2.11 are, thus, a direct consequence of the fact that the arrow on top is an isomorphism (2.21). □

2.23. Suppose H is a $\mathbb{Q}$-mixed Hodge structure. We call a class $v \in H_{\mathbb{Q}}$ *Tate of weight* w if it can be expressed as the image of 1 under a morphism of mixed Hodge structures $\mathbb{Q}(-w/2) \to H$ (for some even integer w).

Theorem 2.24. *Let $\mathcal{H}$ be a $\mathbb{Q}$ variation of pure Hodge structure as in Theorem 2.11. Then, for $s \in \bar{S}$, the class $\sigma_s(\nu) \in \mathrm{IH}^1(\mathcal{H})_s$ is Tate of weight* 0*.*

To prove Theorem 2.24, we are are going to use a general fact about mixed Hodge modules on reduced separated schemes of finite type over $\mathbb{C}$; that is, we use a result from the theory of mixed Hodge modules in the algebraic case. If X is such a scheme, we write $\mathrm{MHM}(X)$ for the category of mixed Hodge modules on X. If $\overline{X}$ is any proper scheme in which X is embedded as an open subscheme, then the category $\mathrm{MHM}(X)$ is equivalent to the category $\mathrm{MHM}(X^{\mathrm{an}})^p_{\overline{X}^{\mathrm{an}}}$. Here, as in [21, p. 313] where this statement is proved, X^{an} denotes the underlying analytic space associated to X.

Lemma 2.25. *Let X be a reduced separated scheme of finite type over $\mathbb{C}$, and let M and N be objects in* $\mathrm{D^b\,MHM(X)}$*. Then there is a natural Hodge structure on the group* $\mathrm{Hom}_{\mathrm{D^b\,Perv(X)}}(\mathrm{rat}\,M, \mathrm{rat}\,N)$ *and the image of the natural map*

$$\mathrm{Hom}_{\mathrm{D^b\,MHM(X)}}(M, N) \xrightarrow{\mathrm{rat}} \mathrm{Hom}_{\mathrm{D^b\,Perv(X)}}(\mathrm{rat}\,M, \mathrm{rat}\,N)$$

consists of Tate classes of weight 0*.*

Proof. Let $\pi : X \to \operatorname{Spec}\mathbb{C}$ denote the structure morphism. Then

$$\mathrm{Hom}_{\mathrm{D^b\,Perv(X)}}(\mathrm{rat}\,M, \mathrm{rat}\,N) = \mathrm{rat}\,H^0\pi_*\underline{\mathrm{Hom}}(M, N) \tag{2.26}$$

where $\underline{\mathrm{Hom}}(M, N)$ denotes the internal Hom in $\mathrm{D^b\,MHM(X)}$. Since $\mathrm{MHM}(\operatorname{Spec}\mathbb{C})$ is equivalent to the category of polarizable mixed Hodge structures with rat taking a Hodge structure to its underlying $\mathbb{Q}$-vector space, the above isomorphism puts a mixed Hodge structure on $\mathrm{Hom}_{\mathrm{D^b\,Perv(X)}}(\mathrm{rat}\,M, \mathrm{rat}\,N)$. We leave the rest of the verification to the reader. □

Proof of Theorem 2.24. Given a $\nu \in \mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$, let $\overline{\nu} \in \mathrm{Ext}^1_{\mathrm{MHM}(\bar{S})^p}(\mathbb{Q}[d], j_{!*}\mathcal{H}[d])$ denote the unique class such that $j^*\overline{\nu} = \nu$ (2.21). Let $i : \{s\} \to \bar{S}$ denote the inclusion morphism. Then, by Theorem 2.11, $\sigma_s(\nu)$ is the image of $\overline{\nu}$ in $\mathrm{IH}^1(\mathcal{H})_s = \mathrm{Ext}^1_{\mathrm{Perv}(\{s\})}(\mathbb{Q}[d], i^*j_{!*}\mathcal{H}[d])$ under the composition

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D^b\,MHM(\bar{S})}}(\mathbb{Q}[d], (j_{!*}\mathcal{H}[d])[1]) &\xrightarrow{i^*} \mathrm{Hom}_{\mathrm{D^b\,MHM(\{s\})}}(\mathbb{Q}[d], i^*(j_{!*}\mathcal{H}[d])[1]) \\ &\xrightarrow{\mathrm{rat}} \mathrm{Hom}_{\mathrm{D^b\,Perv(\{s\})}}(\mathbb{Q}[d], i^*(j_{!*}\mathcal{H}[d])[1]). \end{aligned}$$

By (2.25), the result follows. □

3. Absolute Hodge cohomology

3.1. For a separated scheme Y of finite type over $\mathbb{C}$ let $a_Y : Y \to \operatorname{Spec}\mathbb{C}$ denote the structure morphism and let $\mathbb{Q}(p)$ denote the Tate object in MHS = MHM(Spec $\mathbb{C}$). Let $\mathbb{Q}_Y(p) := a_Y^*\mathbb{Q}(p)$ in $\mathrm{D^b\,MHM(Y)}$. (To simplify notation, we write $\mathbb{Q}(p)$ for $\mathbb{Q}_Y(p)$ when no confusion can arise.) For an object M in $\mathrm{D^b\,MHM(Y)}$, set

$$\mathrm{H}^n_{\mathcal{AH}}(Y, M) = \mathrm{Hom}_{\mathrm{D^b\,MHM(Y)}}(\mathbb{Q}, M[n]).$$

The functor rat : MHM$(Y) \to$ Perv(Y) induces a "cycle class map"

$$\mathrm{rat} : \mathrm{H}^n_{\mathcal{AH}}(Y, M) \to \mathrm{H}^n(Y, M)$$

to the hypercohomology of rat M. Note that $\mathrm{H}^n_{\mathcal{AH}}(Y, \mathbb{Q}(p)) = \mathrm{H}^n_{\mathcal{D}}(Y, \mathbb{Q}(p))$ for Y smooth and projective and in this case rat is simply the cycle class map from Deligne cohomology. Following Beilinson and Saito [22], we will call $\mathrm{H}^n_{\mathcal{AH}}(Y, M)$ the *absolute Hodge cohomology* of M. By abuse of notation, we will also write "rat" for the cycle class map $\mathrm{H}^n_{\mathcal{D}}(Y, \mathbb{Z}(p)) \to \mathrm{H}^n(Y, \mathbb{Z}(p))$.

3.2. Suppose $j : S \to \bar{S}$ is the inclusion of a Zariski open subset of a smooth complex algebraic variety and $s \in \bar{S}(\mathbb{C})$. Let $i : \{s\} \to \bar{S}$ denote the inclusion. If $\mathcal{H}$ is an admissible variation of mixed Hodge structure on S, we adopt the notation of (2.1) and write

$$\mathrm{IH}^n_{\mathcal{AH}}(\bar{S}, \mathcal{H}) = \mathrm{Hom}_{\mathrm{D^b\,MHM(\bar{S})}}(\mathbb{Q}[d_S - n], j_{!*}\mathcal{H}[d_S])$$
$$\mathrm{IH}^n_{\mathcal{AH}}(\mathcal{H})_s = \mathrm{Hom}_{\mathrm{D^b\,MHS}}(\mathbb{Q}[d_S - n], i^* j_{!*}\mathcal{H}[d_S]).$$

We can now amplify Theorem 2.11.

Proposition 3.3. *Let $j : S \to \bar{S}$ be an open immersion of smooth complex varieties and let $\mathcal{H}$ be a variation of pure Hodge structure of weight -1 on S. Then, for $i : \{s\} \to \bar{S}$ the inclusion of a closed point, the diagram*

$$\begin{tikzcd}
\mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}} \otimes \mathbb{Q} \arrow[d, "\sigma_s"'] & \mathrm{IH}^1_{\mathcal{AH}}(\bar{S}, \mathcal{H}_{\mathbb{Q}}) \arrow[l, "="'] \arrow[r, "\mathrm{rat}"] & \mathrm{IH}^1(\bar{S}, \mathcal{H}_{\mathbb{Q}}) \arrow[d, "i^*"] \\
\mathrm{H}^1(\mathcal{H}_{\mathbb{Q}})_s & & \mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}})_s \arrow[ll]
\end{tikzcd}$$

commutes.

Proof. This is consequence of (2.22), Corollary 2.21 and the notation of (3.1) which converts the top line of (2.22) into absolute Hodge cohomology groups. □

Remark 3.4. Since the map $\mathrm{IH}^1(\mathcal{H})_s \to \mathrm{H}^1(\mathcal{H})_s$ is an injection by Lemma 2.3 and the map $\sigma_s : \mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}} \to \mathrm{H}^1(\mathcal{H})_s$ factors through $\mathrm{IH}^1(\mathcal{H})_s$, we can write $\sigma_s(\nu)$ for the class of an admissible normal function ν in $\mathrm{IH}^1(\mathcal{H})_s$.

4. The decomposition of Beilinson-Bernstein-Deligne & Saito

Let $f : X \to S$ denote a projective morphism between smooth complex algebraic varieties. The fundamental theorem alluded to in the title of this section states that there is a direct sum decomposition

$$f_*\mathbb{Q}[d_X] = \oplus H^i(f_*\mathbb{Q}[d_X])[-i] \tag{4.1}$$

in MHM(S) [20, Corollary 1.11]. Moreover, the object $f_*\mathbb{Q}[d_X]$ in $\mathrm{D}^{\mathrm{b}}\,\mathrm{MHM(S)}$ is pure of weight d_X; equivalently, the mixed Hodge modules $H^i(f_*\mathbb{Q}[d_X])$ occurring in the decomposition are pure of weight $d_X + i$ [19, Theorem 1].

4.2. The decomposition of (4.1) is not unique. However, given the choice of a relatively ample line bundle $\mathcal{L}$ for f, Deligne has shown that there exist canonical decompositions that induce the identity on $H^i(f_*\mathbb{Q}[d_X])$. One of these, the decomposition of [8, Proposition 3.5], is constructed by producing an sl_2 triple and is self-dual. It depends on $c_1(\mathcal{L})$: $f_*\mathbb{Q}[d_X] \to f_*[d_X](1)[2]$ but is otherwise canonical. In particular, it is functorial with respect to maps which preserve $c_1(\mathcal{L})$. Although it is not necessary in this paper, we fix this decomposition.

4.3. The summands appearing in (4.1) can be further decomposed by codimension of strict support [20, 3.2.6]: we can write

$$H^i(f_*\mathbb{Q}[d_X]) = \oplus E_{i,Z}(f) \tag{4.4}$$

where Z is a closed subscheme of S and $E_{i,Z}(f)$ is a Hodge module supported on Z with no sub or quotient object supported in a proper subscheme of Z.

Let us set $E_{ij}(f) = \oplus_{\mathrm{codim}_S Z=j} E_{i,Z}(f)$. We then have a decomposition

$$f_*\mathbb{Q}[d_X] = \oplus E_{ij}(f)[-i]. \tag{4.5}$$

We write $E_{i,Z}$ (resp. E_{ij}) for $E_{i,Z}(f)$ (resp. $E_{ij}(f)$) when there is no chance of confusion. We write $\Pi_{ij} : f_*\mathbb{Q}[d_X] \to E_{ij}[-i]$ for the projection map and $\coprod_{ij} : E_{ij}[-i] \to f_*\mathbb{Q}[d_X]$ for the inclusion. (We suppress the indices and write Π and $\coprod$ instead of Π_{ij} and $\coprod_{ij}$ when no confusion can arise.)

Observation 4.6. Let $p \in S(\mathbb{C})$ and form the pullback diagram

$$\begin{tikzcd} X_p \arrow[r] \arrow[d, "f_p"] & X \arrow[d, "f"] \\ \{p\} \arrow[r, "i"] & S. \end{tikzcd} \tag{4.7}$$

The decomposition in (4.5) gives decompositions

$$\oplus\,\Pi_{ij} : \mathrm{H}^n_{\mathcal{AH}}(X, \mathbb{Q}[d_X]) \xrightarrow{\cong} \oplus_{ij}\mathrm{H}^{n-i}_{\mathcal{AH}}(S, E_{ij});$$
$$\oplus\,\Pi_{ij} : \mathrm{H}^n_{\mathcal{AH}}(X_p, \mathbb{Q}[d_X]) \xrightarrow{\cong} \oplus_{ij}\mathrm{H}^{n-i}_{\mathcal{AH}}(i^*E_{ij});$$
$$\oplus\,\Pi_{ij} : \mathrm{H}^n(X, \mathbb{Q}[d_X]) \xrightarrow{\cong} \oplus_{ij}\mathrm{H}^{n-i}(S, E_{ij});$$
$$\oplus\,\Pi_{ij} : \mathrm{H}^n(X_p, \mathbb{Q}[d_X]) \xrightarrow{\cong} \oplus_{ij}\mathrm{H}^{n-i}(E_{ij})_p.$$

The restriction morphisms on cohomology $\mathrm{H}^n(X, \mathbb{Q}[d_X]) \to \mathrm{H}^n(X_p, \mathbb{Q}[d_X])$ and $\mathrm{H}^n_{\mathcal{AH}}(X, \mathbb{Q}[d_X]) \to \mathrm{H}^n_{\mathcal{AH}}(X_p, \mathbb{Q}[d_X])$) are the direct sums of the morphisms

$$\mathrm{H}^{n-i}(S, E_{ij}) \to \mathrm{H}^{n-i}(E_{ij})_p \text{ and}$$
$$\mathrm{H}^{n-i}_{\mathcal{AH}}(S, E_{ij}) \to \mathrm{H}^{n-i}_{\mathcal{AH}}(i^*E_{ij}).$$

Furthermore, the morphism rat commutes with restriction from X to X_p. The above assertions follow from proper base change [19, 4.4.3] for the cartesian diagram (4.7) and the commutativity of rat with the six functors of Grothendieck.

Proposition 4.8. *With the notation of* (4.5)*, let $j : S^{sm} \to S$ denote the largest Zariski open subset of S over which f is smooth, and let $f^{sm} : X^{sm} \to S^{sm}$ denote the pull-back of f to S^{sm}. Then*

$$E_{i0} = j_{!*}((R^{i+d_X-d_S} f^{sm}_* \mathbb{Q})[d_S]).$$

Proof. Set $F = j_{!*}((R^{i+d_X-d_S} f^{sm}_* \mathbb{Q})[d_S])$. Clearly $j^* E_{i0} = (R^{i+d_X-d_S} f^{sm}_* \mathbb{Q})[d_S]$. Since E_{i0} is pure, it follows that E_{i0} contains F as a direct factor. Since any complement of F in E_{i0} would have to be supported on a proper subscheme of S, the proposition follows from the definition of E_{i0}. □

Corollary 4.9. *With the notation as in* (4.8)*, set $\mathcal{H}_i := R^i f^{sm}_* \mathbb{Q}$, a variation of Hodge structures of weight i on S^{sm}. Then*

(i) *The group* $\mathrm{IH}^r(S, \mathcal{H}_i)$ *(resp.* $\mathrm{IH}^r_{\mathcal{AH}}(S, \mathcal{H}_i)$*) is a direct factor in* $\mathrm{H}^{r+i}(X, \mathbb{Q})$ *(resp.* $\mathrm{H}^{r+i}_{\mathcal{AH}}(X, \mathbb{Q})$*);*
(ii) *for $p \in S$,* $\mathrm{IH}^r(\mathcal{H}_i)_p$ *(resp.* $\mathrm{IH}^r_{\mathcal{AH}}(\mathcal{H})_p$*) is a direct factor in* $\mathrm{H}^{r+i}(X_p, \mathbb{Q})$ *(resp.* $\mathrm{H}^{r+i}_{\mathcal{AH}}(X_p, \mathbb{Q})$*).*
(iii) *Moreover the morphism* rat *is compatible with the morphisms Π and $\coprod$ inducing the direct factors.*

Proof. This follows from directly from Observation 4.6. □

4.10. Using the notation of (4.4), write $Z_{ij}(f) = \operatorname{supp} E_{ij}(f)$ (and write Z_{ij} for $Z_{ij}(f)$). Then Z_{ij} is a reduced closed subscheme of S of codimension j. There exists an open dense subscheme $g_{ij} : U_{ij} \hookrightarrow Z_{ij}$ and a variation of pure Hodge structures $\mathcal{H}_{ij}$ on U_{ij} such that $E_{ij} = (g_{ij})_{!*}\mathcal{H}_{ij}[d_S - j]$. Clearly we can take $U_{i0} = S^{sm}$ and

$$\mathcal{H}_{i0} = \mathcal{H}_{i+d_X-d_S}.$$

The variation $\mathcal{H}_{2k-1}(k)$ on S^{sm} is a $\mathbb{Q}$ − VMHS of weight −1 on S for each integer k arising from an integral variation. Then by Corollary 2.21,

$$\mathrm{IH}^1_{\mathcal{AH}}(S, \mathcal{H}_{2k-1}(k)) = \mathrm{NF}(S^{sm}, \mathcal{H}_{2k-1}(k))^{\mathrm{ad}}.$$

By Corollary 4.9, the above is a direct factor in $\mathrm{H}^{2k}_{\mathcal{AH}}(X, \mathbb{Q}(k))$. Therefore, the composition

$$\mathrm{H}^{2k}_{\mathcal{AH}}(X, \mathbb{Q}(k)) = \mathrm{H}^{2k}_{\mathcal{AH}}(S, f_*\mathbb{Q}(k)) \xrightarrow{\Pi} \mathrm{IH}^1_{\mathcal{AH}}(S, \mathcal{H}_{2k-1}(k))$$

associates an admissible $\mathbb{Q}$-normal function to every absolute Hodge cohomology class. Moreover, suppose $p \in S$ and let $i : \{p\} \to S$ denote the inclusion. Set $r = 2k - d_X + d_S - 1$ so that $\mathcal{H}_{r0} = \mathcal{H}_{2k-1}$. Applying the functor i^* to the projection $\Pi : f_*\mathbb{Q}[d_X] \to E_{r0}[-r]$ and using proper base change, we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{H}^{2k}_{\mathcal{AH}}(X, \mathbb{Q}(k)) \arrow[r, "\Pi"] \arrow[d, "i^*"'] & \mathrm{NF}(S^{sm}, \mathcal{H}_{2k-1}(k))^{\mathrm{ad}} \arrow[d, "i^*"] \\ \mathrm{H}^{2k}(X_p, \mathbb{Q}(k)) \arrow[r, "\Pi"] & \mathrm{IH}^1(\mathcal{H}_{2k-1})_p. \end{tikzcd} \tag{4.11}$$

In the next section we will use this diagram to establish Theorem 1.3. To this end, first note the $i^* : \mathrm{NF}(S^{sm}, \mathcal{H}_{2k-1}(k))^{\mathrm{ad}} \to \mathrm{IH}^1(\mathcal{H}_{2k-1})_p$ is nothing other than the map σ_p taking a normal function to its singularity. This follows from Remark 2.7 and Theorem 2.11.

Now, note that we can write $\mathcal{H}_{2k-1} = \mathcal{H}^{\mathrm{inv}}_{2k-1} \oplus \mathcal{H}^{\mathrm{van}}_{2k-1}$ where $\mathcal{H}^{\mathrm{inv}}_{2k-1}$ is constant and $\mathcal{H}^{\mathrm{van}}_{2k-1}$ has no global sections on P. (See [22, 4.1.2]). We then see that $\mathrm{IH}^1(\mathcal{H}^{\mathrm{inv}}_{2k-1})_p = 0$ as

$j_{!*}\mathcal{H}^{\mathrm{inv}}_{2k-1}[d_S]$ is simply a constant sheaf shifted by d_S. Therefore, using the obvious projection map $\mathrm{NF}(S^{sm}, \mathcal{H}_{2k-1}(k))^{\mathrm{ad}} \to \mathrm{NF}(S^{sm}, \mathcal{H}^{\mathrm{van}}_{2k-1})^{\mathrm{ad}} = \mathrm{NF}(S^{sm}, \mathcal{H}_{2k-1})^{\mathrm{ad}} / \mathrm{NF}(S^{sm}, \mathcal{H}^{\mathrm{inv}}_{2k-1})^{\mathrm{ad}}$ we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{H}^{2k}_{\mathcal{AH}}(X, \mathbb{Q}(k)) \arrow[r, "\Pi"] \arrow[d, "i^*"'] & \mathrm{NF}(S^{sm}, \mathcal{H}^{\mathrm{van}}_{2k-1}(k))^{\mathrm{ad}} \arrow[d, "i^*"] \\ \mathrm{H}^{2k}(X_p, \mathbb{Q}(k)) \arrow[r, "\Pi"] & \mathrm{IH}^1(\mathcal{H}_{2k-1}(k))_p. \end{tikzcd} \tag{4.12}$$

5. Vanishing

Let X be a smooth projective variety of dimension $2n$ and $\mathcal{L}$ be a very ample line bundle on X. Let $\pi : \mathcal{X} \to \bar{P}$ be the associated incidence variety constructed in section 1. In this section, we prove an analogue of the Weak Lefschetz theorem for the map π.

Theorem 5.1 (Perverse Weak Lefschetz)**.** *Let $\pi : \mathcal{X} \to \bar{P}$ be as in* (1.1) *with $d = \dim \bar{P}$, and let $E_{ij} = E_{ij}(\pi)$ be as in* (4.5)*. Then*

(i) *$E_{ij} = 0$ unless $i = 0$ or $j = 0$.*
(ii) *$E_{i0} = \mathrm{H}^i(X, \mathbb{Q}[2n-1]) \otimes \mathbb{Q}_{\bar{P}}[d]$ for $i < 0$.*

Proof. Let $\mathrm{pr}_2 : X \times \bar{P} \to \bar{P}$ denote the projection on the second factor and let $g : U \to X \times \bar{P}$ denote the complement of $\mathcal{X}$ in $X \times \bar{P}$. We then have a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "i"] \arrow[dr, "\pi"'] & X \times \bar{P} \arrow[d, "\mathrm{pr}_2"] & U \arrow[l, "g"'] \arrow[dl, "p"] \\ & \bar{P} & \end{tikzcd}$$

where we write $p : U \to \bar{P}$ for $\mathrm{pr}_{2|U}$.

Note that $g : U \to X \times \bar{P}$ is an affine open immersion. Therefore $g_!\mathbb{Q}[2n+d]$ is perverse and we have an exact sequence

$$0 \to i_*\mathbb{Q}[2n+d-1] \to g_!\mathbb{Q}[2n+d] \to \mathbb{Q}[2n+d] \to 0 \tag{5.2}$$

in $\mathrm{MHM}(X \times \bar{P})$ [3, Corollaire 4.1.3].

Applying pr_2 to (5.2) gives a distinguished triangle

$$\pi_*\mathbb{Q}[2n+d-1] \to p_!\mathbb{Q}[2n+d] \to \mathrm{pr}_{2*}\,\mathbb{Q}[2n+d] \to (\pi_*\mathbb{Q}[2n+d-1])[1] \tag{5.3}$$

in $\mathrm{D^b\,MHM}(\bar{\mathrm{P}})$. Since p is affine, $p_!$ is left t-exact [3, Corollaire 4.1.2]. Thus, $H^i(p_!\mathbb{Q}[2n+d]) = 0$ in $\mathrm{MHM}(\bar{P})$ for $i < 0$. It follows then that $H^{i-1}(\mathrm{pr}_{2*}\,\mathbb{Q}[2n+d]) = H^i(\pi_*\mathbb{Q}[2n+d-1])$ for $i < 0$. Since $H^{i-1}(\mathrm{pr}_{2*}\,\mathbb{Q}[2n+d]) = H^i(X, \mathbb{Q}[2n-1]) \otimes \mathbb{Q}[d] = E_{i0}$ by weak Lefschetz, parts (i) and (ii) follow for $i < 0$.

To finish the proof of (i), note that, by the Hard Lefschetz Theorem [19, Theorem 1 (b)],

$$E_{ij} \cong E_{-i,j}(-i). \tag{5.4}$$

Therefore $E_{ij} = 0$ for $i > 0$ unless $j = 0$. □

Lemma 5.5. *Let $p \in \bar{P}(\mathbb{C})$. Then $H^k(E_{ij})_p = 0$ for $k < j - d$.*

Proof. We have $E_{ij} = (g_{ij})_{!*}\mathcal{H}_{ij}[d-j]$ with the notation as in (4.10). The result follows from [3, Proposition 2.1.11]. □

Corollary 5.6. *Let $p \in \bar{P}(\mathbb{C})$, then*

$$\mathrm{H}^{2n}(\mathcal{X}_p, \mathbb{Q}) = \mathrm{H}^{-d}(E_{10})_p \oplus \mathrm{H}^{-d+1}(E_{00})_p \oplus \mathrm{H}^{-d+1}(E_{01})_p.$$

Proof. By (4.6),

$$\begin{aligned}\mathrm{H}^{2n}(\mathcal{X}_p, \mathbb{Q}) &= \mathrm{H}^{1-d}(\mathcal{X}_p, \mathbb{Q}[d_{\mathcal{X}}]) \\ &= \oplus_{ij}\mathrm{H}^{1-d-i}(E_{ij})_p.\end{aligned}$$

By Theorem 5.1 and (5.4), we see that, for $i \neq 0$,

$$\mathrm{H}^k(E_{i0})_p = \begin{cases}\mathrm{H}^i(X, \mathbb{Q}[2n-1]) & k = -d \\ 0 & \text{else.}\end{cases}$$

Therefore, the only summand $\mathrm{H}^{1-d-i}(E_{ij})_p$ contributing to $\mathrm{H}^{2n}(\mathcal{X}_p, \mathbb{Q})$ with $i \neq 0$ is $\mathrm{H}^{-d}(E_{10})_p$. Thus

$$\mathrm{H}^{2n}(\mathcal{X}_p, \mathbb{Q}) = \mathrm{H}^{-d}(E_{10})_p \oplus (\bigoplus_j \mathrm{H}^{1-d}(E_{0j})_p. \tag{5.7}$$

However, by Lemma 5.5, $\mathrm{H}^{1-d}(E_{0j})_p = 0$ for $j > 1$. □

In fact, the term E_{01} is not difficult to compute and often trivial. It is governed by Lefschetz pencils.

Definition 5.8. Let $\mathrm{P}(\mathcal{L})$ be a property of ample line bundles. We say that P *holds for* $\mathcal{L} \gg 0$ if for every ample line bundle $\mathcal{L}$ there is an integer N such that $\mathrm{P}(\mathcal{L}^n)$ holds for $n > N$.

5.9. By [1, XVII, Theorem 2.5], the projective embedding of X via the complete linear system $|\mathcal{L}|$ is a Lefschetz embedding. Therefore, we can find a Lefschetz pencil $\Lambda \subset \bar{P}$. To each $p \in \Lambda \cap X^\vee$ one has vanishing cycles $\delta_p \in \mathrm{H}^{2n-1}(\mathcal{X}_\eta, \mathbb{Q})$ where η denotes a point of $\Lambda(\mathbb{C})$ such that $\mathcal{X}_\eta$ is smooth. We say that *the vanishing cycles are non-trivial* if $\delta_p \neq 0$ for some $p \in \Lambda \cap X^\vee$. Note that this property depends only on $\mathcal{L}$: it is independent of the choice of $\Lambda \subset \bar{P}$. By the well-known fact that the vanishing cycles are conjugates of each other by the global monodromy of the Lefschetz fibration, it is equivalent to saying that $\Lambda \cap X^\vee \neq \emptyset$ and $\delta_p \neq 0$ for all $p \in \Lambda \cap X^\vee$.

Proposition 5.10. *For $\mathcal{L} \gg 0$, the vanishing cycles are non-trivial.*

Proof. See [1, XVIII, Corollaire 6.4]. □

Theorem 5.11. *If the vanishing cycles are non-trivial, we have $E_{01} = 0$; otherwise, $\mathcal{H}_{01}$ is a rank 1 variation of pure Hodge structure supported on a dense open subset of $X^\vee$.*

Proof. See [1, XVIII, Théorèm 6.3 and XV, Théorèm 3.4]. □

Remark 5.12. N. Fakhruddin has shown that, if $\mathcal{L} \gg 0$, we have $E_{ij} = 0$ for all i and for all $j > 0$. The proof, which appears in the Appendix, relies on the fact that, for $\mathcal{L} \gg 0$, the locus of hypersurfaces in $|\mathcal{L}|$ with non-isolated singularities has codimension larger than the dimension of the hypersurfaces.

Example 5.13. Let $X \cong \mathbb{P}^2$ and set $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(2)$. Then $\dim \mathcal{X} = 6$ and $\dim P = 5$. We have $E_{-1,0} = \mathbb{Q}[5], E_{0,0} = 0$ and $E_{1,0} = \mathbb{Q}(-1)[5]$. Since the vanishing cycles are trivial ($H^1(\mathcal{X}_\eta) = 0$ and any Lefschetz pencil contains a singular conic), $\mathcal{H}_{01}$ is non-zero. In fact, let V denote the locus of point $p \in P$ such that $\mathcal{X}_p$ is the union of two distinct lines. Note that V is a dense open subset of $X^\vee$ and $\pi_1(V) \cong \mathbb{Z}/2$. It is easy to see that $\mathcal{H}_{01}$ is the unique non-trivial rank 1 variation of Hodge structure of weight 2 on V. Moreover, it is not difficult to see that $E_{0j} = 0$ for $j > 1$.

6. Hodge Conjecture

In this section, we complete the proofs of the main results of the paper. We begin with the first Theorem of the introduction.

Proof of Theorem 1.3. We want to build the commutative diagram in Theorem 1.3 from the diagram (4.12). To do this, recall that we have an extension

$$0 \to J^n(X) \to \mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)) \xrightarrow{\mathrm{rat}} \mathrm{H}^{n,n}(X, \mathbb{Z}(n)) \to 0$$

and write $\mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))^{\mathrm{prim}}$ for the inverse image of $\mathrm{H}^{n,n}(X, \mathbb{Z}(n))^{\mathrm{prim}}$ in $\mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$. Recall that the map $\mathrm{AJ} : \mathrm{H}^{n,n}(X, \mathbb{Z}(n))^{\mathrm{prim}} \to \mathrm{NF}(P, \mathcal{H})^{\mathrm{ad}} / J^n(X)$ is defined as follows. For $\gamma \in \mathrm{H}^{n,n}(X, \mathbb{Z}(n))^{\mathrm{prim}}$, we choose $\zeta \in \mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$ such that $\mathrm{rat}(\zeta) = \gamma$. Then, for $q \in P$, $\mathrm{AJ}(\gamma)(q)$ is the restriction of ζ to q modulo $J^n(X)$. This restriction, ζ_q, lands in $J^n(\mathcal{H}_q)$ because we have an extension

$$0 \to J^n(\mathcal{H}_q) \to \mathrm{H}^{2n}_{\mathcal{D}}(X_q, \mathbb{Z}(n)) \to \mathrm{H}^{n,n}(X_q, \mathbb{Z}(n)) \to 0$$

and, from the fact that $\gamma \in \mathrm{H}^{n,n}(X, \mathbb{Z}(n))^{\mathrm{prim}}$, the restriction of γ to $\mathrm{H}^{n,n}(X_q, \mathbb{Z}(n))$ is 0. Since ζ is well-defined modulo $J^n(X)$, we see that $\mathrm{AJ}(\zeta)$ is as well.

Now, define $\mathrm{AJ} : \mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))^{\mathrm{prim}} \to \mathrm{NF}(P, \mathcal{H})^{\mathrm{ad}} / J^n(X)$ to be the composition of $\mathrm{AJ} : \mathrm{H}^{n,n}(X, \mathbb{Z}(n))^{\mathrm{prim}} \to \mathrm{NF}(P, \mathcal{H})^{\mathrm{ad}} / J^n(X)$ with $\mathrm{rat} : \mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))^{\mathrm{prim}} \to \mathrm{H}^{n,n}(X, \mathbb{Z}(n))^{\mathrm{prim}}$. We claim that we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))^{\mathrm{prim}} \arrow[r, "\mathrm{AJ}"] \arrow[d, "\mathrm{pr}^*"] & \mathrm{NF}(P, \mathcal{H})^{\mathrm{ad}} / J^n(X) \arrow[d, "\otimes\mathbb{Q}"] \\ \mathrm{H}^{2n}_{\mathcal{AH}}(\mathcal{X}, \mathbb{Q}(n)) \arrow[r, "\Pi"] & \mathrm{NF}(P, \mathcal{H}^{\mathrm{van}}_{\mathbb{Q}}). \end{tikzcd} \tag{6.1}$$

where here $\Pi = \Pi_{0,0}$.

To prove the claim, suppose $\zeta \in \mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))^{\mathrm{prim}}$. Set $\omega := \mathrm{pr}^* \zeta \in \mathrm{H}^{2n}_{\mathcal{AH}}(\mathcal{X}, \mathbb{Q}(n)) = \mathrm{H}^{-d+1}_{\mathcal{AH}}(\mathcal{X}, \mathbb{Q}(n)[2n+d-1])$. We can use the decomposition $\pi_* \mathbb{Q}(n)[2n+d-1] = \oplus E_{ij}[-i]$ and write ω_{ij} for the component of ω in $\mathrm{H}^{-d+1}_{\mathcal{AH}}(\mathcal{X}, \mathbb{Q}(n)[2n + d - 1]) = \oplus \mathrm{H}^{-d+1}_{\mathcal{AH}}(\bar{P}, E_{ij}[-i])$. To conclude that (6.1) commutes, we will use Proposition 4.5 of [22] which directly implies that two $\mathbb{Q}$-normal functions on P are equal if and only if their restrictions to all points $q \in P$ are equal. Therefore, it suffices to show that, for any $q \in P$, $\omega_q = (\omega_{00})_q \in \mathrm{H}^{2n}_{\mathcal{AH}}(X_q, \mathbb{Q}(n))$. In other words, it suffices to show that, for $(i, j) \neq (0, 0)$, $(\omega_{ij})_q = 0$.

Now, for $j \neq 0$ and $q \in P$, it is clear that $(\omega_{ij})_q = 0$. Therefore, we can assume $j = 0$. Pick $i \neq 0$. Then $E_{i0} = K[d]$ where K is a constant variation of pure Hodge structure (determined explicitly in Theorem 5.1). Therefore, we have

$$\begin{aligned} \mathrm{H}^{-d+1}_{\mathcal{AH}}(\mathcal{X}, E_{i0}[-i]) &= \mathrm{Ext}^{-d+1}_{\mathrm{D}^b \mathrm{MHM}(\bar{\mathrm{P}})}(\mathbb{Q}_{\bar{P}}, K[d - i]) \\ &= \mathrm{Ext}^{1-i}_{D^b \mathrm{MHM}(\bar{P})}(\mathbb{Q}_{\bar{P}}, K). \end{aligned}$$

Thus the restriction of ω_{i0} to $q \in P$ lies in $\mathrm{Ext}^{1-i}_{\mathrm{MHS}}(\mathbb{Q}, K_q)$. This group is 0 unless $i = 0$ or 1. Hence $(\omega_{i0})_q = 0$ unless $i = 0$ or $i = 1$. If $i = 1$, then K is a constant variation of pure Hodge structure of weight 0 with $K_q = \mathrm{H}^{2n}(X_q, \mathbb{Q}(n))$. The image of ω_{10} under the map

$$\mathrm{Ext}^{1-i}_{\mathrm{MHS}}(\mathbb{Q}, K_q) = \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, K_q) \xrightarrow{\mathrm{rat}} \mathrm{Hom}_{\mathrm{Vect}}(\mathbb{Q}, K_q)$$

coincides with the image of ζ in $\mathrm{H}^{2n}(X_q, \mathbb{Q}(n))$. Since rat is an injection and ζ is primitive, it follows that the restriction of $(\omega_{10})_q = 0$.

By (4.12), we see that

$$\begin{tikzcd} \mathrm{H}^{2n}_{\mathcal{AH}}(\mathcal{X}, \mathbb{Q}(n)) \arrow[r, "\Pi"] \arrow[d, "i^*\circ\mathrm{rat}"] & \mathrm{NF}(P, \mathcal{H}^{\mathrm{van}}_{\mathbb{Q}}) \arrow[d, "i^*"] \\ \mathrm{H}^{2n}(X_p, \mathbb{Q}(n)) \arrow[r, "\Pi"] & \mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}})_p \end{tikzcd} \tag{6.2}$$

commutes. Joining (6.1) and (6.2), we obtain a commuative diagram

$$\begin{tikzcd} \mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))^{\mathrm{prim}} \arrow[r, "\mathrm{AJ}"] \arrow[d] & \mathrm{NF}(P, \mathcal{H})^{\mathrm{ad}} / J^n(X) \arrow[d] \\ \mathrm{H}^{2n}(X_p, \mathbb{Q}(n)) \arrow[r, "\Pi"] & \mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}})_p \end{tikzcd} \tag{6.3}$$

where the arrows emanating from the top left corner both factor through the quotient $\mathrm{H}^{n,n}(X, \mathbb{Z})^{\mathrm{prim}}$ of $\mathrm{H}^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$. Now, define $\beta_p : \mathrm{H}^{2n}(X_p, \mathbb{Q}(n)) \to (R^1 j_* \mathcal{H}_{\mathbb{Q}})_p$ to be the composition of $\Pi : \mathrm{H}^{2n}(\mathcal{X}_p, \mathbb{Q}(k)) \to \mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}})_p$ with the inclusion $\mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}})_p \to (R^1 j_* \mathcal{H}_{\mathbb{Q}})$. We then obtain the commutative diagram of Theorem 1.3 by using the natural inclusion of $\mathrm{IH}^1(\mathcal{H}_{\mathbb{Q}})_p$ in $(R^1 j_* \mathcal{H}_{\mathbb{Q}})_p$.

To prove part (ii) of Theorem 1.3, first note that, by Proposition 5.10 and Theorem 5.11, we can find k large enough so that, when $\bar{P} = |\mathcal{L}^k|$, we have $E_{01} = 0$. Then proper base change shows that

$$\mathrm{H}^{2n}(X_p, \mathbb{Q}(n)) = \mathrm{IH}^0(\mathcal{H}_{2n}(n))_p \oplus \mathrm{IH}^1(\mathcal{H})_p.$$

If $\zeta \in \mathrm{H}^{n,n}(X, \mathbb{Z})^{\mathrm{prim}}$, then the image of ζ in $\mathrm{H}^0(P, \mathcal{H}_{2n})$ is 0. It follows that the image of ζ under restriction to $\mathrm{IH}^0(\mathcal{H}_{2n}(n))_p$ is 0. Thus $\alpha_p(\zeta) = i^*(\zeta_{\mathbb{Q}}) \in \mathrm{IH}^1(\mathcal{H})$. From this, and Lemma 2.3, we see that we have proved part (ii) of the theorem. □

The proof of Theorem 1.6 in the introduction used a statement that was essentially the Poincaré dual of the remark at the bottom of page 181 of [26]. For the convenience of the reader, we will prove this dual statement and the statement dual to the main result of [26].

6.4. Let Y be a smooth projective complex variety and let $k \in \mathbb{Z}$. We write $\mathrm{Alg}^k\, Y$ for the subspace of $\mathrm{H}^{k,k}(Y, \mathbb{Q})$ consisting of algebraic cycles. The Hodge conjecture for Y is the statement that $\mathrm{Alg}^k\, Y = \mathrm{H}^{k,k}(Y, \mathbb{Q})$ for all k. By Poincaré duality and the Hodge-Riemann bilinear relations, the cup product

$$\cup : \mathrm{H}^{2k}(Y, \mathbb{Q}(k)) \otimes \mathrm{H}^{2(d_Y - k)}(Y, \mathbb{Q}(d_Y - k)) \to \mathrm{H}^{2d_Y}(Y, \mathbb{Q}(d_Y)) = \mathbb{Q}$$

restricts to a give a perfect pairing

$$\mathrm{H}^{k,k}(Y, \mathbb{Q}) \otimes \mathrm{H}^{d_Y - k, d_Y - k}(Y, \mathbb{Q}) \to \mathbb{Q}.$$

Therefore, the Hodge conjecture for Y is equivalent to the assertion that the perpendicular subspace $(\mathrm{Alg}^k\, Y)^\perp \subset \mathrm{H}^{d_Y - k, d_Y - k}(Y, \mathbb{Q})$ is zero.

Theorem 6.5. *The following statements are equivalent:*

(i) *The Hodge conjecture holds for all smooth projective complex varieties Y.*
(ii) *Let $(X, \mathcal{L})$ be a pair as in 1.1. Then, for every class $\zeta \in \mathrm{H}^{n,n}(X, \mathbb{Q}(n))$, there exists an integer k and a hyperplane section $Z \in |\mathcal{L}^k|$ with only ODP singularities such that $0 \neq \zeta_{|Z} \in \mathrm{H}^{2n}(Z, \mathbb{Q}(n))$.*

(iii) *Let* $(X, \mathcal{L})$ *be a pair as in 1.1. Then, for every class* $\zeta \in \mathrm{H}^{n,n}(X, \mathbb{Q}(n))$*, there exists an integer* k *and a hyperplane section* $Z \in |\mathcal{L}^k|$ *such that* $0 \neq \zeta_{|Z} \in \mathrm{H}^{2n}(Z, \mathbb{Q}(n))$*.*

Remark 6.6. To be precise, (i)$\Leftrightarrow$ (ii) is Poincaré dual to the main result of [26] while the equivalence of (i) annd (iii) is dual to an observation of B. Totaro stated as a remark in [26, p. 181].

Proof. (i)$\Rightarrow$ (ii): By assumption, the Hodge conjecture holds for X. Therefore, we can find a subvariety W of codimension n in X such that $[W]\cdot\zeta$ is non-zero. Thus $\zeta_{|W} \neq 0$. Now, for $k \gg 0$, we can find a hyperplane section Z with only ODP singularities containing W [26, Theorem 4.2]. Therefore, $\zeta_{|Z}$ is non-zero.

(ii)$\Rightarrow$ (iii): obvious.

(iii)$\Rightarrow$ (i): Following [23, p. 88], we let $\mathrm{HC}(Y, p)$ denote the Hodge conjeture for codimension p cycles on a smooth, complex variety Y. Then we prove (iii)$\Rightarrow$ (i) by induction on d_Y.

By our induction, we can assume that $p \leq d_Y/2$, since, when $p > d_Y/2$, [23, Remark 1.3 (ii)] shows that $\mathrm{HC}(Y, p)$ holds as long as $\mathrm{HC}(Z, p-1)$ holds for Z any smooth hyperplane section of Y. Similarly, we can assume that $p \geq d_Y/2$, since, when $p < d_Y/2$, [23] also shows that $\mathrm{HC}(Y, p)$ holds as long as $\mathrm{HC}(Z, p)$ and $\mathrm{HC}(Z, p-1)$ hold for Z a generic hyperplane section of Y. Therefore, we are reduced to proving $\mathrm{HC}(X, n)$ for X a variety of dimension $2n$. By (6.4), to do this, it is sufficient to prove that, for every $\zeta \in \mathrm{H}^{n,n}(X, \mathbb{Q}(n))$, there is an n-dimensional subvariety W of X such that $0 \neq \zeta \cup \mathrm{cl}([W]) \in \mathrm{H}^{4n}(X, \mathbb{Q}(2n))$.

By assumption, there is an integer k and a hyperplane section $Z \in |\mathcal{L}^k|$ such that $\zeta_{|Z}$ is non-zero. Let $f : \tilde{Z} \to Z$ denote a resolution of singularities of Z and write $g : \tilde{Z} \to X$ for the composition of f with the inclusion of Z into X. Then $g^*(\zeta) \in \mathrm{H}^{n,n}(\tilde{Z}, \mathbb{Q}(n))$. By (6.4), there is a Hodge class $\omega \in \mathrm{H}^{n-1,n-1}(\tilde{Z}, \mathbb{Q}(n-1))$ such that $0 \neq g^*(\zeta) \cup \omega \in \mathrm{H}^{4n-2}(\tilde{Z}, \mathbb{Q}(2n-2))$. By induction, ω is algebraic. Thus $\omega = \mathrm{cl}(\sum \alpha_i [W_i])$ for some subvarieties W_i of $\tilde{Z}$ of codimension $n-1$ (= dimension n) in $\tilde{Z}$. It follows that $\zeta \cup g_*(\omega) = g_*(g^*\zeta \cup \omega) \neq 0$. Therefore, there exists some i such that $\zeta \cup [g_* W_i] \neq 0$. □

This completes the proof of the main claims in the introduction, Theorems 1.3 and 1.6.

7. Singularities and rational maps

This last section will be devoted to recovering the Theorem of [10]. As mentioned in the introduction, this will entail studying what happens to singularities of normal functions when the base is blown up.

Lemma 7.1. *Let* S *be a smooth complex algebraic variety, let* $\mathcal{H}$ *be a variation of* $\mathbb{Q}$*-Hodge structure of weight* -1 *on* S *and let* $U \subset S$ *be a non-empty Zariski open subset. Then the restriction map*
$$\mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}} \to \mathrm{NF}(U, \mathcal{H}_{|U})^{\mathrm{ad}}$$
is an isomorphism.

Proof. Using resolution of singularities, find an open immersion $j : S \to \bar{S}$ with $\bar{S}$ proper. Let $j_U : U \to \bar{S}$ denote the inclusion. then $j_{U!*}\mathcal{H}[d_U] = j_{!*}\mathcal{H}[d_S]$. Therefore, by Corollary 2.21,
$$\begin{aligned}\mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}} &= \mathrm{Ext}^1_{\mathrm{MHM}(\bar{S})}(\mathbb{Q}[d_S], j_{!*}\mathcal{H}[d_S]) \\ &= \mathrm{Ext}^1_{\mathrm{MHM}(\bar{S})}(\mathbb{Q}[d_S], j_{U!*}\mathcal{H}[d_S]) \\ &= \mathrm{NF}(U, \mathcal{H}_{|U})^{\mathrm{ad}}_{\bar{S}} = \mathrm{NF}(U, \mathcal{H}_{|U})^{\mathrm{ad}}.\end{aligned}$$

□

Let S be a smooth complex algebraic variety. We define a category G_S as follows: Objects of G_S are weight -1 variations of $\mathbb{Q}$-Hodge structure defined on some non-empty Zariski open subset U of S. If $\mathcal{H}$ and $\mathcal{K}$ are objects in G_S defined on open sets U and V respectively, then a morphism $\phi : \mathcal{H} \to \mathcal{K}$ is a morphism of variations of Hodge structure from $\mathcal{H}_{|U\cap V}$ to $\mathcal{K}_{|U\cap V}$. We call G_S the *category of variations of Hodge structure over the generic point of* S. Note that, if we let $\mathrm{MHM}(S)_{a,b}$ denote the full subcategory of $\mathrm{MHM}(S)$ consisting of pure objects of weight a with support of pure codimension b, then G_S is equivalent to $\mathrm{MHM}(S)_{d_S-1,0}$. This equivalence is brought about by the functor sending $\mathcal{H}$ supported on a Zariski open $j : U \hookrightarrow S$ to the mixed Hodge module $j_{!*}\mathcal{H}[d_S]$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two objects in G_S with $\mathcal{H}$ defined on a Zariski open subset $U \subset S$ and $\mathcal{K}$ defined on a Zariski open subset $V \subset S$. A morphism $\phi : \mathcal{H} \to \mathcal{K}$ in G_S induces a morphism

$$\phi_* : \mathrm{NF}(U, \mathcal{H})^{\mathrm{ad}} \to \mathrm{NF}(V, \mathcal{K})^{\mathrm{ad}}$$

via the composition

$$\mathrm{NF}(U, \mathcal{H})^{\mathrm{ad}} \cong \mathrm{NF}(U \cap V, \mathcal{H})^{\mathrm{ad}} \xrightarrow{\phi_*} \mathrm{NF}(U \cap V, \mathcal{K})^{\mathrm{ad}} \cong \mathrm{NF}(V, \mathcal{K})^{\mathrm{ad}}.$$

It follows that the group $\mathrm{NF}(\mathcal{H})^{\mathrm{ad}}_{\mathbb{Q}}$ of admissible $\mathbb{Q}$-normal functions of an object in G_S is an isomorphism invariant.

7.2. Let $f : S \dashrightarrow P$ be a dominant rational map between smooth projective varieties. Then f induces a functor $f^* : G_P \to G_S$ defined as follows. Given $\mathcal{H}$ defined on a Zariski open subset U of P, let V denote the largest Zariski open subset of U over which f is defined. The functor sends $\mathcal{H}$ to $f^*\mathcal{H}_{|V}$. A similar construction defines f^* of a morphism. Note that we have a natural map

$$f^* : \mathrm{NF}(\mathcal{H})^{\mathrm{ad}} \to \mathrm{NF}(f^*\mathcal{H})^{\mathrm{ad}}.$$

defined by pulling back the extension classes. In particular, if f is a birational map, $\mathrm{NF}(\mathcal{H})^{\mathrm{ad}}_{\mathbb{Q}} \cong \mathrm{NF}(f^*\mathcal{H})^{\mathrm{ad}}_{\mathbb{Q}}$.

In an earlier version of this paper we made the following conjecture.

Conjecture 7.3. *Let $f : \bar{S} \dashrightarrow \bar{P}$ be a birational map between smooth projective varieties, let $\mathcal{H}$ be a weight -1 variation of Hodge structure over the generic point of $\bar{P}$ and let $\nu \in \mathrm{NF}(\mathcal{H})^{\mathrm{ad}}$. Then if ν is singular on $\bar{P}$, $f^*\nu$ is singular on $\bar{S}$.*

Unfortunately, this conjecture turns out to be false. N. Fakhruddin and M. Saito have independently provided us with counterexamples very similar to the following.

Example 7.4. Take $\bar{P}$ to be $\mathbb{P}^2$ and $\bar{S}$ to be the blow up of $\mathbb{P}^2$ at the origin in $\mathbb{A}^2$. Let $\pi : \mathbb{P}^2 \setminus \{[1, 0, 0]\} \to \mathbb{P}^1$ be the map $[x_0, x_1, x_2] \mapsto [x_1, x_2]$. Let $\mathcal{H}$ be a variation of weight -1 on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ admitting an admissible normal function ν such that the class of ν in $\mathrm{IH}^1(\mathbb{P}^1, \mathcal{H})$ is non-zero. (For example, one can take $\mathcal{H}$ to be $H^1(E_\lambda)$ where E_λ is a family of elliptic curves admitting a non-torsion section.) If we pull back ν to a Zariski dense open subset of $\bar{P}$ via the map π, we find that $\pi^*\nu$ is singular at $[1, 0, 0]$: One can identify $\mathrm{H}^1(\mathcal{H})_{[1,0,0]}$ with $\mathrm{H}^1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \mathcal{H})$ and, under this identification the class in the later corresponds to the singularity in the former. However, by direct computation we can see that, when pulled back further to $\bar{S}$, ν has no singularities.

Our initial motivation for stating Conjecture 7.3 was the the comparison of Theorem 1.6 with the analogous assertions made in [10]. To explain this motivation, we briefly recall the assertions of [10]. Let $X, P, \bar{P}, \mathcal{X}$ and $X^\vee$ be as in (1.1). In [10], the authors apply resolution of singularities to produce a projective variety $\bar{S}$ equipped with a birational morphism $f : \bar{S} \to \bar{P}$ such that $f^{-1}X^\vee$ is a divisor with normal crossings. Let us call such an $\bar{S}$ a *resolution of the dual variety*. The authors of [10] then make the following conjecture.

Conjecture 7.5. *For every non-torsion primitive Hodge class ζ, there is an integer k and a resolution $f : \bar{S} \to \bar{P} = |\mathcal{L}^k|$ of the dual variety such that $f^* \operatorname{AJ}(\zeta)$ in $\operatorname{NF}(f^*\mathcal{H})^{\mathrm{ad}}/J^n(X)$ has a non-torsion singularity on $\bar{S}$.*

One of the main assertions of [10] is that Conjecture 7.5 holds (for all even dimensional X) if and only if the Hodge conjecture holds (for all smooth projective algebraic varieties). In fact, we will now prove this assertion by proving that Conjecture 7.3 does hold in a special case.

We begin by establishing a special case of Conjecture 7.3.

Proposition 7.6. *Let $\bar{P}$ be a smooth projective variety, $\mathcal{H}$ a variation of pure Hodge structure of weight -1 on the generic point of $\bar{P}$ and $f : S \to \bar{P}$ a dominant morphism. Let $\nu \in \operatorname{NF}(\mathcal{H})^{\mathrm{ad}}$. If $f^*\nu$ is singular on S, then ν is singular on $\bar{P}$.*

Remark 7.7. In the following proof and the rest of this section, we will work with constructible sheaves as opposed to perverse sheaves. To ease the notation, when $\mathcal{F}$ is a constructible sheaf and f is a morphism of complex schemes, we will write $f_*\mathcal{F}$ for the usual (not derived) operation on constructible sheaves and $R^i f_*\mathcal{F}$ for the constructible higher direct image.

Proof. Suppose that $\mathcal{H}$ is smooth over a dense Zariski open subset $j : U \hookrightarrow \bar{P}$. The Leray spectral sequence for $Rj_*\mathcal{H}$ gives an exact sequence

$$0 \to \mathrm{H}^1(\bar{P}, R^0 j_*\mathcal{H}) \to \mathrm{H}^1(U, \mathcal{H}) \xrightarrow{s_j} \mathrm{H}^0(\bar{P}, R^1 j_*\mathcal{H}) \tag{7.8}$$

and ν is singular on $\bar{P}$ if and only if $s_j(\operatorname{cl}\nu) \neq 0$. The proposition follows by functoriality of the Leray spectral sequence applied to the pullback diagram

$$\begin{array}{ccc} f^{-1}U & \xrightarrow{j_S} & S \\ \downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & \bar{P} \end{array} \tag{7.9}$$

□

Corollary 7.10. *Conjecture 7.5 implies Conjecture 1.5.*

We now begin the proof of the reverse implication.

Lemma 7.11. *Let $f : S \to P$ be a morphism of smooth, complex algebraic varieties. Let U be a non-empty Zariski open subset of P such that $V := f^{-1}U$ is Zariski dense in S, and let $\mathcal{V}$ be a $\mathbb{Q}$-local system on U. Form the cartesian diagram*

$$\begin{array}{ccc} V & \xrightarrow{i} & S \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & P \end{array}$$

using the letters on the arrows as the names for the obvious maps. Then the base change map $f^* j_*\mathcal{V} \to i_* g^*\mathcal{V}$ is an injection of constructible sheaves.

Proof. Suppose that $s \in S(\mathbb{C})$ and that $p = f(s) \in P(\mathbb{C})$. We can find a small ball B about $p \in P$ such that $B \cap U$ is connected, and, for $z \in B \cap U$, $(f^* j_* \mathcal{V})_s = \mathcal{V}_z^{\pi_1(B\cap U,z)}$. We can then find a small ball $D \subset f^{-1}B$ containing s such that $D \cap V$ is connected, and then for $w \in D \cap V$, $(i_* g^* \mathcal{V})_s = \mathcal{V}_w^{\pi_1(D\cap V,w)}$. Without loss of generality, we can assume that $f(w) = z$. Since the action of $\pi_1(D \cap V, w)$ on $\mathcal{V}_w$ then factors through $\pi_1(B \cap U, z)$, it follows that the base-change map $f^* j_* \mathcal{V} \to i_* g^* \mathcal{V}$ is injective. $\square$

Theorem 7.12 (M. Saito)**.** *Let $p : \mathcal{X} \to C$ be a proper, flat morphism from an even dimensional complex algebraic variety $\mathcal{X}$ to a smooth curve C. Assume that the generic fiber is smooth and that, for any closed point $c \in C$, $\mathcal{X}_c$ has only finitely many singularities each of which is an ODP. Then the intersection complex of $\mathcal{X}_c$ is given by $\mathbb{Q}[\dim X_c]$.*

Proof. This is Theorem 3 of [25]. $\square$

Lemma 7.13. *Let C be a smooth curve and $c \in C(\mathbb{C})$ and set $C' = C \setminus \{c\}$. Let $\pi : \mathcal{X} \to C$ be a flat, projective morphism from a complex algebraic scheme $\mathcal{X}$, and let π' denote the restriction of π to $\mathcal{X}' := \pi^{-1}C'$. Suppose that π' is smooth of relative dimension $2k-1$ for k an integer and that $\mathcal{X}_c$ has at worst ODP singularities. Set $\mathcal{H} = R^{2k-1}\pi'_*\mathbb{Q}(k)$ and let $j : C' \to C$ denote the open immersion including C' in C. Then*

$$\mathrm{H}^{2k-1}(\mathcal{X}_c, \mathbb{Q}) \xrightarrow{\cong} (j_*\mathcal{H})_c$$

via the natural morphism coming from proper base change.

Proof. By Theorem 7.12, we have $\mathrm{IC}_{\mathcal{X}} = \mathbb{Q}[2k]$. Set $\mathcal{H} := R^{2k-1}\pi'_*\mathbb{Q}$. The lemma then claims that $\mathrm{H}^{2k-1}(\mathcal{X}_c, \mathbb{Q}) \xrightarrow{\cong} i^* j_*\mathcal{H}$.

To prove this, write $i : \{c\} \to C$ for the inclusion, pick a parameter z at C on C and write ϕ_z for the vanishing cycles functor.

Now, we can use the decomposition theorem to write $R\pi_*\mathbb{Q}[2k] = \oplus E_{ij}[-i]$ with $E_{ij} = E_{ij}(\pi)$. The E_{i0} are given by $j_{!*}\mathcal{H}_i[1]$ for $\mathcal{H}_i = R^{k-1+i}\pi'_*\mathbb{Q}$ and the E_{i1} are supported on c. For $j > 1$ clearly $E_{ij} = 0$. Now $R\pi_*\phi_z\mathbb{Q}[2k]$, which is equal to $\phi_z R\pi_*\mathbb{Q}[2k]$, is simply $\mathbb{Q}^V[0]$ for some non-negative integer V because $\phi_z\mathbb{Q}[2k]$ is supported on the singular points of $\mathcal{X}_c$. It follows that $\phi_z E_{ij} = 0$ unless $i = 0$. This immediately implies that $E_{i1} = 0$ unless $i = 0$. It also implies that the $\mathcal{H}_i$ have trivial monodromy about c unless $i = 0$. Therefore, $\mathrm{H}^0(i^*E_{ij}) = 0$ unless $i = 0$. Note that also $\mathrm{H}^1(i^*E_{ij}) = 0$ for all j by the definition of perverse sheaves. From this, we obtain that $\mathrm{H}^{2k-1}(\mathcal{X}_c, \mathbb{Q}) = \mathrm{H}^{-1}(i^*R\pi_*\mathbb{Q}[2k]) = \mathrm{H}^{-1}(i^*E_{00}) = \mathrm{H}^{-1}(i^* j_{!*}\mathcal{H}[1]) = (j_*\mathcal{H})_c$. $\square$

We now consider a situation where we can show that the base change morphism of Lemma 7.11 induces an isomorphism.

Lemma 7.14. *Let $h : \mathcal{X} \to P$ be a proper, flat morphism of relative dimension $2j-1$ between smooth complex varieties such that h is smooth over a dense Zariski open subset $U \subset P$ and, for all $p \in P$, $\mathcal{X}_p$ presents at worst ODP singularities. Set $\mathcal{H} = R^{2k-1}h_*\mathbb{Q}(n)_{|U}$. Let $f : S \to P$ be a morphism from a smooth variety such that $V := f^{-1}U$ is dense in S. Form the cartesian diagram of Lemma 7.11. Then the base change morphism induces an isomorphism $f^* j_*\mathcal{H} \to i_* g^*\mathcal{H}$ of sheaves.*

Proof. We have already shown that the map is an injection. To prove surjectivity, we are going to use the local invariant cycle theorem of [3].

Pick $s \in S(\mathbb{C})$. We can find a smooth curve C passing through s such that $C' := C \cap V$ is dense in C. Since $h : \mathcal{X} \to P$ is flat, $h_C : \mathcal{X}_C \to C$ is also flat. It follows that

$$((i_{|C'})_*\mathcal{H}_{|C'})_c \cong \mathrm{H}^{2k-1}\mathcal{X}_c.$$

On the other hand, since $\mathcal{X}$ is smooth, the local invariant cycle theorem shows that

$$\mathrm{H}^{2k-1}\mathcal{X}_c \twoheadrightarrow (j_*\mathcal{H})_{f(c)}.$$

Therefore we have a sequence

$$\mathrm{H}^{2k-1}\mathcal{X}_c \twoheadrightarrow (j_*\mathcal{H})_{f(c)} \hookrightarrow (i_*g^*\mathcal{H})_c \hookrightarrow ((i_{|C'})_*\mathcal{H}_{|C'})_c \cong \mathrm{H}^{2k-1}\mathcal{X}_c.$$

Since the composition is the identity, the maps in the sequence are all isomorphisms. □

The idea for the proof of the following theorem was communicated to us by N. Fakhruddin.

Theorem 7.15. *Let $f : X \to Y$ be a proper birational morphism between complex varieties with X normal and Y smooth. Then, for any closed point $y \in Y$, X_y is simply connected.*

The theorem will follow from the following result which is essentially proper base change for homotopy groups.

Lemma 7.16. *Let $f : X \to Y$ be a proper morphism between schemes of finite type over $\mathbb{C}$. Let x be a closed point in X and $y = f(x)$. Then, for any positive integer i, the natural map $\pi_i(X_y, x) \to \varprojlim_{y\in V} \pi_i(f^{-1}V, x)$ is an isomorphism where the limit is taken over all open neighborhoods V of y. Similarly, we have $\pi_0(X_y) \cong \varprojlim_{y\in V} \pi_0(f^{-1}V)$.*

Proof of Lemma 7.16. We work with the underlying topological spaces in the analytic topology, and we abuse notation by writing X instead of $X(\mathbb{C})$. We fix x as a base point for all homotopy groups π_i for $i > 0$ of spaces containing x.

By restricting to an affine open neighborhood of Y, we immediately reduce to the case that Y is a Zariski closed subset of $\mathbb{A}^n$. Then, by using the fact that closed immersions are proper, we reduce to the case that $Y = \mathbb{A}^n$. So let B be an open ball of radius 1 centered at 0 in $Y = \mathbb{A}^n$. Then, by [17, Theorem 2] or [13], we can find a triangulation of $f^{-1}\bar{B}$ with X_y as a subcomplex. Thus any open neighborhood U of X_y contained contains an open neighborhood N which deformation retracts onto X_y by [12, Proposition A.5]. This implies that the map $\pi_i(X_y) \to \varprojlim_{U} \pi_i(U)$ is an isomorphism where the limit is taken over all open neighborhoods U of X_y. Similarly, since f is proper, any open neighborhood U of X_y contains an open neighborhood of the form $f^{-1}V$ for V an open neighborhood of y in Y. It follows that the natural map $\varprojlim_{y\in U} \pi_i(U) \to \varprojlim_{y\in V} \pi_i(f^{-1}V)$ is an isomorphism. □

Proof of Theorem 7.15. Let Z be a closed subset of Y of codimension at least 2 such that the rational map $f^{-1} : Y \setminus Z \to X$ is a morphism. Set $E = f^{-1}Z$. Let V be an open ball containing y in Y. Then $V \setminus Z$ is simply conected because Z has codimension 2. Thus $f^{-1}V \setminus E$ is simply connected. It follows that $f^{-1}V$ is connected. Moreover, since X is normal and E has codimension at least 1, the map $f^{-1}V \setminus E \to f^{-1}V$ induces a sujection $\pi_1(f^{-1}V \setminus E, u) \to \pi_1(f^{-1}V, u)$ for any choice of base point $u \in f^{-1}V \setminus E$. Thus $f^{-1}V$ is simply connected. Now, the sets of the form $f^{-1}V$ with V an open ball are left filtering and left final within the system of all $f^{-1}V$ for V an open neighborhood of y. The result then follows from Lemma 7.16. □

Lemma 7.17. *Let $f : X \to Y$ be a projective birational morphism between smooth complex varieties. Let $\mathcal{F}$ be a constructible sheaf of $\mathbb{Q}$-vector spaces on P. Then*

(i) *the map $\mathcal{F} \to f_*f^*\mathcal{F}$ is an isomorphism of constructible sheaves;*

(ii) *we have* $R^1 f_* f^* \mathcal{F} = 0$.

Proof. It suffices to check both statements on the stalks. By using proper base change, we see that the first statement follows from Zariski's main theorem. Similarly, the second statement follows from the fact that the fibers of a projective birational morphism between separated schemes of finite type over $\mathbb{C}$ have trivial first rational cohomology by Theorem 7.15. ◻

Theorem 7.18. *Let* $h : \mathcal{X} \to P$ *be as in Lemma 7.14 and let* $f : S \to P$ *be a projective birational morphism. Let* $\mathcal{H}$ *and* U *be as in Lemma 7.14 and suppose that* $\nu \in \mathrm{NF}(U, \mathcal{H})_P^{\mathrm{ad}}$. *Then* ν *has a non-torsion singularity on* P *if and only if* $f^*\nu$ *has a non-torsion singularity on* S.

Proof. The "if" part follows from Proposition 7.6. To prove the "only if" direction, we can assume without loss of generality that $f : f^{-1}U \to U$ is an isomorphism. In other words, we may replace the diagram (7.9) in the proof of Proposition 7.6 with the following diagram

$$\begin{tikzcd} & S \arrow[d, "f"] \\ U \arrow[ur, "j_S"] \arrow[r, "j"'] & P. \end{tikzcd} \tag{7.19}$$

By the functoriality of the sequence (7.8), we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(P, R^0 j_*\mathcal{H}) \arrow[r] \arrow[d] & \mathrm{H}^1(U, \mathcal{H}) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^0(P, R^1 j_*\mathcal{H}) \arrow[d] \\ 0 \arrow[r] & \mathrm{H}^1(S, R^0 j_{S*}\mathcal{H}) \arrow[r] & \mathrm{H}^1(U, \mathcal{H}) \arrow[r] & \mathrm{H}^0(S, R^1 j_{S*}\mathcal{H}). \end{tikzcd} \tag{7.20}$$

It suffices then to show that the map $\mathrm{H}^1(P, R^0 j_*\mathcal{H}) \to \mathrm{H}^1(S, R^0 j_{S*}\mathcal{H})$ is an isomorphism. For this, we apply the Leray spectral sequence coming from the map $f : S \to P$. We have an exact sequence

$$0 \to \mathrm{H}^1(P, j_*\mathcal{H}) \to \mathrm{H}^1(S, j_{S*}\mathcal{H}) \to \mathrm{H}^0(P, R^1 f_*(j_{S*}\mathcal{H})). \tag{7.21}$$

By Lemma 7.14, $j_{S*}\mathcal{H} = f^* j_*\mathcal{H}$. Therefore, by Lemma 7.17, it follows that

$$\begin{aligned} R^1 f_*(j_{S*}\mathcal{H}) &= R^1 f_* f^* j_*\mathcal{H} \\ &= 0. \end{aligned}$$

From the exactness of (7.21), it follows that the map $\mathrm{H}^1(P, j_*\mathcal{H}) \to \mathrm{H}^1(S, j_{S*}\mathcal{H})$ is an isomorphism. ◻

Corollary 7.22. *Conjectures 7.5 and 1.5 are equivalent.*

Proof. We have already shown that Conjecture 7.5 implies Conjecture 1.5. To prove the converse, we are going to use the main result of [26] in the form stated in Theorem 6.5[(i)⇒(ii)].

Let $X \subset \mathbb{P}^n$ be a projective complex variety of dimension $2n$ with n an integer and let ζ denote a primitive Hodge class on X.

Since Conjecture 1.5 holds, the Hodge conjecture also holds. Therefore, ζ is algebraic. By Thomas' result, it follows that, for $k \gg 0$, we can find a hyperplane section $s \in \mathrm{H}^0(X, \mathcal{O}_X(k))$ such that

(i) $\zeta_{|V(s)}$ is non-zero in $\mathrm{H}^*(V(s), \mathbb{Q})$;
(ii) $V(s)$ has only ODP singularities.

By choosing $k \gg 0$, we can assume that the vanishing cycles of Lefschetz pencils in $|\mathcal{O}_X(k)|$ are non-trivial. Then set $\mathcal{L} = \mathcal{O}_X(k)$ and let $P, \mathcal{X}$ and π be the incidence scheme in (1.1).

Let $\nu = \mathrm{AJ}(\zeta) \in \mathrm{NF}(P, \mathcal{H})^{\mathrm{ad}}/J^n(X)$. By Theorem 1.3 (ii), we see that ν has a non-torsion singularity at the point $[s] \in P$. Now suppose $f : S \to P$ is any proper birational morphism. By restricting the locus in P of hyperplane sections intersecting X with only ODP singularities, we see that $f^*\nu$ has a non-torsion singularity on S as well. □

Appendix A. By Najmuddin Fakhruddin

Let X be a smooth projective variety of dimension n over $\mathbb{C}$ and $\mathscr{L}_k$, $k = 1, 2, \ldots, r$, $r \leq n$, be ample line bundles on X. Let d_k, $k = 1, 2, \ldots, r$, be positive integers such that $\mathscr{L}_k^{d_k}$ is very ample for all k. Let $P_k := |\mathscr{L}_k^{d_k}|$ be the complete linear system associated to $\mathscr{L}_k^{d_k}$ and let $P = \prod_k P_k$. Let $\mathscr{X} \subset X \times P$ be the incidence variety. Since $\mathscr{L}_k^{d_k}$ is very ample for all k, the morphism induced by the first projection $pr : \mathscr{X} \to X$ is smooth, hence $\mathscr{X}$ is a smooth projective variety. We denote by $\pi : \mathscr{X} \to P$ the morphism induced by the second projection, by $d_{\mathscr{X}}$ the dimension of $\mathscr{X}$ and $d_P = d_{\mathscr{X}} - n + r$ that of P.

By the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber [3] we have

$$\pi_*\mathbb{Q}[d_{\mathscr{X}}] \cong \bigoplus_i {}^pH^i(\pi_*\mathbb{Q}[d_{\mathscr{X}}])[-i] \,,$$

in the bounded derived category of complexes with constructible cohomology on P, where ${}^pH$ denotes perverse homology.

The theorem below is a more precise version of the Perverse Weak Lefschetz theorem of Brosnan, Fang, Nie and Pearlstein (Theorem 5.1), but our hypotheses are stronger.

Theorem A.1. If all $d_k \gg 0$, then

$${}^pH^i(\pi_*\mathbb{Q}[d_{\mathscr{X}}]) \cong \begin{cases} H^{n-r+i}(X, \mathbb{Q}) \otimes \mathbb{Q}_P[d_P] & \text{for } i < 0 \\ H^{n-r-i}(X, \mathbb{Q})(-i) \otimes \mathbb{Q}_P[d_P] & \text{for } i > 0 \\ j_{!*}(R^{n-r}\pi_*^{sm}\mathbb{Q}[d_P]) & \text{for } i = 0 \end{cases}$$

where $j : P^{sm} \to P$ is the inclusion of the open subset of P over which π is smooth and $\pi^{sm} : \pi^{-1}(P^{sm}) \to P^{sm}$ is the induced morphism.

Here, and in the sequel, when we say that a statement holds if all $d_k \gg 0$, we mean that there exists an integer N such that the statement holds whenever $d_k > N$ for $k = 1, 2, \ldots, r$.

Remark A.2. By using Remark A.7 and Theorem 6 of [9], it can be seen from the proof that we may take $d_k \geq 2n + 2r - 3$ for all k.

We will prove the theorem after a few lemmas.

Lemma A.3. Let X be a projective variety, $\mathscr{L}$ an ample line bundle on X and $s > 0$ an integer. Then there exists an integer N such that for Z any closed subscheme of X of length $\leq s$, $\mathscr{I}_Z$ the ideal sheaf of Z, the natural map

$$H^0(X, \mathscr{L}^d) \to H^0(X, \mathscr{L}^d \otimes \mathscr{O}_X/\mathscr{I}_Z)$$

is surjective for all $d > N$.

Proof. This is well known, but we include a proof for the reader's convenience.

It clearly suffices to consider subschemes of length s. Let $\mathrm{Hilb}^s(X)$ be the Hilbert scheme of length s subschemes of X and let $\mathscr{I}_s$ be the ideal sheaf of the universal family on $\mathrm{Hilb}^s(X)\times X$. By definition of the Hilbert scheme, $\mathscr{I}_s$ is flat over $\mathrm{Hilb}^s(X)$. Let $\mathscr{L}_s = p_2^*\mathscr{L}$, so $\mathscr{L}_s$ is relatively ample over $\mathrm{Hilb}^s(X)$. Thus there exists N such $R^i p_{1*}(\mathscr{L}_s^d \otimes \mathscr{I}_s) = 0$ for all $i > 0$ and $d > N$. By the cohomology and base change theorem, more precisely [18, Corollary 4, p.53], it follows that $H^1(X, \mathscr{L}^d \otimes \mathscr{I}_Z) = 0$ for all Z as above and $d > N$. Then the long exact sequence of cohomology associated to the short exact sequence of sheaves on X

$$0 \to \mathscr{L}^d \otimes \mathscr{I}_Z \to \mathscr{L}^d \to \mathscr{L}^d \otimes \mathscr{O}_X/\mathscr{I}_Z \to 0$$

implies the desired surjectivity. □

Remark A.4. Morihiko Saito has shown (personal communication) that for subschemes Z of the form $\cup_{i=1}^m \mathrm{Spec}\,(\mathscr{O}_{X,x_i}/m_{x_i}^2)$, where $x_1, x_2, \ldots, x_m$ are distinct points of X (these are the only ones that we use below) and $\mathscr{L}$ very ample, one may take N to be $2m-1$.

Lemma A.5. Let Y be a smooth proper variety of dimension $d+1 > 1$ and let $\mathscr{L}$ be a line bundle on Y. If the base loci of $|K_Y|$ and $|\mathscr{L}|$ do not contain any divisors, then for all smooth divisors D in $|\mathscr{L}|$ the restriction map $H^d(Y, \mathbb{Q}) \to H^d(D, \mathbb{Q})$ is not surjective.

Proof. By Hodge theory it suffices to show that the restriction map $H^d(Y, \mathscr{O}_Y) \to H^d(D, \mathscr{O}_D)$ is not surjective. From the long exact sequence of cohomology associated to the short exact sequence of sheaves

$$0 \to \mathscr{O}_Y(-D) \to \mathscr{O}_Y \to \mathscr{O}_D \to 0$$

one sees that this is equivalent to showing that the map $H^{d+1}(Y, \mathscr{O}_Y(-D)) \to H^{d+1}(Y, \mathscr{O}_Y)$ is not injective. This, by Serre duality, is equivalent to showing that the map $H^0(Y, K_Y) \to H^0(Y, K_Y(D))$ is not surjective, *i.e.* that D is not contained in the base locus of $|K_Y(D)|$. This holds since the base loci of $|K_Y|$ and $|D| = |\mathscr{L}|$ do not contain any divisors, so neither does the base locus of $|K_Y(D)|$. □

From the lemma we see that the vanishing cycles in the setup of Proposition 5.10 are non-trivial whenever K_X is base point free, which is what we use below.

Now let the notation be as at the beginning of §A. For $p \in P$, let $X_p := \pi^{-1}(p)$ and for $m \in \mathbb{Z}_{>0}$ and $\bar{x} = (x_1, x_2, \ldots, x_m) \in X^m$, let $P(\bar{x})$ be the locus of all $p \in P$ such that $x_i \in X_p$ and for all i X_p is *not* smooth of dimension $n-r$ at x_i. Since π is proper, $P(\bar{x})$ is a Zariski closed subset of P. Let

$$Q_m = \{p \in P \mid \dim X_p > n-r \text{ or } \#|\mathrm{Sing}\, X_p| \geq m\}$$

and let

$$Q = \{p \in P \mid \dim X_p > n-r \text{ or } \dim \mathrm{Sing}\, X_p > 0\}\,.$$

Lemma A.6. For any $m \in \mathbb{Z}_{>0}$, if all $d_k \gg 0$ then $\mathrm{codim}_P Q_m \geq m$ and hence $\mathrm{codim}_P Q \geq m$.

Proof. Fix $m \in \mathbb{Z}_{>0}$. By applying Lemma A.3 with $s = m(n+1)$ to each of the $\mathscr{L}_k$, we may assume that all $d_k \gg 0$ so that the surjectivity statement of that lemma holds for $\mathscr{L} = \mathscr{L}_k$ and $d = d_k$, $k = 1, 2, \ldots, r$. For $x \in X$, let Z_x be the subscheme $\mathrm{Spec}\,(\mathscr{O}_{X,x}/m_x^2)$. The locus of $r \times n$ matrices of rank $< r$ is well known to be irreducible of codimension $n-r+1$ in the space of all $r \times n$ matrices [2, p. 67]. Whether or not an intersection of r hypersurfaces is smooth of dimension $n-r$ at x depends only on the hypersurfaces upto first order, so the surjectivity statement above applied to Z_x implies that for $\bar{x} = (x)$, $P(\bar{x})$ is irreducible and $\mathrm{codim}_P P(\bar{x}) = n + 1$.

Now let $Z_{\bar{x}} = \cup_{i=1}^{m} \mathrm{Spec}\,(\mathscr{O}_{X,x_i}/m_{x_i}^2)$. If all the x_i are distinct, the surjectivity condition on sections implies that the conditions for X_p to be singular at x_i, $i = 1, 2, \ldots, m$, are all independent. Thus $\mathrm{codim}_P P(\bar{x}) = \sum_i \mathrm{codim}_P P((x_i)) = m(n + 1)$.

One sees from the definitions that $Q_m = \cup_{\bar{x}\in X_0^m} P(\bar{x})$, where $X_0^m \subset X^m$ consists of m-tuples of distinct elements of X. Since $\mathrm{codim}_P P(\bar{x}) = m(n + 1)$ for all $\bar{x} \in X_0^m$ and $\dim(X_0^m) = mn$, it follows that $\mathrm{codim}_P Q_m \geq m(n + 1) - mn = m$. Since $Q \subset Q_m$ for all m, $\mathrm{codim}_P Q \geq m$ as well. □

Remark A.7. It follows from Remark A.4 that we may take $d_k \geq 2m - 1$ for all k.

Proof of Theorem A.1. By applying Lemma A.6 with $m = n + r - 1$ we may assume that all $d_k \gg 0$ so that $\mathrm{codim}_P Q \geq n + r - 1$.

By the Weak Lefschetz theorem and induction on r, the restriction maps $H^j(X, \mathbb{Q}) \to H^j(X_p, \mathbb{Q})$ are isomorphisms for $p \in P^{sm}$ and $j < n - r$, so $H^{n-r+i}(X, \mathbb{Q}) \otimes \mathbb{Q}[d_P]$ is a direct summand of ${}^pH^i(\pi_*\mathbb{Q}[d_{\mathscr{X}}])$ for $i < 0$. It follows from Poincaré duality for X_p, with $p \in P^{sm}$, that $H^{n-r-i}(X, \mathbb{Q})(-i) \otimes \mathbb{Q}[d_P]$ is a direct summand of ${}^pH^i(\pi_*\mathbb{Q}[d_{\mathscr{X}}])$ for $i > 0$. By restricting to P^{sm} one also sees that $j_{!*}(R^{n-r}\pi_*^{sm}\mathbb{Q}[d_P])$ is a direct summand of ${}^pH^0(\pi_*\mathbb{Q}[d_{\mathscr{X}}])$. The proper base change theorem applied to compute the cohomology of X_p, $p \in P^{sm}$, shows that for each i the above perverse sheaves are the only summands of ${}^pH^i(\pi_*\mathbb{Q}[d_{\mathscr{X}}])$ which have support equal to P.

Suppose for some i, ${}^pH^i(\pi_*\mathbb{Q}[d_{\mathscr{X}}])$ has a non-zero summand E supported in Q. By Verdier duality for the morphism π we may assume that $i \geq 0$. Since E is perverse and supported in codimension $\geq n+r$, it follows that $H^l(\pi_*\mathbb{Q}[d_{\mathscr{X}}]) \neq 0$ for some $l \geq -d_{\mathscr{X}} + 2n$ (since $d_P = d_{\mathscr{X}} - n + r$), where H^l denotes the usual homology sheaf. Since $\dim(X_p) \leq n-1$ for all $p \in P$, we get a contradiction by applying the proper base change theorem to compute the cohomology of X_p, for p a general point in the support of $H^l(\pi_*\mathbb{Q}[d_{\mathscr{X}}])$. (This argument is taken from [5, Appendice A].)

Suppose ${}^pH^i(\pi_*\mathbb{Q}[d_{\mathscr{X}}])$ has a non-zero irreducible summand E with support $S \subsetneq P$. By the previous paragraph, $S \not\subset Q$. If $i > 0$ or $\mathrm{codim}_P S > 1$, an application of the proper base change theorem to compute the cohomology of X_p, with p a general point of S, contradicts the duality theorem of Kaup [16, Theorem 1.8] which implies that $H^{n-r+i}(X_p, \mathbb{Q}) \cong H^{n-r-i}(X_p, \mathbb{Q}) \cong H^{n-r-i}(X, \mathbb{Q})$ for $i > 1$ and all $p \in P\backslash Q$. So we must have $i = 0$ (by Verdier duality) and $\mathrm{codim}_P S = 1$.

If $r = 1$, it is proved in 5.11 that if d_1 is sufficiently large, then such an E cannot exist; we use essentially the same argument. So assume that $r > 1$ and that such an E exists. Let $P' = P_1 \times \{p_2\} \times \cdots \times \{p_r\}$, where p_i, $i = 2, \ldots, r$, are general points of P_i, so $P' \cap S$ is of codimension one in P'. Let Y be the intersection of the hypersurfaces in X corresponding to the p_i, $i = 2, \ldots, r$. It is a smooth projective variety of dimension $n - r + 1$. For p' a general point of $P' \cap S$, the theory of Lefschetz pencils [1, Exposé XVII] applied to Y and $\mathscr{L} := \mathscr{L}_1^{d_1}|_Y$, implies that $X_{p'}$ has a unique singularity which is an ordinary double point. The adjunction formula for the canonical divisor and Lemma A.5 applied to Y and $\mathscr{L}$ imply that if all $d_k \gg 0$ the local system $R^{n-r}\pi_*^{sm}\mathbb{Q}$ restricted to $P^{sm} \cap P'$ is non-constant, hence the vanishing cycle corresponding to the double point is non–trivial. The cohomological theory of Lefschetz pencils [1, Exposé XVIII] then implies that $H^{n-r+1}(X_{p'}, \mathbb{Q}) \cong H^{n-r+1}(X_p, \mathbb{Q})$ where p is a general point of P, which, by the proper base change theorem, implies that the restriction of E to p' is zero. This is a contradiction, since p_i's general implies that p' is a general point of S . □

Remark A.8. The version of Kaup's duality theorem that we need may be deduced from the fact that the constant sheaf $\mathbb{Q}_Y[-\dim Y]$ is perverse on a variety with l.c.i. singularities [14, Lemma 2.1]. Instead of Kaup's theorem one may also use the theory of nearby and vanishing cycles from [1] after restricting the family to a suitable curve in the base. Lemma A.5 may also be avoided by a generalisation of [1, Lemme 6.4.2] to complete intersections. (For $r = 1$, [1, Lemme 6.4.2] is all we need.) With these replacements, it can be seen that Theorem A.1 also holds over fields of positive characteristic.

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Department of Mathematics, The University of British Columbia, 1984 Mathematics Road, Vancouver, B.C., Canada V6T 1Z2
E-mail address: brosnan@math.ubc.ca

Department of Mathematics, College of Liberal Arts & Sciences, University of Iowa, 14 MLH, Iowa City, IA 52242, USA
E-mail address: haofang@math.uiowa.edu

Department of Mathematics, Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, USA
E-mail address: znie@psu.edu

Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA
E-mail address: gpearl@math.msu.edu

School of Mathematics, Tata Instituе of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
E-mail address: naf@math.tifr.res.in