PERIODS AND IGUSA LOCAL ZETA FUNCTIONS

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ABSTRACT. We show that the coefficients in the Laurent series of the Igusa local zeta functions $I(s) = \int_C f^s \omega$ are periods. This is proved by first showing the existence of functional equations for these functions. This will be used to show in a subsequent paper (by P. Brosnan) that certain numbers occurring in Feynman amplitudes (up to Gamma factors) are periods.

We also give several examples of our main result, and one example showing that Euler's constant γ is an exponential period.

1. INTRODUCTION

In their paper [11], Kontsevich and Zagier give an elementary definition of a period integral as an absolutely convergent integral of a rational function over a subset of $\mathbb{R}^n$ defined by polynomial inequalities and equalities. They then show that some of the most important quantities in mathematics are periods and sketch a proof that their notion of a period agrees with the more elaborate notion that algebraic geometers have studied since Riemann and Weirstraß. The last chapter links periods to the "framed motives" studied by A. Goncharov and proposes a structure of a torsor on a certain set of framed motives. The paper is also full of interesting examples given to justify the following:

Philosophical Principle 1.1. *Whenever you meet a new number and have decided (or convinced yourself) that it is transcendental, try to figure out whether it is a period.*

In this paper we show the "periodicity" of certain numbers naturally arising in the theory of Igusa local zeta functions. This result arose out of a desire to show that certain numbers considered in quantum field theory are periods. We briefly explain this motivation.

If $I(D)$ is a Feynman amplitude coming from a scalar field theory corresponding to a Feynman integral with all parameters in $\mathbb{Q}$, then $I(D) = G(D)J(D)$ where $G(D)$ is a relatively simple Gamma factor and $J(D)$ is a meromorphic function which can be written in terms of Igusa local zeta

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functions. The numbers we want to show to be periods are the coefficients in the Laurent series expansion of $J(D)$ at D_0 for D_0 any integer. We remark that this confirms (albeit in a very weak sense) the fact noticed by Kreimer and Broadhurst that the principal parts of the Laurent series for primitive diagrams often have coefficients which are multiple zeta values (this is not expected in general [2]).

To prove this (or even to state it precisely), we will have to explain the functions $I(D)$ in some detail, and this involves explaining the the technique of dimensional regularization. This subject is well-known in physics but unfortunately not in mathematics. However, once dimensional regularization is understood precisely, the proof of the theorem follows from a result on Igusa local zeta functions which is the main theorem of this paper.

The two topics, dimensional regularization and the arithmetic nature of the special values of Igusa local zeta functions, have different flavors. We have, therefore, decided to separate them. In this paper, we consider only the second topic and prove that the the special values of certain Igusa zeta functions are periods, a fact which we believe is interesting in its own right. The connection to Feynman amplitudes will be made in the forthcoming paper [6]. However, in example 3.1, we sketch the connection to Feynman amplitudes for a certain typical class of integrals.

To explain our result on Igusa Zeta functions, we start by describing a special case: Let $\Delta_n \subset \mathbb{R}^n$ be the n-simplex

$$\{(x_1, \dots, x_n) \mid \sum x_1 \leq 1, x_i \geq 0 \text{ for } i = 1, \dots n\}$$

equipped with the n-form

$$\omega = dx_1 \wedge \cdots \wedge dx_n.$$

Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial function which is non-negative on Δ. Then, according to results of Atiyah [1] and Bernstein and Gelfand [3], the function

$$I(s) = \int_{\Delta_n} f^s \omega$$

is meromorphic on the complex s-plane with isolated singularities. These functions are called *Igusa local zeta functions*. Our main theorem concerning them is the following:

Theorem 1.2. *Suppose that $f \in \mathbb{Q}[x_1, \dots, x_n]$ is a polynomial with rational coefficients and let s_0 be an integer. Let*

$$I(s) = \sum_{i \geq N} a_i (s - s_0)^i$$

be the Laurent series expansion of $I(s)$ at s_0. Then the a_i are periods.

The proof of this statement has two components. The first is to show the result for positive integers s_0. The second is to control the Laurent series coefficients for negative integers s_0. We achieve this by showing the existence of functional equations of the form:

$$\sum_{i=0}^{d+r} c_i(s) I(s+i) = 0 \tag{1}$$

with the c_i polynomials in $\mathbb{Q}[s]$.

It will be convenient to prove a version of this result which is more general in the following two senses: (a) the simplex Δ_n can be taken to be a general semi-algebraic set defined over $\overline{\mathbb{Q}}$, and (b) the function f can be taken to lie in the function field $\overline{\mathbb{Q}}(x_0, \ldots, x_n)$. For (b) we will need to use a more general definition of $I(s)$ than the one in [1, 3]. The general result (Theorem 2.8) is stated in section 2.

Theorem 1.3. *Let X be a smooth variety defined over a field k contained in $\mathbb{R} \cap \overline{Q}$. Let $f \in \mathcal{O}(X)$ be a function, and let C be a compact pre-oriented semi-arithmetic subset of $X_{f \geq 0}(\mathbb{R})$ defined over k. Then, if $\omega \in \Omega^n(X)$ is a differential form, the* Igusa zeta function

$$I(s) = \int_C f^s \omega \tag{2}$$

extends meromorphically to all of $\mathbb{C}$ with poles occurring only at negative integers. Moreover, for any $s_0 \in \mathbb{Z}$, the coefficients a_i in the Laurent expansion

$$I(s) = \sum_{i \geq N} a_i (s - s_0)^i \tag{3}$$

are periods.

We will use the symbol $\mathbf{P}$ to denote the $\mathbb{Q}$-algebra of periods and use the definition of a period that appears in [11]. For the convenience of the reader, we also paraphrase this definition in (2.3).

We thank A. Goncharov, D. Kreimer, M. V. Nori, H. Rossi and T. Terasoma for useful communication. The idea of using Picard-Fuchs equations in Theorem 2.8 comes from discussions with Madhav Nori. This idea is standard when studying periods of powers of functions, but it came somewhat of a surprise that there were no Gamma factors at the end.

2. Igusa Zeta Functions

2.1. **Atiyah's Theorem.** Let X be a smooth complex algebraic variety defined over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the real points of X and let G be a subset of

$X(\mathbb{R})$ defined by inequalities

$$G = \{x \in X(\mathbb{R}) \mid g_i(x) \geq 0 \text{ for all } i\} \tag{4}$$

where here the g_i are real-analytic functions on $X(\mathbb{R})$. Let f be a real-analytic function on $X(\mathbb{R})$ which is non-negative and not identically zero on G. Let Γ denote the characteristic function of G. In this notation Atiyah's theorem [1] can be stated as follows.

Theorem 2.1 (Atiyah)**.** *The function $f^s\Gamma$, which is locally integrable for $\Re(s) > 0$, extends analytically to a distribution on X which is a meromorphic function of s in the whole complex plane. Over any relatively compact open set U in X the poles of $f^s\Gamma$ occur at the points of the form $-r/N, r = 1, 2, \cdots$, where N is a fixed integer (depending on f and U) and the order of any pole does not exceed the dimension of X. Moreover, $f^0\Gamma = \Gamma$.*

2.2. **Semi-algebraic Sets.** The following definition is given in [5].

Definition 2.2. A region $C \subset \mathbb{R}^n$ is *semi-algebraic* if it is a union of intersections of sets of the form $\{x \in X(\mathbb{R}) | f(x) > 0\}$ or $\{x \in X(\mathbb{R}) | f(x) = 0\}$ with $f \in \mathbb{R}[x_1, \ldots, x_n]$.

We will say that $C \subset \mathbb{R}^n$ is *semi-arithmetic* if the functions f appearing in the definition are in $\mathbb{R}_{\text{alg}}[X_1, \ldots, X_n]$ with $\mathbb{R}_{\text{alg}} = \mathbb{R} \cap \overline{\mathbb{Q}}$.

Definition 2.3. A period is a number whose real and imaginary parts are given by absolutely convergent integrals of the form $\int_C f d\mu$ where $C \subset \mathbb{R}^n$ is a semi-arithmetic set, $f \in \mathbb{R}_{\text{alg}}(x_1, \ldots, x_n)$ and μ is Lebesgue measure on $\mathbb{R}^n$.

Assume that X is a variety defined over $\mathbb{R}$. For $f \in \mathbb{R}[X]$, let $X_{f \geq 0}$ denote the set

$$\{x \in X(\mathbb{R}) | f(x) \geq 0\}.$$

Every point $x \in X$ has an affine neighborhood V which is isomorphic to a closed subset of $\mathbb{A}^n$. Following [5], we say that a set $C \subset X(\mathbb{R})$ is *semi-algebraic* if $C \cap V$ (considered as a subset of $\mathbb{R}^n$) is semi-algebraic for every such affine neighborhood V. If X and x are defined over $\mathbb{R}_{\text{alg}}$, we can find a V also defined over $\mathbb{R}_{\text{alg}}$. We say that C is *semi-arithmetic* if $C \cap V$ is semi-arithmetic for all such V. Clearly, $X_{f \geq 0}$ is semi-algebraic for $f \in \mathbb{R}[X]$ and semi-arithmetic for $f \in \mathbb{R}_{\text{alg}}[X]$.

Let $C \subset X(\mathbb{R})$ be a semi-algebraic (resp. semi-arithmetic) set contained in a dimension n variety X, which contains an open (in the usual topology) subset of $X(\mathbb{R})$. It is known that the interior of C contains a semi-algebraic (resp. semi-arithmetic) dense open subset $U \subset C$ which is smooth and orientable. (This follows from Proposition 2.9.10 of [5].) By a *pre-orientation* of C, we mean a choice of such a subset U along with an orientation of U. If $\omega \in \Omega^n(X)$ is a differential form and C is pre-oriented, then we make the definition

$$\int_C \omega \stackrel{\text{def}}{=} \int_U \omega. \tag{5}$$

If $C \subset \mathbb{R}^n$ then the interior of C is smooth and comes with a canonical pre-orientation inherited from the standard orientation on $\mathbb{R}^n$. For C compact, the orientation gives a class in $\sigma \in H_n(C, \partial C)$ where ∂C is the topological boundary of C. To use Atiyah's theorem in the context of semi-algebraic sets, we need to be able to convert an integral $\int_C \omega$ over an arbitrary semi-algebraic set into a sum of integrals over sets of the form of the set G in (4). The following lemma is needed to this end.

Lemma 2.4. *Let f_i ($1 \leq i \leq n$) and g_j ($1 \leq j \leq m$) be two sets of functions in $\mathbb{R}[X]$. Let $U = U_1 \cup U_2$ be an oriented open set with*

$$U_1 = \{x \in X | f_i > 0\, 1 \leq i \leq n\}, \tag{6}$$

$$U_2 = \{x \in X | g_i > 0\, 1 \leq j \leq m\}. \tag{7}$$

Consider strings of the form

$$\mathbf{e} = (a_1, \ldots, a_n, b_1, \ldots, b_m)$$

where the a_i, b_j are in $\{+1, -1\}$ and either all the a's are $+1$ or all the b's are $+1$.

Consider

$$U_{\mathbf{e}} = \{x \in X | a_i f_i > 0, b_j g_j > 0, 1 \leq i \leq n; 1 \leq j \leq m\}$$

Then, for a form $\omega \in \Omega^n(X)$ (with $n = \dim X$),

$$\int_U \omega = \sum_{\mathbf{e}} \int_{U_{\mathbf{e}}} \omega \tag{8}$$

where the $\mathbf{e}$ are subject to the above constraints.

2.3. **Periods.** Since our domains of integration are going to be semi-algebraic sets, we need a more flexible way of generating periods. This is given by the following theorem whose proof is sketched in [11] (pp. 3,31).

Theorem 2.5. *The ring $\mathbf{P}$ of periods is exactly the ring generated by numbers of the form $\int_\gamma \omega$ where X a smooth algebraic variety of dimension d defined over $\mathbb{Q}$, $D \subset X$ is a divisor with normal crossings, $\omega \in \Omega^d(X)$ is an algebraic differential form on X of the top degree, and $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$.*

We could have replaced $\mathbb{Q}$ by $\overline{\mathbb{Q}}$ above, and obtained the same ring (as Kontsevich and Zagier remark). This is easy because a variety defined over $\overline{\mathbb{Q}}$ can be viewed as defined over $\mathbb{Q}$, but we get several copies over the algebraic closure. But there is one more modification that one can make which is a bit more subtle. This is to allow for absolutely convergent integrals. Most examples (eg. multiple zeta values) are not directly periods in the above sense, the integrals defining them can have singularities on the boundary. To take care of this we note the following theorem which will be proved in Section 4.

Theorem 2.6. *Let X be a smooth n-dimensional algebraic variety defined over a field $k \subset \mathbb{R}_{\text{alg}}$. Let F be a reduced effective divisor and let $\omega \in \Omega^n(X-F)$ be an n-form. Let $C \subset X(\mathbb{R})$ be a pre-oriented semi-arithmetic set with non-empty interior C^o. Then the integral $\int_C \omega \in \mathbf{P}$ provided that it is absolutely convergent.*

Remark 2.7. The theorem is certainly assumed in [11] and a proof is sketched on pages 3 and 31 of that reference. Our purpose in providing our own proof is to elaborate on their comment that the result follows from resolution of singularities in characteristic 0.

2.4. **Laurent Series Coefficients.** We now turn to the theorem on Igusa Zeta functions.

Theorem 2.8. *Let X be a smooth variety defined over $k \subset \mathbb{R}_{\text{alg}}$ and let $f \in \mathcal{O}(X)$ be a function. Let C be a compact pre-oriented semi-arithmetic subset of $X_{f\geq 0}(\mathbb{R})$ defined over k. Then, if $\omega \in \Omega^n(X)$ is a differential form, the function*

$$I(s) = \int_C f^s \omega \tag{9}$$

extends meromorphically to all of $\mathbb{C}$ with poles occurring only at negative integers. Moreover, for any $s_0 \in \mathbb{Z}$, the coefficients a_i in the Laurent expansion

$$I(s) = \sum_{i\geq N} a_i (s-s_0)^i \tag{10}$$

are periods.

Our first step is to prove the theorem for $s_0 > 0$. In this case, Atiyah's theorem shows that the integral for $I(s)$ converges and is analytic in a neighborhood of s_0. Thus, assuming $f \neq 0$, we can differentiate under the integral sign to obtain

$$I^{(l)}(s_0) = \int_C f^{s_0} \log^l(f)\omega. \tag{11}$$

Now

$$\log f(x) = \int_0^1 \frac{f(x)-1}{(f(x)-1)t+1}\, dt. \tag{12}$$

Thus we can write the $\log$ factors in (11) as period integrals.

To do this explicitly, set $Y = X \times \mathbb{A}^l$, $D = C \times [0,1]^l$ and

$$\eta = \omega \wedge \frac{f(x)-1}{(f(x)-1)t_1+1}\, dt_1 \wedge \cdots \wedge \frac{f(x)-1}{(f(x)-1)t_l+1}\, dt_l.$$

We then have

$$\int_D f^{s_0}\eta = \int_C f^{s_0} \log^l(f)\omega. \tag{13}$$

The left hand side is absolutely convergent. Thus, $I^{(l)}(s_0)$ is a period for all l as long as $s_0 > 0$ (theorem 2.6), and the theorem is verified for $s_0 > 0$.

To verify the theorem for $s_0 \leq 0$, we use an auxiliary function and the Picard-Fuchs equation. Set

$$J(t) = \int_C \frac{\omega}{1-tf} \tag{14}$$

viewing the integrand as an n-form on $X \times \mathbb{A}^1$. Then $J(t) = \sum_{l\geq 0} I(l)t^l$ for all t such that the sum converges. Since C is compact, f is bounded on C by some constant R. Thus, for $t < 1/R$, C does not intersect the divisor $Z = V(1-tf)$ where the integrand may have a pole, and the integral (14) converges.

Using the triangulation theorem for semi-algebraic sets ([5] Theorem 9.2.1), we can assume that C is homeomorphic analytically to an n-simplicial complex with one n-cell and that ∂C is contained in a divisor $D \subset X$ (defined over k) Let $\sigma \in H_n(X(\mathbb{R}) - Z(\mathbb{R}), D(\mathbb{R}) - Z(\mathbb{R}); \mathbb{Z})$ be the class represented by integration over the points of C that are smooth in X. Then, for each t with $|t| < 1/R$,

$$J(t) = \int_\sigma \frac{\omega}{1-tf}. \tag{15}$$

There is an algebraic vector bundle $\mathcal{V} = H^m_{DR}(X-Z, D-Z)$ over $\mathbb{A}^1 - S$ where S is a finite subset of $\mathbb{A}^1$ which can include 0 and is defined over k. The stalks of $\mathcal{V}$ are the de Rham cohomology groups $H^n(X_y - Z_y, Z_y - D_y)$ over the field $k(y)$ for all $y \in \mathbb{A}^1 - S$. The integrand $s = \frac{\omega}{1-tf}$ can be thought of as a global section of $\mathcal{V}$ (because it is an algebraic differential form of the top degree it is closed and vanishes when restricted to $D - Z$).

This bundle $\mathcal{V}$ carries an algebraic connection (Gauss-Manin) ∇, an isomorphism over $\mathbb{A}^1 - S$ (of analytic vector bundles)

$$\mathcal{V}_{\mathbb{A}^1_{\mathbb{C}}} \to \mathcal{L} \bigotimes_{\mathbb{Z}} \mathcal{O}_{Y_{\mathbb{C}}}$$

where $\mathcal{L}$ is the local system whose fiber at $y \in Y_{\mathbb{C}}$ is the singular cohomology of the pair $(X_y - Z_y, D_y - Z_y)$. The connection is integrable, has regular singular points and the sheaf of flat sections is the sheaf $\mathcal{L}$.

If σ is a flat section of the dual local system $\mathcal{L}^*$ (which is the local system of the homology of pairs $H^n(X_y - Z_y, D_y - Z_y)$) over an open set $U \subset \mathbb{A}^1 - S$ in the analytic topology, then we can form a function on U: $g(y) = \int_\sigma s_y$. If T is a tangent vector field on U, we have the formula

$$T(g) = \int_\sigma \nabla_T(s)_y.$$

Now, $\mathcal{V}$ is a vector bundle of finite rank so given any section s over $\mathbb{A}^1 - S$, there is a relation of the form (we use only the algebraicity of ∇ and not the regularity)

$$\sum_{i=0}^{r} q_i(t) \nabla_T^i(s)_y.$$

where the q_i are rational functions in t with coefficients in k. We can assume that they are polynomials by multiplying the equation by a polynomial $\in k[t]$.

Integrating this against the σ obtained from C and $s = \frac{\omega}{1-tf}$ we obtain a nontrivial linear relation of the form

$$\sum_{i=0}^{r} q_i(t) J^{(i)}(t) = 0 \tag{16}$$

where the $q_i(t) \in k(t)$. (See [8] for a complete reference to the Picard-Fuchs theory .)

Clearing denominators in (16), we can assume that the $q_i(t) \in k[t]$. Expanding out $q_i(t) = \sum_{j=0}^{d_i} a_{i,j} t^j$ (for some $a_{i,j} \in k$) and $J^{(i)}(t) = \displaystyle\sum_{j \geq 0} \frac{j!}{(j-i)!} t^{j-i} I(j)$ and equating terms with the same power of t, we obtain a set of relations between the $I(j)'s$. Explicitly, we obtain the relation

$$\sum_{s \geq 0} \sum_{i=1}^{r} \sum_{j=0}^{d} a_{i,j} \frac{(s+i-j)!}{(s-j)!} I(s+i-j)\, t^s = 0 \tag{17}$$

where $d = \max d_i$.

Noting that, for each pair (i, j), the coefficient $a_{i,j}\dfrac{(s+i-j)!}{(s-j)!}$ is a polynomial of degree i in s, we see that we have a relation of the form

$$\sum_{i=0}^{d+r} c_i(s)I(s+i) = 0 \tag{18}$$

with the c_i polynomials in $k[s]$. Note that, as long as f and ω are nonzero the relation (18) is nontrivial.

We wish to show that (18) holds for all complex values of s. By the uniqueness of analytic continuation, it is enough to show that this is so for $\Re(s) > d + r$. We then use the following corollary of a result from [7] (p. 953).

Theorem 2.9 (Carleson)**.** *Let $h(z)$ be holomorphic for $\Re(z) > 0$ and assume $h(n) = 0$ for $n \in \mathbb{N}$. Then $h(z) = 0$ if $h(z) \le Ke^{m\Re(z)}$ for a constants m and K.*

To use Carleson's theorem, let $P(z)$ be the left hand side of (18) viewed as a function of a complex variable $z = s - d - r$. Set $Q(z) = \frac{P(z)}{(z+1)^M}$ where M is a positive integer greater than the degrees of any of $c_i(s)$. Then $Q(z)$ is holomorphic for $\Re(z) > 0$. Moreover, since f is bounded on the semi-algebraic set C by a number R, $|I(s)|$ is bounded by $AR^{\Re(s)}$ for some constant A. Thus $Q(z)$ is bounded by $Ke^{m\Re(s)}$ for some constants m and K. It follows from Carleson's theorem that $Q(z) = 0$ for $\Re(s) > 0$. Thus, by uniqueness of analytic continuation, it follows that $Q(z) = 0$ for all z.

Without loss of generality, we can assume that $c_0(s)$ in (18) is nonzero. Then we have a relation

$$I(s) = \sum_{i=1}^{d+r} l_i(s)I(s+i). \tag{19}$$

where $l_i(s) = \dfrac{-c_i(s)}{c_0(s)}$. Using (19), we can complete the proof of Theorem 2.8 by descending induction on s_0. For $s_0 > 0$, the theorem is established. Suppose then that the theorem is established for $s_0 > M$. We can use the Laurent expansions for the terms on right hand side of (19) to write out the Laurent expansion for the left hand side. Using the fact that the l_i are rational function in $k(t)$ and using the Laurent expansions of $I(s)$ at $s_0 > M$, it is easy to see that the theorem holds for $s = s_0$.

Remark 2.10. The use of Carleson's theorem is intriguing and the question naturally arises if there is a motivic proof . Namely, from the validity of the functional equation for integer $s > 0$, we deduced it for all s. Now consider the case of s rational and positive. The equation

$$\sum_{i=0}^{d+r} c_i(s)I(s+i) = 0$$

is then a nontrivial relation between periods which is obtained only at the level of values (and not in the abstract ring of periods which is conjectured to inject into complex numbers [11]). It is of course expected that the equation holds at a motivic level, but this is not visible from our method.

Remark 2.11. Precedents to the functional equation in theorem 2.8 should be noted. They first appear in Bernstein's paper [4]. Using the theory of $\mathcal{D}$-modules, he shows that if f is non vanishing on $\mathbb{R}^n$, the domain of integration was all of $\mathbb{R}^n$ and the polynomial function f satisfied a growth rate of the form

$$|f(X)| \geq C||X||^A$$

for $A > 0$ and $||(x_1, \ldots, x_n)|| = \sum x_i^2$, then functions of the type

$$H(s) = \int_{\mathbb{R}^n} f^{-s} dx_1, \ldots, dx_n$$

satisfied functional equations. This was achieved beautifully using the theory of $\mathcal{D}-$modules. But this approach fails (or at least we could not make it work) when the domain of integration is an arbitrary semi-algebraic set.

3. EXAMPLES

Example 3.1. Let G be a graph without self loops and with n edges labeled $1, \ldots, n$. Let x_e denote a formal variable placed on the edge $e \in E(G)$ (the edge set of G).

Form the homogenous polynomial in n-variables

$$P = \sum_T \prod_{e \notin T} x_e$$

where the T sum is over spanning trees of G. This polynomial appears in the study of Feynman amplitudes, and the matrix tree theorem shows that this polynomial can be expressed as a determinant [2, 12].

Let $\Delta_n \subset \mathbb{R}^n$ be the standard n-simplex (each coordinate positive and sum of coordinates $= 1$). Consider the function

$$I(s) = \int_{\Delta_n} P^s dx_1 \ldots dx_n$$

This has an meromorphic continuation to all of $\mathbb{C}$ and our theorem proves that the values (residues, Taylor series coefficients) at every integer of this function are periods. These special values are related to Feynman amplitude calculations.

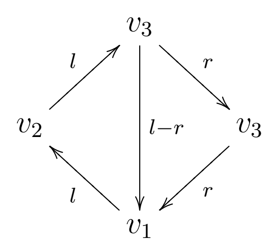
To describe the relation, let V denote the vertex set of G and pick an orientation of G denoting by $h(e)$ (resp. $t(e)$) the head (resp. tail) of the edge e. Form the integral

$$J(D) = \int_{\mathbb{R}^{ED}} \prod_{e\in E} \frac{1}{1+|p_e|^2} \prod_{v\in V} \delta(\sum_{e|h(e)=v} p_e - \sum_{e|t(e)=v} p_e) \prod_{e\in E} d^D p_e \tag{20}$$

where δ denotes the delta function. The integral is gotten by applying the Feynman rules ([10]) to the graph G. The delta function insures conservation of momentum, and it can be removed by replacing the integration over all edges by an integral over the space of loops in the graph.

For example, for the graph

(21)



we have the integral

$$J(D) = \int_{\mathbb{R}^{2D}} \frac{1}{(l^2+1)^2(r^2+1)^2((l-r)^2+1)^2} d^D l \, d^D r \tag{22}$$

(where here we use l^2 to denote $|l|^2$).

When the integral in (20) converges, it is given by

$$J(D) = \frac{\pi^{b_1 D/2}\Gamma(E - b_1 D/2)}{\Gamma(E)} I(-D/2) \tag{23}$$

where b_1 is the first Betti number of G and we write E for the number of edges.

To obtain this expression, one uses the technique of Feynman parameters. See [10] (pp. 206–207, exercises 15 & 16) for a derivation of the expression in the case $D = 4$ (easily generalized to higher dimension). For D such that the integral does not converge (23) can be taken as the definition of the $J(D)$. Moreover, as $I(s)$ is a meromorphic complex-valued function, (23) can be taken as the definition of $J(D)$ for complex D.

In the terminology of quantum field theory, (20) is a Feynman amplitude with zero external momentum and all masses equal to 1. As such, it is a special Feynman amplitude; however, it is typical in the sense that all Feynman amplitudes can be expressed in terms of Igusa zeta functions using considerations similar to the ones used to derive (23). An general theory, will we presented in [6].

Example 3.2. Let $f(x) = x(1-x)$ and

$$I(s) = \int_0^1 f^s dx.$$

In this case, $I(s)$ is the well-known Beta function or Eulerian integral of the first kind $B(s+1, s+1)$ (see [14] p. 253). Expressing the Beta function in terms of Gamma functions, we have

$$I(s) = \frac{\Gamma(s+1)\Gamma(s+1)}{\Gamma(2s+2)}.$$

Using the functional equation, we can write this as

$$I(s) = \frac{\Gamma(s+1)\Gamma(s+1)}{(2s+1)\Gamma(2s+1)}. \tag{24}$$

We can then use the expansion of $\log\Gamma(s+1)$ to write an expansion for $I(s)$ in terms of special values of the Riemann zeta function.

By ([14], 13.6), we have

$$\log\Gamma(s+1) = -\gamma s + \sum_{m=2}^{\infty}(-1)^m\zeta(m)s^m. \tag{25}$$

Substituting this expression into (24), we obtain

$$\log(2s+1) + \log I(s) = \sum_{m=2}^{\infty}\frac{(-2)^{m+1}}{m}(2^{m-1}-1)\zeta(m)s^m \tag{26}$$

Thus,

$$I(s) = \exp(\sum_{m=2}^{\infty}\frac{(-2)^{m+1}}{m}(2^{m-1}-1)\zeta(m)s^m)\sum_{n=0}^{\infty}(-2s)^m. \tag{27}$$

Expanding this out in powers of s, we see that all coefficients can be expressed as finite rational combinations of the $\zeta(m)$. Thus we obtain an explicit example of theorem 2.8.

Note that we could recover the functional equation for the Beta function using using the methods we used to prover theorem 2.8. However, the easiest way to understand the construction is to phrase it in terms of Bernstein's theory. This approach consists of finding a differential operator $\mathcal{D} \in k[x, s, d/dx]$ (no differentiations in s), so that

$$\mathcal{D}f^s = b(s)f^{s-1}$$

and using this to decrease the order. In our case we find

$$[(1-2x)\frac{d}{dx} + 4s]f^s = sf^{s-1}.$$

Using this and integration by parts we get $(4s+2)I(s) = sI(s-1)$. The functional equation turns out to be simple but the Taylor series coefficients at say $\frac{1}{2}$ are not simple (involve π) and at $\frac{1}{3}$ involve elliptic integrals. That a single functional equation captures the growth of these integrals of different genus is surprising.

Let us analyze this example in relation to the algebraic curves $y^N = x(1-x)$ for positive integers N. Let C_N be the smooth model of this equation. It admits a map λ to $\mathbb{P}^1$ with coordinate x. This map is ramified over $0, 1, \infty$. There is a unique lifting of the real path $[0,1]$ to C_N so that y stays real. The values $I(\frac{M}{N})$ are therefore intimately tied with the pairs $(C_N, \{0,1\})$.

If the boundary is irregular, the method of the above example runs into serious difficulties.

This theorem connects with the work of Terasoma [13] who showed that the coefficients of the Taylor expansion of certain Selberg integrals with respect to exponential variables can be expressed as a linear combination of multiple zeta values. We thank Terasoma for bringing this to our attention. This shows the possible complexity of the Taylor series coefficients.

Example 3.3. In section 4.3 of [11], the notion of an *exponential period* is introduced. This is a number that can be written as an absolutely convergent integral of the product of an algebraic function with the exponential of an algebraic function over a semi-algebraic set where all polynomials appearing in the integral have algebraic coefficients. The prototypical example of an exponential period is

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

In the last paragraph of [11], the authors speculate that the set of *exponential periods* can be reasonably extended to include Euler's constant γ. Using (12) and (24), however, it is quite easy to see that γ already is an exponential period. Explicitly,

$$\begin{aligned} \gamma &= -\int_0^\infty e^{-x} \log x\, dx \\ &= -\int_0^\infty \int_0^1 e^{-x} \frac{x-1}{(x-1)t+1}\, dt\, dx. \end{aligned} \tag{28}$$

4. Periods and semi-arithmetic sets

In this section we prove a theorem 2.6. The main tool is the same corollary of resolution of singularities used by Atiyah to prove theorem 2.1. We state it here in the form that we will use.

Theorem 4.1 (Resolution Theorem)**.** *Let $F \in \mathcal{O}(X)$ be a nonzero function on a smooth, complex n-dimensional algebraic variety. Let $\omega \in \Omega^n(X-E)$ be a differential n-form where E is a divisor. Let $Z(\omega)$ denote the zero set of ω. Then there is a proper morphism $\varphi : \tilde{X} \to X$ from a smooth variety $\tilde{X}$ such that*

(i) $\varphi : \tilde{X} - \tilde{A} \to X - A$ *is an isomorphism, where* $A = F^{-1}(0) \cup E \cup Z(\omega)$ *and* $\tilde{A} = \varphi^{-1}(A)$*.*
(ii) *for each* $P \in \tilde{X}$ *there are local coordinates* $(y_1, \ldots, y_n)$ *centered at* P *so that, locally near* P*,*

$$
\begin{aligned}
F \circ \varphi &= \epsilon \prod_{j=1}^{n} y_j^{k_j} \\
\omega &= \delta \prod_{j=1}^{n} y_j^{l_j}\, dy_1 \wedge \cdots \wedge dy_n
\end{aligned}
$$

where ϵ, δ are units in $\mathcal{O}_{X,P}$, the k_j are non-negative integers and the l_j are arbitrary integers.

The theorem, the statement of which is very close to the statement of Atiyah's resolution theorem on p. 147 of [1], is proved by applying Main Theorem II in [9] to the ideals $F\mathcal{O}_X$, E and $Z(\omega)$.

Proposition 4.2. *Let X be a smooth n-dimensional algebraic variety defined over $\mathbb{R}_{\text{alg}}$. Let F be a reduced effective divisor and let $\omega \in \Omega^n(X-E)$ be an n-from. Let*

$$G = \{x \in X(\mathbb{R}) | g_i(x) \geq 0\}$$

for some set $\{g_i\}_{i=1}^m$ of functions in $\mathcal{O}(X)$. be a compact, pre-oriented semi-algebraic set with non-empty interior G^0. Then $\int_G \omega$ converges absolutely only if there is a smooth n-dimensional algebraic variety $\tilde{X}$ with proper, birational morphism $\varphi : \tilde{X} \to X$ and a compact semi-algebraic set $\tilde{G}$ such that

(i) $\int_{\tilde{G}} \varphi^* \omega = \int_G \omega$.
(ii) $\varphi^* \omega$ *is holomorphic on* $\tilde{G}$*.*

Proof. Using the resolution theorem with $F = \prod_{i=1}^m g_i$, we can find a smooth variety $\tilde{X}$ with a proper, birational morphism to X such that for every point $P \in \tilde{X}$ we have local parameters $(y_1, \cdots, y_n)$ defined in a neighborhood of P with

$$\begin{aligned} g_i \circ \varphi &= \epsilon_i \prod_{j=1}^{n} y_j^{k_{ij}} \\ \varphi^* \omega &= \delta \prod_{j=1}^{n} y_j^{l_j} \, dy_1 \wedge \cdots \wedge dy_n. \end{aligned}$$

Here the ϵ_i and δ are invertible near P. Set $\tilde{G}$ equal to the analytic closure of $\varphi^{-1}(G - A)$ with A as in the resolution theorem. Then $\int_{\tilde{G}} \varphi^* \omega = \int_G \omega$ because $\tilde{G}$ and G differ only by measure 0 sets. Moreover, since φ is proper and $\tilde{G}$ is a closed subset of $\varphi^{-1}G$, $\tilde{G}$ is compact.

To see that $\varphi^*\omega$ is holomorphic on $\tilde{G}$, let $P \in \tilde{G}$ be a point and let $\tilde{U}$ be a neighborhood of P with a local coordinate system $(y_1, \cdots, y_n)$ as in the resolution theorem. Since P is in the closure of $\varphi^{-1}(G - A)$, $g_i(P) \geq 0$ for all i. Let s_i be the sign (± 1) of $\epsilon_i(P)$. Then, since $\int_G \omega$ is absolutely convergent, it follows that

$$\int_{0 < s_i y_i(p) < r} \varphi^* \omega = \int_{0 < s_i y_i(p) < r} \prod_{j=1}^{n} y_j^{l_j} \, dy_1 \wedge \cdots \wedge dy_n \tag{29}$$

is absolutely convergent for a sufficiently small r. It is easy to see that this is not possible unless $l_j \geq 0$ for all j. Thus $\varphi^*\omega$ is holomorphic at P. $\square$

Proposition 4.3. *Let X be a smooth algebraic variety over $\mathbb{R}_{\mathrm{alg}}$ and let $G = \{x \in X(\mathbb{R}) | g_i(x) \geq 0\}$ be a compact pre-oriented set. Let $\omega \in \mathcal{O}_X(X)$ be a differential n-form. Then there is a divisor $D \subset X$ and a chain $\sigma \in H_n(X, D)$ such that $\int_G \omega = \int_\sigma \omega$.*

Proof. The pre-orientation on G gives us a dense, smooth, open semi-algebraic subset U in G with an orientation on U. We, therefore, obtain a chain $\sigma \in H_n(X, D)$ where D is the set of zeroes of the functions g_i defining G. This σ corresponds to the orientation on the open subset U so we have $\int_\sigma \omega = \int_G \omega$. $\square$

Using theorem 2.5, theorem 2.6 is then a corollary of the proposition.

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