ZERO LOCI OF ADMISSIBLE NORMAL FUNCTIONS WITH TORSION SINGULARITIES

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ABSTRACT. We show that the zero locus of a normal function on a smooth complex algebraic variety S is algebraic provided that the normal function extends to a admissible normal function on a smooth compactification $\bar{S}$ of S with $\bar{S} \setminus S$ smooth. This result generalizes our previous result for admissible normal functions on curves [4]. It has also been obtained by M. Saito using a different method in a recent preprint [21].

1. INTRODUCTION

Let H be a pure Hodge structure of weight -1 with integral structure $H_{\mathbb{Z}}$. Then, the intermediate Jacobian of H is the complex torus $J(H) = H_{\mathbb{C}}/(F^0 + H_{\mathbb{Z}})$ where $F^\bullet$ is the Hodge filtration of H. If $\mathscr{H}$ is a variation of pure Hodge structure over a complex manifold S with integral structure $\mathscr{H}_{\mathbb{Z}}$, the above construction produces a holomorphic bundle of complex tori $J(\mathscr{H}) \to S$ with fiber $J(\mathscr{H})_s = J(\mathscr{H}_s)$ over s. A normal function ν is a holomorphic section of $J(\mathscr{H})$ which satisfies a version of Griffiths horizontality. Therefore, as a holomorphic section of $J(\mathscr{H})$, the locus of points $\mathscr{Z}$ where ν vanishes is a complex analytic subvariety of S. Furthermore, we have the following conjecture of M. Green and P. Griffiths :

Conjecture 1.1. *Let ν be an admissible normal function* [20] *on a smooth complex algebraic variety S. Then, the zero locus $\mathscr{Z}$ of ν is an algebraic subvariety of S.*

In analogy with the work of Cattani, Deligne and Kaplan [5] on the algebraicity of the locus of a Hodge class, an unconditional proof of this conjecture provides evidence in support of the standard conjectures on the existence of filtrations on Chow groups [11]. In the case where S is a curve, we gave an unconditional proof of (1.1) in [4]. In this paper, we prove the following extension of [4]:

Theorem 1.2. *Let ν be an admissible normal function* [20] *on a smooth complex algebraic variety S. Assume that S has a smooth compactification $\bar{S}$ such that $D = \bar{S} - S$ is a smooth divisor. Then, the zero locus $\mathscr{Z}$ of ν is an algebraic subvariety of S.*

The first step in the proof of Theorem (1.2) is to replace ν by an admissible variation of mixed Hodge structure $\mathscr{V}$ with torsion free integral structure $\mathscr{H}_{\mathbb{Z}}$ and weight graded quotients $Gr_0^W = \mathbb{Z}$ and $Gr_{-1}^W = \mathscr{H}$. This is possible

by [20]. By a standard construction of Deligne, the mixed Hodge structure on the fiber $\mathscr{V}_s$ defines a grading $Y(s)$ of the weight filtration of $\mathscr{V}_s$ which preserves the Hodge filtration. The zero locus $\mathscr{Z}$ is then exactly the set of points where $Y(s)$ is defined over $\mathbb{Z}$.

In analogy with [4], we reduce to the case where the monodromy about D is given by $T = e^N$ for N nilpotent. The two key technical ingredients in the proof of Theorem (1.2) are then the local normal form of a variation of mixed Hodge structure along a normal crossing divisor [19] and the existence of a grading $\hat{Y}(p)$ which is essentially the limit of the gradings $Y(s)$ as $s \to p$ for $p \in D$. The existence of this limit grading, proved in Theorem 3.4, requires the the full strength of the 1-variable SL_2-orbit theorem [18]. Moreover, from the explicit description we give of $\hat{Y}(p)$ in terms of N and the sl_2-splitting of the limit mixed Hodge structure of $\mathscr{V}$ at p, it follows that $\hat{Y}(p)$ is contained in the kernel of $\mathrm{ad}\,N$. Note that the limit mixed Hodge structure of $\mathscr{V}$ at p depends upon the choice of local coordinates of $\bar{S}$ at p. However, because the limit $\hat{Y}(p)$ belongs to the kernel of $\mathrm{ad}\,N$, it well defined independent of the choice of local coordinates.

Alternatively, instead of taking the limit of $Y(s)$ as s accumulates to $p \in D$ along a sequence of points in S, one can twist $Y(s)$ by $e^{-\frac{1}{2\pi i}\log(s)N}$ in analogy with the construction of the limit mixed Hodge structure. This gives a corresponding grading $Y(p)$ which also belongs to the kernel of $\mathrm{ad}\,N$ and has an explicit description in terms of the limit mixed Hodge structure of $\mathscr{V}$ at p. This is stated explicitly in Theorem (4.15) of [18].

In terms of the grading $Y(s)$, the normal function ν is constructed as follows: Let $Y_{\mathbb{Z}}$ be an integral grading of some reference fiber of $\mathscr{V}$. Then, $Y_{\mathbb{Z}}$ extends to a multivalued, integral grading of the weight filtration of $\mathscr{V}$ over S. Therefore, the difference $Y(s) - Y_{\mathbb{Z}}$ is a well defined map from $\mathbb{Z}(0)$ into $J(\mathscr{H}_s)$ for each point $s \in S$. The normal function ν is the image of $1 \in \mathbb{Z}(0)$ under this map. This suggests setting

$$\bar{J}(\mathscr{H})_p = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), K)$$

where K is the induced mixed Hodge structure on $\ker(N : H_{\mathbb{C}} \to H_{\mathbb{C}})$ and defining

(1.3)
$$\nu(p) = (Y(p) - Y_{\mathbb{Z}})(1) \in \bar{J}(\mathscr{H})_p$$

where F_∞ is the limit Hodge filtration of $\mathscr{H}$ at p, and $Y_{\mathbb{Z}}$ is an integral grading of the weight filtration which is invariant under T.

The existence of such a grading $Y_{\mathbb{Z}}$ is obstructed by the class $\sigma_{\mathbb{Z},p}(\nu)$ of ν in the finite group

(1.4)
$$G = \frac{H_{\mathbb{Z}} \cap (T-1)(H_{\mathbb{Q}})}{(T-1)(H_{\mathbb{Z}})}.$$

In analogy with [13], this allows one to construct a "Néron model" which graphs admissible normal functions on a neighborhood of p: In general, the fibers of $\bar{J}(\mathscr{H})$ can not patch together to form a complex analytic space, since the dimension of $\bar{J}(\mathscr{H})_p$ can be less than the dimension of $J(\mathscr{H})_s$ for $s \in S$. For recent work in this direction see [21] which uses the Néron model of [13] to give a proof of Theorem 1.2 independent of ours.

Remark 1.5. In recent joint work with Morihiko Saito [3], we generalize the construction of [13] to higher dimensional parameter spaces using the theory of mixed Hodge modules.

Our original interest in the construction of the limits of normal functions is rooted in the work of Green and Griffiths [12] on singularities of normal functions and the Hodge conjecture. Very briefly, the idea of [12] is to start with a smooth projective variety X of complex dimension $2n$ and a very ample line bundle L on X. Let $|L| = \mathbb{P}H^0(X, L)$ and S be the complement of the dual variety $\hat{X} \subset |L|$ of X. Then (cf. [1]), a primitive Deligne cohomology class $\zeta \in H^{2n}_{\mathscr{D}}(X, \mathbb{Z}(n))$ determines an admissible normal function ν on S with cohomology class $\mathrm{cl}_{\mathbb{Z}}(\nu) \in H^1(S, \mathscr{H}_{\mathbb{Z}})$. We then say that ν is singular on $|L|$ if there is a point $p \in \hat{X}$ such that

$$(1.6) \qquad \sigma_{\mathbb{Z},p}(\nu) = \operatorname*{colim}_{p \in U} \mathrm{cl}_{\mathbb{Z}}(\nu)|_{U \cap S} \in \operatorname*{colim}_{p \in U} H^1(S \cap U, \mathscr{H}_{\mathbb{Z}})$$

is non-torsion, where the colimit is taken over all complex analytic neighborhoods U of p in $|L|$.

Definition 1.7. We call the element $\sigma_{\mathbb{Z},p}(\nu)$ of (1.6) the *singularity* of ν at p.

Remark 1.8. In M. Saito's terminology, $\sigma_{\mathbb{Z},p}$ is the *local invariant* of ν.

The Hodge conjecture is then equivalent to the following statement [12, 1]

Conjecture 1.9. *For each primitive, non-torsion Hodge class $\zeta \in H^{n,n}(X, \mathbb{Z})$ there exists a positive integer k such that ν is singular on $|L^k|$.*

Remark 1.10. The definition of $\sigma_{\mathbb{Z},p}(\nu)$ is valid for any admissible normal function defined on the complement of a divisor $D \subset \bar{S}$. In the case where D is a smooth divisor, admissibility forces $\sigma_{\mathbb{Z},p}(\nu)$ to be torsion. Moreover the finite group (1.4) is exactly the torsion part of the cohomology group appearing in (1.6).

Simple examples show that, in general, unless $\sigma_{\mathbb{Z},p}(\nu) = 0$ the limit of $Y(s)$ along a holomorphic arc γ through p depends upon the multiplicities (assumed finite) of the intersection of γ with the irreducible components of the (normal crossing) boundary divisor at p. However, we will show that if $\sigma_{\mathbb{Z},p} = 0$, the limit $Y(s)$ is independent of γ. Furthermore, modulo one step which we shall defer to [2], we obtain the following result:

Theorem 1.11. *Let ν be an admissible normal function on a smooth complex algebraic variety $S \subset \bar{S}$. Assume that $D = \bar{S} - S$ is a normal crossing divisor and that $\sigma_{\mathbb{Z},p}(\nu)$ is torsion for every point $p \in D$. Then, the zero locus of ν is an algebraic subvariety of S.*

Remark 1.12. In fact, the assumption that D is a normal crossing divisor is not necessary. To see this, suppose that we know the result in the case that D is a normal crossing divisor. Let ν be an admissible normal function on $\bar{S}$ which is smooth over S. By Hironaka, we can find a resolution $p : \bar{T} \to \bar{S}$ such that $p^{-1}S \to S$ is an isomorphism and $p^{-1}(\bar{S} \setminus S)$ is a normal crossing divisor. It is easy to see that, if the singularity of ν is zero at every point in $\bar{S}$, then the singularity of the pullback of ν to $\bar{T}$ is zero on $\bar{T}$ as well. Thus, by the theorem, the zero locus of ν on $S = \bar{T} \setminus p^{-1}S$ is algebraic.

Notation. If G is a group acting on a set S via the action map $G \times S \to S$, we sometimes write the action as $(g, s) \mapsto g.s$. For example, if V is a vector space, we write the action of $\mathrm{GL}(V)$ on the set of filtrations of V by $(g, F) \mapsto g.F$.

Organization. Section 2 of this paper collects several general results about admissible variations. Section 3 proves the existence of limiting gradings of the weight filtration in the case needed for Theorem 1.2; i.e, when the boundary divisor D is smooth. Section 4 proves Theorem 1.2, and section 5 proves Theorem 1.11 modulo Conjecture 5.5 which, as indicated above, we will leave to a the work in progress [2].

2. PRELIMINARY RESULTS

2.1. **Gradings and Splittings.** Let V be a finite dimensional vector space over a field k of characteristic zero, and

$$0 = L_a \subseteq \cdots \subseteq L_i \subseteq L_{i+1} \subseteq \cdots \subseteq L_b = V$$

be an increasing filtration of V indexed by $\mathbb{Z}$. Then, a grading of L is a semisimple endomorphism Y of V such that

$$L_i = E_i(Y) \oplus L_{i-1}$$

for each index i, where $E_i(Y)$ is the i-eigenspace of Y. Elements of $\mathrm{GL}(V)$ which preserve L act on gradings of L by the adjoint action:

$$g.Y = gYg^{-1}.$$

Let (F, W) be a mixed Hodge structure with Hodge filtration F and weight filtration W. Then [9] there exists a unique, functorial bigrading

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$$

of the underlying vector space $V_{\mathbb{C}}$ such that

   (a) $F^p = \oplus_{r \geq p} I^{r,s}$;
   (b) $W_k = \oplus_{r+s \leq k} I^{r,s}$;
   (c) $\bar{I}^{p,q} = I^{q,p} \mod \oplus_{r<q,s<p} I^{r,s}$.

The associated Deligne grading $Y_{(F,W)}$ of W is the semisimple endomorphism of $V_{\mathbb{C}}$ which acts as multiplication by $p+q$ on $I^{p,q}$. In particular, by properties (a)–(c), if g is an element of $\mathrm{GL}(V_{\mathbb{R}})$ which preserves W then

$$I^{p,q}_{(g.F,W)} = g.I^{p,q}_{(F,W)}$$

with respect to the linear action of $\mathrm{GL}(V)$ on filtrations and subspaces. Likewise, for g as above $Y_{(g.F,W)} = g.Y_{(F,W)}$.

The mixed Hodge structure (F, W) induces a mixed Hodge structure on the Lie algebra $\mathfrak{gl}(V_{\mathbb{C}})$ with associated bigrading

(2.1.1) $$\mathfrak{gl}(V_{\mathbb{C}}) = \bigoplus_{p,q} \mathfrak{gl}(V)^{p,q}.$$

Let λ be an element of the subalgebra

$$\Lambda^{-1,-1} = \bigoplus_{a,b<0} \mathfrak{gl}(V)^{a,b}.$$

Then, by properties (a)–(c),

(2.1.2)
$$I^{p,q}_{(e^\lambda.F,W)} = e^\lambda.I^{p,q}_{(F,W)}$$

and hence $Y_{(e^\lambda.F,W)} = e^\lambda.Y_{(F,W)}$.

Definition 2.1.3. A mixed Hodge structure (F,W) is split over $\mathbb{R}$ if $\bar{I}^{p,q} = I^{q,p}$.

Lemma 2.1.4. *Let (F,W) be a mixed Hodge structure. Then, the following are equivalent:*

(a) *(F,W) is split over $\mathbb{R}$;*
(b) *$I^{p,q}_{(F,W)} = F^p \cap \bar{F}^q \cap W_{p+q}$;*
(c) *There exists a grading Y of W which preserves F and is defined over $\mathbb{R}$, in which case $Y = Y_{(F,W)}$.*

If (F,W) is not split over $\mathbb{R}$ we can construct an associated split mixed Hodge structure $(e^{-i\delta}.F,W)$ as follows:

Theorem 2.1.5 (Prop (2.20) [7]). *There exists a unique real element δ in $\Lambda^{-1,-1}$ such that $\bar{Y}_{(F,W)} = e^{-2i\delta}.Y_{(F,W)}$. Moreover, δ commutes with all (r,r)-morphism of (F,W) and*

(2.1.6)
$$(e^{-i\delta}.F,W)$$

is split over $\mathbb{R}$.

Let W be an increasing filtration of V and N be a nilpotent endomorphism of V which preserves W. Assume that the relative weight filtration [22] M of N and W exists, and suppose that there exists a grading Y_M of M which preserves W and satisfies the condition

(2.1.7)
$$[Y_M,N] = -2N.$$

Let Y be a grading of W which preserves M, and
$$N = N_0 + N_{-1} + \cdots$$
be the decomposition of N with respect to $\mathrm{ad}\,Y$ (i.e. $[Y,N_{-j}] = -jN_{-j}$).

Lemma 2.1.8. *(Deligne [8, 14]) Under the hypothesis of the previous paragraph, there exists a unique, functorial grading $Y = Y(N,Y_M)$ of W which commutes with Y_M such that:*

(a) *(N_0,H) is an sl_2-pair where $H = Y_M - Y$;*
(b) *If (N_0,H,N_0^+) is the associated sl_2-triple then $[N - N_0, N_0^+] = 0$.*

Corollary 2.1.9. *For $k > 0$, N_{-k} is either zero or a highest weight vector of weight $k - 2$ with respect the representation of sl_2 constructed in the previous lemma. In particular, $N_{-1} = 0$.*

2.2. **Admissible nilpotent orbits.** Let $\mathscr{V} \to S$ be a variation of mixed Hodge structure over a complex manifold. Then [19, 23], in analogy with a variation of pure Hodge structure, a choice of reference fiber V for $\mathscr{V}$ allows us to represent $\mathscr{V}$ by a period map
$$\varphi : S \to \Gamma \backslash \mathscr{M}$$
where $\mathscr{M}$ is a suitable classifying space of graded-polarized mixed Hodge structures and Γ is the image of the monodromy representation. As in the pure case,

the classifying space $\mathscr{M}$ is a submanifold of a suitable flag variety, and the period map φ is holomorphic, horizontal and locally liftable. If $F : \tilde{S} \to \mathscr{M}$ is a lifting of φ to the universal cover of S, then

$$\frac{\partial F^p}{\partial z_j} \subseteq F^{p-1}, \qquad \frac{\partial F^p}{\partial \bar{z}_j} \subseteq F^p$$

where $(z_1,\dots,z_r)$ are local holomorphic coordinates on $\tilde{S}$.

More precisely, let Q_* be the graded-polarizations of Gr^W and $\mathrm{GL}(V)^W$ denote the subgroup of $\mathrm{GL}(V)$ consisting of elements which preserve W. Define

$$G = \{\, g \in \mathrm{GL}(V)^W \mid Gr(g) \in \mathrm{Aut}_{\mathbb{R}}(Q_*) \,\}$$

to be the subgroup of $\mathrm{GL}(V)^W$ consisting of elements which act by real isometries of Q on Gr^W. Then, in analogy with the pure case, G acts transitively on $\mathscr{M}$ by biholomorphisms. Likewise, we have an embedding of $\mathscr{M}$ into its "compact dual"

$$\mathscr{M} = G/G^F \hookrightarrow G_{\mathbb{C}}/G_{\mathbb{C}}^F = \check{\mathscr{M}}$$

where $G_{\mathbb{C}} = \{\, g \in \mathrm{GL}(V)^W \mid Gr(g) \in \mathrm{Aut}_{\mathbb{C}}(Q_*) \,\}$, and G^F, $G_{\mathbb{C}}^F$ are the corresponding isotopy groups of some point $F \in \mathscr{M}$. The set of points $F \in \mathscr{M}$ for which the corresponding mixed Hodge structure (F,W) is split over $\mathbb{R}$ is a homogeneous space for the Lie group $G_{\mathbb{R}} = G \cap \mathrm{GL}(V_{\mathbb{R}})$. Define $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{C}}$ to be the respective Lie algebras of $G_{\mathbb{R}}$ and $G_{\mathbb{C}}$.

Let $\Delta \subset \mathbb{C}$ be the unit disk and $\mathscr{V}$ be a variation of mixed Hodge structure on the complement Δ^* of the origin with unipotent monodromy $T = e^N$. Then, we have a commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ F\ } & \mathscr{M} \\
{\scriptstyle s=e^{2\pi i z}}\downarrow & & \downarrow \\
\Delta^* & \xrightarrow{\ \varphi\ } & \Gamma\backslash\mathscr{M}
\end{array}
$$

where U denote the upper half-plane. Therefore, $\psi(z) = e^{-zN}.F(z) : \Delta^* \to \check{\mathscr{M}}$ descends to a well defined holomorphic map $\psi : \Delta^*$ into $\check{\mathscr{M}}$. If $\mathscr{V}$ is admissible then [22]

(a) $F_\infty = \lim_{s \to 0} \psi(s) \in \check{\mathscr{M}}$ exists;

(b) The relative weight filtration M of N and W exists.

In this case [22],

(i) (F_∞, M) is a mixed Hodge structure relative to which N is a $(-1,-1)$-morphism;

(ii) $(e^{zN}.F_\infty, W)$ is an admissible nilpotent orbit.

For variations of mixed Hodge structure over a higher dimensional base, Kashiwara defined admissibility via a curve test [15]. In particular, if $\mathscr{V}$ is an admissible variation of mixed Hodge structure defined on the complement of a normal crossing divisor with unipotent monodromy transformations $T_j = e^{N_j}$, then the relative weight filtration $M(N_j, W)$ of W and N_j exists for each j.

The remainder of this section is devoted to the discussion of the 1-variable SL_2-orbit theorem [18] which allows us to approximate the nilpotent orbit $\theta(z)$ by an associated SL_2-orbit $\hat{\theta}(z)$ arising from a representation $\rho : \mathrm{SL}_2(\mathbb{R}) \to G_{\mathbb{R}}$. We start by returning to Lemma (2.1.8):

Lemma 2.2.1. *(Deligne [8, 14]) Let (F, N, W) define an admissible nilpotent orbit with relative weight filtration M. Let $Y_M = Y_{(F,M)}$ and $Y = Y(N, Y_M)$ be the associated grading of Lemma (2.1.8). Then, Y preserves F. If (F, M) is split over $\mathbb{R}$ then $\bar{Y} = Y$.*

Proof. This follows from the functoriality of Deligne's grading together with an explicit computation in the case where (F, M) is split over $\mathbb{R}$. □

Definition 2.2.2. A mixed Hodge structure (F, W) is of type (I) if there exists an index i such that $Gr_k^W = 0$ unless $k = i, i+1$.

Lemma 2.2.3. *Every mixed Hodge structure of type (I) is split over $\mathbb{R}$.*

Proof. This follows directly from the short length of the weight filtration and property (c) of Deligne's bigrading. □

Combining the above result, we now obtain a formula for $Y_{(e^{zN}.F,W)}$ along an admissible nilpotent orbit of type (I), i.e. $(e^{zN}.F, W)$ is mixed Hodge structure of type (I) for $\text{Im}(z) \gg 0$, when the associated limit mixed Hodge structure (F, M) is split over $\mathbb{R}$:

Theorem 2.2.4. *Let $(e^{zN}.F, W)$ be an admissible nilpotent orbit of type (I). Let $Y = Y(N, Y_M)$ be the associated grading of W of Lemma (2.2.1), and suppose that (F, M) is split over $\mathbb{R}$. Then, for $\text{Im}(z) > 0$:*

$$(2.2.5) \qquad\qquad Y = Y_{(e^{zN}.F,W)}.$$

Proof. The fact that $(e^{zN}.F, W)$ is a mixed Hodge structure for $\text{Im}(z) > 0$ follows from the fact that (F, M) is split over $\mathbb{R}$ and [7, Lemma 3.12]. By Corollary (2.1.9) and the short length of W, $N_0 = N$. Therefore, Y preserves $e^{zN}.F$ since $[Y, N] = 0$ and Y preserves F by Lemma (2.2.1). As Y is defined over $\mathbb{R}$, (2.2.5) now follows from part (c) of Lemma (2.1.4). □

The next result allows us to compute the asymptotic behavior of $Y_{(e^{zN}.F,W)}$ along an arbitrary admissible nilpotent orbit of type (I).

Theorem 2.2.6. *(SL$_2$-orbit theorem [18]) Let $(e^{zN}.F, W)$ be an admissible nilpotent orbit of type (I) with relative weight filtration M. Let $(\tilde{F}, M) = (e^{-i\delta}.F, M)$ denote Deligne's δ-splitting (2.1.6) of (F, M). Then, there exists an element*

$$\zeta \in \mathfrak{g}_{\mathbb{R}} \cap \ker(\text{ad} N) \cap \Lambda_{(\tilde{F},M)}^{-1,-1}$$

and a distinguished real analytic function $\tilde{g} : (a, \infty) \to G_{\mathbb{R}}$ such that

   (a) *$e^{iyN}.F = \tilde{g}(y)e^{iyN}.\tilde{F}$ for $y > a$;*
   (b) *$\tilde{g}(y)$ and $\tilde{g}^{-1}(y)$ have convergent series expansions about ∞ of the form*

$$\begin{aligned} \tilde{g}(y) &= e^{\zeta}(1 + \tilde{g}_1 y^{-1} + \tilde{g}_2 y^{-2} + \cdots), \\ \tilde{g}^{-1}(y) &= e^{-\zeta}(1 + \tilde{f}_1 y^{-1} + \tilde{f}_2 y^{-2} + \cdots) \end{aligned}$$

   with $\tilde{g}_k, \tilde{f}_k \in \ker(\text{ad} N)^{k+1}$;
   (c) *δ, ζ and the coefficients $\tilde{g}_k$ are related by the formula*

$$e^{i\delta} = e^{\zeta}\left(1 + \sum_{k>0} \frac{(-i)^k}{k!}(\text{ad} N)^k \tilde{g}_k\right).$$

Let (N_0, H, N_0^+) *be the* sl_2*-triple determined by the* sl_2*-pair of Lemma* (2.2.1) *and the nilpotent orbit* $e^{zN}.\tilde{F}$. *The constant* ζ *can be expressed as a universal Lie polynomial in the Hodge components* $\delta^{r,s}$ *of* δ *with respect to* $(\tilde{F}, M)$. *Likewise the coefficients* $\tilde{g}_k$ *and* $\tilde{f}_k$ *can be expressed as universal Lie polynomials in the Hodge components* $\delta^{r,s}$ *and* $\mathrm{ad}\,N_0^+$.

Remark 2.2.7. As noted in the proof of Theorem (2.2.4), for orbits of type (I), $N = N_0$.

For the purpose of computing the asymptotic behavior of the limit grading in §3, it is useful to renormalize the SL_2-orbit theorem as follows: Let

$$g(y) = \tilde{g}(y)e^{-\zeta}, \qquad \hat{F} = e^\zeta.\tilde{F}.$$

Then, $e^{iyN}.F = g(y)e^{iyN}.\hat{F}$ since $[N, \zeta] = 0$. The mixed Hodge structure $(\hat{F}, M)$ is split over $\mathbb{R}$ since $(\tilde{F}, M)$ is split over $\mathbb{R}$ and $\zeta \in \mathfrak{g}_\mathbb{R}$. Moreover,

$$e^{-\xi} = e^\zeta e^{-i\delta} \in \exp(\Lambda_{(\hat{F}_\infty, M)}^{-1,-1})$$

commutes with N and is a universal polynomial in the Hodge components of δ. Likewise, the coefficients

$$g_k = \mathrm{Ad}\,(e^\zeta)\tilde{g}_k$$

of the series expansion

$$g(y) = 1 + \sum_{k>0} g_k y^{-k}$$

are universal polynomials in the Hodge components of δ and $\mathrm{ad}\,N_0^+$, and satisfy the identity $g_k \in \ker(\mathrm{ad}\,N_0)^{k+1}$.

Definition 2.2.8. Let $(e^{zN}.F, W)$ be a nilpotent orbit of type (I). Then,

$$\hat{F} = e^{-\xi}.F$$

is the sl_2-splitting of (F, M).

Remark 2.2.9. By virtue of the fact that ζ is given by a universal polynomial in the Hodge components of δ, the sl_2-splitting is defined for any mixed Hodge structure. The formula is as follows [17]: Write the Campbell–Baker–Hausdorff formula as $e^\alpha e^\beta = e^{H(\alpha, \beta)}$. Then, δ and ξ are related by the formula

$$\delta = H(\xi, -\bar{\xi})/2\sqrt{-1}.$$

2.3. **Local normal form.** Let Δ^r be a polydisk with local coordinates $(s_1, \ldots, s_r)$ and $\mathcal{V}$ be an admissible variation of mixed Hodge structure on the complement of the divisor $s_1 \cdots s_r = 0$ with unipotent monodromy $T_j = e^{N_j}$ about $s_j = 0$. Then, the $I^{p,q}$'s of the limit mixed Hodge structure (F_∞, M) define a vector space complement

$$\mathfrak{q} = \bigoplus_{a<0} \mathfrak{g}^{a,b}$$

to the isotopy algebra $\mathfrak{g}_\mathbb{C}^{F_\infty}$, and hence by admissibility, near p we can write the Hodge filtration of $\mathcal{V}$ as

(2.3.1) $$F(z) = e^{\sum_j z_j N_j} e^{\Gamma(s)}.F_\infty$$

where $\Gamma(s)$ is a $\mathfrak{q}$-valued function which vanishes at $s = 0$ and $s_j = e^{2\pi i z_j}$. By horizontality,

$$(2.3.2) \qquad \frac{\partial}{\partial z_j} F^p(z) \subseteq F^{p-1}(z).$$

Let $\wp_a = \oplus_b \mathfrak{g}^{a,b}$ and note that:

    (i) $\mathfrak{q} = \oplus_{a<0} \wp_a$;
    (ii) $[\wp_a, \wp_b] \subseteq \wp_{a+b}$;
    (iii) $N_j \in \wp_{-1}$.

Inserting (2.3.1) into (2.3.2) it then follows that

$$(2.3.3) \qquad \mathrm{Ad}(e^{-\Gamma(s)})N_j + 2\pi i s_j e^{-\Gamma(s)} \frac{\partial}{\partial s_j} e^{\Gamma(s)} \in \wp_{-1}.$$

Taking the limit as $s_j \to 0$ in (2.3.3) it then follows by (i)–(iii) that

$$(2.3.4) \qquad [\Gamma^{(j)}, N_j] = 0$$

where $\Gamma^{(j)}$ denotes the restriction of $\Gamma(s)$ to the slice $s_j = 0$.

Remark 2.3.5. In the pure case, this result is due to Cattani and Kaplan [6].

Remark 2.3.6. The results of this subsection remains valid in the case where $\mathscr{V}$ is a variation over $\Delta^{*a} \times \Delta^b$ upon setting $N_j = 0$ for $j = a+1, \ldots, b$.

2.4. **Intersection Cohomology.**

Let $\mathscr{A}_{\mathbb{Q}}$ be a local system of $\mathbb{Q}$-vector spaces over a product of punctured disks Δ^{*r} with unipotent monodromy. Let $A_{\mathbb{Q}}$ be a reference fiber of $\mathscr{A}_{\mathbb{Q}}$ and $N_j \in \mathrm{Hom}(A_{\mathbb{Q}}, A_{\mathbb{Q}})$ denote the monodromy logarithm of $\mathscr{A}_{\mathbb{Q}}$ about the j'th disk. Then, because the N_j's commute, the vector spaces

$$(2.4.1) \qquad B^p(N_1, \ldots, N_r; A_{\mathbb{Q}}) = \bigoplus_{1 \leq j_1 < \cdots < j_p \leq r} N_{j_1} N_{j_2} \cdots N_{j_p}(A_{\mathbb{Q}})$$

form a complex with respect to the differential d which acts on the summands of (2.4.1) by the rule

$$(2.4.2) \qquad d : N_{j_1} \cdots \hat{N}_{j_q} \cdots N_{j_p}(A_{\mathbb{Q}}) \xrightarrow{(-1)^{q-1} N_{j_q}} N_{j_1} \cdots N_{j_p}(A_{\mathbb{Q}}).$$

Let $j : \Delta^{*r} \to \Delta^r$ be a holomorphic embedding (the open inclusion) of Δ^{*r} in a product of disks Δ^r and define

$$\mathrm{IH}^p(\Delta^r, \mathscr{A}_{\mathbb{Q}}) = \mathbb{H}^p(\Delta^r, j_{!*}\mathscr{A}_{\mathbb{Q}}).$$

Then, by [8, 10] or [16, Corollary 3.4.4], $H^p(B^*(N_1, \ldots, N_r; A_{\mathbb{Q}})) \cong \mathrm{IH}^p(\Delta^r, \mathscr{A}_{\mathbb{Q}})$.

The following result follows from Theorem [1, Lemma 2.1.8]. Here we give a proof that is more in the spirit of the calculations done in this paper.

Theorem 2.4.3. *Let $\mathscr{V} \to \Delta^{*r}$ be an admissible variation of graded-polarizable mixed Hodge structure with unipotent monodromy which is an extension of $\mathbb{Q}(0)$ by a variation of Hodge structure $\mathscr{H}$ of pure weight -1. Then, the associated short exact sequence*

$$(2.4.4) \qquad 0 \to \mathscr{H}_{\mathbb{Q}} \xrightarrow{\alpha} \mathscr{V}_{\mathbb{Q}} \xrightarrow{\beta} \mathbb{Q}(0) \to 0$$

induces a long exact sequence

$$\cdots \to \mathrm{IH}^{p-1}(\Delta^r, \mathbb{Q}(0)) \xrightarrow{\partial} \mathrm{IH}^p(\Delta^r, \mathscr{H}_{\mathbb{Q}}) \xrightarrow{\alpha_*} \mathrm{IH}^p(\Delta^r, \mathscr{V}_{\mathbb{Q}}) \xrightarrow{\beta_*} \mathrm{IH}^p(\Delta^r, \mathbb{Q}(0)) \to \cdots$$

in intersection cohomology.

Proof. Let $\mathcal{V}_{\mathbb{Q}}$ underlie an admissible extension of $\mathbb{Q}(0)$ by a variation of pure Hodge structure $\mathcal{H}$ of weight -1 and $B^*(H_{\mathbb{Q}})$, $B^*(V_{\mathbb{Q}})$, $B^*(\mathbb{Q}(0))$ denote the associated complexes (2.4.1) defined by the local monodromy. By abuse of notation, let $\mathrm{IH}^p(H_{\mathbb{Q}})$, etc. denote the cohomology of the the corresponding complex. In particular, since each N_j acts trivially on $\mathbb{Q}(0)$,

$$\mathrm{IH}^0(\mathbb{Q}(0)) = \mathbb{Q}(0), \qquad \mathrm{IH}^p(\mathbb{Q}(0)) = 0, \quad p > 0.$$

Furthermore, since N_j acts trivially on $\mathbb{Q}(0)$ and $Gr_0^W(V_{\mathbb{Q}}) \cong \mathbb{Q}(0)$ it then follows that

$$N_j(V_{\mathbb{Q}}) \subset W_{-1}(V_{\mathbb{Q}}) \cong H_{\mathbb{Q}}$$

By the existence of the relative weight filtration $M_j = M(N_j, W)$ and the short length of W it then follows[22] that

$$N_j(V_{\mathbb{Q}}) = N_j(H_{\mathbb{Q}})$$

and hence $B^p(V_{\mathbb{Q}}) = B^p(H_{\mathbb{Q}})$ for $p > 0$. Consequently,

$$\mathrm{IH}^p(V_{\mathbb{Q}}) = \mathrm{IH}^p(H_{\mathbb{Q}}), \qquad p > 1.$$

Combining the above results, we therefore obtain the exactness of

$$\cdots \to \mathrm{IH}^{p-1}(\mathbb{Q}(0)) \xrightarrow{\partial} \mathrm{IH}^p(\mathcal{H}_{\mathbb{Q}}) \xrightarrow{\alpha_*} \mathrm{IH}^p(\mathcal{V}_{\mathbb{Q}}) \xrightarrow{\beta_*} \mathrm{IH}^p(\mathbb{Q}(0)) \to \cdots$$

for $p > 1$.

Thus, in order to complete the proof, it remains to prove the exactness of the sequence

(2.4.5) $0 \to \mathrm{IH}^0(H_{\mathbb{Q}}) \to \mathrm{IH}^0(V_{\mathbb{Q}}) \to \mathrm{IH}^0(\mathbb{Q}(0)) \xrightarrow{\partial} \mathrm{IH}^1(H_{\mathbb{Q}}) \to \mathrm{IH}^1(V_{\mathbb{Q}}) \to 0.$

By definition,

$$\mathrm{IH}^0(H_{\mathbb{Q}}) = \cap_j \ker(N_j|_{H_{\mathbb{Q}}}), \qquad \mathrm{IH}^0(V_{\mathbb{Q}}) = \cap_j \ker(N_j)$$

and hence the map $\mathrm{IH}^0(H_{\mathbb{Q}}) \to \mathrm{IH}^0(V_{\mathbb{Q}})$ is injective.

To see that (2.4.5) is exact at $\mathrm{IH}^0(V_{\mathbb{Q}})$ observe that since $H_{\mathbb{Q}} = W_{-1}(V_{\mathbb{Q}})$ and $\mathbb{Q}(0) = Gr_0^W(V_{\mathbb{Q}})$, the image of $\mathrm{IH}^0(H_{\mathbb{Q}})$ in $\mathrm{IH}^0(V_{\mathbb{Q}})$ is exactly the kernel of the map

(2.4.6) $Gr_0^W : \mathrm{IH}^0(V_{\mathbb{Q}}) \to \mathrm{IH}^0(\mathbb{Q}(0)).$

For any class $[v] \in IH^0(\mathbb{Q}(0))$,

(2.4.7) $\partial[v] = (N_1(v), \dots, N_r(v)) \mod dB^0(H_{\mathbb{Q}})$

where $v \in V_{\mathbb{Q}}$ is any element which projects onto $[v] \in \mathbb{Q}(0) = Gr_0^W(V_{\mathbb{Q}})$. In particular, $\partial[v] = 0$ if and only if there exists $h \in H_{\mathbb{Q}} = W_{-1}(V_{\mathbb{Q}})$ such that

$$N_j(v) = N_j(h)$$

for all j. In this case, $v_o = v - h$ defines an element of $\mathrm{IH}^0(V_{\mathbb{Q}})$ which projects onto of $[v] \in \mathrm{IH}^0(\mathbb{Q}(0))$ under (2.4.6). As such, (2.4.5) is exact at $\mathrm{IH}^0(\mathbb{Q}(0))$.

To see that (2.4.5) is exact at $IH^1(H_{\mathbb{Q}})$ suppose that

$$(N_1(h_1), \dots, N_r(h_r)), \qquad h_j \in H_{\mathbb{Q}}$$

represents a class $\eta \in \mathrm{IH}^1(H_\mathbb{Q})$ which maps to zero under inclusion in $IH^1(V_\mathbb{Q})$. Then, there exists a vector $v \in V_\mathbb{Q}$ such that

$$N_j(h_j) = N_j(v)$$

for all j, If $v \in H_\mathbb{Q}$ then $\eta = 0$. Otherwise, $[v]$ defines a non-zero class in $\mathrm{IH}^0(\mathbb{Q}(0))$ such that $\eta = \partial[v]$. Finally, to verify the surjectivity of the map

$$\mathrm{IH}^1(H_\mathbb{Q}) \to \mathrm{IH}^1(V_\mathbb{Q})$$

note that $B^p(H_\mathbb{Q}) = B^p(V_\mathbb{Q})$ for $p > 0$ and $dB^0(H_\mathbb{Q}) \subseteq dB^0(V_\mathbb{Q})$. $\qquad\square$

Definition 2.4.8. Let $[1]$ be the class of 1 in $\mathrm{IH}^0(\mathbb{Q}(0))$ and $\partial : \mathrm{IH}^0(\mathbb{Q}(0)) \to \mathrm{IH}^1(\mathscr{H}_\mathbb{Q})$ be the connecting homomorphism. Then, $\mathrm{sing}_0(v) = \partial 1$.

Remark 2.4.9. The results of this section remain valid upon replacing $\mathbb{Q}$ by $\mathbb{R}$.

Remark 2.4.10. There is a natural inclusion $\mathrm{IH}^1(\mathscr{H}_\mathbb{Q}) \to H^1(\Delta^*r, \mathscr{H}_\mathbb{Q})$. The image of $\mathrm{sing}_0(v)$ under this inclusion is $\sigma_{\mathbb{Z},0} v \otimes \mathbb{Q}$. (See [1]).

Proposition 2.4.11. *If $r = 1$, then* $\mathrm{IH}^1(\mathscr{H}_\mathbb{Q}) = 0$. *Thus* $\mathrm{sing}_0 v = 0$.

Proof. This is obvious from (2.4.1). $\qquad\square$

2.5. **Invariant Grading.** Let v be an admissible normal function over a product of punctured disks $\Delta^{*r} \subset \Delta^r$ with associated variation of mixed Hodge structure $\mathscr{V}$, reference fiber V and nilpotent orbit $\theta(\mathbf{z}) = e^{\sum_j z_j N_j}.F_\infty$. Let $H_\mathbb{R} = Gr^W_{-1} V_\mathbb{R}$ and $0 = (0, \ldots, 0) \in \Delta^r$. Let $\hat{\theta}(\mathbf{z}) = e^{\sum_j z_j N_j}.\hat{F}$ be the split (sl_2 or Deligne's δ) orbit attached to $\theta(\mathbf{z})$. Let $\hat{Y}_M$ denote the corresponding grading of $(\hat{F}, M)$ where M is the relative weight filtration of W and the monodromy cone

$$\mathscr{C} = \Big\{ \sum_j a_j N_j \mid a_j > 0 \Big\}.$$

Let $\hat{Y} = Y_{(e^{iN}.\hat{F}, W)}$, where $N = \sum_j N_j$ Then, by Lemma (2.2.1), $\hat{Y}$ is real, preserves $\hat{F}$ and commutes with N.

Suppose that $\mathrm{sing}_0(v) = 0$ and let e_o be the element of $E_0(\hat{Y})$ which projects to $1 \in \mathbb{R}(0)$. Define $e_j = N_j(e_0)$. Then, by (2.4.7),

$$(e_1, \ldots, e_r) \in B^1(H_\mathbb{R})$$

is a representative of $\mathrm{sing}_p(v)$. Therefore, since $\mathrm{sing}_p(v) = 0$ there is an element $f \in H_\mathbb{R} = B^0(H_\mathbb{R})$ such that $e_j = N_j(f)$. Furthermore, since $e_j = N_j(e_0)$ and $e_0 \in \hat{F}^0$ we have $e_j \in \hat{F}^{-1}$. Therefore, by strictness of morphisms of MHS, we can assume $f \in \hat{F}^0$. Then,

$$e_0 - f = e^{iN}.(e_0 - f) \in e^{iN}.\hat{F}^0.$$

Consequently, $e_0 - f$ belongs to $I^{0,0}_{(e^{iN}.\hat{F}, W)}$ since $e_0 - f$ is real. On the other hand, by theorem of Deligne e_0 belongs to $I^{0,0}_{(e^{iN}.\hat{F}, W)}$. Since Gr^W_0 has rank 1, it then follows that $f = 0$.

Corollary 2.5.1. *If $\mathrm{sing}_0(v) = 0$, then $e_0 \in \ker(N_j)$ for all j.*

Corollary 2.5.2. *If $\mathrm{sing}_0(v) = 0$ then*

$$Y_{(e^{\sum_j z_j N_j}.\hat{F}, W)} = Y_{(e^{iN}.\hat{F}, W)}$$

for $\mathrm{Im}(z_1), \ldots, \mathrm{Im}(z_r) > 0$.

Proof. Since $e_0 \in \ker(N_j)$ for all j, the grading $\hat{Y} = Y_{(e^{iN}.\hat{F},W)}$ commutes with $N_1, \ldots, N_r$. Therefore, $\hat{Y}$ is real and preserves $e^{\Sigma_j z_j N_j}.\hat{F}$ since $\hat{Y}$ preserves $\hat{F}$, and hence is the Deligne grading of $(e^{\Sigma_j z_j N_j}.\hat{F}, W)$. $\qquad\qquad\square$

Let $Y_M = Y_{(F,M)}$. Then, $Y_M = e^{i\delta}.\hat{Y}_M$ and $Y = Y(N, Y_M) = e^{i\delta}.\hat{Y}$. Therefore, since $[\delta, N_j] = 0$ for all j, we have:

Corollary 2.5.3. $[Y, N_j] = 0$ *for all j.*

Definition 2.5.4. If $\mathrm{sing}_0(v) = 0$ we define Y_∞ to be the grading Y of Corollary (2.5.3). In particular, since Y_∞ commutes with $N_1, \ldots, N_r$, it is independent of the choice of local coordinates used in its construction.

Remark 2.5.5. If $\mathrm{sing}_{\mathbb{Z},0}(v) = 0$, then there is a $\mathbb{Z}$-grading $Y_{\mathbb{Z}}$ which is invariant under the N_i. Therefore, we can use (1.3) to define the limit value of v in $\bar{J}(\mathscr{H})_0$.

3. LIMIT GRADINGS

Let $D \subset \bar{S}$ be a smooth divisor, $p \in D$ and Δ^r be an analytic polydisk in $\bar{S}$ containing p. Pick local coordinates $(s_1, \ldots, s_r)$ on Δ^r such that $D \cap \Delta^r$ is given by $s_1 = 0$. Represent v by an admissible variation of mixed Hodge structure $\mathscr{V}$ over $\Delta^* \times \Delta^{r-1}$ with weight graded quotients $Gr_0^W = \mathbb{Z}(0)$ and $Gr_{-1}^W = \mathscr{H}$. Assume that the local monodromy of $\mathscr{V}$ about D is given by a unipotent transformation $T = e^N$. Let

$$F(z; s_2, \ldots, s_r) : U \times \Delta^{r-1} \to \mathscr{M}$$

be a lifting of the local period map of $\mathscr{V}$ where U is the upper half-plane.
Let

$$F(z; s_2, \ldots, s_r) = e^{zN} e^{\Gamma(s)}.F_\infty$$

be the local normal form of the period map of $\mathscr{V}$ at p. Let $\Gamma_0(s) = \Gamma(0, s_2, \ldots, s_r)$ and

$$F_\infty(s_2, \ldots, s_r) = e^{\Gamma_0(s)}.F_\infty.$$

Let W be the weight filtration of $\mathscr{V}$, M be the relative weight filtration of N and W. Then,

$$\theta(z; s_2, \ldots, s_r) = e^{zN}.F_\infty(s_2, \ldots, s_r)$$

is an admissible nilpotent orbit in 1-variable which depends complex analytically upon the parameters $(s_2, \ldots, s_r) \in \Delta^{r-1}$. Let

$$(\hat{F}_\infty(s_2, \ldots, s_r), M) = (e^{-\xi(s_2, \ldots, s_r)}.F_\infty(s_2, \ldots, s_r), M)$$

denote the sl_2-splitting of $(F_\infty(s_2, \ldots, s_r), M)$. Then, ξ is real analytic in $(s_2, \ldots, s_r)$ since it is given by universal Lie polynomials in the Hodge components of Deligne's δ-splitting of $(F_\infty(s_2, \ldots, s_r), M)$.

By the SL_2-orbit theorem (2.2.6)

$$\theta(iy; s_2, \ldots, s_r) = g(y; s_2, \ldots, s_r) e^{iyN}.\hat{F}_\infty(s_2, \ldots, s_r)$$

where

$$g(y; s_2, \ldots, s_r) = (1 + \sum_{k>0} g_k(s_2, \ldots, s_r) y^{-k})$$

belongs to $G_{\mathbb{R}}$ and the coefficients $g_k(s_2, \ldots, s_r)$ are real analytic in $(s_2, \ldots, s_r)$ since they are given by universal Lie polynomials.

We now derive an asymptotic formula for $Y_{(F(z;s_2,\ldots,s_r),W)}$. Write $z = x + iy$. Then,

$$
\begin{aligned}
Y_{(F(z;s_2,\ldots,s_r),W)} &= Y_{(e^{xN}e^{iyN}e^{\Gamma(s)}.F_\infty,W)} \\
&= e^{xN}.Y_{(e^{iyN}e^{\Gamma(s)}e^{-\Gamma_0(s)}e^{\Gamma_0(s)}.F_\infty,W)} \\
&= e^{xN}.Y_{(e^{iyN}e^{\Gamma(s)}e^{-\Gamma_0(s)}.F_\infty(s_2,\ldots,s_r),W)}.
\end{aligned}
$$

Let $e^{\Gamma_1(s)} = e^{\Gamma(s)}e^{-\Gamma_0(s)}$ and note that $s_1|\Gamma_1$ in $\mathscr{O}(\Delta^r)$. Then,

$$
\begin{aligned}
Y_{(F(z;s_2,\ldots,s_r),W)} &= e^{xN}.Y_{(e^{iyN}e^{\Gamma_1(s)}.F_\infty(s_2,\ldots,s_r),W)} \\
&= e^{xN}.Y_{(\mathrm{Ad}(e^{iyN})(e^{\Gamma_1(s)}).\theta(iy;s_2,\ldots,s_r),W)} \\
&= e^{xN}.Y_{((\mathrm{Ad}(e^{iyN})(e^{\Gamma_1(s)}))g(y;s_2,\ldots,s_r)e^{iyN}.\hat{F}_\infty(s_2,\ldots,s_r),W)}.
\end{aligned}
$$

Let $F_o(s_2,\ldots,s_r) = e^{iN}.\hat{F}_\infty(s_2,\ldots,s_r)$ and

$$
Y_1 = Y_{(F_o(s_2,\ldots,s_r),W)}, \qquad Y_2 = Y_{(\hat{F}_\infty(s_2,\ldots,s_r),M)}.
$$

Then, by Theorem (2.2.4),

(3.1) $$Y_{(e^{iyN}.\hat{F}_\infty(s_2,\ldots,s_r),W)} = Y_1.$$

Likewise, since N and $H = Y_2 - Y_1$ is an sl_2-pair,

$$
e^{iyN}.\hat{F}_\infty(s_2,\ldots,s_r) = y^{-H/2}.F_o(s_2,\ldots,s_r).
$$

Note that Y_1 and H depend real analytically on $(s_2,\ldots,s_r)$.

Lemma 3.2. *Let* $\gamma(y) = \mathrm{Ad}(e^{-iyN})g(y;s_2,\ldots,s_r)$. *Then,* $\lim_{y\to\infty} \gamma(y)$ *exists, and is real analytic in* $(s_2,\ldots,s_r)$.

Proof. This follows directly from the fact that $g_k(s_2,\ldots,s_r)$ is real analytic in $(s_2,\ldots,s_r)$ and $g_k \in \ker(\mathrm{ad}\,N)^{k+1}$. $\square$

Returning to the calculation of $Y_{(F(z;s_2,\ldots,s_r),W)}$, and abbreviating $g(y;s_2,\ldots,s_r)$ to $g(y)$, we have

$$
\begin{aligned}
Y_{(F(z;s_2,\ldots,s_r),W)} &= e^{xN}.Y_{(\mathrm{Ad}(e^{iyN})(e^{\Gamma_1(s)})g(y)e^{iyN}.\hat{F}_\infty(s_2,\ldots,s_r),W)} \\
&= e^{xN}.Y_{(g(y)e^{iyN}\gamma^{-1}(y)e^{\Gamma_1(s)}\gamma(y).\hat{F}_\infty(s_2,\ldots,s_r),W)}.
\end{aligned}
$$

Let $e^{\Gamma_2} = \mathrm{Ad}(\gamma^{-1}(y))e^{\Gamma_1}$ and recall that $s_1|\Gamma_1$. Therefore,

$$
\begin{aligned}
Y_{(F(z;s_2,\ldots,s_r),W)} &= e^{xN}.Y_{(g(y)e^{iyN}e^{\Gamma_2(s)}.\hat{F}_\infty(s_2,\ldots,s_r),W)} \\
(3.3) \qquad &= e^{xN}g(y)y^{-H/2}.Y_{(e^{iN}\mathrm{Ad}(y^{H/2})(e^{\Gamma_2}).\hat{F}_\infty(s_2,\ldots,s_r),W)}
\end{aligned}
$$

where $|\mathrm{Ad}(y^{H/2})\Gamma_2|$ can be uniformly bounded by a constant times $y^c e^{-2\pi y}$ as $y \to \infty$ for some constant c, independent of $(s_2,\ldots,s_r)$.

We now prove existence of the limit grading $\hat{Y}$ of the introduction, which we shall use in the next section to prove the algebraicity of the zero locus $\mathscr{Z}$. Modulo our discussion of dependence on parameters, this essentially the same calculation use to prove the existence of the limit grading in [4].

Theorem 3.4. *Let $(s_1(m), \ldots, s_r(m))$ be any sequence of points in $\Delta^* \times \Delta^{r-1}$ which converges to $(0, \tilde{s}_2, \ldots, \tilde{s}_r) \in \Delta^r$ as $m \to \infty$. Let $(z(m), s_2(m), \ldots, s_r(m))$ be a lifting of this sequence to $U \times \Delta^{r-1}$ with the real part of z restricted to an interval of finite length. Then,*

$$\lim_{m \to \infty} Y_{(F(z(m); s_2(m), \ldots, s_r(m)), W)} = Y_{(e^{iN}.\hat{F}_\infty(\tilde{s}_2, \ldots, \tilde{s}_r), W)}.$$

Proof. Suppress the dependence of $(z(m); s_2(m), \ldots, s_r(m))$ on m. By (3.3)

$$(3.5) \qquad Y_{(F(z; s_2, \ldots, s_r), W)} = e^{xN} g(y) y^{-H/2}.Y_{(e^{iN} \mathrm{Ad}(y^{H/2})(e^{\Gamma_2(s)}).\hat{F}_\infty(s_2, \ldots, s_r), W)},$$

and $|\mathrm{Ad}(y^{H/2})(e^{\Gamma_2(s)})|$ is uniformly bounded by some constant times $y^c e^{-2\pi y}$. Therefore,

$$(3.6) \qquad Y_{e^{iN} \mathrm{Ad}(y^{H/2})(e^{\Gamma_2(s)}).\hat{F}_\infty(s_2, \ldots, s_r), W} = Y_{(e^{iN}.\hat{F}_\infty(\tilde{s}_2, \ldots, \tilde{s}_r), W)} + \alpha$$

where $|\alpha|$ is uniformly bounded by $y^c e^{-2\pi y}$ as $y \to \infty$. Since $H = H(s_2, \ldots, s_r)$ commutes with $Y_1(s_2, \ldots, s_m)$, the result now follows by inserting (3.6) into (3.5) and taking the limit as $m \to \infty$. $\qquad \square$

In order to construct the limit normal function we need the following analog of Theorem (3.4) where we twist the grading $Y_{(\mathscr{F}, \mathscr{W})}$ by $e^{-\frac{1}{2\pi i} \log(s_1) N}$. Again, modulo dependence on parameters, this is really just a glorified version of Theorem (4.15) in [18].

Theorem 3.7. *Let $(s_1(m), \ldots, s_r(m))$ be a sequence of points in $\Delta^* \times \Delta^{r-1}$ which converges to $(0, s_2, \ldots, s_r) \in \Delta^r$ as $m \to \infty$. Let $(z(m), s_2(m), \ldots, s_r(m))$ be a lifting of this sequence to $U \times \Delta^{r-1}$ with the real part of z restricted to an interval of finite length. Then,*

$$\lim_{m \to \infty} e^{-zN}.Y_{(F(z(m); s_2(m), \ldots, s_r(m)), W)} = Y(N, Y_{(F_\infty(s_2, \ldots, s_r), M)})$$

where $Y(N, Y_{(F_\infty(s_2, \ldots, s_r), M)})$ is the grading of Lemma (2.2.1).

Proof. We repeat the argument of the proof of Theorem (3.4) to obtain

$$
\begin{aligned}
e^{-zN}.Y_{(F(z; s_2, \ldots, s_r), W)} &= e^{-iyN} g(y) y^{-H/2}.(Y_{(e^{iN}.\hat{F}_\infty(s_2, \ldots, s_r), W)} + \alpha) \\
(3.8) \qquad\qquad &= e^{-iyN} \tilde{g}(y) e^{iyN} e^{-\zeta} e^{iyN} y^{-H/2}.(Y_{(e^{iN}.\hat{F}_\infty(s_2, \ldots, s_r), W)} + \alpha).
\end{aligned}
$$

By part (c) of the SL$_2$-orbit theorem (cf. equation (4.19) in [18]), we have

$$\lim_{y \to \infty} \mathrm{Ad}(e^{-iyN}) \tilde{g}(y) = e^\zeta \left(1 + \sum_{k > 0} \frac{(-i)^k}{k!} (\mathrm{ad} N)^k \tilde{g}_k \right) = e^{i\delta}.$$

Therefore, as in the proof of Theorem (3.4) it follows that

$$(3.9) \qquad \lim_{m \to \infty} e^{-zN}.Y_{(F(z; s_2, \ldots, s_r), W)} = e^{i\delta} e^{-\zeta}.Y_{(e^{iN}.\hat{F}_\infty(s_2, \ldots, s_r), W)}$$

where δ and ζ are real-analytic in $(s_2, \ldots, s_r)$. By (2.2.1), (2.2.4) and the functoriality of Deligne's construction;

$$
\begin{aligned}
Y_{(e^{iN}.\hat{F}(s_2, \ldots, s_r), W)} &= Y(N, Y_{(\hat{F}(s_2, \ldots, s_r), M)}) \\
(3.10) \qquad &= Y(N, e^\zeta e^{-i\delta}.Y_{(F_\infty(s_2, \ldots, s_r), M)}) \\
&= e^\zeta e^{-i\delta}.Y(N, Y_{(F_\infty(s_2, \ldots, s_r), M)}).
\end{aligned}
$$

Inserting (3.10) into (3.9) completes the proof. $\qquad \square$

Remark 3.11. By virtue of the functoriality of the grading $Y(N, Y_M)$ with respect to the pair (N, Y_M) and the fact that $Y(N, Y_M) \in \ker(\mathrm{ad}\,N)$ due to the short length of W, it follows that $Y(N, Y_{(F_\infty(s_2, \ldots, s_r), M)})$ is independent of the choice of local coordinates.

4. ALGEBRAICITY OF THE ZERO LOCUS IN THE CASE OF SMOOTH BOUNDARY DIVISOR

We now prove Theorem (1.2). Let $\mathscr{Z}$ be the zero locus of an admissible normal function ν on a smooth complex algebraic variety S which admits a smooth compactification $\bar{S}$ such that $D = \bar{S} - S$ is a smooth divisor. Let $p \in D$ be an accumulation point of $\mathscr{Z}$, and $(s_1, \ldots, s_r)$ be local coordinates on a polydisk $\Delta^r \subset \bar{S}$ containing p, relative to which D is given by the equation $s_1 = 0$. Let $\mathscr{V} \to \Delta^* \times \Delta^{r-1}$ be an admissible variation of mixed Hodge structure which represents ν on $S \cap \Delta^r$. Without loss of generality, assume that $\mathscr{V}$ has unipotent monodromy.

Let $(s_1(m), \ldots, s_r(m))$ be a sequence of points in $\mathscr{Z}$ which converge to p, and

$$F(z; s_2, \ldots, s_r) : U \times \Delta^{r-1} \to \mathscr{M}$$

be a lifting of the period map of $\mathscr{V}$, where U is the upper half-plane. Let $(z(m), s_2(m), \ldots, s_r(m))$ be a lifting of $(s_1(m), \ldots, s_r(m))$ to $U \times \Delta^{r-1}$ with the real part of z restricted to an interval of finite length. Then, by Theorem (3.4),

$$(4.1) \qquad \lim_{m \to \infty} Y_{(F(z(m); s_2(m), \ldots, s_r(m)), W)} = Y_1(0, \ldots, 0).$$

In particular, since the set of integral gradings is discrete, equation (4.1) forces

$$Y_{\mathbb{Z}} = Y_1(0, \ldots, 0).$$

to be an integral grading of W. By Lemma (2.2.1) and Theorem (2.2.4), it then follows that

(a) $Y_{\mathbb{Z}} \in \ker(\mathrm{ad}\,N)$;
(b) $Y_{\mathbb{Z}}$ preserves the Hodge filtration $\hat{F}_\infty = e^{-\xi}.F_\infty$ of the sl_2-splitting the limit mixed Hodge structure (F_∞, M);
(c) $\xi \in \ker(\mathrm{ad}\,N) \cap \Lambda^{-1,-1}_{(\hat{F}_\infty, M)}$.

Let $Y_\infty = e^\xi.Y_{\mathbb{Z}}$. Then, Y_∞ preserves F_∞ and belongs to $\ker(\mathrm{ad}\,N)$. Therefore, due to the short length of the weight filtration, there exists a unique $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$-valued function $f(z; s_2, \ldots, s_r)$ such that

$$Y_{(F(z; s_2, \ldots, s_r), W)} = e^{zN} e^{\Gamma(s)}.(Y_\infty + f).$$

The local defining equation for $\mathscr{Z}$ near p is therefore

$$(4.2) \qquad Y_{\mathbb{Z}} = e^{zN} e^{\Gamma(s)}.(Y_\infty + f).$$

[Note: Here we use the fact that Theorem (3.4) asserts that we get the same limit for every sequence through p, in order to conclude that, after shrinking Δ^r as necessary, $Y_{\mathbb{Z}}$ is the only integral value assumed by $Y_{(F(z; s_2, \ldots, s_r), W)}$]. Transposing the $e^{zN} e^{\Gamma(s)}$ factor over to the other side, we then obtain,

$$(4.3) \qquad e^{-\Gamma(s)}.Y_{\mathbb{Z}} = Y_\infty + f.$$

The subalgebra $\mathfrak{q}$ is closed under the action of $\mathrm{ad}\,Y_\infty$. Consequently,

$$(4.4) \qquad Y_{\mathbb{Z}} = e^{-\xi}.Y_\infty = Y_\infty + \lambda$$

for some element $\lambda \in \mathfrak{q}$. More properly, by equation (2.1.2), $\Lambda_{(\hat{F},M)}^{-1,-1} = \Lambda_{(F,M)}^{-1,-1}$, wherefrom (4.4) follows since $\Lambda_{(F,M)}^{-1,-1}$ is closed under $\mathrm{ad}\,Y_\infty$.

Accordingly, (4.3) reduces to

$$(4.5) \qquad e^{-\Gamma(s)}.(Y_\infty + \lambda) = Y_\infty + f.$$

Again, because Y_∞ grades W and $\mathrm{ad}\,Y_\infty$ preserves $\mathfrak{q}$, we have

$$e^{-\Gamma(s)}.Y_\infty = Y_\infty + \alpha(s)$$

for some holomorphic function $\alpha(s)$ with values in $\mathfrak{q} \cap W_{-1}\mathfrak{g}_\mathbb{C}$. Recalling that $W_{-1}\mathfrak{g}_\mathbb{C}$ acts simply transitively on the gradings of W, it then follows that equation (4.5) simplifies to

$$e^{-\Gamma(s)}.(Y_\infty + \lambda) = Y_\infty$$

since $\mathfrak{g}_\mathbb{C} = \mathfrak{q} \oplus \mathfrak{g}_\mathbb{C}^{F_\infty}$ and f takes values in $\mathfrak{g}_\mathbb{C}^{F_\infty}$. Clearly, this equation is complex analytic on Δ^r. It also forces $\lambda = 0$.

To finish the argument, let Z denote the union of the irreducible components of the analytic subvariety of Δ^r defined by the equation $e^{-\Gamma(s)}.Y_\infty = Y_\infty$ which are not contained in $D \cap \Delta^r$. Then,

$$\mathscr{Z} \cap \Delta^r = Z \cap S$$

and hence Z is the closure of $\mathscr{Z} \cap \Delta^r$.

Remark 4.6. The above arguments also show that if ν is an admissible normal function on $S = \bar{S} - D$ and p is a smooth point of D such that $\sigma_{\mathbb{Z},p}(\nu)$ is non-zero torsion then p can not be an accumulation point of $\mathscr{Z}$.

5. ALGEBRAICITY IN THE CASE OF TORSION SINGULARITY

In connection with the proof of Theorem (1.11), we now consider the case where ν is an admissible normal function, on $\Delta^{*r} \subseteq \Delta^r$ with unipotent monodromy, and $\mathrm{sing}_0(\nu) = 0$. Let $(s_1(m), \ldots, s_r(m))$ be a sequence of points in Δ^{*r} which converge to $0 = (0, \ldots, 0)$. Let $(z_1(m), \ldots, z_r(m))$ be a lifting of this sequence to the product of upper half-planes, with the real parts of each $z_j(m)$ restricted to an interval of finite length. Then, we want to compute

$$\lim_{m\to\infty} Y_{(F(z_1(m),\ldots,z_r(m)),W)}$$

where $F(z_1, \ldots, z_r)$ is a lifting of the local period map to U^r. Suppose that (after passage to a subsequence)

$$(5.1) \qquad \lim_{m\to\infty} \frac{y_{j+1}(m)}{y_j(m)} \in (0, \infty)$$

for $j = 1, \ldots, r-1$. Then, exactly the same arguments as above show that

$$\lim_{m\to\infty} Y_{(F(z_1(m),\ldots,z_r(m)),W)} = Y(N, Y_{(\hat{F}_\infty,M)})$$

where N is any element in the monodromy cone $\mathscr{C} = \{\sum_j a_j N_j \mid a_j > 0\}$. The key point is that:

(a) By Corollary (2.5.2), $\hat{Y} = Y(N, Y_{(\hat{F}_\infty,M)})$ is independent of N.

(b) Under the hypothesis of condition (5.1), the element

$$N(y_1,\ldots,y_r) = N_1 + \frac{y_2}{y_1}N_2 + \cdots + \frac{y_r}{y_1}N_r$$

remains within a compact subset of $\mathscr{C}$ as $m \to \infty$. Therefore,

$$e^{iy_1 N(y_1,\ldots,y_r)}.F_\infty = g(y_1)e^{iy_1 N(y_1,\ldots,y_r)}.\hat{F}_\infty$$

where all the coefficients of g depend real-analytically upon $N(y_1,\ldots,y_r)$, since Deligne's construction (2.1.8) is algebraic in the pair (N,Y_M).

Corollary 5.2. *Let ν be an admissible normal function on $\Delta^{*r} \subseteq \Delta^r$ with unipotent monodromy, and suppose that $\mathrm{sing}_0(\nu) = 0$. Let $\gamma : (\Delta,0) \to (\Delta^r,0)$ be a holomorphically embedded disk which intersects each irreducible component of $D = \Delta^r - \Delta^{*r}$ with finite multiplicity. Then, $\gamma^*(\nu)$ is an admissible normal function on Δ^* with limiting value at $0 \in \Delta$ equal the common value of $Y(N,Y_{(\hat{F}_\infty,M)})$ for any $N \in \mathscr{C}$.*

In general, by reordering the variables if necessary, one can always pass to some subsequence such that

$$(5.3) \qquad \lim_{m\to\infty} \frac{y_{j+1}(m)}{y_j(m)} \in [0,\infty)$$

for $j = 1,\ldots,r-1$. Suppose for simplicity that $\lim_{m\to\infty} \frac{y_{j+1}(m)}{y_j(m)} = 0$. Then, the main theorem of [17] asserts that

$$(5.4) \qquad \lim_{m\to\infty} Y_{(e^{iy_1 N_1 + \cdots + iy_r N_r}.F_\infty, W)}$$

exists (independent of any assumptions about $\mathrm{sing}(\nu) = 0$).

Conjecture 5.5. [2] *Assume that $\mathrm{sing}(\nu) = 0$ and that $(z_1(m),\ldots,z_r(m))$ is a sequence of points in U^r which satisfies condition (5.3). Then,*

$$\lim_{m\to\infty} Y_{(F(z_1(m),\ldots,z_r(m)),W)} = Y\Big(\sum_j N_j, Y_{(\hat{F}_\infty,M)}\Big).$$

Granting Conjecture (5.5), the proof of Theorem (1.11) is identical to that of Theorem 1.2: A sequence of points $(s_1,\ldots,s_r)$ converging to a point $p \in D$ where $\mathrm{sing}_p(\nu) = 0$ forces

$$Y_{\mathbb{Z}} = Y_{(\hat{F}_\infty,M)}.$$

to be an integral grading which in the kernel of $\mathrm{ad}\,N_j$ for each j. Repeating the argument given in §4, it then follows that the local defining equation for the zero locus is $e^{-\Gamma(s)}.Y_\infty = Y_\infty$.

References

[1] Patrick Brosnan, Hao Fang, Zhaohu Nie, and Gregory Pearlstein. Singularities of admissible normal functions, 2007.

[2] Patrick Brosnan and Gregory Pearlstein. The zero locus of an admissible normal function in several variables. In progress.

[3] Patrick Brosnan, Gregory Pearlstein, and Morihiko Saito. A generalization of the Néron models of Green, Griffiths and Kerr. In progress.

[4] Patrick Brosnan and Gregory J. Pearlstein. The zero locus of an admissible normal function, 2006. To appear in Annals of Mathematics.

[5] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995.

[6] Eduardo Cattani and Aroldo Kaplan. Degenerating variations of Hodge structure. *Astérisque*, (179-180):9, 67–96, 1989. Actes du Colloque de Théorie de Hodge (Luminy, 1987).

[7] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. Degeneration of Hodge structures. *Ann. of Math. (2)*, 123(3):457–535, 1986.

[8] Pierre Deligne. Personal communication to A. Kaplan and E. Cattani.

[9] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[10] A. Galligo, M. Granger, and Ph. Maisonobe. $\mathscr{D}$-modules et faisceaux pervers dont le support singulier est un croisement normal. *Ann. Inst. Fourier (Grenoble)*, 35(1):1–48, 1985.

[11] Mark Green and Phillip Griffiths. Hodge-theoretic invariants for algebraic cycles. *Int. Math. Res. Not.*, (9):477–510, 2003.

[12] Mark Green and Phillip Griffiths. Algebraic cycles and singularities of normal functions. In *Algebraic Cycles and Motives*, volume 343 of *London Mathematical Society Lecture Note Series*, pages 206–263. Cambridge University Press, 2007.

[13] Mark Green, Phillip Griffiths, and Matthew Kerr. Neron models and limits of abel-jacobi mappings. Preprint.

[14] Aroldo Kaplan and Gregory Pearlstein. Singularities of variations of mixed Hodge structure. *Asian J. Math.*, 7(3):307–336, 2003.

[15] Masaki Kashiwara. A study of variation of mixed Hodge structure. *Publ. Res. Inst. Math. Sci.*, 22(5):991–1024, 1986.

[16] Masaki Kashiwara and Takahiro Kawai. The Poincaré lemma for variations of polarized Hodge structure. *Publ. Res. Inst. Math. Sci.*, 23(2):345–407, 1987.

[17] Kazuya Kato, Chikara Nakayama, and Sampei Usui. SL_2-orbit theorem for degeneration of mixed Hodge structure. *J. Algebraic Geom.*, 2007. Electronic.

[18] Gregory Pearlstein. SL_2-orbits and degenerations of mixed Hodge structure. *J. Differential Geom.*, 74(1):1–67, 2006.

[19] Gregory J. Pearlstein. Variations of mixed Hodge structure, Higgs fields, and quantum cohomology. *Manuscripta Math.*, 102(3):269–310, 2000.

[20] Morihiko Saito. Admissible normal functions. *J. Algebraic Geom.*, 5(2):235–276, 1996.

[21] Morihiko Saito. Hausdorff property of the zucker extension at the monodromy invariant subspace, 2008.

[22] Joseph Steenbrink and Steven Zucker. Variation of mixed Hodge structure. I. *Invent. Math.*, 80(3):489–542, 1985.

[23] Sampei Usui. Variation of mixed Hodge structure arising from family of logarithmic deformations. II. Classifying space. *Duke Math. J.*, 51(4):851–875, 1984.

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