Kashiwara Conjugation for Twisted $\mathcal{D}$-modules

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Outline

1 Introduction

2 $\mathcal{D}$-modules and Kashiwara conjugation
- $\mathcal{D}$-modules
- Kashiwara conjugation

3 Twisted $\mathcal{D}$ modules

4 The theorem

Kashiwara conjugation is a contravariant functor that sends regular holonomic $\mathcal{D}$-modules on a complex manifold X to regular holonomic $\mathcal{D}$-modules on its complex conjugate $\bar{X}$. Using it, Kashiwara explained how, locally, every regular holonomic $\mathcal{D}$-module can be defined in terms of distributions. Later Barlet and Kashiwara, applied a twisted version of this on flag varieties to questions in representation theory. My goal is to formulate a version of Kashiwara conjugation valid for arbitrary complex manifolds. I have two motivations:

- Formulating the general version gives an explicit presentation of rings of twisted differential operators.
- The general formulation was motivated by a beautiful representation-theoretic conjecture of Schmid and Vilonen.

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- $\bar{X} = (X_{\mathbb{R}}, \mathcal{O}_{\bar{X}})$ is the complex conjugate of X. It has the same underlying smooth manifold $X_{\mathbb{R}}$, but the sheaf of $\mathbb{C}$-algebras is the $\mathbb{C}$-algebra of anti-holomorphic functions on X.
- $\mathfrak{Db}_X = \mathfrak{Db}_{X_{\mathbb{R}}}$ is the sheaf of complex valued distributions on the underlying smooth manifold $X_{\mathbb{R}}$. If $\dim X_{\mathbb{R}} = 2n$, then a section of $\mathfrak{Db}_X$ takes a compactly supported smooth $2n$-form on $X_{\mathbb{R}}$ and returns a complex number. For example, $L^1_{\mathrm{loc}} \subset \mathfrak{Db}_X$.
- $\mathrm{Diff}_{X_{\mathbb{R}}}$ is the sheaf of $\mathcal{C}^\infty$ differential operators on $X_{\mathbb{R}}$. It is a non-commutative sheaf of $\mathbb{C}$-algebras.
- $\mathcal{D}_X$ is the sheaf of holomorphic differential operators on X. It is the subring of $\mathrm{Diff}_{X_{\mathbb{R}}}$ generated by $\mathcal{O}_X$ and the holomorphic tangent vectors.
- Write $F_p\mathcal{D}_X$ for the subsheaf of $\mathcal{D}_X$ generated by differential operators of order $\leq p$. Each $F_p\mathcal{D}_X$ is a locally free sheaf of $\mathcal{O}_X$ modules.

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$\mathcal{D}_X$ modules.

The sheaf $\mathcal{D}_X$ acts on the left on several interesting sheaves. For example:

- $\mathcal{O}_X$ is a $\mathcal{D}_X$-modules with $\mathcal{D}_X$ acting by differentiating holomorphic functions.
- $\mathcal{C}^\infty_{X_\mathbb{R}}$ is $\mathrm{Diff}_{X_\mathbb{R}}$ module, so, since $\mathcal{D}_X$ is a subring of $\mathrm{Diff}_{X_\mathbb{R}}$, $\mathcal{C}^\infty_{X_\mathbb{R}}$ is a $\mathcal{D}_X$ module.
- It's not hard to see that $\mathfrak{Db}_{X_\mathbb{R}}$ is a $\mathrm{Diff}_{X_\mathbb{R}}$-module, and, thus a $\mathcal{D}_X$ module.
- As $\mathcal{D}_X$-modules $\mathcal{O}_X \subset \mathcal{C}^\infty_{X_\mathbb{R}} \subset \mathfrak{Db}_{X_\mathbb{R}}$.
- Note that $\mathcal{D}_X$ and $\mathcal{D}_{\bar{X}}$ commute with each other inside of $\mathrm{Diff}_{X_\mathbb{R}}$. In particular, their actions on $\mathfrak{Db}_{X_\mathbb{R}}$ commute.

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$\mathcal{D}_X$ module examples

Here's a nice way to get a $\mathcal{D}_X$ modules: Take a distribution $\mu \in \Gamma(X, \mathfrak{Db}_{X_{\mathbb{R}}})$. Then set $\mathcal{M} = \mathcal{D}_X \mu \subset \mathfrak{Db}_{X_{\mathbb{R}}}$.
For example, with $X = \mathbb{P}^1$:

- $\mathcal{O}_X = \mathcal{D}_X \cdot 1$. So here μ is the constant function.
- Set $\mathcal{L} = \mathcal{D}_X |z|$. Since $|z| \in L^1_{\mathrm{loc}}$, $\mathcal{L}$ is a $\mathcal{D}_X$-module.
- Let δ denote the Dirac delta function concentrated at 0 in $\mathbb{A}^1 \subset \mathbb{P}^1$. To a 2-form, $\phi\, dx\, dy$, δ associates the valued $\phi(0)$. Then $\mathcal{M} = \mathcal{D}_X \delta$ is a $\mathcal{D}_X$ module supported at the 0.

All of the above are examples of *regular holonomic* $\mathcal{D}_X$ modules. Kashiwara showed that the functor

$$\mathcal{M} \rightsquigarrow \mathrm{DR}_X(\mathcal{M}) := \mathbf{RHom}_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})[\dim X]$$

is an equivalence between from the category $\mathrm{RH}(\mathcal{D}_X)$ to the category $\mathrm{Perv}(\mathbb{C}_X)$ of perverse sheaves on X.

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- Since the actions of $\mathcal{D}_X$ and $\mathcal{D}_{\bar{X}}$ on $\mathfrak{Db}_{X_\mathbb{R}}$ commute, $\mathcal{D}_{\bar{X}}$ acts on $K(\mathcal{M})$ via its action on $\mathfrak{Db}_{X_\mathbb{R}}$. Thus $K(\mathcal{M})$ is a $D_{\bar{X}}$ module.
- It's not hard to see that we get a functor

$$K : (\mathcal{D}_X \,\mathrm{mod})^{\mathrm{op}} \rightsquigarrow (\mathcal{D}_{\bar{X}} \,\mathrm{mod}).$$

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Theorem (Kashiwara)

Let $\mathrm{RH}(X)$ denote the category of regular holonomic $\mathcal{D}_X$ modules on X. Then Kashiwara conjugation gives an equivalence of categories between

$$K : \mathrm{RH}(X)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{RH}(\bar{X}).$$

Call a distribution ϕ on X, *regular holonomic* if the $\mathcal{D}_X$ module $\mathcal{D}_X\phi$ is regular holonomic. Using the theorem, Kashiwara showed that every regular holonomic $\mathcal{D}_X$ module is locally of the form $\mathcal{D}_X\phi$ for some regular holonomic distribution ϕ.

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A tdo is a sheaf of rings $\mathcal{A}$ on X together with a filtration $F_p\mathcal{A}$ by $\mathcal{O}_X$ modules and an isomorphism $i : \mathcal{O}_X \to F_0\mathcal{A}$ such that the triple $(\mathcal{A}, F_p\mathcal{A}, i)$ is locally isomorphic to the obvious triple for $\mathcal{D}_X$ (where $F_p\mathcal{D}_X$ is the filtration by order of operator).

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The filtration on a tdo enjoys several properties:

- $F_p\mathcal{A} \cdot F_q\mathcal{A} \subset F_{p+q}\mathcal{A}$ and $F_p\mathcal{A} = 0$ for $p < 0$.
- The sheaf of rings $\mathrm{Gr}^F \mathcal{A}$ is a commutative $\mathcal{O}_X$ algebra. This induces a map
$$\nabla : \mathrm{Gr}^F_1 \mathcal{A} \to TX$$
sending $P \in F_1\mathcal{A}$ to the derivation $f \mapsto [P, f]$ ($f \in \mathcal{O}_X$). In a tdo, ∇ is an isomorphism.
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- The sheaf of rings $\mathrm{Gr}^F \mathcal{A}$ is a commutative $\mathcal{O}_X$ algebra. This induces a map
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sending $P \in F_1\mathcal{A}$ to the derivation $f \mapsto [P, f]$ ($f \in \mathcal{O}_X$). In a tdo, ∇ is an isomorphism.
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Write $\mathrm{Tw}(\mathcal{D}_X)$ for the set of isomorphism classes of tdos.

Theorem

There is a natural isomorphism $\mathrm{Tw}(\mathcal{D}_X) = H^1(X, d\mathcal{O}_X)$.

- The theorem is not horribly difficult. You can find a proof in Björk's big book on $\mathcal{D}$-modules.
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These are classified by elements of $H^1(X, \Omega_X)$. The class of $F_1\mathcal{A}$ in $H^1(X, \Omega_X)$ agrees with the class of $\mathcal{A}$ in $H^1(X, d\mathcal{O}_X)$ under the natural map $H^1(X, d\mathcal{O}_X) \to H^1(X, \Omega_X)$.

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We have $H^1(X, d\mathcal{O}_X) = Z\mathcal{D}_X / B\mathcal{D}_X$.

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This is the calculation that falls out of the Dolbeault resolution of the complex $\Omega_X^{\geq 1}[1]$ which is quasi-isomorphic to $d\mathcal{O}_X$. □

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The main definition of this talk is a way to embed certain tdo's $\mathcal{A}$ in $\mathrm{Diff}_{X_{\mathbb{R}}}$. The first step is to embed $F_1\mathcal{A}$.

Definition

Suppose $\lambda \in \mathcal{A}^{1,1}_X$ is $\bar{\partial}$ closed. Set

$$E_\lambda := \{(f, v) \in \mathcal{C}^\infty_X \oplus TX : \forall w \in T\bar{X}, w(f) = \lambda(w \wedge v)\}.$$

Proposition

We have a short exact sequence of $\mathcal{O}_X$-modules

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- *If $\mathcal{M}$ is a $\mathcal{D}_{X,\lambda}$ module then*

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- Suppose $\bar{\partial}\partial \log g = \lambda$. Write L_g for left multiplication by g inside $\mathrm{Diff}_{X_{\mathbb{R}}}$. Then $\mathcal{D}_{X,\lambda} = L_g^{-1}\mathcal{D}_X L_g$.
- In the theorem, everything, even the definition $\mathrm{RH}(\mathcal{D}_{X,\lambda})$, is local. So, since $\mathcal{D}_{X,\lambda}$ is locally isomorphic to $\mathcal{D}_X$, the theorem follows almost directly from the untwisted case.
- Define a twisted regular holonomic distribution to be a distribution ϕ such that $\mathcal{D}_{X,\lambda}\phi$ is a regular holonomic $\mathcal{D}_{X,\lambda}$ module. You can use the theorem to see that locally every regular holonomic $\mathcal{D}_{X,\lambda}$ is of the above form.
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There is another functor going from twisted $\mathcal{D}_X$ modules to $\mathcal{D}_{\bar{X}}$ modules. This is induced by the complex anti-linear ring homomorphism

$$\begin{aligned}\mathcal{D}_{\bar{X},\bar{\lambda}} &\to \mathcal{D}_{X,\lambda} \\ P &\mapsto \bar{P}.\end{aligned}$$

It induces an equivalence of categories

$$C : \mathrm{RH}(\mathcal{D}_{X,\lambda}) \to \mathrm{RH}(\mathcal{D}_{\bar{X},\bar{\lambda}}),$$

and we write $C(\mathcal{M}) = \bar{\mathcal{M}}$ for short.

Definition

Suppose $\lambda = -\bar{\lambda}$. A *polarization* of a $\mathcal{D}_{X,\lambda}$ modules $\mathcal{M}$ is an isomorphism $\bar{\mathcal{M}} \to K(\mathcal{M})$.

- By definition, a polarization gives rise to a pairing

$$\mathcal{M} \otimes \bar{\mathcal{M}} \xrightarrow{\langle , \rangle} \mathfrak{Db}_{X_{\mathbb{R}}},$$

which is $\mathcal{D}_{X,\lambda} \times \mathcal{D}_{\bar{X},\bar{\lambda}}$-equivariant.
- Suppose that X has a chosen smooth volume form vol. Then we can integrate the above pairing on global sectinos to get another pairing

$$(\, , \,) : \Gamma(\mathcal{M}) \otimes \Gamma(\bar{\mathcal{M}}) \to \mathbb{C}$$
$$\alpha \otimes \beta \mapsto \int_X \langle \alpha, \beta \rangle d\mathrm{vol}$$

- The motivation for the work above was to understand a conjecture of Schmid and Vilonen about the pairing $(\,,\,)$ in the case that $X = G/B$ is a generalized flag variety and $\mathcal{M}$ is a twisted $\mathcal{D}$-module on X coming from representations of a real form of G.
- In this case $\mathcal{M}$ can be made into a mixed Hodge module in a natural way. So it has a Hodge filtration $F_p\mathcal{M}$. To fix notation, suppose $F_p = 0$ for $p < 0$ and $F_0 \neq 0$ for $\mathcal{M} \neq 0$. Roughly speaking, the Schmid-Vilonen conjecture is the following:

Conjecture (Schmid-Vilonen)

Suppose $\lambda + \rho > 0$. Then (for $\mathcal{M}$ coming from representation theory) the pairing $(\,,\,)$ is $(-1)^p$ definite on $\Gamma(F_p\mathcal{M}) \cap \Gamma(F_{p-1}\mathcal{M})^{\perp}$.

- If $p = 0$, then the conjecture is just saying that $(\alpha, \beta) > 0$ for $\alpha, \beta \in F_0\mathcal{M}$. This follows from old results of Schmid about the asymptotics of the Hodge metric.
- The conjecture implies that $\Gamma(\mathcal{M}) = \oplus H_p$ where $H_p := \Gamma(F_p\mathcal{M}) \cap \Gamma(F_{p-1}\mathcal{M})^\perp$. In this sense you can view it as associating a polarized infinite dimensional Hodge structure to $\mathcal{M}$.
- Schmid and Vilonen have verified the conjecture in many cases. To see what's going on, let's check it for $X = \mathbb{P}^1$.

Untwisted Example

- On $X = \mathbb{P}^1$ we can pick $\mathrm{vol} = \frac{1}{2i}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$
- Take $\mathcal{M} = \mathcal{D}_X\delta$. This is, in fact, the modules coming from the discrete series representation of $\mathrm{SL}_{2\mathbb{R}}$.
- The Hodge filtration is given by $F_p\mathcal{M} = \sum_{k\le p}\mathbb{C}\partial^k\delta$.
- We have

$$\begin{aligned}
(\partial^k\delta, \bar{\partial}^j\delta) &= \int(\partial^k\bar{\partial}^j\delta)d\mathrm{vol} \\
&= (-1)^{k+j}(\delta, \partial^k\bar{\partial}^j\mathrm{vol}) \\
&= (-1)^{k+j}(\delta, \partial^k\partial^j\frac{1}{2i}\sum_{p\ge 0}(-1)^p(p+1)|z|^{2p}) \\
&= \begin{cases} (-1)^k(k+1), & k = j \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

- So the conjecture holds for $\mathcal{M}$.

Twisted Example

- Pick a non-integral real number $c > 0$ and set $g = (1 + |z|^2)^c$ and

$$\lambda = \partial\bar{\partial}g = \frac{2c\, dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

- Let $\mathcal{N} = \mathcal{D}_X|z|^c$. This is an (untwisted) $\mathcal{D}_X$ module corresponding to a local system is monodromy $e^{2\pi ic}$. It is a (complex) Hode module in a natural way.
- Away from ∞, set $\mathcal{M} = L_g^{-1}\mathcal{N}$. This is a $\mathcal{D}_{X,\lambda}$-module isomorphic to

$$D_{X,\lambda}\frac{|z|^c}{(1 + |z|^2)^c}$$

The expresion makes sense near ∞ and we get a $\mathcal{D}_{X,\lambda}$ module on $\mathbb{P}^1$. too.

Continued Twisted Example

- The Hodge filtration on $\mathcal{M}$ comes from that on $\mathcal{N}$. We have
$$\Gamma(F_p\mathcal{M}) = \langle z^k \frac{|z|^c}{(1+|z|^2)^c} : |k| < \frac{c}{2} + p + 1\rangle$$
with $k \in \mathbb{Z}$ and $p \geq 0$.
- To verify the conjecture we have to look at the integral
$$I_k = \int_X \frac{|z|^{c+2k}}{(1+|z|^2)^c}\, d\mathrm{vol} = \int_X \frac{|z|^{c+2k}}{(1+|z|^2)^{c+2}} dx\, dy.$$
- Note that the integrand is L^1 for $|k| < c/2 + 1$ and it is manifestly positive. So on F_0 the integral is positive. But the integrand is really a distribution otherwise.
- By change of variables and other magic we find that
$$I_k = \alpha\Gamma(c/2 + 1 + k)\Gamma(c/2 + 1 - k)$$
where α is an (unimportant) positive constant.
- Now the conjecture is trivial to verify using the sign alternation of that Γ function.