On signal reconstruction from absolute value of frame coefficients

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ABSTRACT

We will construct new classes of Parseval frames for a Hilbert space which allow signal reconstruction from the absolute value of the frame coefficients. As a consequence, signal reconstruction can be done without using phase or its estimation.

Keywords: frames, nonlinear reconstruction, signal processing

1. INTRODUCTION

Reconstruction of a signal using noisy phase or its estimation can be a critical problem in speech recognition technology. But, for many years, engineers have believed that speech recognition should be independent of phase. By constructing new classes of Parseval frames for a Hilbert space, we will show that this allows reconstruction of a signal without using noisy phase or its estimation.

Frames are redundant systems of vectors in a Hilbert spaces. They satisfy the well-known property of perfect reconstruction, in that any vector of the Hilbert space can be synthesized back from its inner products with the frame vectors. More precisely, the linear transformation from the initial Hilbert space to the space of coefficients obtained by taking the inner product of a vector with the frame vectors is injective and hence admits a left inverse. This property has been succesfully used in a broad spectrum of applications, including internet coding, multiple antenna coding, optics, quantum information theory, signal/image processing, and much more. The purpose of this paper is to study what kind of reconstruction is possible if we only have knowledge of the absolute values of the frame coefficients.

In this paper we consider only finite dimensional frames the reason being their direct link to practical applications. Since the same question can be raised for infinite dimensional frames,

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we state the problem in the setting of abstract frames. Furthermore, here we will concentrate our analysis only on real frames, which correspond e.g. to redundant wavelet filter banks.

Consider a Hilbert space H with scalar product $\langle , \rangle$. A finite or countable set of vectors $\mathcal{F} = \{f_i; i \in \mathbb{I}\}$ of H is called a *frame* if there are two positive constants $A, B > 0$ such that for every vector $x \in H$,

$$A\,\|x\|^2 \leq \sum_{i \in I} |\langle x, f_i \rangle|^2 \leq B\,\|x\|^2 \tag{1}$$

The frame is *tight* when the constants can be chosen equal to one another, $A = B$. For $A = B = 1$, $\mathcal{F}$ is called a *Parseval frame*. The numbers $\langle x, f_i \rangle$ are called *frame coefficients*.

To a frame $\mathcal{F}$ we associate the *analysis* and *synthesis operators* defined by:

$$T : H \rightarrow l^2(\mathbb{I}) \quad , \quad T(x) = \{\langle x, f_i \rangle\}_{i \in \mathbb{I}} \tag{2}$$

$$T^* : l^2(\mathbb{I}) \rightarrow H \quad , \quad T^*(c) = \sum_{i \in I} c_i f_i \tag{3}$$

which are well defined due to (1), and are adjoint to one another. The range of T in $l^2(\mathbb{I})$ is called the *range of coefficients*. The *frame operator* defined by $S = T^*T : H \rightarrow H$ is invertible by (1) and provides the perfect reconstruction formula:

$$x = \sum_{i \in \mathbb{I}} \langle x, f_i \rangle S^{-1} f_i \tag{4}$$

For more information on frames we refer the reader to.[6]

Consider now the nonlinear mapping

$$\mathbb{M}_a : H \rightarrow l^2(\mathbb{I}) \ , \ \mathbb{M}_a(x) = \{|\langle x, f_i \rangle|\}_{i \in \mathbb{I}} \tag{5}$$

obtained by taking the absolute value entrywise of the analysis operator. Let us denote by H_r the quotient space $H_r = H/\sim$ obtained by identifying two vectors that differ by a constant phase factor: $x \sim y$ if there is a scalar c with $|c| = 1$ so that $y = cx$. For real Hilbert spaces c can only be $+1$ or -1, and thus $H_r = H/\{\pm 1\}$. For complex Hilbert spaces c can be any complex number of modulus one, $c = e^{i\varphi}$, and then $H_r = H/\mathbb{T}^1$, where $\mathbb{T}^1$ is the complex unit circle. In quantum mechanics these projective rays define quantum states (see[16]). Clearly two vectors of H in the same ray would have the same image through $\mathbb{M}_a$. Thus the nonlinear mapping $\mathbb{M}_a$ extends to H_r as

$$\mathbb{M} : H_r \rightarrow l^2(\mathbb{I}) \ , \ \mathbb{M}(\hat{x}) = \{|\langle x, f_i \rangle|\}_{i \in \mathbb{I}} \ , \ x \in \hat{x} \tag{6}$$

The problem we study in this paper is the injectivity of the map $\mathbb{M}$. When it is injective, $\mathbb{M}$ admits a left inverse, meaning that any vector (signal) in H can be reconstructed up to a constant phase factor from the modulus of its frame coefficients.

2. ANALYSIS OF $\mathbb{M}$ FOR REAL FRAMES

Consider the case $H = \mathbb{R}^N$, and the index set $\mathbb{I}$ has cardinality M, $\mathbb{I} = \{1, 2, \dots, M\}$. Then $l^2(\mathbb{I}) \simeq \mathbb{R}^M$.

The set $Gr(N, M; \mathbb{R})$ of N-dimensional linear subspaces of $\mathbb{R}^M$ has the structure of an $N(M-N)$-dimensional manifold called the *Grassman manifold*.[19] The *frame bundle* $F(N, M; \mathbb{R})$ is the $GL(N, \mathbb{R})$-bundle over $Gr(N, M)$ defined as follows: The fiber of $F(N, M; \mathbb{R})$ over a point of $GL(N, \mathbb{R})$ corresponding to an N-dimensional linear subspace $W \subset \mathbb{R}^M$ is the set of all possible bases for W.

For a frame $\mathcal{F} = \{f_1, \dots, f_M\}$ of $\mathbb{R}^N$ we denote by T the analysis operator,

$$T : \mathbb{R}^N \to \mathbb{R}^M \quad , \quad T(x) = \sum_{k=1}^{M} \langle x, f_k \rangle e_k \quad , \tag{7}$$

where $\{e_1, \dots, e_M\}$ is the canonical basis of $\mathbb{R}^M$. We let W denote the range of the analysis map $T(\mathbb{R}^N)$. It is an N-dimensional linear subspace of $\mathbb{R}^M$ and thus corresponds to a point of the Grassman manifold $Gr(N, M)$. Two frames $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are *equivalent* if there is an invertible operator T on H with $T(f_i) = g_i$, for all $i \in I$. It is known that two frames are equivalent if and only if their associated analysis operators have the same range (see[2,12]). We deduce that M-element frames on $\mathbb{R}^N$ are parametrized by the fiber bundle $F(N, M; \mathbb{R})$.

Recall the nonlinear map we are interested in is

$$\mathbb{M}^{\mathcal{F}} : \mathbb{R}^N/\{\pm 1\} \to \mathbb{R}^M \quad , \quad \mathbb{M}^{\mathcal{F}}(\hat{x}) = \sum_{k=1}^{M} |\langle x, f_k \rangle| e_k, \quad x \in \hat{x} \tag{8}$$

When there is no danger of confusion, we shall drop $\mathcal{F}$ from the notation.

First we reduce our analysis to equivalent classes of frames:

PROPOSITION 1. *For any two frames $\mathcal{F}$ and $\mathcal{G}$ that have the same range of coefficients, $\mathbb{M}^{\mathcal{F}}$ is injective if and only if $\mathbb{M}^{\mathcal{G}}$ is injective.*

Proof. Any two frames $\mathcal{F} = \{f_k\}$ and $\mathcal{G} = \{g_k\}$ that have the same range of coefficients are equivalent, i.e. there is an invertible $R : \mathbb{R}^N \to \mathbb{R}^N$ so that $g_k = Rf_k$, $1 \le k \le M$. Their associated nonlinear maps $\mathbb{M}^{\mathcal{F}}$, and respectively $\mathbb{M}^{\mathcal{G}}$, satisfy $\mathbb{M}^{\mathcal{G}}(x) = \mathbb{M}^{\mathcal{F}}(R^* x)$. This shows that $\mathbb{M}^{\mathcal{F}}$ is injective if and only if $\mathbb{M}^{\mathcal{G}}$ is injective. Consequently the property of injectivity of $\mathbb{M}$ depends only on the subspace of coefficients W in $Gr(N, M)$. □

This result says that for two frames corresponding to two points in the same fiber of $F(N, M; \mathbb{R})$, the injectivity of their associated nonlinear maps would jointly hold true or fail. Because of this result we shall always assume the induced topology by the base manifold $Gr(N, M)$ of the fiber bundle $F(N, M; \mathbb{R})$ into the set of M-element frames of $\mathbb{R}^N$.

If $\{f_i\}_{i\in I}$ is a frame with frame operator S then $\{S^{-1/2} f_i\}_{i\in I}$ is a Parseval frame which is equivalent to $\{f_i\}_{i\in I}$ and called the *canonical Parseval frame* associated to $\{f_i\}_{i\in I}$. Also, $\{S^{-1} f_i\}_{i\in I}$ is a frame equivalent to $\{f_i\}_{i\in I}$ and is called the *canonical dual frame* associated to $\{f_i\}_{i \in I}$. Proposition 2.1 shows that when the nonlinear map $\mathbb{M}^{\mathcal{F}}$ is injective then the same property holds for the canonical dual frame and the canonical Parseval frame.

Given $\phi \subset \{1, \ldots, M\}$, let $\phi(i)$ denote the characteristic function of ϕ defined by the rule that $\phi(i) = 1$ if $i \in \phi$ and $\phi(i) = 0$ if $i \notin \phi$. Define a map $\sigma_\phi : \mathbb{R}^M \to \mathbb{R}^M$ by the formula

$$\sigma_\phi(a_1, \ldots, a_M) = ((-1)^{\phi(1)} a_1, \ldots, (-1)^{\phi(M)} a_M).$$

Clearly $\sigma_\phi^2 = id$ and $\sigma_{\phi^\complement} = -\sigma_\phi$ where $\phi^\complement$ is the complement of ϕ. We let L^ϕ denote the $|\phi|$-dimensional linear subspace of $\mathbb{R}^M$ where $L^\phi = \{(a_1, \ldots, a_M) | a_i = 0, i \in \phi\}$, and we let $P_\phi : \mathbb{R}^M \to L^\phi$ denote the orthogonal projection onto this subspace. Thus $(P_\phi(u))_i = 0$, if $i \in \phi$, and $(P_\phi(u))_i = u_i$, if $i \in \phi^\complement$. For every vector $u \in \mathbb{R}^M$, $\sigma_\phi(u) = u$ iff $u \in L^\phi$. Likewise $\sigma_\phi(u) = -u$ iff $u \in L^{\phi^\complement}$. Note

$$P_\phi(u) = \frac{1}{2}(u + \sigma_\phi(u)) \quad , \quad P_{\phi^\complement}(u) = \frac{1}{2}(u - \sigma_\phi(u))$$

Theorem 2 (Real Frames). *If $M \geq 2N - 1$ then for a generic frame $\mathcal{F}$, $\mathbb{M}$ is injective.* By *generic* we mean an open dense subset of the set of all M-element frames in $\mathbb{R}^N$.

Proof. Suppose that x and x' have the same image under $\mathbb{M} = \mathbb{M}^{\mathcal{F}}$. Let $a_1, \ldots a_M$ be the frame coefficients of x and $a'_1, \ldots a'_M$ the frame coefficients for x'. Then $a'_i = \pm a_i$ for each i. In particular there is a subset $\phi \subset \{1, \ldots, M\}$ of indices such that $a'_i = (-1)^{\phi(i)} a_i$. Then two vectors x, x' have the same image under $\mathbb{M}$ if and only there is a subset $\phi \subset \{1, \ldots, M\}$ such that $(a_1, \ldots a_M)$ and $((-1)^{\phi(1)} a_1, \ldots, (-1)^{\phi(M)} a_M)$ are both in W the range of coefficients associated to $\mathcal{F}$.

To finish the the proof we will show that when $M \geq 2N - 1$ such a condition is impossible for a generic subspace $W \subset \mathbb{R}^N$. This means that the set of such W's is a dense (Zariski) open set in the Grassmanian $Gr(N, M)$. In particular the probability that a randomly chosen W will satisfy this condition is 0.

To finish the proof of the theorem we need the following lemma.

Lemma 3. *If $M \geq 2N - 1$ then the following holds for a generic N-dimensional subspace $W \subset \mathbb{R}^M$. Given $u \in W$ then $\sigma_\phi(u) \in W$ iff $\sigma_\phi(u) = \pm u$.*

Proof. [Proof of the Lemma] Suppose $u \in W$ and $\sigma_\phi(u) \neq \pm u$ but $\sigma_\phi(u) \in W$. Since σ_ϕ is an involution, $u + \sigma_\phi(u)$ is fixed by σ_ϕ and is non-zero. Thus $W \cap L^\phi \neq 0$. Likewise

$$0 \neq u - \sigma_\phi(u) = u + \sigma_{\phi^\complement}(u).$$

Hence $W \cap L^{\phi^\complement} \neq 0$.

Now L^ϕ and L^{ϕ^c} are fixed linear subspaces of dimension $M - |\phi|$ and $|\phi|$. If $M \geq 2N - 1$ then one of these subspaces has codimension greater than or equal to N. However a generic linear subspace W of dimension N has 0 intersection with a fixed linear subspace of codimension greater than or equal to N. Therefore, if W is generic and $x, \sigma_\phi(x) \in W$ then $\sigma_\phi(x) = \pm x$ which ends the proof of Lemma. □

The proof of the theorem now follows from the fact that if W is in the intersection of generic conditions imposed by the proposition for each subset $\phi \subset \{1, \ldots, M\}$ then W satisfies the conclusion of the theorem. $\square$

Note what the above proof actually shows:

COROLLARY 4. *The map* $\mathbb{M}$ *is injective if and only if whenever there is a non-zero element* $u \in W \subset \mathbb{R}^M$ *with* $u \in L^{\phi}$*, then* $W \cap L^{\phi^{\complement}} = \{0\}$.

Next we observe that this result is best possible.

PROPOSITION 5. *If* $M \leq 2N - 2$*, then the result fails for all* M*-element frames.*

Proof. Since $M \leq 2N-2$, we have that $2M - 2N + 2 \leq M$. Let $(e_i)_{i=1}^M$ be the canonical orthonormal basis of $\mathbb{R}^M$. We can write $(e_i)_{i=1}^M = (e_i)_{i=1}^k \cup (e_i)_{i=k+1}^M$ where both k and $M-k$ are $\geq M - N + 1$.

Let W be any N-dimensional subspace of $\mathbb{R}^M$. Since dim $W^{\perp} = M - N$, there exists a nonzero vector $u \in$span $\{e_i\}_{i=1}^k$ so that $u \perp W^{\perp}$, hence $u \in W$. Similarly, there is a nonzero vector v in span $\{e_i\}_{i=k+1}^M$ with $v \perp W^{\perp}$, that is $v \in W$. By the above corollary, $\mathbb{M}$ cannot be injective. In fact $\mathbb{M}(u+v) = \mathbb{M}(u-v)$. $\square$

The next result gives an easy way for frames to satisfy the condition above.

COROLLARY 6. *If* $\mathcal{F}$ *is a* M*-element frame for* $\mathbb{R}^N$ *with* $M \geq 2N-1$ *having the property that every* N*-element subset of the frame is linearly independent, then* $\mathbb{M}$ *is injective.*

Proof. Given the conditions, it follows that W has no elements which are zero in N coordinates and so the Corollary holds. $\square$

COROLLARY 7.

1. *If* $M = 2N - 1$*, then the condition given in Corollary 6 is also necessary.*
2. *If* $M \geq 2N$*, this condition is no longer necessary.*

Proof. (1) For the first part we will prove the contrapositive. Let $M = 2N-1$ and assume there is an N-element subset $(f_i)_{i \in \phi}$ of $\mathcal{F}$ which is not linearly independent. Then there is a non-zero $x \in (span(f_i)_{i\in\phi})^{\perp} \subset \mathbb{R}^N$. Hence, $0 \neq u = T(x) \in L^{\phi} \cap W$. On the other hand, since $dim(span(f_i)_{i \in \phi^{\complement}}) \leq N-1$, there is a non-zero $y \in (span(f_i)_{i\in\phi^{\complement}})^{\perp} \subset \mathbb{R}^N$ so that $0 \neq v = T(y) \in L^{\phi^{\complement}} \cap W$. Now, by Corollary 4, $\mathbb{M}$ is not injective.

(2) If $M \geq 2N$ we construct an M-element frame for $\mathbb{R}^N$ that has an N-element linearly dependent subset. Let $\mathcal{F}' = \{f_1, \ldots, f_{2N-1}\}$ be a frame for $\mathbb{R}^N$ so that any N-element subset is linearly independent. By Corollary 4, the map $\mathbb{M}^{\mathcal{F}'}$ is injective. Now extend this frame to $\mathcal{F} = \{f_1, \ldots, f_M\}$ by $f_{2N} = \cdots = f_M = f_{2N-1}$. The map $\mathbb{M}^{\mathcal{F}}$ extends $\mathbb{M}^{\mathcal{F}'}$ and therefore remains injective, whereas clearly any N-element subset that contains two vectors from $\{f_{2N-1}, f_{2N}, \ldots, f_M\}$ is no longer linearly independent. $\square$

Remark: The frames above can easily be constructed "by hand". Start with an orthonormal basis for $\mathbb{R}^N$, say $(f_i)_{i=1}^N$. Assume we have constructed sets of vectors $(f_i)_{i=1}^M$ such that every subset of N vectors is linearly independent. Look at the span of all of the (N-1)-element subsets of $(f_i)_{i=1}^M$. Pick f_{M+1} not in the span of any of these subsets. Then $(f_i)_{i=1}^{M+1}$ has the property that every N-element subset is linearly independent.

Now we will give a slightly different proof of this result which gives necessary and sufficient conditions for a frame to have the required properties.

THEOREM 8. *Let $(f_i)_{i=1}^M$ be a frame for $\mathbb{R}^N$. The following are equivalent:*

(1) The map $\mathbb{M}$ is injective.

(2) For every subset $\phi \subset \{1, 2, \dots, M\}$, either $\{f_i\}_{i\in\phi}$ spans $\mathbb{R}^N$ or $\{f_i\}_{i\in\phi^c}$ spans $\mathbb{R}^N$.

Proof. (1) $\Rightarrow$ (2): We prove the contrapositive. So assume that there is a subset $\phi \subset \{1, 2, \dots M\}$ so that neither $\{f_i \ ; \ i \in \phi\}$ nor $\{f_i \ ; \ i \in \{\phi^{\complement}\}$ spans $\mathbb{R}^N$. Hence there are non-zero vectors $x, y \in \mathbb{R}^N$ so that $x \perp span(f_i)_{i\in\phi}$ and $y \perp span(f_i)_{i\in\phi^{\complement}}$. Then $0 \neq T(x) \in L^S \cap W$ and $0 \neq T(y) \in L^{\phi^{\complement}} \cap W$. Now by Corollary 4 we have that $\mathbb{M}$ cannot be injective.

(2) $\Rightarrow$ (1): Suppose $\mathbb{M}(\hat{x}) = \mathbb{M}(\hat{y})$ for some $\hat{x}, \hat{y} \in \mathbb{R}^N/\{\pm 1\}$. This means for every $1 \leq j \leq M$, $|\langle x, f_j\rangle| = |\langle y, f_j\rangle|$ where $x \in \hat{x}$ and $y \in \hat{y}$. Let

$$\phi = \{j \ : \ \langle x, f_j\rangle = -\langle y, f_j\rangle\}. \tag{9}$$

Note

$$\phi^{\complement} = \{j \ : \ \langle x, f_j\rangle = \langle y, f_j\rangle\} \tag{10}$$

Now, $x + y \perp span(f_i)_{i\in\phi}$ and $x - y \perp span(f_i)_{i\in\phi^{\complement}}$. Assume that $\{f_i \ ; \ i \in \phi\}$ spans $\mathbb{R}^N$. Then $x + y = 0$ and thus $\hat{x} = \hat{y}$. If $\{f_i \ ; \ i \in \phi^{\complement}\}$ spans $\mathbb{R}^N$ then $x - y = 0$ and again $\hat{x} = \hat{y}$. Either way $\hat{x} = \hat{y}$ which proves $\mathbb{M}$ is injective. □

For $M < 2N - 1$ there are plenty of frames for which $\mathbb{M}$ is not injective. However for a generic frame, we can show the set of rays that can be reconstructed from the image under $\mathbb{M}$ is open dense in $\mathbb{R}^N/\{\pm 1\}$.

THEOREM 9. *Assume $M > N$. Then for a generic frame $\mathcal{F} \in \mathcal{F}[N, M; \mathbb{R}]$, the set of vectors $x \in \mathbb{R}^N$ so that $(\mathbb{M}^{\mathcal{F}})^{-1}(\mathbb{M}_a^{\mathcal{F}}(x))$ consists of one point in $\mathbb{R}^N/\{\pm 1\}$ has dense interior in $\mathbb{R}^N$.*

Proof.

Let $\mathcal{F}$ be a M-element frame in $\mathbb{R}^N$. Then $\mathcal{F}$ is similar to a frame $\mathcal{G}$ which consists of the union of the canonical basis of $\mathbb{R}^N$, $\{d_1, \dots, d_N\}$, with some other set of $M - N$ vectors. Let $\mathcal{G} = \{g_k \ ; \ 1 \leq k \leq M\}$. Thus $g_{k_j} = d_j$, $1 \leq j \leq N$, for some N elements $\{k_1, k_2, \dots, k_N\}$ of $\{1, 2, \dots, M\}$. Consider now the set $\mathbb{B}$ of frames $\mathcal{F}$ so that its similar frame $\mathcal{G}$ constructed above has a vector g_k with all entries non-zero,

$$\mathbb{B} = \{\mathcal{F} \in \mathcal{F}[N, M; \mathbb{R}] \mid \mathcal{F} \sim \mathcal{G} = \{g_k\} \ , \ \{d_1, \dots, d_N\} \subset \mathcal{G} \ ,$$

$$\prod_{j=1}^{N} \langle g_{k_0}, d_j\rangle \neq 0, \text{for some } k_0\}$$

Clearly $\mathbb{B}$ is open dense in $\mathcal{F}[N, M; \mathbb{R}]$. Thus generically $\mathcal{F} \in \mathbb{B}$. Let $\mathcal{G}$ be its similar frame satisfying the condition above. We want to prove the set $X = X^{\mathcal{F}}$ of vectors $x \in \mathbb{R}^N$ so that

$(\mathbb{M}^{\mathcal{G}})^{-1}(\mathbb{M}_a^{\mathcal{G}}(x))$ has more than one point is *thin*, i.e. it is included in a set whose complement is open and dense in $\mathbb{R}^N$. We claim $X \subset \cup_\phi (V_\phi^+ \cup V_\phi^-)$ where $(V_\phi^\pm)_{\phi \subset \{1,2,\ldots,N\}}$ are linear subspaces of $\mathbb{R}^N$ of codimension 1 indexed by subsets ϕ of $\{1, 2 \ldots, N\}$. This claim will conclude the proof of Theorem.

To verify the claim, let $x, y \in \mathbb{R}^N$ be so that $\mathbb{M}_a^{\mathcal{G}}(x) = \mathbb{M}_a^{\mathcal{G}}(y)$ and yet $x \neq y$, nor $x \neq -y$. Since $\mathcal{G}$ contains the canonical basis of $\mathbb{R}^N$, $|x_k| = |y_k|$ for all $1 \leq k \leq N$. Then there is a subset $\phi \subset \{1, 2, \ldots, N\}$ so that $y_k = (-1)^{\phi(k)} x_k$. Note $\phi \neq \emptyset$, nor $\phi \neq \{1, 2, \ldots, N\}$. Denote by D_ϕ the diagonal $N \times N$ matrix $(D_\phi)_{kk} = (-1)^{\phi(k)}$. Thus $y = D_\phi x$, and yet $D_\phi \neq \pm I$. Let $g_{k_0} \in \mathcal{G}$ be so that none of its entries vanishes. Then $|\langle x, g_{k_0}\rangle| = |\langle y, g_{k_0}\rangle|$ implies

$$\langle x, (I \pm D_\phi) g_{k_0} \rangle = 0$$

This proves the set $X^{\mathcal{G}}$ is included in the union of $2(2^N - 2)$ linear subspaces of codimension 1,

$$\cup_{\phi \neq \emptyset, \phi^{\complement} \neq \emptyset} \{(I - D_\phi) g_{k_0}\}^\perp \cup \{(I + D_S) g_{k_0}\}^\perp$$

Since $\mathcal{F}$ is similar to $\mathcal{G}$, $X^{\mathcal{F}}$ is included in the image of the above set through a linear invertible map, which proves the claim.

□

3. IMPLEMENTATION OF THESE RESULTS

For these results to be widely applied they need to run on existing software with only trivial modifications. So there are two critical issues that need to be addressed for implementation of signal reconstruction without phase. (1) Find wavelet frames which work in this setting - so we can use the fast wavelet transform for transforming the signal. (2) Find efficient reconstruction algorithms - preferably algorithms which are close to the inverse wavelet transform. These two problems are the focus of current research on this topic.[3] It appears at this time that small frames near the threshold of our results ($(2N-1)$ elements) may require exponential time for reconstruction. However, it is shown in[3] that generic frames with N^2-elements give polynomial time reconstruction (on the order of at most N^6 calculations). In[3] there are some special classes of frames with N^2 elements which have extremely efficient algorithms for reconstruction in N calculations ($2N$ in the complex case).

In the following we present a nonlinear reconstruction method using neural networks.

A 3-layer neural network with input $c = (c_f)_{1 \leq f \leq M}$ and output $z = (z_n)_{1 \leq n \leq N}$ is defined by:

$$q_k = \sigma\left(\sum_{f=1}^{M} a_{kf} c_f + \theta_k\right) \quad , \quad 1 \leq k \leq L \tag{11}$$

$$z_n = \sigma\left(\sum_{k=1}^{L} b_{nk} q_k + \tau_n\right) \quad , \quad 1 \leq n \leq N \tag{12}$$

where $\sigma : \mathbb{R} \to [0,1]$ is the sigmoid function $\sigma(u) = 1/(1 + e^{-u})$, and $A = (a_{kf})_{1 \leq k \leq L, 1 \leq f \leq M}$, $B = (b_{nk})_{1 \leq n \leq N, 1 \leq k \leq L}$, and $\theta = (\theta_k)_{1 \leq k \leq L}$, $\tau = (\tau_n)_{1 \leq n \leq N}$ are network parameters to be learned in the training phase. We compactly write them as $\pi = (A, B, \theta, \tau)$. When dealing with Parseval frames, the input vector norm can always be recovered from the Euclidian norm of the frame coefficients (hence from their absolute values). Denote by $F : (\mathbb{R}^+)^M \rightarrow \mathbb{R}^N$ the inverse of the nonlinear map (8). Using above neural network, F can be implemented by

$$(F(c))_n = z_n \sqrt{\frac{c_1^2 + \cdots c_M^2}{z_1^2 + \cdots + z_N^2}} \quad , \quad 1 \leq n \leq N \tag{13}$$

The entire theory presented in Section 2 proves the existence of the inverse F. In turn, this fact allows to train a neural network, and control its performance. More results will be presented elsewhere ([3]).

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