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A Spectral Power Factorization

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Abstract

This note is related to Theorem 1 of the ICA-BSS Conference' paper "AR Processes and Sources can be Reconstructed from Degenerate Mixtures" [1]. Here we study further the uniqueness of the decomposition.

Consider the more general statement of the aformentioned result:

THEOREM 1 *Suppose we are given the sum of two independent, stable with no common poles $AR(p_1)$ and $AR(p_2)$ process outputs. Then the second order statistics generically is sufficient to uniquely identify the two AR processes.*

Let us note that the two $AR(p_1)$ and $AR(p_2)$ processes define a $p_1 + p_2 + 2$ dimensional space of parameters. By "generically" in the statement above, we mean the set of "bad" AR processes form an algebraic manifold of positive codimension in this space. We return to the codimensionality of this algebraic manifold after the proof of the theorem.

To fix the notations, the two AR processes have spectral power densities:

$$R_{s_1}(z) = \frac{G_1^2}{P_1(z)P_1(\frac{1}{z})} \quad (1)$$

$$R_{s_2}(z) = \frac{G_2^2}{P_2(z)P_2(\frac{1}{z})} \quad (2)$$

and therefore the sum has the spectral power:

$$R_x(z) = \frac{G_1^2}{P_1(z)P_1(\frac{1}{z})} + \frac{G_2^2}{P_2(z)P_2(\frac{1}{z})} \tag{3}$$

Proof of Theorem 1

The proof uses (3) above recast into:

$$R_x(z) = \frac{G_1^2 P_2(z)P_2(\frac{1}{z}) + G_2^2 P_1(z)P_1(\frac{1}{z})}{P_1(z)P_2(z)P_1(\frac{1}{z})P_2(\frac{1}{z})} \tag{4}$$

Since the two processes have no common poles, and all the zeros of P_1, P_2 are inside the unit circle, we can spectrally factorize the numerator:

$$G_1^2 P_2(z)P_2(\frac{1}{z}) + G_2^2 P_1(z)P_1(\frac{1}{z}) = Q(z)Q(\frac{1}{z}) \tag{5}$$

where Q is a polynomial with stable zeros of degree $q_0 = max(p_1, p_2)$. Let $P = P_1P_2$ be the polynomial of degree $p_0 = p_1 + p_2$. Then

$$R_x(z) = \frac{Q(z)Q(\frac{1}{z})}{P(z)P(\frac{1}{z})}.$$

Thus, if we identify $\{x(n)\}_n \in Z$ as an $ARMA(p_0, q_0)$ process using the second order statistics, it follows we can obtain P and Q. Now the problem becomes an algebraic-combinatorial problem. Given P and Q, two polynomials of degree p_0, respectively q_0, we have to find all possible factorization of P into a product of two polynomials P_1P_2 (all having the first term 1) such that there are two real numbers G_1, G_2 that satisfy (5). This decomposition has at least one solution, because P and Q are associated to the sum of two AR processes. On the other hand note there are only a finite number of possible partitions (and therefore factorizations) of zeros of P. Thus, if the number of solutions can only be finite. We prove now that, generically, the factorization is unique. Assume (G_1, P_1) and (G_2, P_2) is a pair of AR processes for which the factorization is not unique. First assume P_1 has a simple zero. If this is not the case, perturb its coefficients until this happens. We prove that for every $\varepsilon > 0$, there is a $\varepsilon > \varepsilon' > 0$ so that the factorization for (G_1, P_1) and $(G_2 + \varepsilon'', P_2)$ is unique, for every $0 < \varepsilon'' < \varepsilon'$. Indeed, suppose this is not the case. Then there is a sequence $\varepsilon_n \to 0$ so that (G_1, P_1) and $(G_2 + \varepsilon_n, P_2)$ admit a nonunique factorization. Since there are only finitely many partitions of zeros of P_1 and P_2, there are two ε_n's that correspond to the same partition. Let us rename these epsilons by ε_1 and ε_2. Thus we have:

$$G_1^2 P_2(z)P_2(\frac{1}{z}) + (G_2 + \varepsilon_1)^2 P_1(z)P_1(\frac{1}{z}) = C_1^2 A_1(z)A_1(\frac{1}{z}) + C_2^2 A_2(z)A_2(\frac{1}{z})$$

$$G_1^2 P_2(z)P_2(\frac{1}{z}) + (G_2 + \varepsilon_2)^2 P_1(z)P_1(\frac{1}{z}) = D_1^2 A_1(z)A_1(\frac{1}{z}) + D_2^2 A_2(z)A_2(\frac{1}{z})$$

with $A_1 A_2 = P_1 P_2$ and $\varepsilon_1 \neq \varepsilon_2$. Now subtract the two equations. We get:

$$GP_1(z)P_1(\frac{1}{z}) = (C_1 - D_1)A_1(z)A_1(\frac{1}{z}) + (C_2 - D_2)A_2(z)A_2(\frac{1}{z})$$

Now, the single zero of P_1 will be either in A_1, or in A_2, but not in both. Assume A_1 collects that zero. Since the left hand side and A_1 vanishe on that point, we obtain that necessarily $C_2 - D_2 = 0$. Hence

$$GP_1(z)P_1(\frac{1}{z}) = (C_1 - D_1)A_1(z)A_1(\frac{1}{z})$$

and thus $A_1 = P_1$, remaining $A_2 = P_2$. Thus we proved for any pair of AR processes for which the decomposition (5) is not unique, and any $\varepsilon > 0$, there is a perturbed pair for which the decomposition is uniqe (hence the system is uniquely identifiable). Now we show the set of "good" parameters (i.e. those AR processes that are uniquely identifiable) is open. Assume $\xi = (G_1, G_2, \underline{a}, \underline{b})$ is such a set of parameters ($P_1(z) = 1 + \sum_{k=1}^{p_1} a_k z^{-k}$, $P_2(z) = 1 + \sum_{k=1}^{p_2} b_k z^{-k}$). Then we claim there is $\varepsilon_0 > 0$ so that for every set of parameters $\xi^{'} = (G_1^{'}, G_2^{'}, \underline{a}^{'}, \underline{b}^{'})$ within ε_0 to the previous set ($|\xi - \xi^{'}| < \varepsilon_0$), the factorization is unique. Indeed, assume this is not the case. Then for every $\varepsilon_n > 0$ we find a set ξ_n within ε_n for which the factorization is not unique. Then we obtain a sequence of polynomials $A_1^n(z), A_2^n(z)$ and D_1^n, D_2^n so that:

$$(G_1^n)^2 P_2^n(z)P_2^n(\frac{1}{z}) + (G_2^n)^2 P_1(z)P_1(\frac{1}{z}) = (D_1^n)^2 A_2(z)A_2^n(\frac{1}{z}) + (D_2^n)^2 A_1^n(z)A_1^n(\frac{1}{z})$$

and

$$P_1^n(z)P_2^n(z) = A_1^n(z)A_2^n(z)$$

with $A_1(z) \neq P_1(z)$, $A_2(z) \neq P_2(z)$, and $(G_1^n, G_2^n, \underline{a^n}, \underline{b^n})$ (associated to the left hand-side) converges to ξ. Since the zeros of A_1^n, A_2^n are bounded, their coefficients are bounded as well, and also D_1^n, D_2^n. Hence we can extract a subsequence for which the coefficients of A_1^n, A_2^n are convergent (together with D_1^n, D_2^n), and they all are associated to the same partition of zeros of $P_1 P_2$. Hence we get a factorization of ξ into $\lim_n(D_1^n, D_2^n, A_1^{\cdot} A_2^n)$ different from (G_1, G_2, P_1, P_2) which contradicts the exactness assumption of identification for ξ. Hence the set of identifiable processes is open. Overall we obtained the set of identifiable *AR* processes is open and (everywhere) dense. In this sense it is generic. □

Next we look to how "fat" the set of "bad" AR processes is. To have a nonunique decomposition of $R_x(z)$, we have to have nontrivial polynomials A, B, C, D and four positive numbers $G_1, G_2, G_1^{'}, G_2^{'}$ so that:

$$P_1 = A\,C \quad , \quad P_2 = B\,D \tag{6}$$

$$P_1^{'} = A\,D \quad , \quad P_2^{'} = B\,C \tag{7}$$

and satisfy

$$\begin{aligned} G_1^2B(z)D(z)B(\frac{1}{z})D(\frac{1}{z}) \quad &+ \quad G_2^2A(z)C(z)A(\frac{1}{z})C(\frac{1}{z}) \\ = (G_1^{'})^2B(z)C(z)B(\frac{1}{z})C(\frac{1}{z}) \quad &+ \quad (G_2^{'})^2A(z)D(z)A(\frac{1}{z})D(\frac{1}{z}) \end{aligned} \tag{8}$$

This can be rearranged into:

$$\begin{aligned} B(z)B(\frac{1}{z})[G_1^2D(\frac{1}{z}) \quad &- \quad (G_1^{'})^2C(z)C(\frac{1}{z})] \\ = A(z)A(\frac{1}{z})[(G_2^{'})^2D(z)D(\frac{1}{z}) \quad &- \quad G_2^2C(z)C(\frac{1}{z})] \end{aligned} \tag{9}$$

Since A and B should have no common zeros, it follows there is a Laurent polynomial $R(z)$ such that:

$$G_1^2D(z)D(\frac{1}{z}) - (G_1^{'})^2C(z)C(\frac{1}{z}) = R(z)A(z)A(\frac{1}{z}) \tag{10}$$

$$(G_2^{'})^2D(z)D(\frac{1}{z}) - G_2^2C(z)C(\frac{1}{z}) = R(z)B(z)B(\frac{1}{z}) \tag{11}$$

and $R(z) = R(\frac{1}{z})$. Solving for $C(z)C(\frac{1}{z})$ and $D(z)D(\frac{1}{z})$, we obtain:

$$\begin{aligned} C(z)C(\frac{1}{z}) &= \frac{(G_2^{'})^2}{G_1^2G_2^2 - (G_1^{'}G_2^{'})^2}R(z)A(z)A(\frac{1}{z}) \\ &\quad -\frac{G_1^2}{(G_1G_2)^2 - (G_1^{'}G_2^{'})^2}R(z)B(z)B(\frac{1}{z}) \end{aligned} \tag{12}$$

$$\begin{aligned} D(z)D(\frac{1}{z}) &= \frac{G_2^2}{G_1^2G_2^2 - (G_1^{'}G_2^{'})^2}R(z)A(z)A(\frac{1}{z}) \\ &\quad -\frac{(G_1^{'})^2}{(G_1G_2)^2 - (G_1^{'}G_2^{'})^2}R(z)B(z)B(\frac{1}{z}) \end{aligned} \tag{13}$$

This implies the zeros of R are common between C and D, which is not possible. Therefore R should be a constant, $R(z) = R_0$. Note $R_0 = 0$ implies $G_1 = G_1^{'}$ and $D = C$, and $G_2 = G_2^{'}$. Hence $A = B$, i.e. unique factorization.

There are now two cases: $G_1G_2 = G_1^{'}G_2^{'}$, or $G_1G_2 \neq G_1^{'}G_2^{'}$.

Assume $G_1G_2 = G_1^{'}G_2^{'}$. This implies

$$(G_2^{'})^2R_0^2A(z)A(\frac{1}{z}) = G_1^2R_0B(z)B(\frac{1}{z})$$

and

$$G_2^2A(z)A(\frac{1}{z}) = (G_1^{'})^2B(z)B(\frac{1}{z})$$

which in turn implies a trivial solution.

Consider now the other case, $G_1G_2 \neq G_1^{'}G_2^{'}$. Since P_1 and P_2 have degrees p_1, respetively p_2, it follows that C and D should have the same degree, $deg\, C = deg\, D$. By (12,13) we obtain:

$$
\begin{aligned}
deg\, A &\leq max(deg\, C, deg\, D) = deg\, C \\
deg\, B &\leq max(deg\, C, deg\, D) = deg\, C \\
deg\, C &\leq max(deg\, A, deg\, B) \leq C
\end{aligned}
$$

Therefore

$$max(deg\, A, deg\, B) = deg\, C \tag{14}$$

and thus

$$max(p_1, p_2) = 2deg\, C \tag{15}$$

should be an even number. Hence in the case p_1, p_2 are odd we obtain:

COROLLARY 1 *If the larger of the two AR process lengths is odd, then the second order statistics of the sum uniquely identifies the AR parameters.*

In general we can construct $AR(p_1)$ and $AR(p_2)$ processes whose second order statistics of the sum is not sufficient for identification. This is done as follows: Fix G_1, G_2, C and D with $deg\, C = \deg\, D = \tilde{p}$. Parametrize by $G_1^{'}, G_2^{'}$ the family of solutions of the spectral factorization (10,11) for $G_2{'}$ large enough and $G_1^{'}$ small enough; R is obtained from the monotonicity condition of both A and B. Thus the condition to have the same R reduces the solution to a one parameter family. Hence for every G_1, G_2, C and D we obtain at most a 1-dimensional algebraic manifold of factors A and B. In the total space of parameters $(G_1, G_2, \underline{a}, \underline{b})$ of dimension $2 + p_1 + p_2$, this corresponds to a family of 1-dimensional algebraic curves parametrized by $2 + deg\, C + deg\, D$ parameters, thus a totla of $3 + 2\tilde{p}$-dimensional algebraic manifold. Note the dimension condition: $max(p_1, p_2) = 2\tilde{p}$. Thus we proved:

COROLLARY 2 *Assume $max(p_1, p_2)$ is even. Then the set of independent and stable $AR(p_1)$ and $AR(p_2)$ processes not identifiable by the second order statistics of their sum is parametrized as union of $3 + max(p_1, p_2)$ dimensional algebraic manifolds.*

References

[1] R. Balan, A. Jourjine, and J Rosca. Ar processes and sources can be reconstructed from degenerate mixtures. In *Proceedings ICA'99, Aussois*, pages 467–472, 1999. Aussois, France.