A Nonlinear Reconstruction Algorithm from Absolute Value of Frame Coefficients for Low Redundancy Frames

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Abstract:

In this paper we present a signal reconstruction algorithm from absolute value of frame coefficients that requires a relatively low redundancy. The basic idea is to use a nonlinear embedding of the input signal Hilbert space into a higher dimensional Hilbert space of sesquilinear functionals so that absolute values of frame coefficients are associated to relevant inner products in that space. In this space the reconstruction becomes linear and can be performed in a polynomial number of steps.

1. Introduction

Let us denote by $\mathbf{E}^n$ the n-dimensional space of signals (e.g. $E^n = \mathbf{R}^n$ or $E^n = \mathbf{C}^n$), and assume we are given a frame of m vectors $\{f_1, \ldots, f_m\} \subset \mathbf{E}^n$ that span E^n. Thus necessarily $m \geq n$. In this paper we look at the following problem: Given $c_l = |\langle x, f_l \rangle|$, $1 \leq l \leq m$, reconstruct the original signal $x \in \mathbf{E}^n$ up to a constant phase ambiguity, that is, obtain a signal $y \in \mathbf{E}^n$ such that $y = e^{i\varphi}x$ for some $\varphi \in [0, 2\pi)$.
This problem arises in several areas of signal processing (see [BCE06] for a more detailed discussion of these issues). In particular, in X-Ray Crystallography (see [LFB87]) it is known as the *phase retrieval problem*. In speech processing it is related to the use of cepstral coefficients in Automatic Speech Recognition as well as direct reconstruction from denoised spectogram (see [NQL82]). By the same token the solution posed here can be viewed as a new, nonlinear signal generating model.
Recently ([BBCE09]) we proposed a quasi-linear reconstruction algorithm that requires the frame to have high redundancy ($m = O(n^2)$). The algorithm works as follows. First note that two vectors $x, y \in \mathbf{E}^n$ that are equivalent (i.e. equal to one another up to a constant phase) generate the same rank-one operators $K_x = K_y$, where

$$K_u : \mathbf{E}^n \to \mathbf{E}^n \ , \ K_u(z) = \langle z, u \rangle u \tag{1}$$

with $u = x$ or $u = y$. Conversely, if $K_x = K_y$ then necessarily there exists a phase φ so that $y = e^{i\varphi}x$. Thus the reconstruction problem reduces to obtaining first K_x, and then a representative of the class $\hat{x}$. Next notice that the absolute value of frame coefficient $|\langle x, f_l \rangle|$ is related to the Hilbert-Schmidt inner product between K_x and K_{f_l}:

$$\langle K_x, K_{f_l} \rangle := trace(K_x K_{f_l}^*) = |\langle x, f_l \rangle|^2$$

Hence, if $\{K_{f_l} \ , \ 1 \leq l \leq m\}$ form a frame for the set of Hilbert-Schmidt operators (this is the same as the set of quadratic forms), then K_x can be reconstructed from d_l^2 with a linear algorithm, from where a vector $y \in \hat{x}$ can be obtained. Explicitely, the algorithm is as follows: First denote by $\{\widetilde{K_l} : \mathbf{E}^n \to \mathbf{E}^n \ , \ 1 \leq l \leq m\}$ the canonical dual frame of $\{K_{f_l} \ , \ 1 \leq l \leq m\}$.
1. Compute:

$$K_x = \sum_{l=1}^{m} c_l^2 \widetilde{K_l} \tag{2}$$

2. Assume $e \in \mathbf{E}^n$, $\|e\| = 1$ is so that $\|K_x e\| \neq 0$. Then:

$$y = \frac{1}{\sqrt{\langle K_x(e), e \rangle}} K_x(e) \tag{3}$$

is a vector in $\mathbf{E}^n$ equivalent to x.
While very appealing from a computational perspective, this algorithm requires the set $\{K_{f_l} \ , \ 1 \leq l \leq m\}$ to be complete (spanning) in the Hilbert space of $n \times n$ quadratic forms. In the real case ($\mathbf{E} = \mathbf{R}$) this latter Hilbert space is of dimension $n(n+1)/2$. In the complex case ($\mathbf{E} = \mathbf{C}$) the dimension becomes n^2. Thus the algorithm requires the original frame set $\{f_l \ , \ 1 \leq l \leq m\}$ to have $m = O(n^2)$ vectors. In practice this requirement may not be feasible. Furthermore, in [BCE06] we obtained that generically $m \geq 4n - 2$ should suffice in the complex case, and $n \geq 2n - 1$ should suffice in the real case. In this paper we present an algorithm that applies to a generic frame set of $m = 5.394n - 4.394$ vectors in the complex case, and $m = 2n - 1$ in the real case. The main ingredient of this algorithm is the nonlinear embeding of $\mathbf{E}^n$ into a linear space $\Lambda_{d,d}$ of (d, d)-sesquilinear symmetric forms where the absolute value of frame coefficients provide the inner products with a frame set.

2. Nonlinear Embeddings

Let $\mathbf{E}^n$ be the signal n-dimensional Hilbert space. Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ be a spanning set of m vectors in $\mathbf{E}^n$. Its *redundancy* is $r = m/n \geq 1$. Fix an integer $d \geq 1$ which is going to measure the embedding *depth*. Let $\Lambda_{d,d}(\mathbf{E}^n)$ denote the linear space of (d, d)-sesquilinear functionals, that is

$$\Lambda_{d,d}(\mathbf{E}^n) = \{ \ \alpha : \underbrace{\mathbf{E}^n \times \cdots \mathbf{E}^n}_{2d} \to \mathbf{C} \ \} \tag{4}$$

where $\alpha(y_1,\ldots,y_d,z_1,\ldots,z_d)$ is linear in $y_1,\ldots,y_d$, and antilinear in $z_1,\ldots,z_d$. Note $\Lambda_{d,d}(\mathbf{E}^n)$ is a vector space of dimension n^{2d}. Let $\{e_k\,,\,1\leq k\leq n\}$ be an orthonormal basis of $\mathbf{E}^n$. For each $2d$-tuple $(k_1,\ldots,k_{2d})$ of integers from $1,\ldots,n$ (repetitions are allowed) define

$$\delta_{k_1,\ldots,k_{2d}}(y_1,\ldots,y_d,z_1,\ldots,z_d) = \langle y_1,e_{k_1}\rangle\cdots\langle y_d,e_{k_d}\rangle\cdot \langle e_{k_{d+1}},z_1\rangle\cdots\langle e_{k_{2d}},z_d\rangle \quad (5)$$

Note $\Delta = \{\delta_{k_1,\ldots,k_{2d}}\,;\,1\leq k_l\leq n, 1\leq l\leq 2d\}$ forms a basis in $\Lambda_{d,d}(\mathbf{E}^n)$. We define an inner product on $\Lambda_{d,d}(\mathbf{E}^n)$ so that this basis is orthonormal. Consider two sesquilinear functionals in $\Lambda_{d,d}(\mathbf{E}^n)$:

$$\alpha(y_1,\ldots,y_d,z_1,\ldots,z_d) = \langle y_1,a_1\rangle\cdots\langle y_d,a_d\rangle\langle b_1,z_1\rangle\cdots\langle b_d,z_d\rangle$$

$$\beta(y_1,\ldots,y_d,z_1,\ldots,z_d) = \langle y_1,g_1\rangle\cdots\langle y_d,g_d\rangle\langle h_1,z_1\rangle\cdots\langle h_d,z_d\rangle$$

Then their inner product is defined as

$$\langle\alpha,\beta\rangle := \langle g_1,a_1\rangle\cdots\langle g_d,a_d\rangle\langle b_1,h_1\rangle\cdots\langle b_d,h_d\rangle \quad (6)$$

Extend this binary operation to an inner product on $\Lambda_{d,d}(\mathbf{E}^n)$. With this inner product Δ becomes an orthonormal basis for the Hilbert space $\Lambda_{d,d}(\mathbf{E}^n)$.

Now we are ready to define the nonlinear embedding of the input Hilbert space $\mathbf{E}^n$ in $\Lambda_{d,d}(\mathbf{E}^n)$. This is given by the map $\Phi:\mathbf{E}^n\rightarrow\Lambda_{d,d}(\mathbf{E}^n)$

$$\Phi(x)(y_1,\ldots,y_d,z_1,\ldots,z_d) = \langle y_1,x\rangle\cdots\langle y_d,x\rangle\cdot \cdot\langle x,z_1\rangle\cdots\langle x,z_d\rangle \quad (7)$$

Let $E_d = span(\Phi(\Lambda_{d,d}(\mathbf{E}^n)))$ be the linear span of the embedding. Note in general $E_d\subsetneq\Lambda_{d,d}(\mathbf{E}^n)$ unless $d=1$. Let P denote the orthogonal projection onto E_d, $P:\Lambda_{d,d}(\mathbf{E}^n)\rightarrow E_d$.

Define now the following sesquilinear functionals associated to the frame set $\mathcal{F}$. Fix $1\leq j_1,\ldots,j_d\leq m$.

$$\psi_{j_1,\ldots,j_d}(y_1,\ldots,y_d,z_1,\ldots,z_d) = \langle y_1,f_{j_1}\rangle\cdots\langle y_d,f_{j_d}\rangle\cdot \cdot\langle f_{j_1},z_1\rangle\cdots\langle f_{j_d},z_d\rangle \quad (8)$$

Note there are m^d distinct such functionals, however the number of distinct projections onto E_d is much smaller. Notice

$$\langle\Phi(x),\psi_{j_1,\ldots,j_d}\rangle = |\langle x,f_{j_1}\rangle|^2\cdots|\langle x,f_{j_d}\rangle|^2 \quad (9)$$

Thus if $(k_1,\ldots,k_d)$ is a permutation of $(j_1,\ldots,j_d)$ then $P\psi_{k_1,\ldots,k_d} = P\psi_{j_1,\ldots,j_d}$. For converse we need to assume first that frame vectors belong to distinct equivalence classes (that is, for any two $1\leq l<j\leq m$ and any $a\in[0,2\pi)$, $f_l\neq e^{ia}f_j$). Then we get that $P\psi_{k_1,\ldots,k_d} = P\psi_{j_1,\ldots,j_d}$ if and only if $(k_1,\ldots,k_d)$ is a permutation of $(j_1,\ldots,j_d)$. Thus we obtain that for frames with frame vectors in distinct equivalence classes the set

$$\Psi = \{\psi_{j_1,\ldots,j_d}\,,\,1\leq j_1\leq j_2\leq\cdots\leq j_d\leq m\} \quad (10)$$

is a maximal set of sesquilinear functionals of type (8) that have distinct projections through P.

For our algorithm to work we need to assume:

Assumption A. The set $P\Psi := \{P\psi\,,\,\psi\in\Psi\}$ is spanning in E_d.

In section 4. we analyze the dimensionality constraint $|P\Psi|\geq dim(E_d)$, and in section 5. we present numerical results supporting Assumption A for a generic frame.

3. The Reconstruction Algorithm

Under Assumption A, let us denote by $\{\widetilde{\psi_{j_1,\ldots,j_d}}\,,\,1\leq j_1\leq\cdots\leq j_d\leq m\}$ the canonical dual frame to $P\Psi$. This dual frame allows us to recover $\Phi(x)$. Recall $\{e_1,\ldots,e_n\}$ is an orthonormal basis of $\mathbf{E}^n$. Notice the following relations:

$$\Phi(x)(e_k,\ldots,e_k) = |\langle x,e_k\rangle|^{2d} \quad (11)$$

$$\sum_{k=1}^{n}(\Phi(x)(e_k,\ldots,e_k))^{1/d} = \|x\|^2 \quad (12)$$

$$\Phi(x)(\underbrace{e_j,\ldots,e_j}_{2d-1},e_k) = |\langle x,e_j\rangle|^{2d-2}\langle e_j,x\rangle\langle x,e_k\rangle \quad (13)$$

From (11) and (13) we obtain:

$$\langle x,e_k\rangle = \frac{\langle x,e_j\rangle}{|\langle x,e_j\rangle|}\frac{\Phi(x)(e_j,\ldots,e_j,e_k)}{(\Phi(x)(e_j,\ldots,e_j,e_j))^{(2d-1)/2d}} \quad (14)$$

The Reconstruction Algorithm is as follows.

Reconstruction Algorithm

Input: Coefficients $c_1=|\langle x,f_1\rangle|$, ... $c_m=|\langle x,f_m\rangle|$.

Step 0. If $\sum_{k=1}^m c_k^2=0$ then $y=0$ and stop. Otherwise continue.

Step 1. Construct the following sesquilinear functional

$$\alpha = \sum_{1\leq j_1\leq\cdots\leq j_d\leq m} c_{j_1}^2\cdots c_{j_d}^2\widetilde{\psi_{j_1,\ldots,j_d}} \quad (15)$$

Step 2. Find a $1\leq j_0\leq n$ so that $\alpha(e_{j_0},\cdots,e_{j_0})>0$. This is possible due to (12). Set

$$\nu = \sqrt[2d]{\alpha(e_{j_0},\ldots,e_{j_0})} \quad (16)$$

Step 3. Set

$$y = \frac{1}{\nu^{2d-1}}\sum_{k=1}^{n}\alpha(\underbrace{e_{j_0},\ldots,e_{j_0}}_{2d-1},e_k)e_k \quad (17)$$

Summarizing all results obtained so far we obtain:

Theorem 3..1 *For every $x\in\mathbf{E}^n$ there is $z\in\mathbf{C}$ so that $|z|=1$ and the output of the Reconstruction Algorithm satisfies $x=zy$. Specifically $z=\frac{\langle x,e_{j_0}\rangle}{|\langle x,e_{j_0}\rangle|}$, with j_0 obtained in Step 2.*

4. Redundancy Constraint

In this section we analyse the necessary condition $|\Psi|\geq dim(E_d)$.

4.1 The Cardinal of Set Ψ

The set Ψ given in (10) has the same cardinal as

$$\{(k_1,\ldots,k_d)\,,\,1\leq k_1\leq\cdots\leq k_d\leq m\} \quad (18)$$

Let us denote this number by $M_{m,d}$. In order to compute it, consider the following cardinal equivalent set:

$$\{(n_1,\ldots,n_m)\,,\,0\leq n_1,\ldots,n_m\leq d, n_1+\cdots+n_m=d\} \quad (19)$$

The bijective correspondence between d-tuples of (18) and m-tuples of (19) is given by the following interpretation: n_l is the number of times l is presented in the d-tuple $(k_1,\ldots,k_d)$. Then, one can obtain the following recursion:

$$M_{m+1,d} = \sum_{r=0}^{d} M_{m,d}$$

where we set $M_{m,0}=1$. Since $M_{1,d}=1$, one obtains by induction that:

$$M_{m,d} = \left(\begin{array}{c} m+d-1 \\ m-1 \end{array} \right) = \frac{m(m+1)\cdots(m+d-1)}{d!} \quad (20)$$

4.2 The Dimension of E_d

Recall E_d is the linear span of vectorx $\Phi(x)$ in $\Lambda_{d,d}(\mathbf{E}^n)$. Recall also that Δ whose n^{2d} vectors are defined in (5) is an orthonormal basis in $\Lambda_{d,d}(\mathbf{E}^n)$. Let us denote by $N_{n,d}$ the dimension of E_d. We will describe an orthonormal basis in E_d. Fix $t_1,\ldots,t_n\in\mathbf{C}$ and expand:

$$\Phi(t_1e_1+\cdots t_ne_n) = \sum_{1\le k_1,\ldots,k_{2d}\le n} t_{k_1}\cdots t_{k_d}\overline{t_{k_{d+1}}}\cdots\overline{t_{k_{2d}}}\cdot \cdot\delta_{k_1,\ldots,k_{2d}} \quad (21)$$

We shall group together terms containing same t_k terms. The real case will be treated separately from the complex case.

To simplify the exposition, we introduce notation common to both cases. Let us denote by $\underline{k}=(k_1,\ldots,k_r)$ an ordered r-tuple of integers each from 1 to n, where the length r is equal to $2d$ (in the real case), or d (in the complex case). Let us denote by $\mathcal{P}_r$ the set of r-permutations, and by $\mathcal{P}_{\underline{k}}$ the quotient set $\mathcal{P}_{\underline{k}}=\mathcal{P}/\sim_{\underline{k}}$ where $\pi',\pi''\in\mathcal{P}_r$ are equivalent $\pi'\sim_{\underline{k}}\pi''$ if and only if $\pi'(\underline{k})=\pi''(\underline{k})$. Note

$$|\mathcal{P}_{\underline{k}}| = \frac{r!}{m_1!\cdots m_n!}$$

where m_l denotes the number of repetitions of l in $\underline{k}$.

The Complex Case

In the complex case, t_k and $\overline{t_k}$ can be treated as indepedent (real) variables. Then terms in (21) are grouped using two independent d-tuples, $\underline{j}=(j_1,\ldots,j_d)$ and $\underline{l}=(l_1,\ldots,l_d)$ as follows

$$\sum_{1\le j_1\le\cdots\le j_d\le n}\ \sum_{1\le l_1\le\cdots\le l_d\le n} t_{j_1}\cdots t_{j_d}\overline{t_{l_1}}\cdots\overline{t_{l_d}}\times \times\sum_{\pi\in\mathcal{P}_{\underline{j}}}\sum_{\rho\in\mathcal{P}_{\underline{l}}}\delta_{\pi(j_1),\ldots,\pi(j_d),\rho(l_1),\ldots,\rho(l_d)}$$

Then the following sesquilinear functionals are orthonormal and form a basis in E_d:

$$d_{\underline{j},\underline{l}} = \frac{1}{\sqrt{|\mathcal{P}_{\underline{j}}|}\sqrt{|\mathcal{P}_{\underline{l}}|}}\sum_{\pi\in\mathcal{P}_{\underline{j}}}\sum_{\rho\in\mathcal{P}_{\underline{l}}}\delta_{\pi(j_1),\ldots,\pi(j_d),\rho(l_1),\ldots,\rho(l_d)} \quad (22)$$

Their number (and hence dimension of E_d) is equal to the number of ordered d-tuples $\underline{j}$ times the number of ordered d-tuples $\underline{l}$:

$$N_{n,d} = (M_{n,d})^2 = \left(\frac{n(n+1)\cdots(n+d-1)}{d!}\right)^2 \quad (23)$$

where we used (20). Note $N_{n,1}=n^2$ and we recover the complex case considered in [BBCE09].

The Real Case

In the real case, t_k and $\overline{t_k}$ are the same variables. Then the independent terms in (21) are indexed by $2d$-tuples $\underline{k}=(k_1,\ldots,k_{2d})$ as follows:

$$\sum_{1\le k_1\le k_{2d}\le n} t_{k_1}\cdots t_{k_{2d}}\sum_{\pi\in\mathcal{P}_{\underline{k}}}\delta_{\pi(k_1),\ldots,\pi(k_{2d})}$$

and an orthonormal basis of E_d is given by the following vectors indexed by ordered $2d$-tuples $\underline{k}$:

$$d_{\underline{k}} = \frac{1}{\sqrt{|\mathcal{P}_{\underline{k}}|}}\sum_{\pi\in\mathcal{P}_{\underline{k}}}\delta_{\pi(k_1),\ldots,\pi(k_{2d})} \quad (24)$$

The dimension of E_d in real case is then:

$$N_{n,d} = M_{n,2d} = \frac{n(n+1)\cdots(n+2d-1)}{(2d)!} \quad (25)$$

Note $N_{n,1}=\frac{n(n+1)}{2}$ and this recovers the real case in [BBCE09].

4.3 The Optimal Depth and Redundancy Condition

For given n we would like to find the minimum $m=m^*$ so that $M_{m,d}\ge N_{n,d}$ for some $d\ge 1$.

The Complex Case

We need to solve

$$\frac{m(m+1)\cdots(m+d-1)}{d!} \ge \left(\frac{n(n+1)\cdots(n+d-1)}{d!}\right)^2$$

or, completing the factorials:

$$(m+d-1)!\,d!\,((n-1)!)^2 \ge (m-1)!\,((n+d-1)!)^2$$

Let us denote

$$R(n,m,d) = \frac{(m+d-1)!d!((n-1)!)^2}{(m-1)!((n+d-1)!)^2} \quad (26)$$

Ideally we would like to solve:

(1) $\quad d^*(n,m) = argmax_d R(n,m,d)$

(2) $\quad m^*(n) = min_{R(n,m,d^*(n,m))\ge 1}\, m$

Instead we make the following choices for $d=d(n)$ and $m=m(n)$, and then optimize using Stirling's formula:

$$d = n-1 \quad (27)$$
$$m = A(n-1)+1. \quad (28)$$

Using Stirling's formula $n!=\sqrt{2\pi n}n^ne^{-n}$ we obtain for $R(n+1,An+1,n)$,

$$R(n+1,An+1,n) = \sqrt{\frac{8\pi(A+1)n}{A}}\left[\frac{A+1}{16}(1+\frac{1}{A})^A\right]^n$$

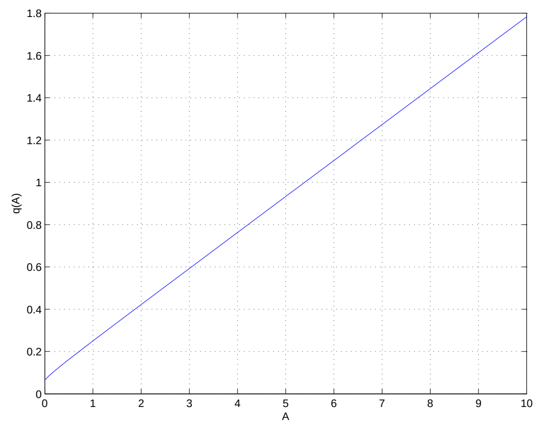

Figure 1: The plot of $q = q(A)$ from (29).

To obtain $R \geq 1$ for large n, we need

$$q(A) = \frac{A+1}{16}(1+\frac{1}{A})^A \geq 1 \qquad (29)$$

In Figure 1 we plot the function $q = q(A)$. Numerically we obtain $A = 5.394$. The remaining factor in $R(n+1, An+1, n)$ becomes $5.376\sqrt{n} \geq 1$ for all n. Thus we obtain as sufficient conditions:

$$d = n - 1 \qquad (30)$$

$$m = 5.394n - 4.394 \qquad (31)$$

The Real Case

In the real case we need to solve

$$\frac{m(m+1)\cdots(m+d-1)}{d!} \geq \frac{n(n+1)\cdots(n+2d-1)}{(2d)!}$$

Following the same approach we obtain the following ratio function that we need to make supraunital:

$$R(n,m,d) = \frac{(m+d-1)!(n-1)!(2d)!}{(m-1)!(n+2d-1)!d!} \qquad (32)$$

It follows:

$$R(n+1, 2n+1, n) = 1$$

Hence a possible choice is

$$d = n - 1 \qquad (33)$$

$$m = 2n - 1 \qquad (34)$$

It is interesting to note that in the real case we recover the critical case $m \geq 2n - 1$.

5. Numerical Evidence Supporting Genericity of the Assumption A.

While the previous section computed necessary conditions for Assumption A to hold true, we still need to prove (or check) that $P\Psi$ is frame in E_d. In this section we plot the distribution of eigenvalues of the frame operator associated to $P\Psi$ for one randomly generated example.

Using (22), each vector $P\psi_{\underline{k}}$ is represented by a $N_{n,d}$-vector whose components are indexed by a pair $(\underline{j}, \underline{l})$, $F_{(\underline{j},\underline{l}),\underline{k}} = \langle \psi_{\underline{k}}, d_{\underline{j},\underline{l}} \rangle$. Explicitely this becomes

$$F_{(\underline{j},\underline{l}),\underline{k}} = \frac{1}{\sqrt{|\mathcal{P}_{\underline{j}}|}\sqrt{|\mathcal{P}_{\underline{l}}|}} \sum_{\pi \in \mathcal{P}_{\underline{j}}} \sum_{\rho \in \mathcal{P}_l u} \langle e_{\pi(j_1)}, f_{k_1} \rangle \cdots \cdot \langle e_{\pi(j_d)}, f_{k_d} \rangle \langle f_{k_1}, e_{\rho(l_1)} \rangle \cdots \langle f_{k_d}, e_{\rho(l_d)} \rangle \qquad (35)$$

Thus $P\Psi$ is frame for E_d if and only if the $N_{n,d} \times M_{m,d}$ matrix F is of full rank. The frame operator is given by $S = FF^*$.

We considered the complex case ($\mathbf{E} = \mathbf{C}$) with the following parameters $n = 5$ and $d = 3$. For $m = 21$ the ratio function (26) takes the value $R(5, 21, 3) = 1.4457 > 1$. Note for the algorithm in [BBCE09] to work m has to be greater than or equal to n^2, that is $m \geq 25$. For a frame with 21 vectors in dimension 5 whose vectors are obtained as realizations of complex valued normal random variables of zero mean and variance 2 (each real and imaginary part is i.i.d. $\mathcal{N}(0,1)$), the distribution of eigenvalues of its frame operator is plotted in Figure 2. Note the conditioning number is $cond(S) = 6267.7$. While relatively large, the important thing to note is that the realization $P\Psi$ is frame (spanning) for E_d. While this result is by no means a proof, or even an exhaustive experiment, it suggests the Assumption A might be generically true whenever $R(n,m,d) > 1$.

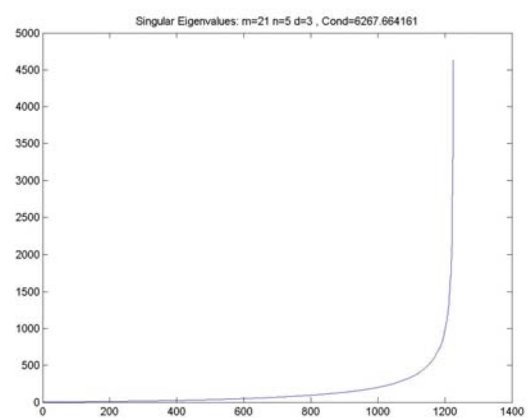

Figure 2: Distribution of eigenvalues for a random frame..

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