# Topological Obstructions to Localization Results*

Radu V. Balan

Siemens Corporate Research, 755 College Road East, Princeton NJ 08540

## ABSTRACT

In this paper we present topology aspects in non-localization results. A well-known such no-go result is the Balian-Low theorem that states the generator of a Weyl-Heisenberg Riesz basis cannot be well time-frequency localized. More general, the statement applies to multi WH Riesz bases, or super frames as well. These results turn out to be connected to non-triviality of a complex vector bundle. Another class of problem is related to optimality of coherent approximations of stochastic signals. More specific, for a given deficit ($\alpha\beta > 1$), find the best Riesz sequence generator optimal to respect to the mean square approximation error. A topological obstruction turns out to be responsable for ill-localization of the optimal generator.

**Keywords:** Gabor analysis, Non-localization results, non-trivial vector bundles

## 1. INTRODUCTION

For $\alpha, \beta > 0$ and $g \in L^2(\mathbf{R})$, we denote by $(g; \alpha, \beta)$ the *Weyl-Heisenberg set* (or, *WH set*) defined by:

$$(g; \alpha, \beta) = \{g_{mn;\alpha,\beta} \ ; \ m, n \in \mathbf{Z}\} \quad , \quad g_{mn;\alpha,\beta}(x) = e^{2\pi i m\alpha x} g(x - n\beta) \tag{1}$$

We shall also use $g_{mn}$ to denote $g_{mn;\alpha,\beta}$ when there is no danger of confusion. The WH set $(g; \alpha, \beta)$ is called *frame* with bounds $A, B > 0$ if for every $f \in L^2(\mathbf{R})$,

$$A||f||^2 \leq \sum_{m,n \in \mathbf{Z}} |\langle f, g_{mn}\rangle|^2 \leq B||f||^2 \tag{2}$$

The WH set $(g; \alpha, be)$ is called *Riesz sequence*, or *Riesz basis for its span* (shorthanded by *s-Riesz basis*) with bounds $A, B > 0$ if for every finite secuence $c = (c_{mn})$ (and by boundedness, for every square summable $c \in l^2(\mathbf{Z}^2)$),

$$A||c||^2 \leq || \sum_{mn} c_{mn} g_{mn} ||^2 \leq B||c||^2 \tag{3}$$

When $(g; \alpha, \beta)$ is simultaneously frame and Riesz sequence, it is called *Riesz basis*. A remarcable result due to R.Balian and F.Chow (see,[1 2]) states that if $(g; \alpha, \beta)$ is an orthonormal basis for $L^2(\mathbf{R})$, then

$$\int x^2 |g(x)|^2 \, dx \int \xi^2 |\hat{g}(\xi)|^2 \, d\xi = \infty \tag{4}$$

where $\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} g(x) dx$ is the Fourier transform of $g$. Equation (4) roughly says the window (or generator) $g$ cannot be well localized in time-frequency domain. The original proof contained a slight gap, which was later filled in by Coiffman and Semmes (cf.[3]). An independent and interesting proof was later given by G.Battle in[4] who also extended the result to the more general case of Riesz bases. We also have to mention[5] that presented a complex vector bundle argument for a no-go theorem. The case of multi WH sets has been considered in[8] where the authors showed that at least one generator has to suffer of the time-frequency ill localization property. In[9] we have obtained a similar statement related to WH super frames. The initial Balian's proof shows, instead, another non-localization form. This, so called *amalgam BL theorem* (see[6]) says that, if $(g; \alpha, \beta)$ is a Riesz basis generator, then neither $g$ nor $\hat{g}$ cannot belong to the following amalgam space (called *Wiener algebra*):

$$W(C, l^1) = \{f : \mathbf{R} \to \mathbf{C} \ ; \ \ f \ continuous \ and \ ||f||_{W(C,l^1)} = \sum_{n \in \mathbf{Z}} \sup_{x \in [n,n+1]} |f(x)| < \infty \ \} \tag{5}$$

---

*Presented at SPIE Conference in San Diego, July 29 - August 3, 2001

E-mail address: radu.balan@scr.siemens.com

The proof shows, as an intermediary result, the Zak transform of $g$ cannot be continuous. The *Zak transform* is defined by:

$$G(t,s) = \sqrt{\beta} \sum_{k \in \mathbf{Z}} e^{2\pi i k t} g(\beta(s+k)) \tag{6}$$

and is a unitary operator from $L^2(\mathbf{R})$ to $L^2([0,1] \times [0,1])$ (for more details see[7]).

Separately and coming from a completely different domain, in[10] we have obtained a non-localization result for the generator of an optimal WH Riesz sequence approximating a continuous-time stochastic signal. This result suggests the following ansatz: generically, optimal WH sets are associated to ill-localized generators. Again, the proof shows the Zak transform cannot be continuous. Recently, in,[11] we have started exploring a broader class of optimality problems related to signal approximation and encoding schemes.

The purpose of this paper is to explore the common link between these two classes of results from a topology theory point of view. The organization of the paper is as follows: in section 2 we revisit the multi and super WH set concepts mentioning the duality between them; next, in section 3 we present the continuous-time stochastic signal approximation problem and its optimal solution; section 4 contains the complex vector bundle construction and some of its properties; in particular it is shown to be non-trivial, and, as a consequence, in section 5 the nonlocalization results for multi and super WH sets are obtained; conclusions are presented in 6 and are followed by the bibliograph.

The anlysis can be easily carried out in higher dimension spaces, but for the sake of clarity of notation we present only the one-dimensional theory.

## 2. MULTI AND SUPER WEYL-HEISENBERG SETS

Consider a collection of $p$ WH sets $(g^1;\alpha,\beta)$,...,$(g^p;\alpha,\beta)$. More generally we may consider WH sets with different time-frequency shift parameters, but for the purposes of this paper the equal-parameter case is sufficient. The *multi WH set* denoted by $(g^1,\ldots,g^p;\alpha,\beta)_m$ is simply the union of such sets:

$$(g^1,\ldots,g^p;\alpha,\beta)_m = \{g^1_{m_1 n_1;\alpha,\beta},\ldots,g^p_{m_p n_p;\alpha,\beta}\ ;\ m_1,n_1,\ldots,m_p,n_p \in \mathbf{Z}\ \} \tag{7}$$

The WH multisets may be a frame, Riesz sequence or Riesz basis for $L^2(\mathbf{R})$, whenever (2) respectively (3), or both, hold for the union set defined in (7).

Another concept that can be used with the collection of $p$ WH sets above is the *super WH set*. A *super WH set* is denoted by $(g^1,\ldots,g^p;\alpha,\beta)_s$ and represents the collection of $p$ (ordered) WH sets. With a fixed ordering relation, an auxiliary set is constructed, namely the pointwise direct sum of vectors:

$$\mathcal{G} = \{g^1_{mn} \oplus \cdots \oplus g^p_{mn}\ ;\ m,n \in \mathbf{Z}\} \tag{8}$$

that "lives" in the $p$-direct sum of Hilbert spaces $L^2(\mathbf{R})$,

$$L^{2,p}(\mathbf{R}) = L^2(\mathbf{R}) \oplus \cdots \oplus L^2(\mathbf{R}) \tag{9}$$

Accordingly, the WH super set $(g^1,\ldots,g^p;\alpha,\beta)_s$ is said to be a *super WH frame*, *Riesz sequence*, or *Riesz basis*, if the frame, Riesz sequence or Riesz basis conditions hold for $\mathcal{G}$ in $L^{2,p}(\mathbf{R})$.

A number of results are known about both sets. Among these we mention density and localization results that parallel the corresponding single generator case. For density of multisets see,[8],[12] whereas a non-localization result appeared in.[8] For superset case see[9] for non-localization results, and.[13] See also[14] and[15] for a separate and independent development of superset theory (called *disjoint sets* therein).

For the framework used in this paper (namely, uniform lattice and equal parameter case), the density and localization results for multi and super WH sets turn out to be equivalent due to the following duality result (see[16]):

**Theorem** 2.1 (Duality of supercoherent WH multi and super sets). 1. The multi WH set $(g^1,g^2,\ldots,g^p;\alpha,\beta)_m$ is a multi Riesz sequence if and only if the super WH set $(g^1,g^2,\ldots,g^p;\frac{1}{\beta},\frac{1}{\alpha})_s$ is a super frame.

2. The multi WH set $(g^1,g^2,\ldots,g^p;\alpha,\beta)_m$ is a multi Riesz basis if and only if the super WH set $(g^1,g^2,\ldots,g^p;\frac{1}{\beta},\frac{1}{\alpha})_s$ is a super Riesz basis.

This result can be proved either by using von Neumann algebra representation techniques, by using the GNS construction or by using the Wexler-Raz identity (see[17]). Using this theorem, it is clear the super set case reduces to the WH multiset situation. Because of this, in the following section I consider only the latter case (i.e. the multi set case). The density results mentioned before imply $\alpha\beta = p$. Denote by $\Gamma$ the following $\mathbf{C}^{p\times p}$-valued function over the rectangle $[0,1)\times[0,\frac{1}{p})$:

$$\Gamma = \left[ \begin{array}{cccc} G^1(t,s) & G^2(t,s) & \cdots & G^p(t,s) \\ G^1(t,s+\frac{1}{p}) & G^2(t,s+\frac{1}{p}) & \cdots & G^p(t,s+\frac{1}{p}) \\ \vdots & \vdots & & \vdots \\ G^1(t,s+\frac{p-1}{p}) & G^2(t,s+\frac{p-1}{p}) & \cdots & G^p(t,s+\frac{p-1}{p}) \end{array} \right] \tag{10}$$

Then, the Riesz basis condition and bounds are given in the following theorem:

**Theorem** 2.2. $(g^1,g^2,\ldots,g^p;\alpha,\beta)_m$ is a multi WH Riesz basis with bounds $A,B$ if and only if:

$$pA \le \lambda_{min}(\Gamma^*\Gamma) \le \lambda_{max} \le pB \quad , \quad a.e.\,(t,s) \tag{11}$$

where $\lambda_{min}(\cdot)$, $\lambda_{max}(\cdot)$ are the minimum, respectively maximum of the $p$ real eigenvalues. This result has as a corollary the following equivalent condition:

**Corollary** 2.3. Suppose $(g^1,g^2,\ldots,g^p;\alpha,\beta)_m$ is a multi WH Bessel sequence (i.e. the upper bound is satisfied). Then it is a multi Riesz basis as well if and only if:

$$|det(\Gamma)| \ge A_0 > 0 \quad , \quad a.e.\,(t,s) \tag{12}$$

for some $A_0 > 0$.

As mentioned before, the extension of the Balian-Low theorem to the multiset case has been done by M.Zibulski and Y.Y.Zeevi in[8] where they obtained the following result (here I slightly extend their statement to cover both the amalgam BL theorem and superset case):

**Theorem** 2.4. Assume $(g^1,g^2,\ldots,g^p;\alpha,\beta)_m$, or $(g^1,g^2,\ldots,g^p;\alpha,\beta)_s$ is a multi, respectively super WH Riesz basis for $L^2(\mathbf{R})$. Then at least one generator is ill-localized in the following sense: Assume $g^k$ is the generator in question.

1. $g^k$ cannot belong to the following space:

$$H^{1,1} = \{f \in L^2(\mathbf{R}) \mid \|f\|^2_{H^{1,1}} := \int (1+x^2)|f(x)|^2\,dx + \int \xi^2|\hat{f}(\xi)|^2\,dx < \infty\} \tag{13}$$

2. Neither $g^k$, nor $\hat{g^k}$ can belong to the Wiener algebra $W(C,l^1)$ defined by (5).

This theorem follows by a standard technique from the following lemma:

**Lemma** 2.5. Assume $(g^1,g^2,\ldots,g^p;\alpha,\beta)_m$ is a multi WH Riesz basis. Then for at least one $1 \le k \le p$, $G^k$, the Zak transform of $g^k$, is discontinuous.

We analyze in Section 5 the topological reason of this result and its consequences.

## 3. OPTIMAL CONTINUOUS-TIME STOCHASTIC SIGNAL APPROXIMATIONS

Assume $f$ is a stationary continuous-time stochastic signal of zero average and known autocovariance function $R(t)$. Thus:

$$\begin{aligned} \mathbf{E}[f(t_1)\overline{f(t_2)}] &= R(t_1-t_2) \quad (14) \\ \mathbf{E}[f(t)] &= 0 \quad (15) \end{aligned}$$

We want to approximate $f$ by a coherent expansion of the form $S_{g^1,g^2;\alpha,\beta}f = \sum_{mn}\langle f, g^1_{mn}\rangle g^2_{mn}$. To distinguish among different approximation solutions, we consider a measure of the approximation error. Obviously this question is trivial when $(g^1;\alpha,\beta)$ is a frame and $(g^2;\alpha,\beta)$ a dual. In general we are interested in the case when both $(g^1;\alpha,\beta)$ and $(g^2;\alpha,\beta)$ are incomplete sets, such as s-Riesz bases. Consider now a nonnegative bounded summable weight

$w \geq 0$, $w \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$. Typical such weights are characteristic functions of intervals. Then the weighted $L^2(\mathbf{R})$ norm of the approximation error measures how well $S_{g^1,g^2;\alpha,\beta} f$ approximates $f$ and its expectation is a measure of the *stochastic approximation of the continuous-time signal* $f$ *by the WH pair* $(g^1, g^2; \alpha, \beta)$:

$$J_{ca}(g^1, g^2; \alpha, \beta, w, R) = \int_{-\infty}^{\infty} \mathbf{E}[|f(x) - S_{g^1,g^2;\alpha,\beta} f(x)|^2] w(x)\, dx \tag{16}$$

The problem is to find the best WH pairs of s-Riesz bases $(g^1, g^2; \alpha, \beta)$ that minimizes (16), i.e.

$$\inf_{(g^1,g^2;\alpha,\beta)\ pair\ of\ s-Riesz\ bases} J_{ca}(g^1, g^2; \alpha, \beta, w, R) \tag{17}$$

In[18] and[10] we have analyzed this problem in details and obtained its optimal solution for the rational case, namely when $\alpha\beta = \frac{p}{q}$, for $(p, q)$ relatively prime. The interested reader can find in the aformentioned papers (and in[11]) computational details. Here we give only the end result. First we need to introduce several objects.

First, for a generic window $g$, and its Zak transform $G$, we denote by:

$$\Gamma(t,s) = \begin{bmatrix} G(t,s) & G(t+\frac{1}{q}, s) & \cdots & G(t+\frac{q-1}{q}, s) \\ G(t, s+\frac{q}{p}) & G(t+\frac{1}{q}, s+\frac{q}{p}) & \cdots & G(t+\frac{q-1}{q}, s+\frac{q}{p}) \\ \vdots & \vdots & & \vdots \\ G(t, s+(p-1)\frac{q}{p}) & G(t+\frac{1}{q}, s+(p-1)\frac{q}{p}) & \cdots & G(t+\frac{q-1}{q}, s+(p-1)\frac{q}{p}) \end{bmatrix} \tag{18}$$

and obtain accordingly $\Gamma^1$ and $\Gamma^2$. Let $E(t) \in \mathbf{C}^{p\times p}$, $Q, D \in \mathbf{C}^{q\times q}$ be defined by:

$$E(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ e^{-2\pi i q t} & 0 & 0 & \cdots & 0 \end{bmatrix} \ , \ Q = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{19}$$

$$D = \begin{bmatrix} 1 & & & & \\ & e^{-2\pi i \frac{1}{q}} & & & \\ & & e^{-2\pi i \frac{2}{q}} & & \\ & & & \ddots & \\ & & & & e^{-2\pi i \frac{q-1}{q}} \end{bmatrix} \tag{20}$$

Then the standard quasiperiodicity relations of Zak transform turn into:

$$\Gamma(t+\frac{1}{q}, s) = \Gamma(t,s)\cdot Q \quad , \quad \Gamma(t, s+\frac{1}{p}) = e^{-2\pi i n_0 t} E(t)^{r_0} \cdot \Gamma(t,s) \cdot D^{n_0} \tag{21}$$

where $(r_0, n_0)$ are coprime factors of $(q, p)$, i.e. $r_0 q + n_0 p = 1$.

Let us denote by

$$\omega(s) = \sum_k w(\beta(s+k)) \tag{22}$$

and by $W(s)$ the following 1-periodic diagonal matrix valued function:

$$W(s) = \begin{bmatrix} \omega(s) & 0 & \cdots & 0 \\ 0 & \omega(s+\frac{q}{p}) & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \omega(s+(p-1)\frac{q}{p}) \end{bmatrix} \tag{23}$$

Finally, denote by:

$$\rho_r(t) = \sum_m e^{2\pi imqt} R(\frac{mp+r}{\alpha}) \tag{24}$$

and by $M(t)$ the following $\frac{1}{q}$-periodic Toeplitz self-adjoint matrix:

$$M(t) = \begin{bmatrix} \rho_0(t) & \rho_{-1}(t) & \cdots & \rho_{-(p-1)}(t) \\ \rho_1(t) & \rho_0(t) & \ddots & \rho_{-(p-2)}(t) \\ \cdots & \ddots & \ddots & \vdots \\ \rho_{p-1}(t) & \rho_{p-2}(t) & \cdots & \rho_0(t) \end{bmatrix} \tag{25}$$

Note also the following properties:

$$\begin{aligned} W(s+\frac{1}{p}) &= E(t)^{r_0} W(s) E(t)^{*r_0} && (26) \\ M(t) &= E(t) M(t) E(t)^* && (27) \end{aligned}$$

With these notations, and $\Lambda = [0, \frac{1}{q}] \times [0, \frac{1}{p}]$, the criterion $J_{ca}$ becomes:

$$J_{ca} = \beta q \int\int_\Lambda trace\{W(s)(I - \frac{1}{p}\Gamma^2(t,s)\Gamma^{1*}(t,s))M(t)(I - \frac{1}{p}\Gamma^1(t,s)\Gamma^{2*}(t,s))\} \tag{28}$$

and the optimal solution $\Gamma^1_o$, $\Gamma^2_o$ is parametrized by:

$$\begin{aligned} \Gamma^1(t,s) &= W(s)^{1/2} V(t,s) L^*(t,s) && (29) \\ \Gamma^2(t,s) &= W(s)^{-1/2} V(t,s) L^{-1}(t,s) && (30) \end{aligned}$$

where $L(t,s)$ is an arbitrary $q \times q$ invertible matrix so that $sup_{(t,s)}\|L(t,s)\| < \infty$, $sup_{(t,s)}\|L^{-1}(t,s)\| < \infty$, and $V(t,s)$ is a $p \times q$ matrix whose columns are eigenvectors of

$$R(t,s) = W(s)^{1/2} M(t) W(s)^{1/2}, \tag{31}$$

corresponding to the largest $q$ eigenvalues, so that $V^*(t,s)V(t,s) = I_q$. Examples of such functions are given in,[10] and.[18] It turns out these exemples exhibit a specific discontinuity property that has been proved in[10] for a special class of stochastic signals. In section 5 we generalize non-localization results obtained in.[10]

## 4. THE COMPLEX VECTOR BUNDLE

The new element of this paper is a new vector bundle and associated exterior forms algebra that is going to play an essential role in the following derivations.

Let $\bar{\xi} = (\mathbf{C}^p \times \mathbf{R}^2, \bar{\pi}, \mathbf{R}^2)$ be the trivial complex vector bundle over the 2-dimensional (Euclidian) plane $\mathbf{R}^2$ whose fiber is $\mathbf{C}^p$, and $\bar{\pi} : \mathbf{C}^p \times \mathbf{R}^2 \to \mathbf{R}^2$, the canonical projection $(v,t,s) \mapsto \pi(v,t,s) = (t,s)$. This vector bundle is *trivial* in the sense that the base manifold, $\mathbf{R}^2$, admits a global parametrization and coordinate system, which is simply $\mathbf{R} \times \mathbf{R} \to \mathbf{R}^2$, so that globally, the total manifold $\mathbf{C}^p \times \mathbf{R}^2$ is diffeomorphic to $\mathbf{C}^p \times \mathbf{R} \times \mathbf{R}$. On this vector bundle, consider the following equivalence relation:

$$\begin{aligned} (v, t+1, s) &\sim (v,t,s) && (32) \\ (v,t,s+\frac{1}{p}) &\sim (e^{-2\pi i n_0 t} E(t)^{r_0} v, t, s) && (33) \end{aligned}$$

Now we denote by $\xi = \bar{\xi}/\sim$ the quotient bundle obtained by identifying the equivalent pairs. The base manifold becomes $[0,1) \times [0, \frac{1}{p}) \sim \mathbf{T}^2$ instead of $\mathbf{R}^2$, and the total manifold is denoted by $C_p$. The new projection map is denoted $\pi : C_p \to \mathbf{T}^2$. Thus $\xi = (C_p, \pi, \mathbf{T}^2)$. Consider $i : \bar{\xi} \to \xi$ which associates to every $(v,t,s) \in \bar{\xi}$, the corresponding equivalence class via (32,33) whose representant is in $\mathbf{C}^p \times [0,1) \times [0, \frac{1}{p})$. Any map $\psi : C_p \to V$ (some space $V$), extends naturally to a map $\bar{\psi} : \mathbf{C}^p \times \mathbf{R}^2 \to V$ through $\bar{\psi} = \psi \circ i$, that is a $\sim$-invariant map over $\mathbf{C}^p \times \mathbf{R}^2$, that is $\bar{\psi}(v,t+1,s) = \bar{\psi}(v,t,s)$ and $\bar{\psi}(v,t,s+\frac{1}{p}) = \bar{\psi}(E(t)^{r_0}v,t,s)$. Denote by $\mathcal{F}(\mathbf{T}^2;\mathbf{C})$ the ring of complex valued continuous functions over the 2-torus $\mathbf{T}^2$.

Next consider the $\mathcal{F}(\mathbf{T}^2;\mathbf{C})$-module $\mathrm{X}(\xi)$ of *sections* of $\xi$. A *section* $v$ is defined as a continuous map $v:\mathbf{T}^2\to C_p$ so that $\pi\circ v = 1_{T^2}$ is the identity of $T^2$. We define also the more general $L^a$*-sections*, where we require only that the components of $v$ are in $L^a(\mathbf{T}^2;\mathbf{C})$ and $\pi\circ v = 1_{\mathbf{T}^2}$. We denote by $\mathrm{X}_a(\xi)$ the $L^\infty(\mathbf{T}^2;\mathbf{C})$-module of $L^a$-sections. The rational behind this vector bundle construction is that each column of $\Gamma$ in (10) defines a $L^2$-section of $\xi$ for $n_0=0, r_0=1$, whereas if $g\in W(C,l^1)$, then the sections are continuous, and thus in $\mathrm{X}(\xi)$. For the purposes of this paper, we are going to consider only those WH sets whose Zak transforms are continuous. Hence we deal with $\mathrm{X}(\xi)$ only. Thus, general properties of this vector bundle yield several results for WH multi and super sets.

Consider now the algebra of exterior forms associated to $\xi$. This is constructed as follows. Each fiber is isomorphic to $\mathbf{C}^p$. Let $\Lambda^k(\mathbf{C}^p)$ be the complex vector space of $k$-forms over $\mathbf{C}^p$ and $\Lambda(\mathbf{C}^p) = \oplus_{k=0}^p\Lambda^k(\mathbf{C}^p)$ the exterior algebra over $\mathbf{C}^p$. Then the exterior algebra of $\bar{\xi}$, $\Lambda(\bar{\xi})$, is given by the $\mathcal{F}(\mathbf{T}^2;\mathbf{C})$-module of continuous sections of the trivial bundle $(\Lambda(\mathbf{C}^p),\bar{\pi},\mathbf{R}^2)$. The equivalence relations (32,33) induce on exterior forms of $\bar{\xi}$ the following relations:

$$(\omega,t+1,s) \sim (\omega,t,s) \ , \tag{34}$$

$$(\omega,t,s+\frac{1}{p}) \sim (\phi,t,s) \ , \ \phi(v_1,\cdots,v_k) = \omega(e^{2\pi i n_0 t}E(t)^{*r_0}v_1,\cdots e^{2\pi i n_0 t}E(t)^{*r_0}v_k,) \tag{35}$$

for all *omega* $\in\Lambda^k(\mathbf{C}^p)$. We denote by $\Lambda^k(\xi)$ the $\mathcal{F}(\mathbf{T}^2;\mathbf{C})$-module thus obtained, and $\Lambda(\xi) = \oplus_{k=0}^p\Lambda^k(\xi)$, where, by convention, $\Lambda^0(\xi) := \mathcal{F}(\mathbf{T}^2;\mathbf{C})$. Note that any exterior differential form in $\Lambda(\xi)$ can be lifted to $\Lambda(\bar{\xi})$ and extended to the whole 2-plane $\mathbf{R}^2$, via (34,35).

To a given WH set $(g;\alpha,\beta)$ with $\alpha\beta=p$, we associate the following exterior 1-form over $\bar{\xi}$ and then, since satisfies (34,35), also a 1-form over $\xi$:

$$\omega|_{(t,s)} = \overline{G(t,s)}dz_1 + \overline{G(t,s+\frac{1}{p})}dz_2 + \ldots + \overline{G(t,s+\frac{p-1}{p})}dz_p \tag{36}$$

(where the bar denotes the complex conjugation). One can easily check that $\omega$ satisfies (34,35) for $k=1$, $n_0=0$ and $r_0=1$, and thus is corresponds as well to a 1-form in $\Lambda^1(\xi)$. The usefulness of this construction lays in the following result:

**Theorem** 4.1. Assume $\alpha\beta=p$ and $g^l,h^l$ are generators whose Zak transforms are continuous (for instance in $W(C,l^1)$), $1\le l\le q$.

1. Assume $(g^1,\ldots,g^q;\alpha,\beta)_m$ is a WH multi Bessel sequence. Denote by $\omega_1,\ldots,\omega_q\in\Lambda^1(\xi)$ their associated exterior 1-forms as constructed before. Then $(g^1,\ldots,g^q;\alpha,\beta)_m$ is a WH multi Riesz sequence if and only if:

$$\omega_1\wedge\ldots\wedge\omega_q\neq 0 \ , \ \forall (t,s)\in\mathbf{T}^2 \tag{37}$$

Moreover, given an exterior $q$-form $\Omega\in\Lambda^q(\xi)$, it corresponds to a WH multi Riesz sequence if and only if $dim\, ker\,\Omega = p-q$, where

$$ker\,\Omega = \{v\in\mathrm{X}(\xi) \mid \Omega(v,v_2,\cdots,v_q)=0 \ , \ \forall v_2,\ldots,v_q\in\mathrm{X}(\xi)\}$$

2. Assume $(g^1,\ldots,g^q;\alpha,\beta)_m$ and $(h^1,\ldots,h^q;\alpha,\beta)_m$ are two multi WH riesz sequences. Denote by $\omega_1,\ldots,\omega_q$, respectively $\chi_1,\cdots,\chi_q$ their corresponding exterior 1-forms. Then, the two multi Riesz bases have the same span if and only if there is a $f\in\Lambda^0(\xi)=\mathcal{F}(\mathbf{T}^2;\mathbf{C})$, $f(t,s)\neq 0$ for every $(t,s)$ so that

$$\omega_1\wedge\ldots\wedge\omega_q = f\,\chi_1\wedge\ldots\wedge\chi_q \tag{38}$$

Moreover, they are unitary equivalent if and only if $|f|=1$.

3. Assume $(g^1,\ldots,g^q;\frac{1}{\beta},\frac{1}{\alpha})_s$ is a WH super Bessel sequence and denote by $\omega_1,\ldots,\omega_q$ their associated 1-forms. Then $(g^1,\ldots,g^q;\frac{1}{\beta},\frac{1}{\alpha})_s$ is a WH super frame if and only (37) holds true.

4. Assume $(g^1,\ldots,g^q;\frac{1}{\beta},\frac{1}{\alpha})_s$ and $(h^1,\ldots,h^q;\frac{1}{\beta},\frac{1}{\alpha})_s$ are two WH super frames. Denote by $\omega_1,\ldots,\omega_q,\chi_1,\ldots,\chi_q$ their associated 1-forms. Then the two WH super frames are equivalent, in the sense that their direct-sum frames are quivalent as frames in $L^{2,q}$, if and only if (38) holds true form some nonzero $f$ in $\mathcal{F}(\mathbf{T}^2;\mathbf{C})$.

**Proof** 1. The proof of this result uses the following remarks: Denote by $\Gamma$ the $p \times q$ complex matrix constructed as in (18) for $(g^1, \ldots, g^q; \alpha, \beta)$ and by $\Upsilon$ the corresponding matrix for $(h^1, \ldots, h^q; \alpha, \beta)$. Then condition (37 is equivalent to $rank(\Gamma) = q$ for every $(t, s)$. Use next the continuity of $G$ to deduce first part of 1. For the second part, note that whenever $dim\, ker\, \Omega = q$, there are $\omega_1, \ldots, \omega_q \in \Lambda^1(\xi)$ so that $\Omega = \omega_1 \wedge \cdots \wedge \omega_q$.

2. Note (38) is equivalent to $Ran\, \Gamma = Ran \Upsilon$, or, the fact there is an invertible and bounded $L$ so that $\Gamma = \Upsilon L$. Note next the $q$-form associated to $\Upsilon$ is $\overline{det(L)} \chi_1 \wedge \ldots \wedge \chi_q$. The statement follows from the previous remarks.

3. The super frame case reduces to the multi Riesz sequence, through Theorem 2.1. Then apply 1.

4. Again, equivalence of super frames is equivalent to multi Riesz sequences having the same span ([19]). Then apply 2. □

## 5. NON-LOCALIZATION RESULTS

In this section we present the connection between the non-localization results stated in the introduction and the vector bundles introduced above.

First we consider the Balian-Low type phenomenon. We want to analyze the statement of Lemma 2.5. Theorem 4.1 implies that $(g^1, \ldots, g^p; \alpha, \beta)_m$ with $\alpha\beta = p$ is a Riesz basis if and only if it corresponds to a nonvanishing exterior $p$-form of $\Lambda^p(\xi)$. Thus the no-go statement of Lemma 2.5 implies that all continuous sections of $(\Lambda^p(\mathbf{C}^p), \pi\mathbf{T}^2)$ have to vanish at least in one point of the 2-torus $\mathbf{T}^2$. In turn, this means that globally there are no $p$ sections in $X(\xi)$ so that at every point of $\mathbf{T}^2$ to form a basis in the local fiber of $\xi$. This statement represents (by definition) the non-triviality condition of $\xi$. Thus we obtained:

**Theorem** 5.1. The vector bundle $\xi$ is non-trivial. Moreover, the nontriviality of this vector bundle is precisely the reason of the amalgam BL statement as stated in part 2 of Theorem 2.4.

Let us now consider the optimization problem of $J_{ca}$. As shown by (29) and (30), the optimizers of this problem depend strongly on the class of invariant spaces of $R(t, s)$ from (31). Let us first analyze the case when $W(s) = I$. Then, by (27), $R(t, s) = E(t)R(t, s)E(t)^*$. The eigenvectors of $R(t, s) = M(t)$ coincides with the eigenvectors of $E(t)$. An explicit computation shows that:

$$x_r = \frac{1}{p}\begin{bmatrix} 1 & \varepsilon_r & \varepsilon_r^2 & \cdots & \varepsilon_r^{p-1} \end{bmatrix}^T \tag{39}$$

are the eigenvectors, for $0 \le r \le p - 1$, where $\varepsilon_r = e^{-2\pi i \frac{q}{p}(t + \frac{r}{q})}$ are the $p^{th}$ root of $e^{-2\pi i q t}$. Moreover, we can explicitely compute the corresponding eigenvalue. A little algebra shows that:

$$\mu_r(t) = \sum_{l \in \mathbf{Z}} e^{2\pi i l \frac{q}{p}(t + \frac{r}{q})} R(\frac{l}{\alpha}) \tag{40}$$

Now assume the set $t$ so that the $q^{th}$ eigenvalue of $M(t)$ is degenerate is discrete. Let $I_r$ be the $q$ subset of $\{0, 1, \ldots, p-1\}$ of indices corresponding to the largest $q$ eigenvalues of $M(t)$. Then, if $V(t, s)$ is continuous, necessarily:

$$V(t, s) = [x_{r_1} \;\; x_{r_2} \;\; \cdots \;\; x_{r_q}]$$

with $r_1, \cdots, r_q \in I_r$. But this does not satisfy the periodicity relation (21), namely:

$$V(t + \frac{1}{q}, s) \neq V(t, s)Q$$

The only solution is that $V(t, s)$ cannot equal the projection corresponding to the largest eigenvalues for all $t$, unless it loses the continuity. In particular we proved:

**Theorem** 5.2. Assume $W(s) = I$ and the $q^{th}$ eigenvalue of $M(t)$ is degenerate only for a finite number of points in the interval $[0, \frac{1}{q}]$. Then the optimal windows $g^1, g^2$ and their Fourier transforms $\hat{g}^1$, $\hat{g}^2$ cannot be in $W(C, l^1)$.

Now let us consider the general case for $W(s)$. The approach we use here is the following. Consider:

$$R_u(t, s) = e^{\frac{u}{2}\, log\, W(s)} M(t) e^{\frac{u}{2}\, log\, W(s)}$$

where $0 \leq u \leq 1$. Clearly $R_0(t,s) = M(t)$ and $R_1(t,s) = R(t,s)$ and $R_u(t,s)$ depends continuously on $u$. It follows the spectrum will change continuously with $(u,t,s)$. In particular, unless a "flat" are appears (an open domain in $(t,s)$ domain where the $q^{th}$ eigenvalue is degenerate) the initial eigenvalues $V(t,s)$ starts with, at some particular $(t,s)$, will not be the same to the eigenvalues at $(t+\frac{1}{q}, s)$, and hence cannot be maximal. Again the consequence of Theorem 5.2 is obtained.

## 6. CONCLUSIONS

In this paper we have constructed several vector bundles associated to Weyl-Heisenberg sets. First we have characterized multi WH Riesz sequences and super frames as well as their equivalence, in terms of some exterior forms. In particular, those sets correspond to nonvanishing and $q$-dimensional kernel exterior forms; moreover, the WH super frame sets are equivalent, and WH multi Riesz sequences have the same span, if and only if the corresponding exterior forms are dependent at every point of the 2-torus. The non-localization phenomen as known in the amalgam verion of the BL theorem turned out to be related to the nontriviality property of the aformentioned vector bundles.

Next we analyzed the (non)localization property of some optimal generator. To do this, we have considered spectral properties of the operator $R(t,s)$. It turned out there cannot be a continuous invariant projector corresponding to the largest eigenvalues of $R(t,s)$.

## REFERENCES


1. R. Balian, "Un principe d'incertitude fort en théorie du signal on en mécanique quantique", *C.R.Acad.Sci. Paris* , vol. 292, pp.1357–1362, 1981.
2. F. Low, *Complete sets of wave packets*, A Passion for Physics - Essays in Honor of Geoffrey Chew, World Scientific, Singapore, pp.17–22, 1985.
3. I. Daubechies, " The Wavelet Transform, Time-Frequency Localizatin and Signal Analysis", *IEEE Trans.Inform.Theory*, vol. 36, no.5, pp.961–1005, 1990.
4. G. Battle, "Heisenberg Proof of the Balian-Low Theorem", *Letters in Mathematical Physics* , vol. 15, pp. 175–177, 1988.
5. L. Auslander, R. Tolimieri, **Abelian Harmonic Analysis, Theta Functions and Function Algebras on a Nilmanifold**, Lecture Notes in Mathematics, vol.436, Springer-Verlag, New York 1975.
6. J. J. Benedetto, C. Heil, D. F. Walnut, "Differentiation and the Balian-Low Theorem", *J.Fourier Anal.Appl.* vol. 1, no.4, pp.355–402, 1995.
7. A. J. E. M. Janssen, " The Zak Transform: a Signal Transform for Sampled Time-Continuous Signals", *Philips J.Res.*, vol. 43, pp.23–69, 1988.
8. M. Zibulski, Y. Y. Zeevi, "Analysis of Multi-Window Gabor-Type Schemes by Frame Methods", Appl.Comp.Harm.Anal., vol. 4, pp.188–221, 1997.
9. R. Balan, "Extensions of No-Go Theorems to Many Signal Systems" , in Contemporary Mathematics 216, Proceedings of the AMS Special Session on Wavelets, Multiwavelets and their Applications, San Antonio 1997, A. Aldroubi, E. Lin Eds., pp.3–15, 1997.
10. R. Balan, I. Daubechies, V. Vaishampayan, "The Analysis and Design of Windowed Fourier Frame based Multiple Description Encoding Schemes", *IEEE Trans.Inform.Theory*, **46** no.7 , pp.2491–2537 , 2000.
11. R. Balan, I. Daubechies, "Optimal Stochastic Approximations and Encoding Schemes using Weyl-Heisenberg Sets", to appear in *Gabor Analysis II: Theory and Applications*, Eds. H. Feichtinger and T. Strohmer, 2002.
12. O. Christensen, B. Deng, C. Heil , "Density for Gabor Frames", Appl. Comput. Harmon. Anal. **7**, pp.292–304, 1999.
13. R. Balan, *Density and Redundancy of the Non-Coherent Weyl-Heisenberg Superframes*, in Contemporary Mathematics 247, Proceedings of the AMS Special Session on the Functional and Harmonic Analysis of Wavelets, San Antonio 1999, L. Bagget and D. Larson Eds., pp.29–43, 1999.
14. D. Han, D. Larson, "Frames, bases and group representations", Memoirs AMS 1998.
15. J. P. Gabardo, D. Han, *Subspace Weyl-Heisenberg Frames*, preprint 1999.
16. R. Balan, "Multi Riesz bases and Multiframes: II. The Weyl-Heisenberg case", preprint 1998.
17. I.Daubechies, H.J.Landau, Z.Landau, "Gabor Time-Frequency Lattices and the Wexler-Raz Identity", *J.Fourier Anal.Applic.* **1**, no.4 (1995), pp.437–478
18. R. Balan, *A Study of Weyl-Heisenberg and Wavelet Frames*, Ph.D. thesis, Princeton University 1998.
19. R. Balan, Z. Landau, "Equivalence between Weyl-Heisenberg Frames and Riesz Sequences", preprint 2000.