Absolute Stability Criteria for Nonlinear Affine Systems : A Kalman-Yakubovich-Popov Type Approach

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Abstract

In this paper we present nonlinear versions of the Circle and Popov Criteria of absolute asymptotic stability for nonlinear systems. We use a Kalman-Yakubovich-Popov type approach for nonlinear systems which involves a Hamilton-Jacobi equation and a nonlinear KYP system. Our criteria give sufficiency conditions of absolute asymptotic stability as well as uniform estimations of the attraction basin of the origin for the closed loop systems.

1 The Hamilton-Jacobi Equation and The Nonlinear Kalman-Yakubovich-Popov System

Let us consider the *nonlinear Popov system* composed by a nonlinear affine dynamics and a quadratic criterion:

$$P\left\{\begin{array}{lcl} \dot{x} & = & f(x)+g(x)\cdot u \quad , \; x(0)=x_0 \\ J(t_1) & = & \int_0^{t_1}[q(x)+2l(x)\,u+u^*R(x)u]dt \end{array}\right. \tag{1}$$

where $x(t)\in D\subset \mathbf{R}^n$, $u(t)\in\mathbf{R}^m$, $q:D\rightarrow\mathbf{R}$, $l^*:D\rightarrow\mathbf{R}^m$ ($*$ denotes the transpose), $R:D\rightarrow\mathbf{R}^{m\times m}$, $f,g_i:D\rightarrow TD$ are vector fields of class $\mathcal{C}^1$ on D a domain of $\mathbf{R}^n$ with $0\in D$, $f(0)=0$ and $g_i(0)=0$ (we have denoted $g(x)\cdot u=\sum_{i=1}^m g_i(x)u_i$). We denote by $P=(f,g;q,l,R)$ a nonlinear Popov system.

We shall suppose the following assumptions on criterion: $q(0)=0$, $q'(0)=0$, $l(0)=0$, $R(x)>0$, $\forall x\in D$ and they are functions of class $\mathcal{C}^1$ on D.

Starting with system (1) we define two objects: the HJ equation and the nonlinear KYP system.

The Hamilton-Jacobi Equation (The nonlinear Riccati equation) has the form:

$$\nabla V\cdot f-(\frac{1}{2}\nabla V\cdot g+l)R^{-1}(\frac{1}{2}\nabla V\cdot g+l)^*+q=0 \tag{2}$$

*on leave from University "Politehnica" of Bucharest where the solution is a scalar function $V : U \subset D \to \mathbf{R}$, $V(0) = 0$ of class $\mathcal{C}^1$. We call a solution $V_s(x)$ *stabilizable* if the feedback:

$$u = -R^{-1}(\frac{1}{2}\nabla V \cdot g + l)^* \tag{3}$$

asymptotically stabilizes the system (1) (i.e. the vector field:

$$\tilde{f} = f - gR^{-1}(\frac{1}{2}\nabla V \cdot g + l)^*$$

has at $\bar{x} = 0$ an asymptotic stable equilibrium point).

We call a solution $V_a(x)$ *antistabilizable* if the feedback (3) makes the origin stable in reverse time for the system (1).

The Nonlinear Kalman-Yakubovich-Popov System is defined by the following set of equations:

$$\begin{array}{rcl} R(x) & = & G^*(x)G(x) \\ \frac{1}{2}\nabla V(x) \cdot g(x) + l(x) & = & W^*(x)G(x) \\ \nabla V(x) \cdot f(x) + q(x) & = & W^*(x)W(x) \end{array} \tag{4}$$

where a solution is a triplet of the form $(G(x), W(x), V(x))$ composed by real matrix valued functions defined on $U \subset D$ with $V(0) = 0$ and V of class $\mathcal{C}^1$. We call a solution $(G_s(x), W_s(x), V_s(x))$ *stabilizable* if the feedback:

$$u = -G_s^{-1}(x)W_s(x) \tag{5}$$

makes the origin asymptotically stable for the system (1) (i.e. the vector field

$$\tilde{f} = f - gG_s^{-1}W_s$$

has at $\bar{x} = 0$ an asymptotically stable equilibrium point). We call a solution (G_a, W_a, V_a) *antistabilizable* if the feedback (5) stabilizes the system (1) in reverse time.

We see that under the assumption $R(x) > 0$ both objects (the HJ equation and the nonlinear KYP system) are equivalent.

In this paper we shall work only under the assumption $R(x) > 0$.

Let (G, W, V) be a solution of (4). Then one can obtain the following factorization of the criterion:

$$J(t_1) = -[V(x)]|_{x(0)}^{x(t_1)} + \int_0^{t_1} \| W(x) + G(x)u\|^2 dt \tag{6}$$

The connexion between the nonlinear Popov system and HJ equation is given by a variational problem and is well known. Actually, the equation (2) has the form $H(x, \nabla V(x)) = 0$ where the Hamiltonian is given by:

$$H(x,p) = p \cdot f(x) - (\frac{1}{2}p \cdot g(x) + l(x))R^{-1}(x)(\frac{1}{2}p \cdot g(x) + l(x))^* + q(x) \tag{7}$$

Furthermore, it is well known that any solution of (2) corresponds to an invariant manifold of the Hamiltonian vector field given in the space (x, p) by:

$$X_H = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial x})^T$$

(see [Isid91] for details).

On another hand, the nonlinear KYP system and relation (6) are natural associated to the nonlinear Popov system when someone studies the property of (1). Such approaches are investigated, for example, in [Will72] , [Moyl74] and [HiMo76].

2 Nonlinear Popov Systems

In this section we are going to study some properties of the nonlinear Popov systems of the form (1).

We start by recalling the definition of *minimal stability* (see [Popov73] for instance):

Definition The nonlinear Popov system (1) has the property of *minimal stability* if there is a neighborhood of the origin $0 \in \mathcal{U}$ such that for any $x_0 \in \mathcal{U}$ there is a piecewise continuous control $u : [0, \infty) \to \mathbf{R}^m$ such that the solution of the dynamics from (1) remains in D ($x(t) \in D$), goes to the origin $\lim_{t\to\infty} x(t) = 0$, $J(\infty)$ is finite and $J(\infty) \leq 0$.

We stress out that the control u may depend on x_0 as well as on $\mathcal{U}$.

It is worth noting that the criterion is finite under the following requirements: $u \in L^{2,m}[0, \infty)$, $x \in L^{2,n}[0, \infty) \cap L^{\infty,n}[0, \infty)$.

Indeed, since $q'(0) = 0$, $q(0) = 0$ and q is of class $\mathcal{C}^2$ on D we get: $q(x) = x^* \cdot q_2(x) \cdot x$. In a similar way, since $l(0) = 0$ we get $l(x) = x^* \cdot l_1(x)$. Now, $q_2(x)$ and $l_1(x)$ are both of class $\mathcal{C}^0$ on D. From the boundedness of the state $x(t)$ there exist positive constants $M_1, M_2, M_3 > 0$ such that:

$$\| q_2(x(t)) \| \leq M_1 \quad \| l_1(x(t)) \| \leq M_2 \quad \| R(x(t)) \| \leq M_3$$

for any $t \in [0, \infty)$. From $\| x(t) \|, \| u(t) \| \in L^2[0, \infty)$ using Cauchy-Buniakowsky-Schwartz inequality we also get $x^* \cdot N \cdot u \in L^2[0, \infty)$ for any constant matrix $N \in \mathbf{R}^{n\times m}$. Now we obtain:

$$|q(x) + 2l(x)u + u^* R(x)u| \leq \| q_2(x)\| \; \| x\|^2 + 2 \| l_1(x)\| \; \| x\| \; \| u\| + \| R(x)\| \; \| u\|^2 \leq$$

$$\leq M_1 \| x\|^2 + 2M_2 \| x\| \; \| u\| + M_3 \| u\|^2$$

from where we conclude that $J(\infty)$ is finite.

The definition of the minimal stability is useful because of the following Lemma:

LEMMA 1 *Let us consider the nonlinear Popov system (1) and assume it has the property of minimal stability. Then any solution of the HJ equation (2) such that $V(0) = 0$ is negative semidefinite: $V(x) \leq 0$, whereas the antistabilizable solution (if it exists) is strict definite: $V_a(x) < 0$, $\forall x \neq 0$.*

Proof

We rewrite (6) as:

$$V(x_0) = J(t_1) + V(x(t_1)) - \int_0^{t_1} \| W(x) + G(x)u\|^2 dt$$

Now we choose $u = u_{x_0}$ as in the definition of minimal stability and we obtain (for $t_1 = \infty$):

$$V(x_0) = J(\infty) - \int_0^{\infty} \| W(x) + G(x)u\|^2 dt \leq 0$$

This proves the first part of lemma. For the second part we can see that $V_a(x_0) = 0$ for $x_0 \neq 0$ iff $J(\infty) = 0$ and $W_a(x) + G_a(x)u \equiv 0$. Then $u = -G_a^{-1}(x)W_a(x)$ on the trajectory starting from $x_0 \neq 0$ and $\lim_{t\to\infty} x(t) = 0$. On the other hand, because of (5), this control must render origin of the closed loop stable in reverse time. Now we come very simple to a contradiction. Let $\varepsilon = \| x_0 \| /2$. Then for any $\delta > 0$ we can choose a moment t_δ such that $\| x(t_\delta) \| < \delta$, because of the convergence. Take now $\overline{x_0} = x(t_\delta)$ an initial condition for the reverse time system. Let us denote by $\bar{x}(t)$ the solution of the reverse time dynamics. Then $\bar{x}(t_\delta) = x_0$ and $\| \bar{x}(t_\delta) \| > \varepsilon$. □

As in the linear case we try to analyze the equivalence of two Popov systems (see [Popov73]). This will be useful in the proof of Popov criterion.

Definition We say that two Popov systems $P_1 = (f, g; q_1, l_1, R_1)$ and $P_2 = (f, g; q_2, l_2, R_2)$ are *weak local equivalent* if there is a neighborhood of the origin $\mathcal{U} \subset D_1 \cup D_2$ and a scalar function $\tilde{V} : \mathcal{U} \to \mathbf{R}$ such that:

$$\begin{aligned} R_2(x) &= R_1(x) \\ l_2(x) &= l_1(x) + \frac{1}{2}\nabla\tilde{V}(x)\cdot g(x) \\ q_2(x) &= q_1(x) + \nabla\tilde{V}(x)\cdot f(x) \end{aligned} \tag{8}$$

for any $x \in \mathcal{U}$.

The nonlinear KYP systems as well as the HJ equations of two weak local equivalent nonlinear Popov systems are very similar. This similitude is established by the following Lemma:

LEMMA 2 *Suppose $P_1 = (f, g; q_1, l_1, R_1)$ and $P_2 = (f, g; q_2, l_2, R_2)$ are two nonlinear Popov systems. Then P_1 and P_2 are weak local equivalent iff one of the following is true:*

i) For any solution of the HJ equation associated to P_1 there is a solution of the HJ equation associated to P_2 such that:

$$V_2(x) = V_1(x) - \tilde{V}(x) \quad , \quad x \in \mathcal{U}$$

ii) For any solution of the nonlinear KYP system associated to P_1 there is a solution of the nonlinear KYP system associated to P_2 such that:

$$\begin{aligned} G_2(x) &= G_1(x) \\ W_2(x) &= W_1(x) \\ V_2(x) &= V_1(x) - \tilde{V}(x) \quad , \quad x \in \mathcal{U} \end{aligned}$$

iii) Between the two criteria J_1 and J_2 there exists the following relation:

$$J_2(t_1) = J_1(t_1) + [\tilde{V}(x)]|_{x(0)}^{x(t_1)}$$

Proof

The equivalence between the condition of weak local equivalence and one of the conditions (i)-(iii) comes from the following identity:

$$J_1(t_1) + \int_0^{t_1} \frac{d\tilde{V}}{dt}dt = \int_0^{t_1} [q_1(x) + \nabla\tilde{V}\cdot f + (2l_1(x) + \nabla\tilde{V}\cdot g)u + u^* R_1(x)u]dt$$

and:

$$[\tilde{V}(x)]|_{x(0)}^{x(t_1)} = \int_0^{t_1} \frac{d\tilde{V}}{dt}dt$$

Then by simple algebraic computations, using (2), (4) and (8), we obtain the statement. $\square$.

We point out that a complete study of local equivalence must take into account the change of coordinates as well as the state feedback. But the equations involving by these conditions would be much messyer than (8). Actually, in the present study we do not need this type of equivalency.

From the second point of the above Lemma we obtain the following result:

CORROLARY 3 *If P_1 and P_2 are two weak local equivalent nonlinear Popov systems as in the above Lemma then:*

i) If V_{1s} is a stabilizable solution of P_1 then $V_{2s} = V_{1s} - \tilde{V}$ is a stabilizable solution of P_2;

ii) If V_{1a} is an antistabilizable solution of P_1 then $V_{2a} = V_{1a} - \tilde{V}$ is an antistabilizable solution of P_2;

iii) For any solution V_1 of P_1 and V_2 of P_2 connected by $V_1 - \tilde{V} = V_2$ the feedbacks are the same:

$$u = -G_1^{-1}(x)W_1(x) = -G_2^{-1}(x)W_2(x)$$

Proof

The first two points (i) and (ii) come from (iii) and this one comes from relation (5) and point (ii) of Lemma 2. □

3 Statement of The Absolute Stability Criteria

Let us consider a SISO nonlinear affine system of the form:

$$(ND)\left\{\begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x) \end{array}\right. \tag{9}$$

where f, g are vector fields of classe $\mathcal{C}^1$ on $D \subset \mathbf{R}^n$, h is a continuous scalar function defined on D and $f(0) = 0$, $h(0) = 0$.

Let us consider two continuous scalar functions $\alpha, \beta : \mathbf{R} \to \mathbf{R}$ having the properties: $\alpha(0) = 0, \beta(0) = 0$ and $y\alpha(y) \leq y\beta(y)$. With these functions we set the following sectors:

$$\begin{array}{l} N_{\alpha,\beta} = \{\varphi : \mathbf{R} \to \mathbf{R} | \varphi \text{ continuous}, \varphi(0) = 0,\ \alpha(y)y < \varphi(y)y < \beta(y)y \text{ for } y \neq 0\} \\ N_{\alpha,\beta}(t) = \{\varphi : \mathbf{R} \times \mathbf{R} \to \mathbf{R} | \varphi(y,t) \text{ piecewise continuous in t and } \varphi(.,t) \in N_{\alpha,\beta}\} \end{array} \tag{10}$$

Our problem could be formulated as follows:

The Problem Find conditions of local absolute asimptotic stability as well as an estimation of attraction domain of the origin for nonlinear system (9) with respect to one sector defined above (see figure 1).

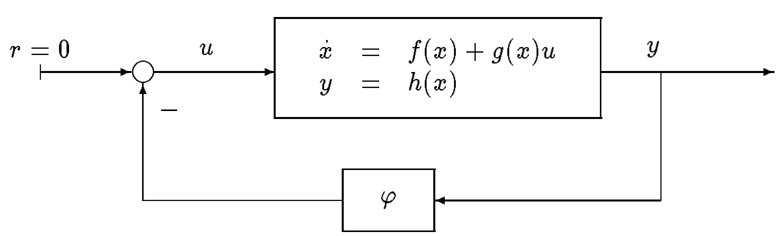

Figure 1: Closed-loop system

Before to state the theorems we need a definition.

Definition The pair (h, f) is called *zero-state detectable* if from $y(t) \equiv 0$ for $t \geq 0$ we conclude $x(t) \in D$ for $t \geq 0$ and $\lim_{t\to\infty} x(t) = 0$.

In the sequel we present the nonlinear versions of Circle and Popov Criteria.

THEOREM 4 (Circle Criterion - the nonlinear version) *We consider the following HJ equation:*

$$\nabla V(x) \cdot f(x) - \frac{1}{4}(\nabla V(x) \cdot g(x) + \alpha(h(x)) + \beta(h(x)))^2 + \alpha(h(x))\beta(h(x)) = 0 \tag{11}$$

If the following conditions are fulfilled:

1) There exists an antistabilizable solution $V_a : \mathcal{U} \subset D \to \mathbf{R}$ of class $\mathcal{C}^1$ of (11) ($\mathcal{U}$ being a neighborhood of the origin);

2) There exists a feedback $\varphi(.,.) \in N_{\alpha,\beta}(t)$ that asymptotically stabilizes the closed loop and has $\mathcal{U}$ included in the attraction domain;

3) The pair (h, f) is zero-state detectable;

then the system (9) is local absolute asymptotic stable with respect to the class $N_{\alpha,\beta}(t)$ and, furthermore, the largest connected compact set of the form $V_a^{-1}([-a, 0]) \subset \mathcal{U}$ is included in the attraction domain of the origin for any nonlinearity in the sector $N_{\alpha,\beta}(t)$. □

For the Popov Criterion we give two versions: the first one can be applied to general nonlinear sectors while the second one can be applied only to some special sectors.

THEOREM 5 (Popov Criterion - nonlinear version) *We consider the nonlinear system (9) for which h is of class $\mathcal{C}^1$. Suppose the denominator of the following HJ equation does not vanish on D:*

$$\nabla V(x) \cdot f(x) - \frac{1}{4}\frac{\left(\nabla V(x) \cdot g(x) + \alpha(h(x)) + \beta(h(x)) + \alpha_0 \nabla h(x) \cdot f(x)\right)^2}{1 + \alpha_0 \nabla h(x) \cdot g(x)} + \alpha(h(x))\beta(h(x)) = 0 \quad (12)$$

where $\alpha_0 \in \mathbf{R}$ is a real parameter and:

1) There exists an antistabilizable solution $V_a : \mathcal{U} \subset D \to \mathbf{R}$ of class $\mathcal{C}^1$ of the equation (12) ($\mathcal{U}$ being a neighborhood of the origin);

2)

- *For $\alpha_0 < 0$ the feedback $\varphi(y) = \beta(y)$ asymptotically stabilizes the closed loop and $\mathcal{U}$ is included in the attraction domain;*
- *For $\alpha_0 > 0$ the feedback $\varphi(y) = \alpha(y)$ asymptotically stabilizes the closed loop and $\mathcal{U}$ is included in the attraction domain;*
- *For $\alpha_0 = 0$ there exists a feedback $\varphi(.) \in N_{\alpha,\beta}$ that asymptotically stabilizes the closed loop, with $\mathcal{U}$ included in the attraction domain;*

3) The pair (h, f) is zero-state detectable;

then the system (9) is local absolute asymptotic stable with respect to the class $N_{\alpha,\beta}$. Set

$$V_{2a}(x) = V_a(x) + \alpha_1 \int_0^{h(x)} \alpha(u)du - \alpha_2 \int_0^{h(x)} \beta(u)du \quad (13)$$

and:

$$W_c(x) = -V_{2a}(x) + |\alpha_0| \int_0^{h(x)} (\beta(u) - \alpha(u))du \quad (14)$$

where $\alpha_1 - \alpha_2 = \alpha_0$, $\alpha_1\alpha_2 = 0$, $\alpha_1, \alpha_2 \geq 0$. Then, for any positive number $a > 0$ such thatthe connected component of the origin of $V_{2a}^{-1}([-a, 0])$ is a compact set included in $\mathcal{U}$, the compact set $W_c^{-1}([0, a])$ is included in the attraction domain of the origin for any feedback in the sector $N_{\alpha,\beta}$. □

In Theorem 5 the found domain of attraction may depend on α_0. Then a better choose of α_0 may imply a larger set. The choice of α_0 could be suggested by the linearized system and then using the Popov criterion (the linear version).

We point out that for $\alpha_0 = 0$ the HJ equation (12) turns into the HJ equation (11) and then we obtain the Circle Criterion, but only for time-invariant feedbacks.

In a special case when $\beta(y) - \alpha(y)$ is a linear map we can relax condition 2) of the above statement. In this case we require that at least one feedback in the convex hull of $[\alpha, \beta]$ stabilizes the closed loop. The statement is given below:

THEOREM 6 (Popov Criterion - a special case) *We consider the nonlinear system (12) for which* h *is of class* $\mathcal{C}^1$ *and a nonlinear sector* $N_{\alpha,\beta}$ *for which* $\beta - \alpha$ *is a linear map* ($\beta(y) - \alpha(y) = ky,\ k > 0$). *Suppose the denominator of the HJ equation (12) does not vanish on* D *and:*

1) There exists an antistabilizable solution $V_a : \mathcal{U} \subset D \to \mathbf{R}$ *of class* $\mathcal{C}^1$ *of the equation (12) (*$\mathcal{U}$ *being a neighborhood of the origin);*

2) There exists $\xi \in [0,1]$ *such that the feedback:*

$$\varphi_s(y) = \xi\alpha(y) + (1-\xi)\beta(y)$$

asymptotically stabilizes the closed loop and $\mathcal{U}$ *is included in the attraction domain of the origin;*

3) The pair (h, f) *is zero-state detectable;*

then the system (9) is local absolute asymptotic stable with respect to the class $N_{\alpha,\beta}$*. Set*

$$V_{2a}(x) = V_a(x) + \alpha_1 \int_0^{h(x)} \alpha(u)du - \alpha_2 \int_0^{h(x)} \beta(u)du \tag{15}$$

and:

$$W_c(x) = -V_{2a}(x) + \frac{k|\alpha_0|}{2} h^2(x) \tag{16}$$

where $\alpha_1 - \alpha_2 = \alpha_0$, $\alpha_1\alpha_2 = 0$, $\alpha_1, \alpha_2 \geq 0$. *Then, for any positive number* $a > 0$ *such that the connected component of the origin of* $V_{2a}^{-1}([-a, 0])$ *is a compact set included in* $\mathcal{U}$*, the compact set* $W_c^{-1}([0, a])$ *is included in the attraction domain of the origin for any feedback in the sector* $N_{\alpha,\beta}$. □

We have to point out that our results are local. If the functions V_a from Theorem 4 or $W_c(x)$ from Theorems 5 or 6 are radially unbounded (i.e. $\lim_{R\to\infty} \min_{||x||=R}(-V_a) = \infty$, $\lim_{R\to\infty} \min_{||x||=R} W_c(x) = \infty$) then the asymptotic stability is in the whole and we obtain a global absolute asymptotic stability with respect to the considered class.

4 Proof of Circle Criterion

Let us consider the nonlinear Popov system associated to the HJ equation (4):

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ J(t_1) &= \int_0^{t_1}[\alpha(h(x))\beta(h(x)) + (\alpha(h(x)) + \beta(h(x)))u + u^2]dt \end{cases} \tag{17}$$

One can see that the criterion could be rewritten as:

$$J(t_1) = \int_0^{t_1} (\alpha(h(x)) + u)(\beta(h(x)) + u)dt \tag{18}$$

We claim that (17) has the property of minimal stability. Let us choose the neighborhood $\mathcal{U}$ from the definition of minimal stability the same with the attraction domain of the feedback φ. Let $x_0 \in \mathcal{U}$ be an arbitrarly initial state. We set $u(t) = \varphi(h(x(t)), t)$ which is continuous as function of t. Now, obviously $x(t) \in D$ and $\lim_{t\to\infty} x(t) = 0$. Since $\varphi \in N_{\alpha,\beta}(t)$ and taking into account (18) we conclude that $J(t_1) \leq 0$ for any $t_1 \geq 0$. Now we use the factorization (10) written as:

$$J(t_1) + V(x(t_1)) - V(x_0) = \int_0^{t_1} \| W(x) + g(x)u\|^2 dt \tag{19}$$

Here, since the control system (9) is SISO, we have $\| W(x) + G(x)u \| = |W(x) + G(x)u|$; for the MIMO systems the norm sign is more appropriate and the proof goes in the same way. This is the reason for which we prefer this notation. The right-hand side is positive and $V(x(t_1))$ is bounded (recall that

$\lim_{t\to\infty} x(t) = 0$ and $V(0) = 0$) then we can tend $t_1 \to \infty$ and obtain finite values for $J(\infty)$ and $\int_0^\infty \| W(x) + G(x)u\|^2 dt$. This proves the minimal stability of (17).

Now we apply Lemma 1 and we conclude that $V_a : \mathcal{U} \to \mathbf{R}$, the antistabilizable solution of (4) is negative definite $V_a(x) < 0$, for $x \neq 0$. From (19) and (18), by deriving with respect to t_1 we get:

$$\frac{dV_a}{dt}|_x = \| W(x) + G(x)u\|^2 - (\alpha(h(x)) + u)(\beta(h(x)) + u) \tag{20}$$

We can see that for any feedback $\Psi \in N_{\alpha,\beta}(t)$ the control $u(t) = \Psi(h(x(t)), t)$ makes negative the second term of the right-hand side of (20). Then $\frac{dV_a}{dt}|_x \geq 0$, for any $x \in \mathcal{U}$.

This proves that we can choose as a Liapunov candidate $Liap(x) = -V_a(x)$. Furthermore, from (20) we can see that $Q(x) = \frac{dLiap(x(t))}{dt} \equiv 0 \Rightarrow h(x) \equiv 0$ *and* $u(t) \equiv 0$ (recall that the sector is open). Using the zero-state detectability property of (h, f) we find that, if there exists a trajectory in $K = \{x \in D | Q(x) = 0\}$, then it must go to the origin. Thus $V_a(x(t)) \equiv 0$ on such trajectory. Since $V_a(x) < 0$, $\forall x \neq 0$ we obtain that the maximal invariant set to the vectorfield $f(x) + g(x)\varphi(h(x), t)$ included in K is $N = \{0\}$. Now, using LaSalle's Invariance Principle we obtain that the closed-loop system with the feedback Ψ has the origin an asymptotical stable equilibrium point. Furthermore, because of the Liapunov function, the largest compact set of the form $V_a^{-1}([-a, 0]) \subset \mathcal{U}$, with $a > 0$, is included in the attraction domain of the origin for any feedback in the sector $N_{\alpha,\beta}(t)$ (see, for instance, Theorem 8.1 from [Will70] or Theorem 6, §13 from [SalLe61]). This ends the proof of Theorem 4. □

5 Proofs of Popov Criteria

The proofs of Popov Criteria follow a similar manner than the Circle Criterion proof. We shall prove both Theorems. The difference between proofs concerns only the claiming that P_2 (a Popov system that will be descibed below) has the property of minimal stability.

Firstly, we define two Popov systems:

$$P_1 \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ J_1(t_1) & = & \int_0^{t_1} [\alpha(h(x))\beta(h(x)) + (\alpha(h(x)) + \beta(h(x)) + \alpha_0 \nabla h(x) \cdot f(x))u + (1 + \alpha_0 \nabla h(x) \cdot g(x))u^2] dt \end{array} \right.$$

$$P_2 \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u \\ J_2(t_1) & = & \int_0^{t_1} [\alpha(h(x))\beta(h(x)) + (\alpha_1\alpha(h(x)) - \alpha_2\beta(h(x)))\nabla h(x) \cdot f(x) + (\alpha(h(x)) + \beta(h(x)) \\ & & +\alpha_0 \nabla h(x) \cdot f(x) + (\alpha_1\alpha(h(x)) - \alpha_2\beta(h(x)))\nabla h(x) \cdot g(x))u + (1 + \alpha_0 \nabla h(x) \cdot g(x))u^2] dt \end{array} \right.$$

where: $\alpha_0 \in \mathbf{R}$, $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 - \alpha_2 = \alpha_0$, $\alpha_1\alpha_2 = 0$.

We are going to prove first that P_1 and P_2 are weak local equivalent. For, we see that $D_1 = D_2 = D$ and we can choose $\mathcal{U} = D_1 = D_2 = D$ and:

$$\tilde{V} = -(\alpha_1 \int_0^{h(x)} \alpha(u)du - \alpha_2 \int_0^{h(x)} \beta(u)du) \tag{21}$$

Then relations (8) are fulfilled and the two Popov systems are equivalent. Using Corrolary 3 we obtain that the Popov system P_2 has also an antistabilizable solution. We shall denote this solution by V_{2a}.

On one hand we can see that the HJ equation associated to P_1 is exactly (12), on the other hand the criterion of P_2 could be rewritten as:

$$J_2(t) = \int_0^t (\alpha(h(x)) + u)(\beta(h(x)) + u)d\tau + \alpha_1 \int_0^t (\alpha(h(x)) + u)\frac{dh(x)}{d\tau}d\tau - \alpha_2 \int_0^t (\beta(h(x)) + u)\frac{dh(x)}{d\tau}d\tau$$

Let us choose a feedback $\varphi \in N_{\alpha,\beta}$. Let us consider $u = -\varphi(y)$ and denote by:

$$\Psi_1(y_1) = \int_0^{y_1} (u + \alpha(y))dy \tag{22}$$

$$\Psi_2(y_1) = \int_0^{y_1} (u + \beta(y))dy \tag{23}$$

Then the following inequalities hold:

$$\int_0^t (u + \alpha(y))(u + \beta(y))d\tau \leq 0$$

$$\Psi_1(y_1) \leq 0$$

$$\Psi_2(y_1) \geq 0$$

and the quadratic criterion becomes:

$$J_2(t) = \int_0^t (u + \alpha(y))(u + \beta(y))d\tau + \alpha_1\Psi_1(y(t)) - \alpha_1\Psi_1(y(0)) - \alpha_2\Psi_2(y(t)) + \alpha_2\Psi_2(y(0)) \tag{24}$$

With $\alpha_1, \alpha_2 \geq 0$ we have the following boundedness:

$$J_2(t) \leq -\alpha_1\Psi_1(y(0)) + \alpha_2\Psi_2(y(0)) \tag{25}$$

We are going to prove now that P_2 has the property of minimal stability. In the case $\alpha_0 = 0$ Popov's Criterion turns into Circle Criterion and the minimal stability property is already proved. Then we assume that $\alpha_0 \neq 0$.

In this point we can go on in different ways for Theorem 5 and Theorem 6.

Firstly the proof of minimal stability under the assumptions of Theorem 5: Let us consider the case $\alpha_0 > 0$. Then $\alpha_1 = \alpha_0$ and $\alpha_2 = 0$. We choose $u = -\alpha(y)$ that stabilizes the closed loop and moreover: $J_2(t) = 0$ for any $t > 0$. Then $J_2(\infty)$ is finite, $J_2(\infty) = 0 \leq 0$ and $\lim_{t\to\infty} x(t) = 0$ for any $x(0) \in \mathcal{U}$. In the case $\alpha_0 < 0$ we follow the same scheme. Now $\alpha_1 = 0$ and $\alpha_2 = -\alpha_0$ and we choose $u = -\beta(y)$. We obtain the same conclusions. This proves that P_2 has the property of minimal stability.

Now the proof of minimal stability uder the conditions of Theorem 6. We consider the case $\alpha_0 > 0$ (the case $\alpha_0 < 0$ is similar). Then $\alpha_1 = \alpha_0$ and $\alpha_2 = 0$. Now, consider $u = -\varphi_s(y) + v$. Then the dynamics becomes:

$$\dot{x} = f(x) - g(x)\varphi_s(h(x)) + g(x)v$$

and the criterion J_2 becomes:

$$J_2(t) = \int_0^t (-(1-\xi)ky + v)(\xi ky + v + \alpha_0\frac{dy}{d\tau})d\tau$$

If $\xi = 1$ then we take $v = 0$ and then we obtain as above the minimal stability. Let us set a new variable: $\tilde{y} = y - \frac{1}{(1-\xi)k}v$. With a little algebra one can bring the criterion into the form:

$$J_2(t) = -\xi(1-\xi)k^2\int_0^t \tilde{y}^2 d\tau - \frac{(1-\xi)k\alpha_0}{2}\tilde{y}^2|_0^t - k\int_0^t \tilde{y}(v + \frac{\alpha_0}{k}\frac{dv}{d\tau})d\tau$$

We choose v to be the solution of the differential equation: $\frac{\alpha_0}{k}\frac{dv}{dt} + v = 0$ with initial condition $v(0)$ such that $\tilde{y}(0) = 0$: $v(0) = (1-\xi)kh(x_0)$. Then:

$$v(t) = (1-\xi)kh(x_0)exp(-\frac{k}{\alpha_0}t)$$

and

$$J_2(t) = -\xi(1-\xi)k^2 \int_0^t \tilde{y}^2 d\tau - \frac{(1-\xi)k\alpha_0}{2}\tilde{y}^2(t) \leq 0$$

It remains to prove that $\lim_{t\to\infty} x(t) = 0$. (since there exists a solution of the HJ equation associated to P_2 one can prove that $J_2(\infty)$ is finite, using the factorization (6)). For, we observe that the dynamics can be brought into the following form:

$$\begin{array}{rcl} \dot{x} &=& f_s(x) + g(x)v \\ \dot{v} &=& -\frac{k}{\alpha_0}v \end{array} \tag{26}$$

where $f_s = f - g \cdot (\varphi_s \circ h)$ is a vector field having at the origin an asymptotically stable equilibrium and for any $x(0) = x_0$ we take $v(0) = (1-\xi)kh(x_0)$. We apply now a well-known property of cascade systems (see for instance [Vidy80]) and obtain that the extended system (26) has at $(\bar{x}, \bar{\xi}) = (0,0)$ a local asymptotic stable equilibrium. Now we can shrink enough the neighborhood of the origin to obtain that $\lim_{t\to\infty} x(t) = 0$ when we initialize the system in this neighborhood. Then P_2 has the property of minimal stability.

Now we apply Lemma 1 and we get that $V_{2a}(x) < 0$ for $x \neq 0$. Furthermore, the connection between the antistabilizable solution of P_2 and that of (12) (or P_1) is given by (13).

As Liapunov candidate we consider:

$$Liap(x) = -V_{2a}(x) - \alpha_1\Psi_1(h(x)) + \alpha_2\Psi_2(h(x)) \tag{27}$$

Since $\Psi_1 \leq 0$, $\Psi_2 \geq 0$ (given by (22) and (23)) and $V_{2a} < 0$ we have $Liap(x) > 0$, for $x \neq 0$, $Liap(0) = 0$. The factorization (6) becomes (using (24)):

$$Liap(x) = Liap(x_0) + \int_0^t (u + \alpha(h(x)))(u + \beta(h(x)))d\tau - \int_0^t \| W_a(x) + G(x)u\|^2 d\tau$$

Then, for the derivative we get:

$$\frac{dLiap(x)}{dt} = (u + \alpha(y))(u + \beta(y)) - \| W_a(x) + G(x)u\|^2$$

Since $u = -\varphi(y)$ and $\varphi \in N_{\alpha,\beta}$ we obtain that $\frac{dLiap(x)}{dt} \leq 0$, for any x and $\frac{dLiap(x)}{dt} \equiv 0 \Rightarrow h(x) \equiv 0$ on trajectory. Using the zero-state detectability property of our system and LaSalle's Invariance Principle (as in the proof of the Circle Criterion) we obtain that the origin of the closed loop system with the feedback $u = -\varphi(y)$ is asymptotically stable.

In order to obtain an uniformley estimation of the attraction basin of the origin we try to uniformly bound the Liapunov functions. Since $\Psi_1 \leq 0$ and $\Psi_2 \geq 0$ we get: $-V_{2a}(x) \leq Liap(x)$ for any $x \in \mathcal{U}$ and $\varphi \in N_{\alpha,\beta}$. Next we see that:

$$-\Psi_1(x) = \int_0^{h(x)} (\varphi(u) - \alpha(u))du \leq \int_0^{h(x)} (\beta(u) - \alpha(u))du$$

$$\Psi_2(x) = \int_0^{h(x)} (\beta(u) - \varphi(u))du \leq \int_0^{h(x)} (\beta(u) - \alpha(u))du$$

and $\alpha_1 + \alpha_2 = |\alpha_0|$. Then:

$$Liap(x) \leq -V_{2a}(x) + |\alpha_0| \int_0^{h(x)} (\beta(u) - \alpha(u))du$$

We have denoted by $W(x)$ the right hand side of the above inequality. So we have obtained the following boundedness:

$$-V_{2a}(x) \leq Liap(x) \leq W_c(x)$$

for any $x \in \mathcal{U}$ and $\varphi \in N_{\alpha,\beta}$.

Now if $V_{2a}^{-1}([-a,0])$ is a connected compact set then also are $Liap^{-1}([0,a])$ and $W_c^{-1}([0,a])$ and the last one is included in every $Liap^{-1}([0,a])$. Now the proofs of both Theorems are complete.

6 An Example

Let us consider the following system:

$$\begin{cases} \dot{x} = -x^3 - xu \\ y = x^2 \end{cases} \tag{28}$$

and a sector of the form:

$$\alpha(x) = 0 \quad , \quad \beta(x) = kx \quad , \quad k > 0$$

We look for the largest k such that the closed loop system is absolute asymptotic stable with respect to the sector $N_{\alpha,\beta}$. First of all we see that the closed loop with some feedback $\varphi \in N_{\alpha,\beta}$ has the form:

$$\dot{x} = -x^3 + x\varphi(x^2)$$

and around the origin (if φ is of class $\mathcal{C}^1$): $\varphi(x^2) = \varphi'(0)x^2 + \mathcal{O}(x^4)$. We get:

$$\dot{x} = -(1 - \varphi'(0))x^3 + \mathcal{O}(x^4)$$

and the origin is asymptotically stable if $\varphi'(0) < 1$. Then we expect that the largest value of k to be 1.

On the other hand, Theorem 7 can not be applied here because the linearization of (28) is trivial. Then we use the Circle Criterion (4) to find conditions on k.

The equation (11) takes the form:

$$-x^2((\frac{dV}{dx})^2 + 2(2-k)x\frac{dV}{dx} + k^2x^2) = 0$$

and the solutions of this differential equation are:

$$V_{s,a}(x) = \frac{k - 2 \pm 2\sqrt{1-k}}{2}x^2$$

In order to check which one is the unstabilizable solution, we compute the feedback (3) and we get:

$$u_{s,a}(y) = (-1 \pm \sqrt{1-k})y$$

Then, we plug in the dynamics (28) and we obtain that the antistabilizable solution coresponds to the minus sign: $V_a(x) = (\frac{k}{2} - 1 - \sqrt{1-k})x^2$. We see that this solution is defined on the whole axis and, furtermore, it is radially unbounded (all of these for $k < 1$).

The second condition of Theorem 4 is fulfilled by , for instance, $\varphi(y,t) = \frac{k}{2}y$ and we also see that the pair $(h(x) = x^2, f(x) = -x^3)$ is zero-state observable.

Then all conditions of Theorem 4 are fulfilled and we conclude that the system (28) is globally absolute asymptotic stable with respect to the class $N_{\alpha,\beta}(t)$ for $k = 1$.

7 Connexions with the Linearized System

Let us consider the linearized system of (9) in the form:

$$\begin{cases} \dot{x} &= Ax + bu \\ y &= c^T x \end{cases}$$

and for the sector: $k_1 = \alpha'(0)$, $k_2 = \beta'(0)$. The equations (11) and (12) require quadratic solutions of the form: $V(x) = \frac{1}{2}x^T X x$ where X is solution of one of the following Riccati equations:

$$A^T X + XA - (Xb + \frac{k_1 + k_2}{2}c)(Xb + \frac{k_1 + k_2}{2}c)^T + k_1 k_2 cc^T = 0 \tag{29}$$

$$A^T X + XA - \frac{1}{1 + \alpha_0 c^T b}(Xb + \frac{k_1 + k_2}{2}c + \frac{\alpha_0}{2}A^T c)(Xb + \frac{k_1 + k_2}{2}c + \frac{\alpha_0}{2}A^T c)^T + k_1 k_2 cc^T = 0 \tag{30}$$

The existence condition of a solution for Riccati equation reduces, in fact, to a frequence condition via Popov's Positivity Theorem (we refer the reader to [Popov73] or to [IoWe93]). For instance, the Popov function associated to the above Riccati equations are:

$$\Pi_1(s) = 1 + \frac{k_1 + k_2}{2}(H(-s) + H(s)) + k_1 k_2 H(-s)H(s)$$

$$\Pi_2(s) = 1 + \frac{k_1 + k_2}{2}(H(-s) + H(s)) + k_1 k_2 H(-s)H(s) + \frac{\alpha_0}{2}s(H(s) - H(-s))$$

where $H(s) = c^T(sI - A)^{-1}b$, and the frequence condition is $\Pi(j\omega) \geq 0$, $\forall \omega \in \mathbf{R}$. If, moreover, the frequence inequality is strict and a controllability condition on the imaginary axis is fulfilled, then it results the dichotomy of the associated linear Hamiltonian and hyperbolicity of the nonlinear Hamiltonian vector field. Then the antistabilizable solution of the HJ equation is exactly the unstable manifold associated to the Hamiltonian vector field (for details see [Schaf91]). To be more specific we consider the following realisations of the above Popov functions:

$$\Pi_1(s) = \left[\begin{array}{cc|c} \mathrm{A} & 0 & \mathrm{b} \\ k_1 k_2 cc^T & -A^T & -\frac{k_1+k_2}{2}c \\ \hline \frac{k_1+k_2}{2}c^T & b^T & 1 \end{array}\right]$$

$$\Pi_2(s) = \left[\begin{array}{cc|c} \mathrm{A} & 0 & \mathrm{b} \\ k_1 k_2 cc^T & -A^T & -\frac{k_1+k_2}{2}c - \frac{\alpha_0}{2}A^T c \\ \hline \frac{k_1+k_2}{2}c^T + \frac{\alpha_0}{2}c^T A & b^T & 1 + \alpha_0 c^T b \end{array}\right]$$

So we get the following result:

THEOREM 7 *If the pair (A, b) is controllable, the frequence inequality is strict and the realisation of Popov function has no uncontrollable modes on the imaginary axis (all these conditions equivalent with the existence of both the stabilizable and anstistabilizable solutions of Riccati equation (29) or (30)) then the HJ equation of the nonlinear system (equation (11) or (12)) has locally both antistabilizable and stabilizable solutions.* □

8 Criteria of Absolute Stability for MIMO Systems

We can apply the same approach to nonlinear affine systems with many feedbacks. In this case we consider a MIMO nonlinear affine system with a same number of inputs and outputs, say m:

$$\begin{cases} \dot{x} &= f(x) + g(x) \cdot u \\ y_i &= h_i(x) \end{cases} \tag{31}$$

where f, g_i are vector fields of class $\mathcal{C}^1$ on $D \subset \mathbf{R}^n$, h_i are continuous scalar functions defined on D and $f(0) = 0$, $h(0) = 0$. We also consider two families of continuous scalar functions $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$ $\alpha_i, \beta_i : \mathbf{R} \to \mathbf{R}$ such that: $\alpha_i(0) = 0$, $\beta(0) = 0$ and $y\alpha_i(y) \leq y\beta_i(y)$. With these families we consider the following sectors:

$$\begin{array}{c} N^{(m)}_{\alpha,\beta}(t) = N_{\alpha_1,\beta_1}(t) \times N_{\alpha_2,\beta_2}(t) \times \ldots \times N_{\alpha_m,\beta_m}(t) \\ N^{(m)}_{\alpha,\beta} = N_{\alpha_1,\beta_1} \times N_{\alpha_2,\beta_2} \times \ldots \times N_{\alpha_m,\beta_m} \end{array} \quad (32)$$

The problem can be formulated in the same way as in 3 with the remark that the feedback $\varphi = (\varphi_1, \ldots, \varphi_m)$ belongs now to one of the classes defined above (see figure 2). We are going now to state

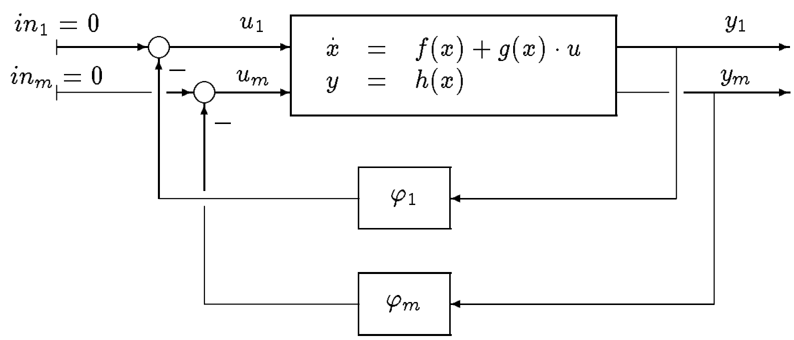

Figure 2: MIMO Closed-loop System

in the case of MIMO systems the corresponding theorems to 4 - 6. Since the proofs of these results are similar with those presented before we do not prove them but we point out, for the convenience of the reader, the nonlinear Popov systems considered in the proof of each one.

THEOREM 8 (Circle Criterion - nonlinear MIMO systems) *Let us consider the MIMO system given in (31) and a sector $N^{(m)}_{\alpha,\beta}(t)$ as in (32). For m positive real numbers $r_1, \ldots, r_m > 0$ we consider the following HJ equation:*

$$\nabla V(x) \cdot f(x) - \frac{1}{4} \sum_{i=1}^{m} \frac{1}{r_i} (\nabla V(x) \cdot g_i(x) + r_i(\alpha_i(h_i(x)) + \beta_i(h_i(x))))^2 + \sum_{i=1}^{m} r_i \alpha_i(h_i(x)) \beta_i(h_i(x)) = 0 \quad (33)$$

If the following conditions are fulfilled:

1) There exists an antistabilizable solution $V_a : \mathcal{U} \subset D \to \mathbf{R}$ of class $\mathcal{C}^1$ of (33) ($\mathcal{U}$ being a neighborhood of the origin);

2) There exists a feedback $\varphi(.,.) \in N^{(m)}_{\alpha,\beta}(t)$ ($\varphi = (\varphi_1, \ldots, \varphi_m)$) that asymptotically stabilizes the closed loop and has $\mathcal{U}$ included in the attraction domain;

3) The pair (h, f) is zero-state detectable ($h = (h_1, \ldots, h_m)$);

then the system (31) is local absolute asymptotic stable with respect to the class $N^{(m)}_{\alpha,\beta}$ and, furthermore, the largest connected compact set of the form $V_a^{-1}([-a, 0]) \subset \mathcal{U}$ is included in the attraction domain of the origin for any nonlinearity in the sector $N^{(m)}_{\alpha,\beta}(t)$. □

Remark. In the proof of Theorem 8 we need consider the following nonlinear Popov system:

$$P \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x) \cdot u \\ J(t_1) & = & \int_0^{t_1} \sum_{i=1}^{m} r_i(\alpha_i(h_i(x)) + u_i)(\beta_i(h_i(x)) + u_i) dt \end{array} \right.$$

We see that we can scale the constants r_i in (33) without lose of generality; then we can require, for instance, $\sum_{i=1}^{m} r_i = 1$. We state now the two versions of Popov criterion for the MIMO systems:

THEOREM 9 (Popov Criterion - nonlinear MIMO systems) *We consider the nonlinear MIMO system (31) for which h is of class $\mathcal{C}^1$, and $N^{(m)}_{\alpha,\beta}$ a sector. Suppose the denominators of the following HJ equation do not vanish on D:*

$$\nabla V(x)\cdot f(x) - \frac{1}{4}\sum_{i=1}^{m}\frac{1}{r_i}\frac{\left(\nabla V(x)\cdot g_i(x) + r_i(\alpha_i(h_i(x)) + \beta_i(h_i(x)) + \alpha_0^i \nabla h_i(x)\cdot f(x))\right)^2}{1+\alpha_0^i\nabla h_i(x)\cdot g_i(x)} + \sum_{i=1}^{m} r_i\alpha_i(h_i(x))\beta_i(h_i(x)) = 0 \tag{34}$$

where $\alpha_0^1,\ldots,\alpha_0^m \in \mathbf{R}$ are real parameters and $r_1,\ldots,r_m > 0$ are positive numbers as above. If the following conditions are fulfilled:

1) There exists an antistabilizable solution $V_a : \mathcal{U} \subset D \to \mathbf{R}$ of class $\mathcal{C}^1$ of the equation (34) ($\mathcal{U}$ being a neighborhood of the origin);

2) There exists a feedback $\varphi = (\varphi_i)$ whose entries fulfiles the following rules:

- *If $\alpha_0^i < 0$ then $\varphi_i(y) = \beta_i(y)$;*
- *If $\alpha_0^i > 0$ then $\varphi_i(y) = \alpha_i(y)$;*
- *If $\alpha_0^i = 0$ then $\varphi_i(.) \in N_{\alpha_i,\beta_i}$;*

that asymptotically stabilizes the closed loop and $\mathcal{U}$ is included in the attraction domain;

3) The pair (h, f) is zero-state detectable;

then the system (31) is local absolute asymptotic stable. Set:

$$V_{2a}(x) = V_a(x) + \sum_{i=1}^{m}\alpha_1^i\int_0^{h_i(x)}\alpha_i(u)du - \sum_{i=1}^{m}\alpha_2^i\int_0^{h_i(x)}\beta_i(u)du$$

and

$$W_c(x) = -V_{2a}(x) + \sum_{i=1}^{m}|\alpha_0^i|\int_0^{h_i(x)}(\beta_i(u)-\alpha_i(u))du$$

where $\alpha_1^i - \alpha_2^i = \alpha_0^i$, $\alpha_1^i\alpha_2^i = 0$, $\alpha_1^i, \alpha_2^i \geq 0$. Then, for any positive number $a > 0$ such that the connected component of the origin of $V_{2a}^{-1}([-a,0])$ is a compact set included in $\mathcal{U}$, the compact set $W_c^{-1}([0,a])$ is included in the attraction domain of the origin for any feedback in the sector $N^{(m)}_{\alpha,\beta}$. □

THEOREM 10 (Popov Criterion - MIMO systems, special case) *Let us consider the nonlinear system (31) for which h is of class $\mathcal{C}^1$ and a nonlinear sector $N^{(m)}_{\alpha,\beta}$ for which $\beta_i - \alpha_i$ are linear maps ($\beta_i(y) - \alpha_i(y) = k_i y$, $k_i > 0$). Suppose the denominators of the HJ equation (34) do not vanish on D and α_0^i, r_i are as above. Then, if:*

1) There exists an antistabilizable solution $V_a : \mathcal{U} \subset D \to \mathbf{R}$ of class $\mathcal{C}^1$ of the equation (34) ($\mathcal{U}$ being a neighborhood of the origin);

2) There exists $\xi_i \in [0,1]$ such that the feedback:

$$\varphi_s(y) = (\xi_i\alpha_i(y) + (1-\xi_i)\beta_i(y))$$

asymptotically stabilizes the closed loop and $\mathcal{U}$ is included in the attraction domain of the origin;

3) The pair (h, f) is zero-state detectable;

then the system (31) is local absolute asymptotic stable. Set:

$$V_{2a}(x) = V_a(x) + \sum_{i=1}^{m}\alpha_1^i\int_0^{h_i(x)}\alpha_i(u)du - \sum_{i=1}^{m}\alpha_2^i\int_0^{h_i(x)}\beta_i(u)du$$

and

$$W_c(x) = -V_{2a}(x) + \sum_{i=1}^{m} \frac{k_i|\alpha_0^i|}{2} h_i^2(x)$$

where $\alpha_1^i - \alpha_2^i = \alpha_0^i,\ \alpha_1^i\alpha_2^i = 0,\ \alpha_1^i, \alpha_2^i \geq 0$. *Then, for any positive number* $a > 0$ *such that the connected component of the origin of* $V_{2a}^{-1}([-a, 0])$ *is a compact set included in* $\mathcal{U}$, *the compact set* $W_c^{-1}([0, a])$ *is included in the attraction domain of the origin for any feedback in the sector* $N_{\alpha,\beta}^{(m)}$. □

Remarks. The equivalent nonlinear Popov systems considered in the proof of previous theorems are:

$$P_1 \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x) \cdot u \\ J_1(t_1) & = & \int_0^{t_1} \sum_{i=1}^m [r_i \alpha_i(h_i(x))\beta_i(h_i(x)) + r_i(\alpha_i(h_i(x)) + \beta_i(h_i(x)) + \alpha_0^i \nabla h_i(x) \cdot f(x))u_i + \\ & & + r_i(1 + \alpha_0^i \nabla h_i(x) \cdot g_i(x))u_i^2] dt \end{array} \right.$$

$$P_2 \left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x) \cdot u \\ J_2(t_1) & = & \int_0^{t_1} \sum_{i=1}^m r_i(\alpha_i(h_i(x)) + u_i)(\beta_i(h_i(x)) + u_i) dt + \int_0^{t_1} \sum_{i=1}^m \alpha_1^i r_i(\alpha_i(h_i(x)) + u_i) \frac{dh_i(x)}{dt} dt - \\ & & - \int_0^{t_1} \sum_{i=1}^m \alpha_2^i r_i(\beta_i(h_i(x)) + u_i) \frac{dh_i(x)}{dt} dt \end{array} \right.$$

The other remarks concerning the global absolute asymptotic stability also hold here in the same form.

9 Conclusions

In this paper the problem of absolute asymptotic stability for nonlinear affine control systems is presented and sufficiency conditions of absolute asymptotic stability are given.

The main tools for proving these criteria are the Hamilton-Jacobi equation and the nonlinear Kalman-Yakubovich-Popov system associated to a nonlinear Popov system. Connexions between these objects are presented in the first section.

In the second section we are dealing with nonlinear Popov system composed by a nonlinear affine dynamics and a quadratic criterion in control. We describe two properties of this system, namely the minimal stability and the equivalence of nonlinear Popov systems. The first property guarantees the sign of the antistabilizable solution of HJ equation, whereas the second property enables us to obtain some simpler equivalent HJ equations.

In the next section we present the statement of the criteria. These correspond, in the liniar case, to Circle and Popov Criteria. We point out that our criteria give not only conditions of (local) absolute asymptotic stability, but also they give some uniform estimations of the attraction domain of the origin (which is supposed to be the equilibrium point). The Circle Criterion corresponds to a feedback independent Liapunov function obtained from the solution of some HJ equation (equation (11)). For the linear Popov Criterion we obtain two versions for nonlinear statement. They differ by some condition which achieves the minimal stability of a certain nonlinear Popov system (P_2 in the proof): the first version applyes to general sectors and requires that a boundary feedback asymptotically stabilizes the closed loop, whereas the second version is useful only for special sectors, namely for those which fulfills the condition $\beta - \alpha$ is a linear map.

In the next two sections we prove these criteria using the properties of nonlinear Popov systems and, particularly, relation (6).

Then we give an example of dimension two.

In the seventh section we present connexions with the linearized system. Actually, for linearized systems the HJ equations turn into the classical Algebraic Riccati Equation and the quadratic form constructed with the hermitic solutions of ARE are lower order approximations of solutions of the original HJ equation. Furthermore, as it was shown by van der Schaft, the antistabilizable (stabilizable) solution of HJ equation corresponds, in the case of dichotomy, to the unstable (stable) manifold of the Hamiltonian vector field associated to the nonlinear Popov system.

Further on, we generalize these criteria to the case of multiple feedbacks. We present only the statement of these criteria and we give the form of nonlinear Popov systems involved in the proofs.

From this point one can study some specific cases, for instance the case of bilinear systems, and develop a theory of the HJ equations and nonlinear Popov systems for these systems.

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