Lipschitz Embeddings and Riemannian Properties of Spaces of Low-Rank Symmetric Matrices

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Overview

The Complex Phase Retrieval Problem: Variants

- Continuous Fourier/Windowed Fourier: Recover $f \in \mathcal{B} \subset \{f \in S'(\mathbb{R}) | \hat{f} \in L^1_{loc}(\mathbb{R})\}$ from $|\hat{f}|$ or $|V_g f|$ (for some known window g). Only possible if $\mathcal{B}$ is sufficiently restrictive - for example if $\hat{f}$ is taken to have compact support or is supported in the half line [KST95, Jam14, AW21, GL22]
- Discrete Fourier/Windowed Fourier: Recover $f = (f[0], \ldots, f[n-1]) \in \mathbb{C}^n$ from the (typically squared) magnitudes of its DFT coefficients $y[k] = |\sum_{j=0}^{n} y[j] e^{2\pi i kj/n}|^2$ [Fie82, Hay82, IPSV20, IMP19, PS19].
- Separable Hilbert space: Take H a separable complex Hilbert space. Recover $z \in H$ from $(|\langle z, f_k \rangle|)_{k \in I}$ where $(f_k)_{k \in I} \subset H$ is a frame for H [CCD16].
- Finite Hilbert space: Recover $z \in H = \mathbb{C}^n$ from $(|\langle z, f_k \rangle|)_{k=1}^m$ where $(f_k)_{k=1}^m$ is a frame for $\mathbb{C}^n$ [...].
- Phase Retrieval with generalized frames: Recover $z \in H = \mathbb{C}^n$ from $\langle z, A_j z \rangle$ where $(A_j)_{j=1}^m$ is a generalized frame of Hermitian matrices (termed measurement matrices). Note that $A_j = f_j f_j^*$ gives the finite Hilbert space case. [WX19]

In all such cases recovery is only ever possible up to an overall phase - that is to say modulo the action of $U(1)$.

Applications

- Inverse Problem in Potential Scattering - Determine potential / surface structure from (typically x-ray or neutron) scattering matrix.[KST95]
- Thin film optics - Inferring dielectric permittivity $\epsilon(z)$ of medium from the frequency dependence of the ratio $R(k)$ of the strength of transmitted and reflected tangential components.[KST95]
- Coherent Diffraction Imaging - infer shape of object in imaging plane from the diffraction pattern it produces under a coherent beam.[HTSL20]
- X-ray crystallography - infer electron density function $\rho(r) = \sum_{i=1}^{N} r_i \delta(r - r_i)$ of a single crystal cell from the measured diffraction pattern. [KH91]
- Speech recognition - the human ear is quite reliably "phase deaf," determining what has been said only from the magnitude spectrum of a signal. [DJPH93]
- Pure state quantum tomography - inferring the state of a quantum system (represented by a vector in a Hilbert space) from potentially noisy measurements.[BBCE09][KW15]

Motivating Application: Mixed Quantum Tomography

A mixed state quantum system is modeled as a statistical ensemble over pure quantum states living in a Hilbert space H. The standard example is unpolarized light. In such cases, all of the measurable information in the system is contained in a density matrix:

$$\rho = \sum_{j\in\mathcal{I}} p_j \psi_j \psi_j^*$$

- p_j - ensemble probability of being in pure state ψ_j: $\sum_{i\in\mathcal{I}} p_j = 1$.
- $\psi_j \in H$ - a pure state: Given an observable (Hermitian matrix) A with eigenpair (v,λ) we have $\Pr_{\psi_j}[A \text{ takes value } \lambda] = |\langle v, \psi_j\rangle|^2$.

If we take $H = \mathbb{C}^n$ and $|\mathcal{I}| = r$ then ρ is a positive semi-definite matrix of rank at most r and having unit trace, we write $\rho \in S^{r,0}(\mathbb{C}^n) \cap \{x \in \mathrm{Sym}(\mathbb{C}^n) | \mathrm{tr}\{x\} = 1\}$, where $S^{r,0}(\mathbb{C}^n)$ denotes the set of PSD matrices of rank at most r. The goal of quantum tomography is to infer ρ from measurements $(tr\{\rho A_j\})_{j\in[m]}$ given by a collection of observables $(A_j)_{j=1}^m$.

Motivating Application: Mixed Quantum Tomography

The expectation of an observable A_j in mixed state ρ is

$$\mathbb{E}_\rho[A_j] = \sum_{k=1}^{r} p_k \langle \psi_k, A_j \psi_k \rangle = \sum_{k=1}^{r} p_k \mathrm{tr}\{\psi_k \psi_k^* A_j\} = \mathrm{tr}\{\rho A_j\} = \langle \rho, A_j \rangle$$

By repeatedly measuring our observables and allowing the system to "relax" we may obtain these expectations to within a small error. Since $\rho \in S^{r,0}(\mathbb{C}^n)$ we may write via Cholesky factorization for some $z \in \mathbb{C}^{n\times r}$

$$\rho = zz^*$$

Note ρ is unchanged by $z \mapsto zU$ for $U \in U(r)$, so the problem becomes to stably recover z modulo $U(r)$ (a "unitary phase") from $(\langle zz^*, A_j \rangle)_{j=1}^m$. In particular we would like the following map to be injective (and indeed lower Lipschitz):

$$\beta : \mathbb{C}^{n\times r}/U(r) \to \mathbb{R}^m$$
$$\beta(z) = (\langle zz^*, A_j \rangle)_{j=1}^m$$

A generalized frame $(A_j)_{j=1}^m$ for which β is injective is called $U(r)$ phase retrievable.

$U(r)$ phase retrievability

A generalized frame $\mathcal{A} = (A_j)_{j=1}^m$ for which β is injective is called $U(r)$ phase retrievable.

- As for $U(1)$, $U(r)$ phase retrievability is a stronger condition than being a generalized frame for $\mathbb{C}^{n\times r}$.
- If $\mathcal{A}$ is a frame for $\mathrm{Sym}(\mathbb{C}^n)$ itself then it is automatically $U(r)$ phase retrievable.
- if $\mathcal{A}$ is $U(r)$ phase retrievable then it is $U(k)$ phase retrievable for any $1 \leq k \leq r$, in particular it is phase retrievable.

Thus the concept of being $U(r)$ phase retrievable is an intermediate between being phase retrievable for $\mathbb{C}^n$ and being a frame for $\mathrm{Sym}(\mathbb{C}^n)$. Another way to think about $U(r)$ phase retrieval is as low rank positive semi-definite matrix recovery [Xu18].

In analogy with the pure state case in which one is also interested in the stable recovery properties of the non-linear measurement map $\alpha_j(x) = |\langle x, f_j\rangle|$ we define

$$\alpha : \mathbb{C}^{n\times r}/U(r) \to \mathbb{R}^m$$

$$\alpha(z) = (\langle zz^*, A_j\rangle^{\frac{1}{2}})_{j=1}^m$$

The problem

$$\beta, \alpha : \mathbb{C}^{n\times r}/U(r) \to \mathbb{R}^m$$
$$\beta(z) = (\langle zz^*, A_j\rangle)_{j=1}^m$$
$$\alpha(z) = (\langle zz^*, A_j\rangle^{\frac{1}{2}})_{j=1}^m$$

The problem is then to

- Identify appropriate distances on $\mathbb{C}^{n\times r}/U(r)$ to use for stability analysis of α and β.
- Find out whether β (resp. α) is lower Lipschitz on its range whenever $(A_j)_{j=1}^m$ is $U(r)$ phase retrievable.
- If so, provide a means of computing the lower Lipschitz constant for β (resp. α).
- Give if and only if criteria for a given frame of observables to be phase retrievable.

Metric Space Structures

We define the equivalence relation $\sim$ on $\mathbb{C}^{n\times r}$ via

$$x \sim y \iff \exists U \in U(r) | x = yU$$

and denote by $[x]$ the equivalence class of $x \in \mathbb{C}^{n\times r}$, and by $\mathbb{C}^{n\times r}/U(r)$ the set of equivalence classes $\{[x]|x \in \mathbb{C}^{n\times r}\}$. We define $D, d, \delta : \mathbb{C}^{n\times r} \times \mathbb{C}^{n\times r} \to \mathbb{R}$:

$$D(x,y) = \min_{U\in U(r)} ||x - yU||_2 = \sqrt{||x||_2^2 + ||y||_2^2 - 2||x^*y||_1}$$

$$d(x,y) = \min_{U\in U(r)} ||x - yU||_2||x + yU||_2 = \sqrt{(||x||_2^2 + ||y||_2^2)^2 - 4||x^*y||_1^2}$$

$$\delta(x,y) = ||xx^* - yy^*||_2$$

- D is known as the Bures-Wasserstein distance, or the "natural" distance. Note for $\lambda \in \mathbb{C}$, $D(\lambda x, \lambda y) = |\lambda| D(x,y)$, so D is appropriate for analyzing the α map.
- d scales like $d(\lambda x, \lambda y) = |\lambda|^2 d(x,y)$ and is appropriate for analyzing β and so is δ, the matrix norm induced distance.
- d, D, δ are not Lipschitz equivalent but they do generate the same topology on $\mathbb{C}^{n\times r}/U(r)$.

The new (and mysterious) distance d

It is easy to show that D, $d(x,y) = \min_{U \in U(r)} ||x - yU||_2$ is a (semi)distance.
For d, $d(x,y) = \min_{U \in U(r)} ||x - yU||_2 ||x + yU||_2$, it is easy to show positivity and symmetry. The tricky part is the triangle inequality.
For $r = 1$ real-case see [EM14, BCMN14, Sal19] (the last paper analyzes the complex-case as well).
However it has never been explicitly mentioned it is a metric.
In the real-case the triangle inequality reduces to a statement about the analytic geometry of parallelipipeds in $\mathbb{R}^3$, namely that the sum of the products of face diagonals on any two sides sharing a vertex exceeds the product of the third side sharing the vertex.
On the other hand, in the real-case, for $x, y \in \mathbb{R}^n$,
$||x - y||_2 ||x + y||_2 = ||xx^T - yy^T||_1$; in the complex case, for $x, y \in \mathbb{C}^n$,

$$\min_{\varphi} ||x - e^{i\varphi} y||_2 ||x + e^{i\varphi} y||_2 = ||xx^* - yy^*||_1.$$

This identity implies the triangle inequality in the case $r = 1$.
For $r > 1$, the identity $\min_{U \in U(r)} ||x - yU||_2 ||x + yU||_2 = ||xx^* - yy^*||_1$ fails in general for $x, y \in \mathbb{C}^{n \times r}$! However both sides define (inequivalent) distances!

Lipschitz Embeddings

We would like to embed the metric spaces $(\mathbb{C}^{n\times r}/U(r), D \text{ or } d)$ into $(\mathrm{Sym}(\mathbb{C}^n), ||\cdot||_2)$ in a (bi)Lipschitz way. Defining invertible maps (modulo $\sim$) $\theta, \pi, \psi : \mathbb{C}^{n\times r} \to S^{r,0}(\mathbb{C}^n)$

$$\theta(x) = (xx^*)^{\frac{1}{2}} \qquad \pi(x) = xx^* \qquad \psi(x) = ||x||_2(xx^*)^{\frac{1}{2}}.$$

Theorem ([BD22])

(i) $\theta : (\mathbb{C}^{n\times r}/U(r), D) \to (S^{r,0}(\mathbb{C}^n), ||\cdot||_2)$ *is a bi-Lipschitz map:*

$$\frac{1}{\sqrt{2}}||\theta(x)-\theta(y)||_2 \leq D(x,y) \leq ||\theta(x)-\theta(y)||_2 \qquad \forall x,y \in \mathbb{C}^{n\times r}/U(r)$$

(ii) $\pi, \psi : (\mathbb{C}^{n\times r}/U(r), d) \to (S^{r,0}(\mathbb{C}^n), ||\cdot||_1)$ *are upper and lower Lipschitz:*

$$||\pi(x)-\pi(y)||_1 \leq d(x,y) \leq 2||\psi(x)-\psi(y)||_2 \qquad \forall x,y \in \mathbb{C}^{n\times r}/U(r)$$

(iii) *For* $r=1$ *we have* $d(x,y) = ||\pi(x)-\pi(y)||_1$

(iv) *For* $r>1$*, there is no constant* C *satisfying* $d(x,y) \leq C||\pi(x)-\pi(y)||_2$ *for all* $x,y \in \mathbb{C}^{n\times r}/U(r)$ *(hence the use of the alternate embedding* ψ*).*

Lipschitz Constants

With these embeddings in mind we define lower Lipschitz bounds:

$$a_0 = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{||\beta(x)-\beta(y)||_2^2}{||\pi(x)-\pi(y)||_2^2} = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{\sum_{j=1}^m (\langle xx^*, A_j\rangle_{\mathbb{R}} - \langle yy^*, A_j\rangle_{\mathbb{R}})^2}{||xx^*-yy^*||_2^2}$$

$$A_0 = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{||\alpha(x)-\alpha(y)||_2^2}{||\theta(x)-\theta(y)||_2^2} = \inf_{\substack{x,y\in\mathbb{C}^{n\times r}\\ [x]\neq[y]}} \frac{\sum_{j=1}^m (\langle xx^*, A_j\rangle_{\mathbb{R}}^{\frac{1}{2}} - \langle yy^*, A_j\rangle_{\mathbb{R}}^{\frac{1}{2}})^2}{||(xx^*)^{\frac{1}{2}}-(yy^*)^{\frac{1}{2}}||_2^2}$$

Assume $(A_j)_{j\in[m]}$ is $U(r)$ phase retrievable for $\mathbb{C}^{n\times r}$. Then we showed that:

1. The bound $a_0 > 0$ and provided a means of computing it for any $r \geq 1$; hence $\beta : (\mathbb{C}^{n\times r}/U(r), \delta) \to (\mathrm{Sym}(\mathbb{C}^n), ||\cdot||_2)$ is bi-Lipschitz, where $\delta(x,y) = ||xx^* - yy^*||_2$.
2. However $A_0 = 0$ for $r > 1$! Thus the α map is not Lipschitz invertible for $r > 1$ with respect to any of the three metrics d, D or δ, nor matrix norm induced distances via θ, π, ψ.

Geometry of $S^{r,0}(\mathbb{C}^n)$

To compute a_0 and A_0 we need to linearize π and θ and find their actions on the "tangent bundle" of $S^{r,0}(\mathbb{C}^n)$. $S^{r,0}(\mathbb{C}^n)$ is only a semi-algebraic variety, however, so we need to understand its singular structure and whether the linearized problem suffices when "boundary manifolds" are encountered. We showed that $S^{r,0}(\mathbb{C}^n)$ has a Whitney stratification over the smooth Riemannian manifolds $\mathring{S}^{i,0}(\mathbb{C}^n)$ (PSD matrices of rank exactly i) for $i = 0, \ldots, r$ having real dimension $2ni - i^2$.

Definition (a-regular, b-regular, from [Kal00])

Let V_i, V_j be disjoint real manifolds embedded in $\mathbb{R}^d$ such that $\dim V_j > \dim V_i$ and $V_i \cap \overline{V_j}$ non-empty. Let $x \in V_i \cap \overline{V_j}$. Then a triple (V_j, V_i, x) is called *a*− (resp. *b*−) regular if

- (a) If a sequence $(y_n)_{n\geq 1} \subset V_j$ converges to x in $\mathbb{R}^d$ and $T_{y_n}(V_j)$ converges in the Grassmannian $\mathrm{Gr}_{\dim V_j}(\mathbb{R}^d)$ to a subspace τ_x of $\mathbb{R}^d$ then $T_x(V_i) \subset \tau_x$.
- (b) If sequences $(y_n)_{n\geq 1} \subset V_j$ and $(x_n)_{n\geq 1} \subset V_i$ converge to x in $\mathbb{R}^d$, the unit vector $(x_n - y_n)/||x_n - y_n||_2$ converges to a vector $v \in \mathbb{R}^d$, and $T_{y_n}(V_j)$ converges in the Grassmannian $\mathrm{Gr}_{\dim V_j}(\mathbb{R}^d)$ to a subspace τ_x of $\mathbb{R}^d$ then $v \in \tau_x$.

Geometry of $S^{r,0}(\mathbb{C}^n)$

Definition (Whitney stratification, from [Kal00])

Let V be a real semi-algebraic variety. A disjoint decomposition

$$V = \bigsqcup_{i \in I} V_i, \qquad V_i \cap V_j = \emptyset \text{ for } i \neq j$$

into smooth manifolds $\{V_i\}_{i \in I}$, termed strata, is a Whitney stratification if

(a) Each point has a neighborhood intersecting only finitely many strata

(b) The boundary sets $\overline{V_j} \setminus V_j$ of each stratum V_j are unions of other strata.

(c) Every triple (V_j, V_i, x) such that $x \in V_i \subset \overline{V_j}$ is *a*-regular and *b*-regular.

The point is that there is a compatibility between the stratifying manifolds - if you are in the tangent space of lower dimensional strata you are in a limiting sense also in the tangent space of higher strata. This gives the semi-algebraic variety more structure, and as we'll see in this case enables us to find what almost looks like a Riemannian geometry on the whole of $S^{r,0}(\mathbb{C}^n)$ (even though it isn't a manifold).

Geometry of $S^{r,0}(\mathbb{C}^n)$

We will stratify $S^{r,0}(\mathbb{C}^n)$ as $\sqcup_{i=0}^{r}\mathring{S}^{i,0}(\mathbb{C}^n)$, where $\mathring{S}^{i,0}(\mathbb{C}^n)$ is the set of positive semi-definite matrices of rank exactly i.

Theorem ([BD22]; see also [BJL19])

Let D be the Bures-Wasserstein (a.k.a., the natural) distance. Then

1. $\mathring{S}^{p,q}(\mathbb{C}^n)$ *is a real analytic manifold with* $dim_{\mathbb{R}}(\mathring{S}^{p,q}(\mathbb{C}^n)) = 2n(p+q)-(p+q)^2$.
2. $\pi : \mathbb{C}_*^{n\times r} \to \mathring{S}^{r,0}(\mathbb{C}^n)$ *can be made into a Riemannian submersion by choosing the unique Riemannian metric on* $\mathring{S}^{r,0}(\mathbb{C}^n)$*, for* $Z_1, Z_2 \in T_X(\mathring{S}^{r,0}(\mathbb{C}^n))$*:*
$$h_X^r(Z_1,Z_2) = tr\{Z_2^{||}\int_0^\infty e^{-uX}Z_1^{||}e^{-uX}du\} + \Re tr\{Z_1^{\perp *}Z_2^{\perp}X^{\dagger}\}$$
where $Z_i^{||} = \mathbb{P}_{Ran(X)}Z_i\mathbb{P}_{Ran(X)}$, $Z_i^{\perp} = \mathbb{P}_{Ran(X)^{\perp}}Z_i\mathbb{P}_{Ran(X)}$ *and the real Hilbert-Schmidt inner product as metric on* $T_{\pi^{-1}(X)}(\mathbb{C}_*^{n\times r})$.
3. $(\mathring{S}^{r,0}(\mathbb{C}^n), h^r)$ *is a Riemannian manifold with* $D\circ(\pi^{-1}\times\pi^{-1})$ *as its geodesic distance.*
4. $S^{r,0}(\mathbb{C}^n)$ *admits as a Whitney stratification* $(\mathring{S}^{i,0})_{i=0}^{r}$.

Geometry of $S^{r,0}(\mathbb{C}^n)$

We will stratify $S^{r,0}(\mathbb{C}^n)$ as $\sqcup_{i=0}^{r}\mathring{S}^{i,0}(\mathbb{C}^n)$, where $\mathring{S}^{i,0}(\mathbb{C}^n)$ is the set of positive semi-definite matrices of rank exactly i.

Theorem ([BD22])

The geometry associated to h is compatible with the Whitney stratification in the following sense: If $(A_i)_{i\geq 1}, (B_i)_{i\geq 1} \subset \mathring{S}^{p,0}$ have limits A and B respectively in $\mathring{S}^{q,0}$ for $q < p$ and if $\gamma_i : [0,1] \to \mathring{S}^{p,0}$ are geodesics in $\mathring{S}^{p,0}$ connecting A_i to B_i chosen in such a way that the limiting curve $\delta : [0,1] \to \overline{\mathring{S}^{p,0}}$ given by

$$\delta(t) = \lim_{i\to\infty} \gamma_i(t)$$

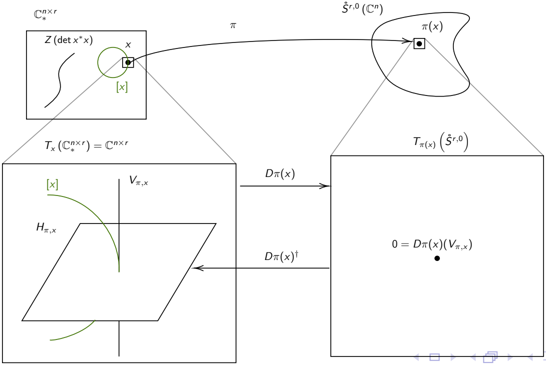
exists, then the image of δ lies in $\mathring{S}^{q,0}$ and is a geodesic curve in $\mathring{S}^{q,0}$ connecting A to B.

Another way to look at this is if $0 \leq q \leq p \leq r$ and $X = xx^* \in \mathring{S}^{p,0}$, $Y = yy^* \in \mathring{S}^{q,0}$ and $\gamma_{X_1,X_2} : [0,1] \to \mathring{S}^{p,0}$ is the geodesic connecting X_1 to X_2 then

$$D(x,y)^2 = \min_{U\in U(r)} ||x - yU||_2^2 = \lim_{\substack{Z\to Y \\ Z\in \mathring{S}^{p,0}(\mathbb{C}^n)}} \int_0^1 h^p_{\gamma_{X,Z}(t)}(\gamma'_{X,Z}(t), \gamma'_{X,Z}(t))dt$$

Geometry of $S^{r,0}(\mathbb{C}^n)$ via $\mathbb{C}^{n\times r}$

We may view $S^{r,0}(\mathbb{C}^n)$ as the image under $\pi : x \mapsto \pi(x) = xx^*$ of $\mathbb{C}^{n\times r}$, and each stratifying manifold $\mathring{S}^{i,0}(\mathbb{C}^n)$ as the image of $[\mathbb{C}_*^{n\times i}|0^{n\times(r-i)}]$ (the $*$ means full rank). This is surjective, but not injective owing to the ambiguity $U(r)$. We can compute the differential $D\pi(z)(w) = zw^* + wz^*$, its kernel $V_{\pi,x}(\mathbb{C}_*^{n\times r})$ (the vertical space), and the orthogonal complement of its kernel $H_{\pi,x}(\mathbb{C}_*^{n\times r})$ (the horizontal space) which maps one to one onto the tangent space of $\mathring{S}^{i,0}(\mathbb{C}^n)$.



Geometry of $S^{r,0}(\mathbb{C}^n)$ via $\mathbb{C}^{n\times r}$

The spaces $V_{\pi,x}(\mathbb{C}_*^{n\times r})$, $H_{\pi,x}(\mathbb{C}_*^{n\times r})$ and $T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n))$ may be computed as

Lemma ([BD22])

Let $\pi : \mathbb{C}_*^{n\times r} \to \mathring{S}^{r,0}(\mathbb{C}^n)$ be as before and let $V_{\pi,x}(\mathbb{C}_*^{n\times r})$ and $H_{\pi,x}(\mathbb{C}_*^{n\times r})$ denote the vertical and horizontal spaces of the manifold $\mathbb{C}_*^{n\times r}$ at x with respect to the embedding π. Let $T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n))$ denote the tangent space of $\mathring{S}^{r,0}(\mathbb{C}^n)$ at $\pi(x)$. Then

$$\begin{aligned}
V_{\pi,x}(\mathbb{C}_*^{n\times r}) &= \{xK | K \in \mathbb{C}^{r\times r}, K^* = -K\} \\
H_{\pi,x}(\mathbb{C}_*^{n\times r}) &= \{Hx + X | H \in \mathbb{C}^{n\times n}, H^* = H = \mathbb{P}_{Ran(x)} H, \\
& X \in \mathbb{C}^{n\times r}, \mathbb{P}_{Ran(x)} X = 0\} \\
T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n)) &= \{W \in Sym(\mathbb{C}^n) | \mathbb{P}_{Ran(x)^\perp} W \mathbb{P}_{Ran(x)^\perp} = 0\} \\
&= D\pi(x)(H_{\pi,x}(\mathbb{C}_*^{n\times r}))
\end{aligned}$$

Note that $\dim_{\mathbb{R}}(V_{\pi,x}(\mathbb{C}_*^{n\times r})) = r^2$ and $\dim_{\mathbb{R}}(T_{\pi(x)}(\mathring{S}^{r,0}(\mathbb{C}^n))) = \dim_{\mathbb{R}}(H_{\pi,x}(\mathbb{C}_*^{n\times r})) = 2nr - r^2$.

The tangent space Lipschitz bounds

In our effort to obtain or at least control the global Lipschitz constant a_0 we define the following local lower Lipschitz constants:

$$a_1(z) = \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r} \\ ||\pi(x)-\pi(z)||_2<R}} \frac{||\beta(x)-\beta(z)||_2^2}{||\pi(x)-\pi(z)||_2^2}$$

$$a_2(z) = \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times r} \\ ||\pi(x)-\pi(z)||_2<R \\ ||\pi(y)-\pi(z)||_2<R}} \frac{(||\beta(x)-\beta(y)||_2^2}{||\pi(x)-\pi(y)||_2^2}$$

As well as the following geometric constant

$$a(z) := \min_{\substack{W\in T_{\pi(\hat{z})}(\mathring{S}^{k,0}(\mathbb{C}^n)) \\ ||W||_2=1}} \sum_{j=1}^{m} |\langle W, A_j\rangle_{\mathbb{R}}|^2$$

Where here $\hat{z}\in\mathbb{C}_*^{n\times k}$ is such that $z=[\hat{z}|0]U$ for some $U\in U(r)$ ($\hat{z}=z$ if z has rank r, and moreover the tangent space doesn't depend on the choice of $\hat{z}$).

The tangent space Lipschitz bounds

Given $z \in \mathbb{C}^{n\times r}$ having rank $k>0$ define $Q_z \in \mathbb{R}^{(2nk-k^2)\times(2nk-k^2)}$ as follows. Let $U_1 \in \mathbb{C}^{n\times k}$ be a matrix whose columns are left singular vectors of z corresponding to non-zero singular values of z, so that $U_1U_1^* = \mathbb{P}_{\mathrm{Ran}(z)}$. Let $U_2 \in \mathbb{C}^{n\times(n-k)}$ be a matrix whose columns are left singular vectors of z corresponding to the zero singular values of z, so that $U_2U_2^* = \mathbb{P}_{\mathrm{Ran}z^\perp}$. Then

$$Q_z := Q_{[U_1|U_2]} = \sum_{j=1}^m \begin{bmatrix} \tau(U_1^*A_jU_1) \\ \mu(U_2^*A_jU_1) \end{bmatrix} \begin{bmatrix} \tau(U_1^*A_jU_1) \\ \mu(U_2^*A_jU_1) \end{bmatrix}^T$$

where the isometric isomorphisms τ and μ are given by

$$\tau : \mathrm{Sym}(\mathbb{C}^k) \to \mathbb{R}^{k^2} \qquad \mu : \mathbb{C}^{p\times q} \to \mathbb{R}^{2pq}$$

$$\tau(X) = \begin{bmatrix} D(X) \\ \sqrt{2}T(\Re X) \\ \sqrt{2}T(\Im X) \end{bmatrix} \qquad \mu(X) = \mathrm{vec}(\begin{bmatrix} \Re X \\ \Im X \end{bmatrix})$$

where if $A \in \mathrm{Sym}(\mathbb{R}^n)$ $D(A)$ is the vectorization of its diagonal and and $T(A)$ is the vectorization of its upper triangular part.

The tangent space Lipschitz bounds

Theorem ([BD22])

- $(A_j)_{j=1}^m$ *is* $U(r)$ *phase retrievable if and only if* $a_0 > 0$.
- *The global lower bound* a_0 *is given as* $a_0 = \inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} a(z)$.
- *The local lower bounds* $a_1(z)$ *and* $a_2(z)$ *are squeezed between* $a_0 \leq a_2(z) \leq a_1(z) \leq a(z)$ *so that in particular* $a_0 = \inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} a_i(z)$.
- *The infimization problem in* $a(z)$ *may be reformulated as an eigenvalue problem. Let* Q_z *be as above. Then*

$$a(z) = \lambda_{2nk-k^2}(Q_z)$$

Corollary

Fix any $U_2 \in \mathbb{C}^{n\times n-r}$ *with orthonormal columns. We may compute* a_0 *as*

$$a_0 = \min_{\substack{U_1\in\mathbb{C}^{n\times r} \\ U=[U_1|U_2]\in U(n)}} \lambda_{2nr-r^2}(Q_{[U_1|U_2]})$$

The horizontal space Lipschitz bounds

An alternate method of controlling a_0 is to use the (new) distance d. We define for $z \in \mathbb{C}^{n\times r}$ with rank k the local lower Lipschitz constants

$$\hat{a}_1(z) = \lim_{R\to 0} \inf_{\substack{x\in\mathbb{C}^{n\times r} \\ d(x,z)<R \\ \text{rank}(x)\le k}} \frac{||\beta(x)-\beta(z)||_2^2}{d(x,z)^2}$$

$$\hat{a}_2(z) = \lim_{R\to 0} \inf_{\substack{x,y\in\mathbb{C}^{n\times r} \\ d(x,z)<R \\ d(y,z)<R \\ \text{rank}(x)\le k \\ \text{rank}(y)\le k}} \frac{||\beta(x)-\beta(y)||_2^2}{d(x,y)^2}$$

Unfortunately the rank constraints are necessary here - without them the constants would be zero. We also define the geometric constant

$$\hat{a}(z) = \min_{\substack{w\in H_{\pi,\hat{z}}(\mathbb{C}_*^{n\times k}) \\ ||w||_2=1}} \sum_{j=1}^{m} |\langle D\pi(\hat{z})(w), A_j\rangle_{\mathbb{R}}|^2$$

The horizontal space Lipschitz bounds

Given $z \in \mathbb{C}^{n\times r}$ having rank $k > 0$ define $\hat{Q}_z \in \mathbb{R}^{2nk\times 2nk}$ as follows. Let $F_j = \mathbb{I}_{k\times k} \otimes j(A_j) \in \mathbb{R}^{2nk\times 2nk}$ where

$$j : \mathbb{C}^{m\times n} \to \mathbb{R}^{2m\times 2n}$$

$$j(X) = \begin{bmatrix} \Re X & -\Im X \\ \Im X & \Re X \end{bmatrix}$$

is an injective homomorphism. Then

$$\hat{Q}_z := 4\sum_{j=1}^{m} F_j \mu(\hat{z})\mu(\hat{z})^T F_j$$

The horizontal space Lipschitz bounds

Theorem ([BD22])

- *For* $r = 1$ $\hat{a}(z)$ *differs from* $a(z)$ *by a constant factor hence* $\inf_{z\in\mathbb{C}^{n\times r}\setminus\{0\}} \hat{a}(z) > 0$. *For* $r > 1$ *this infimum is zero and there is no non-trivial global lower bound* $\hat{a}_0$ *analogous to* a_0 *for the natural metric* d.
- *The local lower bounds with respect to the alternate metric* d *satisfy*

$$\hat{a}_1(z) = \hat{a}_2(z) = \frac{1}{4||z||_2^2}\hat{a}(z)$$

- *The infimization problem in* $\hat{a}(z)$ *may be reformulated as an eigenvalue problem. Let* $\hat{Q}_z$ *be as above. Then* $\hat{a}(z)$ *is directly computable as*

$$\hat{a}(z) = \lambda_{2nk-k^2}(\hat{Q}_z)$$

- *We have the following local inequality relating* $a(z)$ *and* $\hat{a}(z)$.

$$\frac{1}{4||z||_2^2}\hat{a}(z) \leq a(z) \leq \frac{1}{2\sigma_k(z)^2}\hat{a}(z)$$

The horizontal space Lipschitz bounds

Theorem ([BD22])

- *While the fact that* $\hat{a}_0 = 0$ *makes clear that* a_0 *cannot be upper bounded by* $\inf_{z \in \mathbb{C}^{n\times r}\setminus\{0\}} \hat{a}(z)$, *we can achieve a similar end by constraining* z *to have orthonormal columns. Namely*

$$\frac{1}{4} \inf_{\substack{z \in \mathbb{C}_*^{n\times r} \\ z^* z = \mathbb{I}_{r\times r}}} \hat{a}(z) \leq a_0 \leq \frac{1}{2} \inf_{\substack{z \in \mathbb{C}_*^{n\times r} \\ z^* z = \mathbb{I}_{r\times r}}} \hat{a}(z)$$

Phase retrievability criteria

The last two theorems give criteria for a frame to be $U(r)$ phase retrievable.

Theorem ([BD22])

Let $\{A_j\}_{j=1}^m$ be a frame for $\mathbb{C}^{n\times r}$. Then the following are equivalent:

1. $\{A_j\}_{j=1}^m$ *is $U(r)$ phase retrievable.*
2. *For all $U_1 \in \mathbb{C}^{n\times r}$, $U_2 \in \mathbb{C}^{n\times(n-r)}$ such that $[U_1|U_2] \in U(n)$ the matrix*

$$Q_{[U_1|U_2]} = \sum_{j=1}^{m} \begin{bmatrix} \tau(U_1^* A_j U_1) \\ \mu(U_2^* A_j U_1) \end{bmatrix} \begin{bmatrix} \tau(U_1^* A_j U_1) \\ \mu(U_2^* A_j U_1) \end{bmatrix}^T$$

   is invertible.

3. *For all $z \in \mathbb{C}^{n\times r}$ such that z has orthonormal columns, the matrix*

$$\hat{Q}_z = 4\sum_{j=1}^{m} (\mathbb{I}_{k\times k} \otimes j(A_j))\mu(z)\mu(z)^T(\mathbb{I}_{k\times k} \otimes j(A_j))$$

Phase retrievability criteria

Theorem ([BD22])

(Continued)

1. *For all* $U_1 \in \mathbb{C}^{n\times r}$, $U_2 \in \mathbb{C}^{n\times(n-r)}$ *such that* $[U_1|U_2] \in U(n)$, $H \in Sym(\mathbb{C}^r)$, $B \in \mathbb{C}^{(n-r)\times r}$ *there exist* $c_1, \ldots c_m \in \mathbb{R}$ *such that*

$$U_1^*(\sum_{j=1}^m c_j A_j)U_1 = H \tag{1a}$$

$$U_2^*(\sum_{j=1}^m c_j A_j)U_1 = B \tag{1b}$$

2. *For all* $U_1 \in \mathbb{C}^{n\times r}$ *with orthonormal columns*

$$span_{\mathbb{R}}\{A_j U_1\}_{j=1}^m = \{U_1 K | K \in \mathbb{C}^{r\times r}, K^* = -K\}^\perp$$

The second criterion is a generalization of the result in [BCMN14] which says that a frame $(\phi_i)_{i=1}^m$ for $\mathbb{C}^n$ is phase retrievable iff

Other results in the paper

- We give a purely topological proof that $(A_j)_{j=1}^m$ phase retrievable implies $a_0 > 0$ (we do this before computing a_0).
- We prove using continuity of eigenvalues with respect to matrix entries that $A_0 = 0$ for $r > 1$.
- We compute local lower Lipschitz constants for α.
- We compute Lipschitz upper bounds b_0 and B_0.
- We show that our results reduce to those in [BZ16] for the case $r = 1$.
- We verify the lower Lipschitz theorems numerically.

Summary of differences between mixed and pure state case

$r = 1$ (pure state case)	$r > 1$ (mixed state case)
Phase ambiguity is scalar $e^{i\theta}$	Phase ambiguity is in $U(r)$
$(z_i)_{i\geq 1} \subset \mathbb{C}^n/U(1)$ with $\|\|z_i\|\|_2 = 1$ cannot approach zero	$(z_i)_{i\geq 1} \subset \mathbb{C}^{n\times r}/U(r)$ with $\|\|z_i\|\|_2 = 1$ can come ϵ close to dropping rank
$d(x,y) = \|\|xx^* - yy^*\|\|_1$	$\nexists C$ st. $d(x,y) \leq C\|\|xx^* - yy^*\|\|_p$
β is bi- Lipschitz wrt. d and $\delta(x,y) = \|\|xx^* - yy^*\|\|_2$	β is bi-Lipschitz wrt. $\|\|xx^* - yy^*\|\|_2$ Only locally lower Lipschitz wrt. d
$A_0 > 0$, α is bi-Lipschitz wrt. D and $\|\|(xx^*)^{\frac{1}{2}} - (yy^*)^{\frac{1}{2}}\|\|_2$	$A_0 = 0$, α is locally lower Lipschitz wrt. D and $\|\|(xx^*)^{\frac{1}{2}} - (yy^*)^{\frac{1}{2}}\|\|_2$

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