Phase Retrieval using the Iterative Regularized Least-Squares Algorithm

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IMA, University of Minnesota, Minneapolis, MN

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Joint work with Naveed Haghani (UMD).

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Phase Retrieval

The phase retrieval problem

Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and measurements

$$y_k = |\langle x, f_k \rangle|^2 + \nu_k \quad , \quad 1 \leq k \leq m.$$

The frame is said *phase retrievable* (or that it gives phase retrieval) if $\hat{x} \mapsto (|\langle x, f_k \rangle|)_{1 \leq k \leq m}$ is injective.

The general *phase retrieval problem* a.k.a. *phaseless reconstruction*: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover x from $y = (y_k)_k$ up to a global phase factor.

Our problem today: A reconstruction algorithm.

General Purpose Algorithms
Unstructured Frames. Unstructured Data

1. Iterative Algorithms:
   - Gerchberg-Saxton [Gerchberg&all]
   - Wirtinger flow - gradient descent [CLS14]
   - IRLS [B13]
2. Rank 1 Tensor Recovery:
   - PhaseLift; PhaseCut [CSV12]; [WdAM12]
   - Higher-Order Tensor Recovery [B09]

Specialized Algorithms

Structured Frames and/or Structured Data

1. Structured Frames:
   - Fourier Frames: 4n-4 [BH13]; Masking DFT [CLS13]; STFT/Spectograms [B.][Eldar&all][Hayes&all]; Alternating Projections [GriffinLim][Fannjiang]; Hybrid I-O [Fienup82]
   - Polarization: 3-term [ABFM12], masking [BCM]
   - Shift-Invariant Spaces: Bandlimited [Thakur11]; Filterbanks/Circulant Matrices [IVW2]; Other spaces [Chen&all]
   - X-Ray Crystallography – over 100 years old, lots of Nobel prizes ...
2. Special Signals:
   - Sparse general case: GESPAR[SBE14];
   - Specialized: sparse [IVW1]; speech [ARF03]

... and others – "phase retrieval" in title: 2680 papers

Graduation Method. Homotopic Continuation

First Motivation

Our algorithm (IRLS and variants) belongs to the class of *Graduation Methods* , or *Homotopic Continuations*.

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However we know how to optimize a closely related criterion $J_0(x)$, $argmin_{x\in D_0}J_0(x)$.

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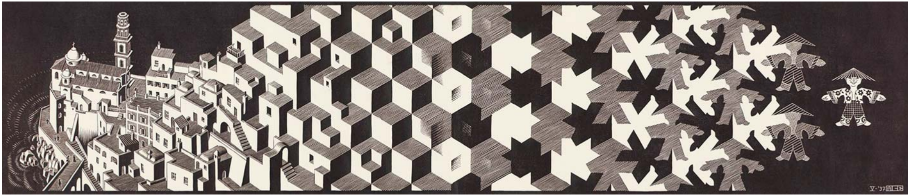
Then we introduce a monotonic sequence $0 \leq t_n \leq 1$ with $t_0 = 1$ and $t_n \to 0$ and solve iteratively

$$x^{n+1} = argmin_{x \in D_n} F(t_n, J(x), J_0(x))$$

using x^n as starting point. Here F is a continue function so that $F(1, J(x), J_0(x)) = J_0(x)$ and $F(0, J(x), J_0(x)) = J(x)$.

Graduation Method. Homotopic Continuation

First Motivation



M.C.Escher (1937) - Metamorphosis I
online at: http://www.mcescher.com/gallery/

Graduation Method. Homotopic Continuation

Second Motivation: LARS Algorithm

Least Angle Regression (LARS) [EHJT04] designed to solve LASSO, or variants:

$$argmin_x \|y - Ax\|_2^2 + \lambda \|x\|_1$$

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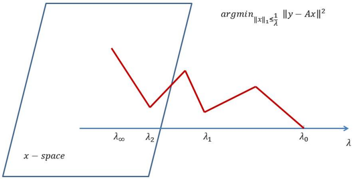
Then LARS finds monotonically decreasing λ values where the slope (and support) of $x(\lambda)$ changes. The algorithm ends at the desired value of $\lambda = \lambda_\infty$ (see also Hierarchical Decompositions of Tadmor&all).



Homotopy Method
Our Main Problem

The ultimate goal is to find the global minimum of the following functional:

$$I(x) = \sum_{k=1}^{m} |y_k - |\langle x, f_k \rangle|^2|^2$$

over $x \in \mathbb{C}^n$, given the set of real numbers $y_1, \cdots, y_m$ and frame vectors $f_1, \cdots, f_m \in \mathbb{C}^n$. The problem is hard because the criterion is non-convex (it is a quartic multivariate polynomial).

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$$J(x; \lambda) = \frac{1}{4} \sum_{k=1}^{m} |y_k - |\langle x, f_k \rangle|^2|^2 + \frac{\lambda}{2} \|x\|^2$$

Homotopy Method

Quartic Criteria: The Convex Regime

$$J(x;\lambda) = \frac{1}{4}\sum_{k=1}^{m} |y_k - |\langle x, f_k\rangle|^2|^2 + \frac{\lambda}{2}\|x\|^2$$

Homotopy Method
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Since

$$J(x;\lambda) = \frac{1}{4}\sum_{k=1}^{m} |\langle x, f_k\rangle|^4 + \frac{1}{2}\langle(\lambda I - R_0)x, x\rangle + \frac{1}{4}\sum_{k=1}^{m} y_k^2$$

with $R_0 = \sum_{k=1}^{m} y_k f_k f_k^*$, it follows for $\lambda > \lambda_0 = \lambda_{max}(R_0)$ the criterion is strongly convex and $x = 0$ is the global minimum.

Homotopy Method

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with $R_0 = \sum_{k=1}^{m} y_k f_k f_k^*$, it follows for $\lambda > \lambda_0 = \lambda_{max}(R_0)$ the criterion is strongly convex and $x = 0$ is the global minimum.
A good candidate for the homotopy method is to start with $J(x;\lambda_0 - \varepsilon)$ whose global minimum is along the principal eigenvector of R_0, and then decrease λ until desired value (e.g. 0).

Homotopy Method
Characteristic Equation

At each $\lambda \geq 0$ consider the set of critical points: $\nabla_x J(x;\lambda) = 0$. To illustrate the method, restrict to the real case. The characteristic equation (of critical points) is given by:

$$\sum_{k=1}^{m} (|\langle x, f_k\rangle|^2 - y_k)\langle x, f_k\rangle f_k + \lambda x = 0$$

or

$$R(x)x + (\lambda I - R_0)x = 0 \tag{3.1}$$

where $R_0 = \sum_{k=1}^{m} y_k f_k f_k^*$ and $R(x) = \sum_{k=1}^{m} |\langle x, f_k\rangle|^2 f_k F_k^*$.

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where $R_0 = \sum_{k=1}^{m} y_k f_k f_k^*$ and $R(x) = \sum_{k=1}^{m} |\langle x, f_k \rangle|^2 f_k F_k^*$.

Note, (3.1) is a system of cubic equations in n variables. Assume the number of roots is always finite (true, unless a degenerate case). Then the number of critical points is at most 3^n.

Homotopy Method
Bifurcation Diagrams

An example of λ-dependent characteristic roots:

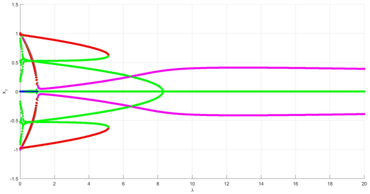

Figure: Plot of $x_1 = x_1(\lambda)$ in a low-dimensional case $n = 3$, $m = 5$. $\lambda_0 = 55.84$

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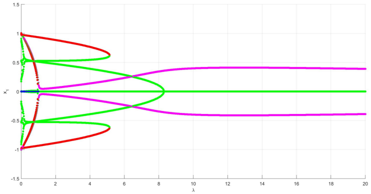

Figure: Plot of $x_1 = x_1(\lambda)$ in a low-dimensional case $n = 3$, $m = 5$. $\lambda_0 = 55.84$

$$y = \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 1.01 \end{bmatrix} \begin{bmatrix} 1 \\ 1.5 \\ -1 \end{bmatrix} \right|^2$$

Homotopy Method
Bifurcation Diagrams

Strategy: Start with

$$J(x;\lambda) = \frac{1}{4}\sum_{k=1}^{m} |y_k - |\langle x, f_k\rangle|^2|^2 + \frac{\lambda}{2}\|x\|^2$$

at $(\lambda_0 - \varepsilon, s(\varepsilon)e_0)$ and then continually track the critical point branch, while decreasing the criterion.

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at $(\lambda_0 - \varepsilon, s(\varepsilon)e_0)$ and then continually track the critical point branch, while decreasing the criterion. Thus:

$$\frac{1}{4}\sum_{k=1}^{m} |\langle x, f_k\rangle|^4 + \frac{1}{2}\langle(\lambda I - R_0)x, x\rangle + \frac{1}{4}\sum_{k=1}^{m} y_k^2 = J(x;\lambda) \leq J(0,\lambda_0) = \frac{1}{4}\sum_{k=1}^{m} y_k^2$$

Thus

$$\sum_{k=1}^{m} |\langle x, f_k\rangle|^4 \leq 2\langle(R_0 - \lambda I)x, x\rangle$$

Let $a_{24} = \min_{\|x\|_2=1} \|Tx\|_4 > 0$. We obtain:

$$\|x\| \leq \frac{\sqrt{\|R_0\| - \lambda}}{a_{24}^2}$$

Homotopy Method

Gradient Level Set

Consider a parametrization of the characteristic curves $(\lambda = \lambda(t), x = x(t))$:

$$\nabla_x J(x(t); \lambda(t)) = 0 \Leftrightarrow R(x(t))x(t) + (\lambda(t)I - R_0)x(t) = 0$$

Differentiate to obtain:

$$\left[\begin{array}{ccc} x & \vdots & H(x,\lambda) \end{array}\right] \left[\begin{array}{c} \frac{d\lambda}{dt} \\ \frac{dx}{dt} \end{array}\right] = 0 \quad (Diff.Sys.)$$

with the Hessian $H(x, \lambda) = 3R(x) + \lambda I - R_0$.

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with the Hessian $H(x,\lambda) = 3R(x) + \lambda I - R_0$.
If Hessian nonsingular, we can parametrize $x = x(\lambda)$ and

$$\frac{dx}{d\lambda} = -\left(H(x,\lambda)\right)^{-1} x.$$

The IRLS Algorithm

IRLS Algorithm

The Iterative Regularized Least-Squares Algorithm attempts to find the global minimum of the non-convex problem

$$argmin_x \sum_{k=1}^{m} |y_k - |\langle x, f_k\rangle|^2|^2 + 2\lambda_\infty \|x\|_2^2$$

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using a sequence of iterative least-squares problems:

$$x^{(t+1)} = argmin_x \sum_{k=1}^{m} |y_k - |\langle x, f_k\rangle|^2|^2 + 2\lambda_t \|x\|_2^2 + \mu_t \left\|x - x^{(t)}\right\|^2$$

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together with a polarization relaxation:

$$|\langle x, f_k \rangle|^2 \approx \frac{1}{2}(\langle x, f_k \rangle \langle f_k, x^{(t)} \rangle + \langle x^{(t)}, f_k \rangle \langle f_k, x \rangle)$$

The IRLS Algorithm
Main Optimization

The optimization problem:

$$\begin{aligned} x^{(t+1)} &= argmin_x \sum_{k=1}^{m} \left| y_k - \frac{1}{2}(\langle x, f_k\rangle\langle f_k, x^{(t)}\rangle + \langle x^{(t)}, f_k\rangle\langle f_k, x\rangle) \right|^2 + \\ & +\lambda_t \|x\|_2^2 + \mu_t \|x - x^{(t)}\|_2^2 + \lambda_t \|x^{(t)}\|_2^2 \\ &= argmin_x \;\; J(x, x^{(t)}; \lambda, \mu) \end{aligned}$$

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Note:

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Note:

- $J(x, .; ., .)$ is quadratic in $x \Rightarrow$ hence a least-squares problem!

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Note:

- $J(x, .; ., .)$ is quadratic in x $\Rightarrow$ hence a least-squares problem!
- $J(x, x; \lambda, \mu) = \sum_{k=1}^{m} |y_k - |\langle x, f_k\rangle|^2|^2 + 2\lambda\|x\|_2^2$ $\Rightarrow$ Fixed points of IRLS are local minima of the original problem.

The IRLS Algorithm
Second Motivation: Relaxation of Constraints

Another motivation: seek $X = xx^*$ that solves

$$\min_{X \geq 0, rank(X)=1} \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda trace(X).$$

The IRLS Algorithm

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PhaseLift algorithm removes the condition $rank(X) = 1$ and shows (for large λ) this produces the desired result with high probability.
Another way to relax the problem is to search for X in a larger space. The IRLS is essentially equivalent to optimize a convex functional of X on the larger space

$$\mathcal{S}^{1,1} = \{T = T^* \in \mathbb{C}^{n \times n} \,,\; T \text{ has at most one positive eigenvalue and at most one negative eigenvalue}\}.$$

The IRLS Algorithm
Second Formulation

Consider the following three convex criteria:

$$
\begin{aligned}
J_1(X;\lambda,\mu) &= \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2(\lambda+\mu)\|X\|_1 - 2\mu trace(X) \\
J_2(X;\lambda,\mu) &= \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda eig_{max}(X) - (2\lambda+4\mu) eig_{min}(X) \\
J_3(X;\lambda,\mu) &= \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda\|X\|_1 - 4\mu eig_{min}(X)
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which coincide on $\mathcal{S}^{1,1}$.

The IRLS Algorithm
Second Formulation

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$$J_3(X;\lambda,\mu) = \sum_{k=1}^{m} |y_k - \langle X, f_k f_k^* \rangle_{HS}|^2 + 2\lambda\|X\|_1 - 4\mu eig_{min}(X)$$

which coincide on $\mathcal{S}^{1,1}$. Consider the optimization problem

$$(Jopt, X) = \min_{X\in\mathcal{S}^{1,1}} J_k(X;\lambda,\mu) \quad , \quad 1 \leq k \leq 3$$

The IRLS Algorithm
Second Formulation -2

The following are true:

1. Optimization in $\mathcal{S}^{1,1}$:

$$\min_{X\in\mathcal{S}^{1,1}} J_k(X;\lambda,\mu) = \min_{u,v\in\mathbb{C}^n} J(u,v;\lambda,\mu)$$

If $\hat{X}$ and $(\hat{u},\hat{v})$ denote optimizers so that $imag(\langle\hat{u},\hat{v}\rangle)=0$, then $\hat{X}=\frac{1}{2}(\hat{u}\hat{v}^*+\hat{v}\hat{u}^*)$.

The IRLS Algorithm

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2. Optimization in $\mathcal{S}^{1,0}$:

$$\min_{X\in\mathcal{S}^{1,0}} J_k(X;\lambda,\mu) = \min_{x\in\mathbb{C}^n} J(x,x;\lambda,\mu)$$

   If $\hat{X}$ and $\hat{x}$ denote optimizers, then $\hat{X}=\hat{x}\hat{x}^*$. $\mathcal{S}^{1,0}=\{xx^*\}$.

The IRLS Algorithm
Initialization

For $\lambda \geq eig_{max}(R(y))$, where $R(y) = \sum_{k=1}^{m} y_k f_k f_k^*$, $J(x;\lambda) = \sum_{k=1}^{m} |y_k - |\langle x, f_k\rangle|^2|^2 + 2\lambda\|x\|_2^2$ is convex. The unique global minimum is $x^0 = 0$.

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Initialization Procedure:

- Solve the principal eigenpair (e, eig_{max}) of matrix $R(y)$ using e.g. the power method;
- Set
$$\lambda_0 = (1-\varepsilon)eig_{max} \ , \ x^0 = \sqrt{\frac{\varepsilon eig_{max}}{\sum_{k=1}^{m} |\langle e, f_k\rangle|^4}}\, e.$$
  Here $\varepsilon > 0$ is a parameter that depends on the frame set as well as the spectral gap of $R(y)$.
- Set $\mu_0 = \lambda_0$ and $t = 0$.

The IRLS Algorithm
Iterations

Repeat the following steps until stopping:

- Optimization: Solve the least-square problem:

$$\begin{aligned} x^{(t+1)} &= argmin_x \sum_{k=1}^{m} \left| y_k - \frac{1}{2}(\langle x, f_k\rangle\langle f_k, x^{(t)}\rangle + \langle x^{(t)}, f_k\rangle\langle f_k, x\rangle)\right|^2 + \\ &\quad +\lambda_t\|x\|_2^2 + \mu_t\|x - x^{(t)}\|_2^2 + \lambda_t\|x^{(t)}\|_2^2 \\ &= argmin_x \;\; J(x, x^{(t)}; \lambda, \mu) \end{aligned}$$

- Update: $\lambda_{t+1} = \gamma\lambda_t$, $\mu_{t+1} = \max(\gamma\mu_t, \mu^{\min})$, $t = t + 1$. Here γ is the learning rate, and μ^{min} is related to performance.

The IRLS Algorithm
Performance

Let $y_k = |\langle x, f_k \rangle|^2 + \nu_k$. Assume the algorithm is stopped at some T so that

$$J(x^{(T)}, x^{(T-1)}; \lambda, \mu) \leq J(x, x; \lambda, \mu).$$

Denote $\hat{X} = \frac{1}{2}(x^{(T)} x^{(T-1)*} + x^{(T-1)} x^{(T)*})$ and $\hat{x}\hat{x}^* = P_+(\hat{X})$.

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Denote $\hat{X} = \frac{1}{2}(x^{(T)}x^{(T-1)*} + x^{(T-1)}x^{(T)*})$ and $\hat{x}\hat{x}^* = P_+(\hat{X})$. Then the following hold true:

1. Matrix norm error:
$$\|\hat{X} - xx^*\|_1 \leq \frac{\lambda}{C_0} + \sqrt{C_0}\|\nu\|$$
2. Natural distance:
$$D(\hat{x}, x)^2 = \|\hat{X} - xx^*\|_1 + |eig_{min}(\hat{X})| \leq \frac{\lambda}{C_0} + \sqrt{C_0}\|\nu\| + \frac{\|\nu\|^2}{4\mu} + \frac{\lambda\|x\|^2}{2\mu}$$

where C_0 is a frame dependent constant (lower Lipschitz constant in $\mathcal{S}^{1,1}$).

Numerical Simulations

Setup

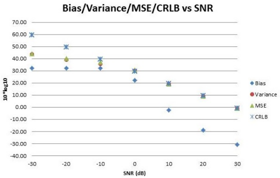
The algorithm requires $O(m)$ memory. Simulations with $m = Rn$ (complex case) with $n = 1000$ and $R \in \{4, 6, 8, 12\}$. Frame vectors corresponding to masked (windowed) DFT:

$$f_{jn+k} = \frac{1}{\sqrt{Rn}} \left(w_l^j e^{2\pi i k(l-1)/n} \right)_{0 \le l \le n-1} \ , \ 1 \le j \le R, 1 \le k \le n$$

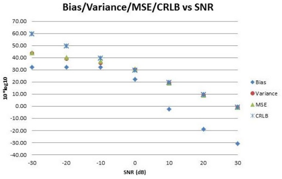
$$\begin{bmatrix} f_1 & f_2 & \cdots & f_m \end{bmatrix} = \begin{bmatrix} Diag(w^1) & \cdots & Diag(w^R) \end{bmatrix} \begin{bmatrix} DFT_n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & DFT_n \end{bmatrix}$$

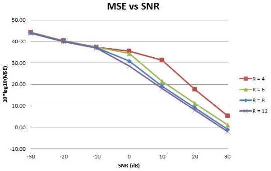
Parameters: $\varepsilon = 0.1$, $\gamma = 0.95$, $\mu^{min} = \frac{\mu^0}{10}$. Power method tolerance: 10^{-8} Conjugate gradient tolerance: 10^{-14}.

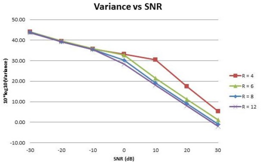
Numerical Simulations
MSE Plots

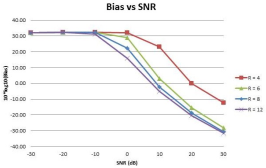


Numerical Simulations
MSE Plots

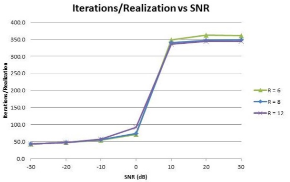


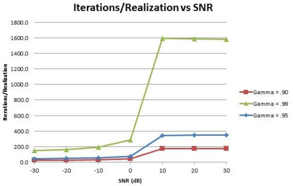






Numerical Simulations
Performance





Numerical Simulations
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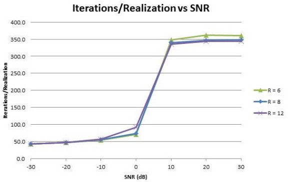
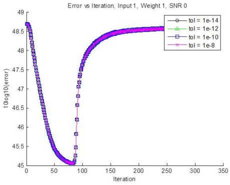
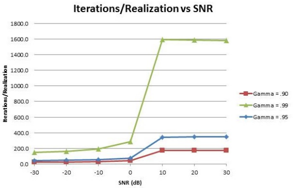
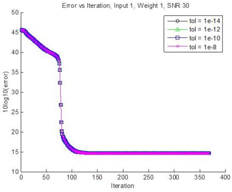
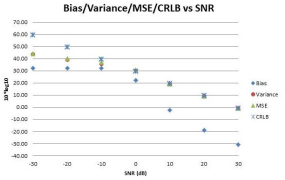






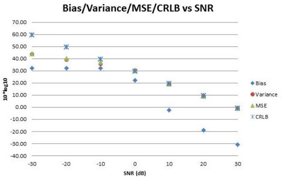


Numerical Simulations
Performance - 2



SNR	$10 \cdot log_{10}(Bias)$	$10 \cdot log_{10}(Variance)$	$10 \cdot log_{10}(MSE)$	CRLB
-30	32.13	43.78	44.07	59.39
-20	32.39	39.29	40.09	49.39
-10	32.27	35.56	37.23	39.39
0	22.17	30.24	30.87	29.39
10	-2.21	19.16	19.19	19.39
20	-18.88	9.05	9.05	9.39
30	-30.63	-0.96	-0.96	-0.61

Numerical Simulations
Performance - 2



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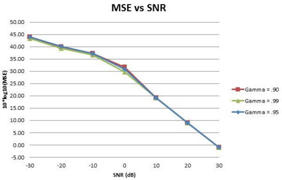

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