ZERO SET OF ZAK TRANSFORM AND THE HRT CONJECTURE

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Abstract. The HRT (Heil-Ramanathan-Topiwala) posits the linear independence of any set of nonzero square-integrable vectors obtained from a single nonzero vector f by applying a finite set of time-frequency shift operators. In this short note, we present findings centered on the zero set of the Zak transform of f, and a distinct arrangement involving a finite set of N points on an integer lattice in the time-frequency plane, excluding a specific point.

1. Introduction

The HRT Conjecture [3] posits that a finite system of time-frequency shifted copies of a nonzero square-integrable function will always be linearly independent. For an updated overview of this conjecture, please refer to our latest work [5].

Following up on [5], the primary objective of this short note is to present a number of new results for this conjecture. Let's consider a function f that is nonzero in $L^2(\mathbb{R}^n)$. Choose an arbitrary natural number N. Following this, we construct a finite system of vectors, which we will denote as $\mathcal{F}_{(N,1)}$ in the following manner

$$\left\{t \mapsto e^{-2\pi i\left\langle y^{(j)},t\right\rangle} f\left(t-x^{(j)}\right) : \left(x^{(1)},y^{(1)}\right),\cdots,\left(x^{(N)},y^{(N)}\right) \in \mathbb{Z}^{2n}\right\} \cup \left\{t \mapsto e^{-2\pi i\langle y,t\rangle} f(t-x)\right\}.$$

This system includes functions of the form

$$t \mapsto e^{-2\pi i\left\langle y^{(j)},t\right\rangle} f\left(t-x^{(j)}\right) \in L^2(\mathbb{R}^n)$$

where each $\left(x^{(1)},y^{(1)}\right),\cdots,\left(x^{(N)},y^{(N)}\right)$ belongs to the integer full-rank lattice $\mathbb{Z}^{2n}$. Furthermore, it includes a time-frequency-shifted vector of the form $t \mapsto e^{-2\pi i\langle y,t\rangle} f(t-x) \in L^2(\mathbb{R}^n)$ where (x,y) can be identified with a point on the time-frequency plane. We establish the following as valid:

Proposition 1. *Suppose that the Zak transform of f is continuous. Then the system of vectors $\mathcal{F}_{(N,1)}$ is linearly independent under the condition that the group which is generated by $(-x,y)$ modulo $\mathbb{Z}^{2n}$ is a dense subset of the $2n$-dimensional torus $[0,1)^n \times [0,1)^n$.*

The subsequent finding indicates that for functions that decay at a sufficiently rapid rate, the zero set of their Zak transform remains unchanged under a specific symmetry.

Proposition 2. *If $\mathcal{F}_{(N,1)}$ is linearly dependent then the zero set of the Zak transform of f is invariant under the action*

$$[0,1)^n \times [0,1)^n \ni z \mapsto z+(-x,y) \text{ modulo } \mathbb{Z}^{2n} \in [0,1)^n \times [0,1)^n.$$

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Proposition 3. *Suppose that the Zak transform of f is continuous and has a finite number of zeros in the unit square $[0,1)^n \times [0,1)^n$. The system of vectors $\mathcal{F}_{(N,1)}$ is linearly independent under the condition that the group which is generated by $(-x,y)$ modulo $\mathbb{Z}^{2n}$ forms a countably infinite subset (not necessarily dense) of the $2n$-dimensional torus $[0,1)^n \times [0,1)^n$.*

As a straightforward application of Proposition 3, we settle the HRT Conjecture with respect to the configuration at hand for a certain class of totally positive functions.

Corollary 4. *Let f be a totally positive nonzero function without Gaussian factor in its Fourier transform. Then the system of vectors*

$$\mathcal{F} = \left\{ t \mapsto e^{-2\pi i y^{(j)} t} f\left(t - x^{(j)}\right) : x^{(1)}, y^{(1)}, \cdots, x^{(N)}, y^{(N)} \in \mathbb{Z} \right\} \cup \left\{ t \mapsto e^{-2\pi i y t} f(t-x) \right\}$$

is linearly independent.

Proof. Let's consider a function, f, which is totally positive and does not have a Gaussian factor in its Fourier transform. According to [4], the Zak transform of f is continuous and admits only one zero on its fundamental domain of quasi-periodicity. If $\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}x + \mathbb{Q}y) = 1$ then we are in the case of rational time-frequency shifts, and in this case, the HRT Conjecture is known to always be true. Suppose next that $\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}x + \mathbb{Q}y) > 1$. Then the group generated by $(-x,y)$ modulo $\mathbb{Z}^2$ is infinite and the stated corollary is immediate. □

2. Proof of Propositions 1, 2, and 3

Let $\mathfrak{h}_n$ be the $(2n+1)$-dimensional Heisenberg algebra. For a concrete realization [1], one may choose an algebra constructed on the basis set $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n, Z\}$. This algebra can be expressed as follows: $\sum_{k=1}^n x_k X_k + \sum_{k=1}^n y_k Y_k + zZ$ where

$$\sum_{k=1}^n x_k X_k + \sum_{k=1}^n y_k Y_k + zZ = \begin{bmatrix} 0 & x_1 & \cdots & x_n & z \\ & & & & y_1 \\ & & \ddots & & \vdots \\ & & & & y_n \\ 0 & & & & 0 \end{bmatrix}.$$

Its associated Lie group denoted as H_n is given by:

$$H_n = \left\{ \exp(zZ) \exp\left(\sum_{k=1}^n y_k Y_k\right) \exp\left(\sum_{k=1}^n x_k X_k\right) = \begin{bmatrix} 1 & x_1 & \cdots & x_n & z \\ & & & & y_1 \\ & & \ddots & & \vdots \\ & & & & y_n \\ 0 & & & & 1 \end{bmatrix} : x_j, y_k, z \in \mathbb{R} \right\}.$$

It is a well-established fact that H_n acts unitarily and irreducibly on $L^2(\mathbb{R}^n)$ via π as follows: Given points $y = (y_1, \cdots, y_n), x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and a vector $f \in L^2(\mathbb{R}^n)$, one has: (a) multiplication by characters

$$[\pi \exp(zZ) f]\left(\exp\left(\sum_{k=1}^n t_k X_k\right)\right) = e^{2\pi i z} f\left(\exp\left(\sum_{k=1}^n t_k X_k\right)\right)$$

(b) a modulation action of the form

$$\left[\pi\left(\exp\left(\sum_{k=1}^{n} y_k Y_k\right)\right) f\right]\left(\exp\left(\sum_{k=1}^{n} t_k X_k\right)\right) = e^{-2\pi i\langle y,t\rangle} f\left(\exp\left(\sum_{k=1}^{n} t_k X_k\right)\right)$$

and (c) a translation action defined as

$$\left[\pi\left(\exp\left(\sum_{k=1}^{n} x_k X_k\right)\right) f\right](t) = f\left(\exp\left(\sum_{k=1}^{n} (t_k - x_k) X_k\right)\right).$$

Suppose that there exists a nonzero vector $f \in L^2(\mathbb{R}^n)$ such that

$$\left\{\pi\left(\exp\left(\sum_{k=1}^{n} \ell_k^{(j)} Y_k\right)\exp\left(\sum_{k=1}^{n} m_k^{(j)} X_k\right)\right) f : 1 \leq j \leq N\right\} \cup \left\{\pi\left(\exp\left(\sum_{k=1}^{n} y_k Y_k\right)\exp\left(\sum_{k=1}^{n} x_k X_k\right)\right) f\right\}$$

is linearly dependent for some finite points lattice points of the form

$$\sum_{k=1}^{n} \ell_k^{(j)} Y_k \in \sum_{k=1}^{n} \mathbb{Z} Y_k,$$
$$\sum_{k=1}^{n} m_k^{(j)} X_k \in \sum_{k=1}^{n} \mathbb{Z} X_k, 1 \leq j \leq N$$

and

$$\sum_{k=1}^{n} y_k Y_k \in \sum_{k=1}^{n} \mathbb{R} Y_k,$$
$$\sum_{k=1}^{n} x_k X_k \in \sum_{k=1}^{n} \mathbb{R} X_k.$$

In other words, there exist nonzero complex numbers $c_1, \cdots, c_N$ such that

$$\sum_{j=1}^{N} c_j \pi\left(\exp\left(\sum_{k=1}^{n} \ell_k^{(j)} Y_k\right)\exp\left(\sum_{k=1}^{n} m_k^{(j)} X_k\right)\right) f = \pi\left(\exp\left(\sum_{k=1}^{n} y_k Y_k\right)\exp\left(\sum_{k=1}^{n} x_k X_k\right)\right) f.$$

Next, let $Z : L^2(\mathbb{R}^n) \to L^2([0,1)^n \times [0,1)^n)$ be the Zak transform [2], formally defined as follows:

$$Zf(t,\omega) = \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} f\left(\exp\left(\sum_{k=1}^{n} (t_k + \tau_k) X_k\right)\right) e^{-2\pi i\langle \omega,\tau\rangle}$$

where $\tau = (\tau_1, \cdots, \tau_n)$. Then it is known that the Zak transform is a unitary operator whose range consists of quasi-periodic functions (see [2], Section 8.1 on Page 147 and Theorem 8.2.5 on Page 154.)

Furthermore, it is perhaps worth noting the following properties of the Zak transform:

- If f is integrable then $Zf \in L^1([0,1)^n \times [0,1)^n)$
- If f is continuous and additionally,

$$\sum_{\ell \in \mathbb{Z}^n} \mathrm{esssup}_{t \in [0,1)^n} |f(t+\ell)| < \infty$$

  then Zf is continuous.

- The Zak transform maps Schwartz functions to smooth function on $[0,1)^n \times [0,1)^n$. Also, if F is quasiperiodic on $\mathbb{R}^n \times \mathbb{R}^n$ then $F = Zf$ for some unique Schwartz function f.

Applying the Zak transform to each side of the following equation:

$$\sum_{j=1}^{N} c_j \pi \left(\exp\left(\sum_{k=1}^{n} \ell_k^{(j)} Y_k \right) \exp\left(\sum_{k=1}^{n} m_k^{(j)} X_k \right) \right) f = \pi \left(\exp\left(\sum_{k=1}^{n} y_k Y_k \right) \exp\left(\sum_{k=1}^{n} x_k X_k \right) \right) f$$

we obtain:

$$\begin{aligned} &\sum_{j=1}^{N} c_j \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} e^{-2\pi i \left\langle t+\tau, \ell^{(j)} \right\rangle} f\left(\exp\left(\sum_{k=1}^{n} \left(t_k + \tau_k - m_k^{(j)} \right) X_k \right) \right) e^{-2\pi i \langle \omega, \tau \rangle} \\ &= \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} e^{-2\pi i \langle t+\tau, y \rangle} f\left(\exp\left(\sum_{k=1}^{n} (t_k + \tau_k - x_k) X_k \right) \right) e^{-2\pi i \langle \omega, \tau \rangle}. \end{aligned}$$

On the one hand,

$$\begin{aligned} &\sum_{j=1}^{N} c_j \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} e^{-2\pi i \left\langle t+\tau, \ell^{(j)} \right\rangle} f\left(\exp\left(\sum_{k=1}^{n} \left(t_k + \tau_k - m_k^{(j)} \right) X_k \right) \right) e^{-2\pi i \langle \omega, \tau \rangle} \\ &= \sum_{j=1}^{N} c_j e^{-2\pi i \left\langle t, \ell^{(j)} \right\rangle} \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} e^{-2\pi i \left\langle \tau, \ell^{(j)} \right\rangle} f\left(\exp\left(\sum_{k=1}^{n} \left(t_k + \tau_k - m_k^{(j)} \right) X_k \right) \right) e^{-2\pi i \langle \omega, \tau \rangle} \\ \left(e^{-2\pi i \left\langle \tau, \ell^{(j)} \right\rangle} = 1 \right) \quad &= \sum_{j=1}^{N} c_j e^{-2\pi i \left\langle t, \ell^{(j)} \right\rangle} \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} f\left(\exp\left(\sum_{k=1}^{n} \left(t_k + \tau_k - m_k^{(j)} \right) X_k \right) \right) e^{-2\pi i \langle \omega, \tau \rangle}. \end{aligned}$$

Moreover, the change of variable $\tau_k \mapsto \tau + m_k^{(j)}$ yields

$$\begin{aligned} &\sum_{j=1}^{N} c_j \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} e^{-2\pi i \left\langle t+\tau, \ell^{(j)} \right\rangle} f\left(\exp\left(\sum_{k=1}^{n} \left(t_k + \tau_k - m_k^{(j)} \right) X_k \right) \right) e^{-2\pi i \langle \omega, \tau \rangle} \\ &= \sum_{j=1}^{N} c_j e^{-2\pi i \left\langle t, \ell^{(j)} \right\rangle} \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} f\left(\exp\left(\sum_{k=1}^{n} (t_k + \tau_k) X_k \right) \right) e^{-2\pi i \left\langle \omega, \tau + m^{(j)} \right\rangle} \\ &= \sum_{j=1}^{N} c_j e^{-2\pi i \left\langle t, \ell^{(j)} \right\rangle} e^{-2\pi i \left\langle \omega, m^{(j)} \right\rangle} \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} f\left(\exp\left(\sum_{k=1}^{n} (t_k + \tau_k) X_k \right) \right) e^{-2\pi i \langle \omega, \tau \rangle} \\ &= \left(\sum_{j=1}^{N} c_j e^{-2\pi i \left\langle t, \ell^{(j)} \right\rangle} e^{-2\pi i \left\langle \omega, m^{(j)} \right\rangle} \right) \cdot Zf(t, \omega). \end{aligned}$$

Furthermore,

$$\begin{aligned} &\sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} e^{-2\pi i \langle t+\tau, y \rangle} f\left(\exp\left(\sum_{k=1}^{n} (t_k - x_k + \tau_k) X_k \right) \right) e^{-2\pi i \langle \omega, \tau \rangle} \\ &= e^{-2\pi i \langle t, y \rangle} \sum_{\tau_1 \in \mathbb{Z}} \cdots \sum_{\tau_n \in \mathbb{Z}} e^{-2\pi i \langle \tau, y \rangle} f\left(\exp\left(\sum_{k=1}^{n} (t_k - x_k + \tau_k) X_k \right) \right) e^{-2\pi i \langle \omega, \tau \rangle} \end{aligned}$$

$$= e^{-2\pi i\langle t,y\rangle}\sum_{\tau_1\in\mathbb{Z}}\cdots\sum_{\tau_n\in\mathbb{Z}} f\left(\exp\left(\sum_{k=1}^{n}\left(t_k - x_k+\tau_k\right)X_k\right)\right)e^{-2\pi i\langle \omega+y,\tau\rangle}$$
$$= e^{-2\pi i\langle t,y\rangle} Zf\left(t-x,\omega+y\right).$$

In summary, we have

$$\left(\sum_{j=1}^{N} c_j e^{-2\pi i\left\langle t,\ell^{(j)}\right\rangle} e^{-2\pi i\left\langle \omega,m^{(j)}\right\rangle}\right)\cdot Zf\left(t,\omega\right) = e^{-2\pi i\langle t,y\rangle}Zf\left(t-x,\omega+y\right).$$

For more compact notation, let $(t,\omega) = z \in [0,1)^n \times [0,1)^n$, $F = Zf$, $\gamma = (-x, y)$ and let P be the trigonometric polynomial given by

$$P\left(t,\omega\right) = \sum_{j=1}^{N} c_j e^{-2\pi i\left\langle t,\ell^{(j)}\right\rangle} e^{-2\pi i\left\langle \omega,m^{(j)}\right\rangle}.$$

Given these definitions, we can see that the product of the absolute values of P and F at the point z equals the absolute value of F at the point z shifted by γ. In other words, for $z \in [0,1)^n \times [0,1)^n$,

$$\left|P\left(z\right)\right|\cdot\left|F\left(z\right)\right| = \left|F\left(z+\gamma\right)\right|.$$

Furthermore, $|F|$ is a $\mathbb{Z}^{2n}$-periodic square-integrable function.

As referenced in Lemma 8.4.2 of [2], if F is a continuous function, it is guaranteed to have at least one zero within the unit square defined by $[0,1)^n \times [0,1)^n$. By successively applying the equation

$$\left|P\left(z\right)\right|\cdot\left|F\left(z\right)\right| = \left|F\left(z+\gamma\right)\right|$$

we deduce that for any natural number m,

$$\left|F\left(z+m\gamma\right)\right| = \prod_{j=0}^{m-1}\left|P\left(z+j\gamma\right)\right|\cdot\left|F\left(z\right)\right|.$$

Let Γ be the group generated by γ modulo $\mathbb{Z}^{2n}$. Then Γ is a countable subgroup of the $2n$-dimensional torus $[0,1)^n \times [0,1)^n$.

2.1. **Proof of Proposition 1.** If the set Γ is dense in the torus $[0,1)^n \times [0,1)^n$, it implies that the zero set of the function F must also be densely distributed in this unit square as well. Considering that F is a continuous function, it logically follows that F must be identically zero across its domain. This validates Proposition 1.

2.2. **Proof of Proposition 2.** Fix $\lambda \in \mathrm{Zero}\left(Zf\right)$ then for any natural number m,

$$\left|F\left(\lambda+m\gamma\right)\right| = \prod_{j=0}^{m-1}\left|P\left(\lambda+j\gamma\right)\right|\cdot\left|F\left(\lambda\right)\right| = 0.$$

The calculation above suggests that if λ belongs to the zero set of Zf (denoted as $\mathrm{Zero}\left(Zf\right)$), then the term $(\lambda + m\gamma) \mod \mathbb{Z}^{2n}$ is also a member of the zero set of Zf. This highlights that the zero set of Zf is invariant under the operation of Γ. More specifically, it underlines that the zero set of Zf remains unaltered under the action of γ.

2.3. **Proof of Proposition 3.** Suppose that Γ is an infinite set and let $\text{Zero}\,(Zf)$ be the zero set of the Zak transform of f. Fix $\lambda \in \text{Zero}\,(Zf)$ then for any natural number m,

$$|F(\lambda + m\gamma)| = \prod_{j=0}^{m-1} |P(\lambda + j\gamma)| \cdot |F(\lambda)| = 0.$$

This means that

$$\lambda \in \text{Zero}\,(Zf) \Rightarrow (\lambda + m\gamma) \text{ modulo } \mathbb{Z}^{2n} \in \text{Zero}\,(Zf)$$

and this shows that $|\text{Zero}\,(Zf)| = \infty$, contradicting the assumption that the zero set of the Zak transform of f is finite.

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