The HRT conjecture from the point of view of the Fock space

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Abstract

In this talk, we will introduce the HRT conjecture and prove it for two simple cases. Then, we will introduce the Fock space of entire functions and use it to show that the HRT conjecture holds for point configurations where all but one point lie on a line. This talk is based on a book chapter by Daniel W. Stroock [4].

1 The HRT conjecture

Given $(x, \omega) \in \mathbb{R}^2$, define the *time-frequency shift* $\pi_{(x,\omega)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\pi_{(x,\omega)} f(t) := \mathrm{e}^{-\pi \mathrm{i} x \omega} \cdot f(t - x) \mathrm{e}^{2\pi \mathrm{i} t \omega}.$$

We can directly compute that

$$\pi_{(x',\omega')} \circ \pi_{(x,\omega)} = \mathrm{e}^{\pi \mathrm{i}(x\omega' - x'\omega)} \cdot \pi_{(x+x',\omega+\omega')}.$$

Therefore, (or, alternatively, by the unitarity of time-frequency shifts) we have $(\pi_{(x,\omega)})^{-1} = \pi_{(-x,-\omega)}$.

Conjecture 1 (HRT conjecture; [2, p. 2790])**.** *Let $(\lambda_j)_{j=1}^n \in \mathbb{R}^2$ be distinct and let $f \in L^2(\mathbb{R})$ be non-trivial. Then, $(\pi_{\lambda_j} f)_{j=1}^n \in L^2(\mathbb{R})$ is linearly independent.*

Let us start by looking at some easy cases.

Proposition 2. *The HRT conjecture holds for $n = 2$.*

Proof. Let $\lambda_1, \lambda_2 \in \mathbb{R}^2$ be distinct and $f \in L^2(\mathbb{R})$ non-trivial. Write

$$\lambda_1 = (x_1, \omega_1), \qquad \lambda_2 = (x_2, \omega_2)$$

and suppose by contradiction that there exists $(c_1, c_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that

$$c_1 \pi_{\lambda_1} f + c_2 \pi_{\lambda_2} f = 0.$$

We may assume that $c_1 \neq 0$ by relabelling if necessary. Therefore,

$$\begin{aligned} f &= -\frac{c_2}{c_1} \left(\pi_{\lambda_1}^{-1} \circ \pi_{\lambda_2} \right) f = -\frac{c_2}{c_1} \left(\pi_{(-x_1,-\omega_1)} \circ \pi_{(x_2,\omega_2)} \right) f \\ &= -\frac{c_2}{c_1} \mathrm{e}^{\pi \mathrm{i}(x_1\omega_2 - x_2\omega_1)} \pi_{(x_2-x_1,\omega_2-\omega_1)} f = c \pi_{(x,\omega)} f, \end{aligned}$$

for correctly chosen $c \in \mathbb{C}$ and $(x, \omega) \in \mathbb{R}^2$ non-trivial. If $x = 0$, then we have

$$\left(1 - c\mathrm{e}^{2\pi \mathrm{i} t\omega}\right) f(t) = 0$$

which implies that $f = 0$ a.e.: a contradiction. Therefore, $x \neq 0$ and

$$|f(t)|^2 = |c|^2 \, |f(t-x)|^2 .$$

If $|c| \neq 1$, then iterating the above equality shows that

$$|f(t)|^2 = |c|^{2n} \, |f(t-nx)|^2 ,$$

which implies that $f = 0$: a contradiction. Therefore, $|c| = 1$ and $|f|^2$ is periodic as well as integrable. Hence, $f = 0$ which is our final contradiction. ☐

Another case in which the HRT conjecture is easily proven is when the $(\lambda_j)_{j=1}^n$ lie on a vertical line. To show this, we consider the following lemma.

Lemma 3 ([4, Lemma 2.1 on p. 605])**.** *Let $(z_j)_{j=1}^n \in \mathbb{C}$ and $(c_j)_{j=1}^n \in \mathbb{C}$ not all zero. Define*

$$\psi(t) := \sum_{j=1}^{n} c_j \mathrm{e}^{z_j t}, \qquad t \in \mathbb{R}.$$

Then, ψ extends to a non-trivial entire function on $\mathbb{C}$ and therefore vanishes at most countably often.

Proof. The extension is

$$\Psi(z) := \sum_{j=1}^{n} c_j \mathrm{e}^{z_j z}, \qquad z \in \mathbb{C},$$

which is a well-defined non-trivial entire function. Therefore, ψ vanishes at most countably often[1] ☐

Proposition 4. *The HRT conjecture holds when the $(\lambda_j)_{j=1}^n \in \mathbb{R}^2$ lie on a vertical line.*

Proof. Let us write $\lambda_j = (x_j, \omega_j)$ for $j \in [n]$. We may then without loss of generality assume that $x_j = 0$: indeed, note that

$$\sum_{j=1}^{n} c_j \pi_{(x_j,\omega_j)} f = 0 \iff \pi_{(-x_1,0)} \sum_{j=1}^{n} c_j \pi_{(x_j,\omega_j)} f = 0.$$

The latter can be expressed as

$$\begin{aligned} \sum_{j=1}^{n} c_j \left(\pi_{(-x_1,0)} \circ \pi_{(x_j,\omega_j)}\right) f &= \sum_{j=1}^{n} c_j \mathrm{e}^{\pi \mathrm{i} x_1 \omega_j} \pi_{(x_j - x_1, \omega_j)} f \\ &= \sum_{j=1}^{n} c_j \mathrm{e}^{\pi \mathrm{i} x_1 \omega_j} \pi_{(0,\omega_j)} f = 0. \end{aligned}$$

[1] This is well-known in complex analysis: suppose by contradiction that ψ vanishes on an uncountable set and let us consider open balls $(B_n)_{n\in\mathbb{N}} \subset \mathbb{C}$. Then, some B_n contains uncountably many zeroes of ψ (because, if not, then the zeroes of ψ form a countable sequence). By the Bolzano–Weierstrass theorem the zeroes in B_n have a limit point. Therefore, the identity theorem implies that $\psi = 0$ which is the desired contradiction.

So, if $(\pi_{(0,\omega_j)} f)_{j=1}^n$ is linearly independent, then $c_j \mathrm{e}^{\pi \mathrm{i} x_1 \omega_j} = 0$ which implies $c_j = 0$ for $j \in [n]$ and thus $(\pi_{(x_j,\omega_j)} f)_{j=1}^n$ is linearly independent.

Now, consider

$$\sum_{j=1}^{n} c_j \pi_{(0,\omega_j)} f(t) = \sum_{j=1}^{n} c_j \mathrm{e}^{2\pi \mathrm{i} t \omega_j} f(t) = 0.$$

By the prior lemma, we have that

$$\psi(t) := \sum_{j=1}^{n} c_j \mathrm{e}^{2\pi \mathrm{i} t \omega_j}, \qquad t \in \mathbb{R},$$

either has countably many zeroes or that $c_j = 0$ for all $j \in [n]$. The former cannot be true, however, since then $\psi f = 0$ implies that $f = 0$ almost everywhere. $\square$

Remark 5. *Some readers may now note that the case in which the* $(\lambda_j)_{j=1}^n$ *lie on a general line follows from an application of the fractional Fourier transform. We will present a similar but slightly more general argument after introducing the Fock space.*

2 The Fock space

The *Bargmann transform* of a function $f \in L^2(\mathbb{R})$ is

$$\mathcal{B}f(z) := 2^{1/4} \int_{\mathbb{R}} f(t) \mathrm{e}^{2\pi t z - \pi t^2 - \frac{\pi}{2} z^2} \, \mathrm{d}t, \qquad z \in \mathbb{C}.$$

The *Fock space* $\mathcal{F}^2(\mathbb{C})$ is the Hilbert space of all entire functions F for which the norm

$$\|F\|_{\mathcal{F}} := \left(\int_{\mathbb{C}} |F(z)| \, \mathrm{e}^{-\pi |z|^2} \, \mathrm{d}z \right)^{1/2}$$

is finite. The inner product on $\mathcal{F}^2(\mathbb{C})$ is

$$\langle F, G \rangle_{\mathcal{F}} := \int_{\mathbb{C}} F(z) \overline{G}(z) \mathrm{e}^{-\pi |z|^2} \, \mathrm{d}z.$$

Theorem 6 ([1, Theorem 3.4.3 on p. 56])**.** *The Bargmann transform is a unitary operator from* $L^2(\mathbb{R})$ *onto* $\mathcal{F}^2(\mathbb{C})$*.*

Therefore, the Bargmann transform identifies square-integrable signals with entire functions of certain growth (at infinity) and vice versa. Specifically, we can consider the monomials

$$E_n(z) := \sqrt{\frac{\pi^n}{n!}} \cdot z^n, \qquad z \in \mathbb{C},$$

for $n \in \mathbb{N}_0$ which form an orthonormal basis for the Fock space [1, Theorem 3.4.2 on p. 54]. If we take their inverse Bargmann transform, we obtain the Hermite functions $h_n = \mathcal{B}^{-1} E_n \in L^2(\mathbb{R})$ which must also form an orthonormal basis for $L^2(\mathbb{R})$. The expansion of the Bargmann transform into these bases then is

$$\mathcal{B}f = \sum_{n=0}^{\infty} \langle f, h_n \rangle E_n.$$

Using the definition of the Bargmann transform, we may note that it turns a time-frequency shift into

$$\Pi_\mu F(z) := \mathrm{e}^{\pi \mu z - \frac{\pi}{2}|\mu|^2} F(z - \overline{\mu}), \qquad \mu \in \mathbb{C}.$$

More precisely, we have

$$\mathcal{B} \circ \pi_{(x,\omega)} = \Pi_{x+\mathrm{i}\omega} \circ \mathcal{B}.$$

Finally, let $\mathbb{D} \subset \mathbb{C}$ denote the closed unit disk and define the contraction $\mathrm{C}_\tau : \mathcal{F}^2(\mathbb{C}) \to \mathcal{F}^2(\mathbb{C})$ by $\mathrm{C}_\tau F(z) := F(\tau z)$ for $\tau \in \mathbb{D}$. Note that C_τ is unitary if and only if $|\tau| = 1$. The operation $\mathrm{c}_\tau : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ which corresponds to C_τ — i.e., for which $\mathrm{C}_\tau \circ \mathcal{B} = \mathcal{B} \circ \mathrm{c}_\tau$ — is given by

$$\mathrm{c}_\tau f := \sum_{n=0}^{\infty} \tau^n \langle f, h_n \rangle h_n$$

in the Hermite basis.

3 Rotations and the HRT conjecture

In the following, we will consider $\tau = \mathrm{e}^{\mathrm{i}\theta}$. Then, $\mathrm{C}_\tau \circ \Pi_\mu = \Pi_{\tau\mu} \circ \mathrm{C}_\tau$ such that

$$\mathrm{c}_\tau \circ \pi_{(x,\omega)} = \pi_{(\mathrm{Re}[\tau\mu], \mathrm{Im}[\tau\mu])} \circ \mathrm{c}_\tau, \qquad \mu = x + \mathrm{i}\omega.$$

Alternatively, we can introduce the rotation matrix/operator

$$\mathbf{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and obtain that $\mathrm{c}_\tau \circ \pi_{(x,\omega)} = \pi_{\mathbf{R}_\theta(x,\omega)} \circ \mathrm{c}_\tau$.

Lemma 7 ([4, Lemma 4.1 on p. 610]). *Let $(\lambda_j)_{j=0}^n \in \mathbb{R}^2$ be distinct points such that $(\lambda_j)_{j=1}^n \in \mathbb{R}^2$ lie on a line. Then, there are real numbers $x_0, (\omega_j)_{j=1}^n \in \mathbb{R}$ and an angle $\theta \in \mathbb{R}$ such that*

$$\begin{aligned} &(\pi_{\lambda_j} f)_{j=0}^n \text{ is linearly dependent} \\ &\qquad \iff \pi_{(x_0,0)} \mathrm{c}_\tau f, \pi_{(0,\omega_1)} \mathrm{c}_\tau f, \ldots, \pi_{(0,\omega_n)} \mathrm{c}_\tau f \text{ is linearly dependent,} \end{aligned}$$

where $\tau = \mathrm{e}^{\mathrm{i}\theta}$, for all $f \in L^2(\mathbb{R})$.

Proof. Rotate the line on which the $(\lambda_j)_{j=1}^n$ lie until it is vertical. Then, apply a time-frequency shift to reduce to the case in which the vertical line crosses the origin and the point that is not on the line lies on the temporal axis as in the proof of Proposition 4. □

Now, the followinging two theorems follow.

Theorem 8 ([4, Theorem 4.2 on p. 610]). *Let $(\lambda_j)_{j=0}^n \in \mathbb{R}^2$ be distinct points such that $(\lambda_j)_{j=1}^n \in \mathbb{R}^2$ lie on a line, assume that $x_0, (\omega_j)_{j=1}^n \in \mathbb{R}$ are as in Lemma 7, and let $f \in L^2(\mathbb{R})$ be non-trivial. Then, $(\pi_{\lambda_j} f)_{j=0}^n$ is linearly independent if $x_0\omega_j \in \mathbb{Q}$ for all $j \in [n]$.*

Proof. We assume by contradiction that $(\pi_{\lambda_j} f)_{j=0}^n$ are linearly dependent. Now, Lemma 7 shows that $\pi_{(x_0,0)} \mathrm{c}_\tau f, \pi_{(0,\omega_1)} \mathrm{c}_\tau f, \ldots, \pi_{(0,\omega_n)} \mathrm{c}_\tau f$ are linearly dependent for some $\tau = \mathrm{e}^{\mathrm{i}\theta}$. Thereby, there exist coefficients $(c_j)_{j=0}^n \in \mathbb{C}$ which are not all zero such that

$$c_0 \, \mathrm{c}_\tau f(t - x_0) = \left(\sum_{j=1}^n c_j \mathrm{e}^{2\pi \mathrm{i} t \omega_j} \right) \mathrm{c}_\tau f(t).$$

Let us denote $f_\tau := \mathrm{c}_\tau f$ for ease of notation. If $c_0 = 0$ or $x_0 = 0$, then $f_\tau = 0$ (and thus $f = 0$) following the same argument as in the proof of Proposition 4. Therefore, $c_0 \neq 0$ and $x_0 \neq 0$ and we may assume that $c_0 = 1$ (by scaling appropriately).

So, using the short-hand from Lemma 3, we have $f_\tau(t - x_0) = \psi(t) f_\tau(t)$. We may now use this formula recursively to see that

$$f_\tau(t - kx_0) = \psi_k(t) f_\tau(t), \qquad \psi_k(t) := \prod_{j=0}^{k-1} \psi(t - jx_0),$$

for all $k \in \mathbb{N}$. Interestingly, $\psi_k(t)$ can be expressed as a power of itself for certain values of k. Specifically, we can find $q \in \mathbb{N}$ such that $qx_0\omega_j \in \mathbb{Z}$ for $j \in [n]$ by assumption. We can now see that ψ is qx_0 periodic such that $\psi_{kq}(t) = \psi_q(t)^k$ which implies two things: first, we have

$$\begin{aligned}
\int_{-\infty}^{\ell q x_0} |f_\tau(t)|^2 \, \mathrm{d}t &= \sum_{k \in \mathbb{N}} \int_{(\ell-k)qx_0}^{(\ell-k+1)qx_0} |f_\tau(t)|^2 \, \mathrm{d}t \\
&= \sum_{k \in \mathbb{N}} \int_{\ell q x_0}^{(\ell+1)qx_0} |f_\tau(t - kqx_0)|^2 \, \mathrm{d}t \\
&= \int_{\ell q x_0}^{(\ell+1)qx_0} \left(\sum_{k \in \mathbb{N}} |\psi_q(t)|^{2k} \right) |f_\tau(t)|^2 \, \mathrm{d}t
\end{aligned}$$

for $\ell \in \mathbb{Z}$ if $x_0 > 0$ which shows that $|\psi_k(t)| < 1$ for almost every $t \in \mathbb{R}$ at which $f_\tau(t) \neq 0$ (and a similar argument works if $x_0 < 0$); secondly,

$$\int_{\mathbb{R}} |f_\tau(t)|^2 \, \mathrm{d}t = \int_{\mathbb{R}} |f_\tau(t - kqx_0)|^2 \, \mathrm{d}t = \int_{\mathbb{R}} |\psi_q(t)|^{2k} |f_\tau(t)|^2 \, \mathrm{d}t,$$

where the latter tends to zero as $k \to \infty$ by Lebesgue monotone convergence. Therefore, $f_\tau = 0$ which implies $f = 0$ and is the desired contradiction. $\square$

Theorem 9 ([4, Theorem 4.3 on p. 611]). *Consider distinct points in the time-frequency plane parametrised by*

$$\begin{aligned}
(x_0, \omega_0) &= \lambda + \alpha_0(-\sin\theta, \cos\theta) + \beta_0(\cos\theta, \sin\theta), \\
(x_j, \omega_j) &= \lambda + \alpha_j(-\sin\theta, \cos\theta), \qquad j \in [n],
\end{aligned}$$

where $\beta_0 \neq 0$*. Let* $(\overline{x}_0, \overline{\omega}_0)$ *be the result of mirroring* (x_0, ω_0) *on the line crossing all* (x_j, ω_j)*; i.e.,* $(\overline{x}_0, \overline{\omega}_0)$ *is the result or replacing* β_0 *by* $-\beta_0$ *in the formula parametrising* (x_0, ω_0)*.*

Assume that all the sums $\alpha_j + \alpha_k$ *for* $(j,k) \in [n]^2$ *are distinct; i.e., that* $\alpha_j + \alpha_k = \alpha_{j'} + \alpha_{k'}$ *implies* $(j,k) = (j',k')$*. Then, either*

$$\pi_{(x_0,\omega_0)} f, \ldots, \pi_{(x_n,\omega_n)} f \qquad \text{or} \qquad \pi_{(\overline{x}_0,\overline{\omega}_0)} f, \pi_{(x_1,\omega_1)} f, \ldots, \pi_{(x_n,\omega_n)} f$$

is linearly independent for non-trivial $f \in L^2(\mathbb{R})$*.*

Proof. The reader is referred to the original source [4, pp. 611–612]. $\square$

Abstract

This talk is a continuation of a presentation with the same name last week. By viewing the HRT conjecture from the point of view of the Fock space, we show that it holds for a dense subset of the square-integrable signals. Thereafter, we present a characterisation of this dense subset. This talk is based on a book chapter by Daniel W. Stroock [4] and discussions with Fushuai Jiang.

4 The HRT conjecture holds on a dense set of square-integrable functions

The following is an argument due to Lawrence W. Baggett: suppose that $(\Pi_{\mu_j} F)_{j=0}^n \in \mathcal{F}^2(\mathbb{C})$ is linearly dependent for some $F \in \mathcal{F}^2(\mathbb{C})$ and distinct $(\mu_j)_{j=0}^n \in \mathbb{C}$. Then, there exist constants $(c_j)_{j=0}^n \in \mathbb{C}$ not all zero such that

$$c_0 \Pi_{\mu_0} F + \cdots + c_n \Pi_{\mu_n} F = 0.$$

Let us get rid of all terms in which $c_j = 0$, relable, and write

$$c_0 \Pi_{\mu_0} F + \cdots + c_n \Pi_{\mu_n} F = 0$$

with $c_j \neq 0$ for $j \in \{0, \ldots, n\}$. After a potential reordering, we may assume that $c_0 = -1$ and

$$\operatorname{Re} \mu_0 \leq \operatorname{Re} \mu_j \qquad \text{and} \qquad \operatorname{Re} \mu_0 = \operatorname{Re} \mu_j \implies \operatorname{Im} \mu_0 < \operatorname{Im} \mu_j,$$

for $j \in [n]$. Therefore, we have

$$\Pi_{\mu_0} F = c_1 \Pi_{\mu_1} F + \cdots + c_n \Pi_{\mu_n} F.$$

Note that $\Pi_{-\mu_0}$ is the inverse of Π_{μ_0} such that

$$\begin{aligned} F &= c_1 \left(\Pi_{-\mu_0} \circ \Pi_{\mu_1} \right) F + \cdots + c_n \left(\Pi_{-\mu_0} \circ \Pi_{\mu_n} \right) F \\ &= c_1 \mathrm{e}^{-\pi \mathrm{i} \operatorname{Im}[\mu_1 \overline{\mu}_0]} \Pi_{\mu_1 - \mu_0} F + \cdots + c_n \mathrm{e}^{-\pi \mathrm{i} \operatorname{Im}[\mu_n \overline{\mu}_0]} \Pi_{\mu_n - \mu_0} F. \end{aligned}$$

By our ordering, we have that $\mu_j - \mu_0 = x_j + \mathrm{i}\omega_j$ satisfy

$$x_j \geq 0 \qquad \text{and} \qquad x_j = 0 \implies \omega_j > 0,$$

for $j \in [n]$. Therefore, we have that, for all $\alpha > 0$, there exists $\epsilon > 0$ such that, for all $j \in [n]$, $\epsilon \leq \alpha x_j + \omega_j$. We can also choose $\beta > 0$ large enough such that

$$2 \sum_{j=1}^n |c_j| \leq \mathrm{e}^{\pi \beta \epsilon}.$$

If we set $\mu := \beta(\alpha + \mathrm{i})$, then we obtain

$$\begin{aligned} \mathrm{T}_\mu F(z) := F(z - \overline{\mu}) &= \sum_{j=1}^n c_j \mathrm{e}^{-\pi \mathrm{i} \operatorname{Im}[\mu_j \overline{\mu}_0]} \cdot \Pi_{\mu_j - \mu_0} F(z - \overline{\mu}) \\ &= \sum_{j=1}^n c_j \mathrm{e}^{-\pi \mathrm{i} \operatorname{Im}[\mu_j \overline{\mu}_0] - \pi \overline{\mu}(\mu_j - \mu_0)} \cdot \Pi_{\mu_j - \mu_0} \mathrm{T}_\mu F(z). \end{aligned}$$

By construction, we have

$$\sum_{j=1}^{n}\left|c_j \mathrm{e}^{-\pi\overline{\mu}(x+\mathrm{i}\omega_j)}\right| = \sum_{j=1}^{n}|c_j|\,\mathrm{e}^{-\pi\,\mathrm{Re}[\overline{\mu}(x+\mathrm{i}\omega_j)]} = \sum_{j=1}^{n}|c_j|\,\mathrm{e}^{-\pi\beta(\alpha x_j+\omega_j)}$$
$$\leq \sum_{j=1}^{n}|c_j|\,\mathrm{e}^{-\pi\beta\epsilon} \leq \frac{1}{2}.$$

Therefore, $\|\mathrm{T}_\mu F\|_{\mathcal{F}} \leq \frac{1}{2}\|\mathrm{T}_\mu F\|_{\mathcal{F}}$ which implies $F = 0$ provided that $\mathrm{T}_\mu F \in \mathcal{F}^2(\mathbb{C})$.

Let us analyse the condition $\mathrm{T}_\mu F \in \mathcal{F}^2(\mathbb{C})$. Clearly,

$$\int_{\mathbb{C}} |F(z-\overline{\mu})|^2\,\mathrm{e}^{-\pi|z|^2}\,\mathrm{d}z = \int_{\mathbb{C}} |F(z)|^2\,\mathrm{e}^{-\pi|z+\overline{\mu}|^2}\,\mathrm{d}z$$
$$= \mathrm{e}^{-\pi|\mu|^2}\cdot\int_{\mathbb{C}}\left|F(z)\mathrm{e}^{-\pi\mu z}\right|^2\mathrm{e}^{-\pi|z|^2}\,\mathrm{d}z$$

such that $\mathrm{T}_\mu F \in \mathcal{F}^2(\mathbb{C})$ if and only if $z \mapsto F(z)\mathrm{e}^{-\pi\mu z} \in \mathcal{F}^2(\mathbb{C})$. The latter is true under interesting conditions.

Proposition 10. *Let $F \in \mathcal{F}^2(\mathbb{C})$ be non-trivial such that*

$$\int_{\mathbb{C}} |F(z)|^p\,\mathrm{e}^{-\pi|z|^2}\,\mathrm{d}z < \infty$$

for some $p > 2$. (We write $F \in \mathcal{F}^p(\mathbb{C})$.) Then, the HRT conjecture holds for $f := \mathcal{B}^{-1}F$.

Proof. Indeed,

$$\int_{\mathbb{C}} |F(z)\mathrm{e}^{\pi\mu z}|^2\,\mathrm{e}^{-\pi|z|^2}\,\mathrm{d}z$$
$$= \int_{\mathbb{C}} |F(z)|^2\,\mathrm{e}^{-\frac{2\pi}{p}|z|^2}\cdot\mathrm{e}^{2\pi\,\mathrm{Re}[\mu z]}\mathrm{e}^{-\frac{(p-2)\pi}{p}|z|^2}\,\mathrm{d}z$$
$$\leq \left(\int_{\mathbb{C}}|F(z)|^p\,\mathrm{e}^{-\pi|z|^2}\,\mathrm{d}z\right)^{2/p}\cdot\left(\int_{\mathbb{C}}\mathrm{e}^{\frac{2\pi p}{p-2}\,\mathrm{Re}[\mu z]}\mathrm{e}^{-\pi|z|^2}\right)^{(p-2)/p} < \infty.$$

□

The proposition above can alternatively be stated in the following way.

Proposition 11. *Let $F \in \mathcal{F}^2(\mathbb{C})$ either of order $\rho < 2$, or of order $\rho = 2$ and type $\tau < \pi/2$. Then, the HRT conjecture holds for $f := \mathcal{B}^{-1}F$.*

Since the complex polynomials are dense in the Fock space, the above proposition immediately implies that the HRT conjecture holds for a dense set of $f \in L^2(\mathbb{R})$.

Remark 12. *We know that the HRT conjecture holds for $f \in L^2(\mathbb{R})$ whose Bargmann transform is in $\mathcal{H} := \bigcup_{p>2}\mathcal{F}^p(\mathbb{C}) \subset \mathcal{F}^2(\mathbb{C})$ (where the inclusion follows from the fact that the constants are in $\mathcal{F}^p(\mathbb{C})$ and Hölder's inequality). We also know that this tells us that the HRT conjecture holds on a dense subset of $L^2(\mathbb{R})$. We would be interested in saying a little bit more than that; e.g., the*

HRT conjecture holds everywhere except for (potentially) on a nowhere dense set. Unfortunately, the derivation above does not prove this because $\mathcal{H}^{c}$ is not *nowhere dense in $\mathcal{F}^2(\mathbb{C})$. We can see this by showing that $\mathcal{H}$ has empty interior in $\mathcal{F}^2(\mathbb{C})$. A natural follow up question is whether $\mathcal{H}$ is meager. We have not been able to conclude so far.*

5 Understanding the dense set better

D. W. Stroock presents the following characterisation for a function $F \in \mathcal{F}^2(\mathbb{C})$ whose translate by $\mu \in \mathbb{C}$ is in the Fock.

Theorem 13 ([4, Theorem 6.1 on p. 615])**.** *Let $F \in \mathcal{F}^2(\mathbb{C})$ and set $f := \mathcal{B}^{-1}F \in L^2(\mathbb{R})$. Then, $\mathrm{T}_\mu F \in \mathcal{F}^2(\mathbb{C})$ if and only if*

$$\liminf_{\tau\uparrow 1} \left\| \mathrm{e}^{-\pi\mu(\cdot)}\, \mathrm{c}_\tau\, f\left(\cdot - \frac{\mu}{2}\right)\right\|_2 < \infty. \tag{1}$$

Moreover, if equation (1) *holds, then $\mathrm{e}^{-\pi\mu(\cdot)}\,\mathrm{T}_{\mu/2}\,\mathrm{c}_\tau\, f \to \mathrm{e}^{-\pi\mu^2/4}\mathcal{B}^{-1}\,\mathrm{T}_\mu\, F$ in $L^2(\mathbb{R})$ as $\tau \uparrow 1$.*

Proof. The reader is referred to the original source [4, pp. 611–612]. □

The translation of $\mathrm{c}_\tau\, f$ by $\mu/2$ is defined by extending $\mathrm{c}_\tau\, f$ to an analytic function and evaluating the extension. We want to present everything from a different point of view and try to find necessary (and sufficient) conditions for $\mathrm{T}_\mu\, F \in \mathcal{F}^2(\mathbb{C})$ that are more readily interpretable.

In order to do so, we will introduce the *Gabor transform*:

$$\mathcal{G}f(x,\omega) := 2^{1/4} \cdot \int_{\mathbb{R}} f(t)\mathrm{e}^{-\pi(t-x)^2}\mathrm{e}^{-2\pi\mathrm{i}t\omega}\,\mathrm{d}t, \qquad (x,\omega) \in \mathbb{R}^2,$$

for $f \in L^2(\mathbb{R})$. The Gabor transform is closely related to the Bargmann transform. In particular, one can show that

$$\mathcal{B}f(x+\mathrm{i}\omega) = \mathrm{e}^{-\pi\mathrm{i}x\omega}\mathcal{G}f(x,-\omega)\mathrm{e}^{\frac{\pi}{2}\left(x^2+\omega^2\right)}, \qquad (x,\omega) \in \mathbb{R}^2.$$

Remember that $\mathrm{T}_\mu\,\mathcal{B}f \in \mathcal{F}^2(\mathbb{C})$ if and only if $\mathcal{B}f \cdot \mathrm{e}^{-\pi\mu(\cdot)} \in \mathcal{F}^2(\mathbb{C})$ which in turn happens if and only if

$$\int_{\mathbb{R}^2} |\mathcal{G}f(x,\omega)|^2\,\mathrm{e}^{-2\pi(tx+\xi\omega)}\,\mathrm{d}(x,\omega) < \infty, \tag{2}$$

where $t = \operatorname{Re}\mu$ and $\xi = \operatorname{Im}\mu$. Heuristically, the above means that we are asking the Gabor transform to decay exponentially in time and frequency. Intuitively, this should correspond to an exponential decay in the signal and analyticity of the signal on a strip around $\mathbb{R}$. Let us flesh this out a little bit in the following. We will assume that $t > 0$ and $\xi > 0$. We may get similar results when t and/or ξ are negative. However, we are not really interested in these cases because $\mu = \alpha\beta + \beta\mathrm{i}$ is in the first quadrant of the complex plane.

Lemma 14. *Let $t, \xi > 0$ and $f \in L^2(\mathbb{R})$. Then, $\mathrm{T}_{t+\mathrm{i}\xi}\, \mathcal{B}f \in \mathcal{F}^2(\mathbb{C})$ if and only if $f \cdot \mathrm{e}^{-\pi t(\cdot)}$ "extends" to a holomorphic function f_t on the strip $\{z \in \mathbb{C} \,|\, 0 < \operatorname{Im} z < \xi/2\}$ such that*

$$\int_{\mathbb{R}} |f_t(x + \mathrm{i}y)|^2 \, \mathrm{d}x \lesssim 1, \qquad 0 \le y < \xi/2,$$

where the implicit constant is independent of y, and

$$\lim_{y \downarrow 0} \int_{\mathbb{R}} \left| f_t(x + \mathrm{i}y) - f(x)\mathrm{e}^{-\pi t x} \right|^2 \, \mathrm{d}x = 0.$$

Proof. We have already established that $\mathrm{T}_{t+\mathrm{i}\xi}\, \mathcal{B}f \in \mathcal{F}^2(\mathbb{C})$ is equivalent to equation (2). Following an idea by Fushuai Jiang, we will now try to split the integrand:

$$\int_{\mathbb{R}^2} |\mathcal{G}f(x,\omega)|^2 \, \mathrm{e}^{-2\pi(tx+\xi\omega)} \, \mathrm{d}(x,\omega) = \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{G}f(x,\omega)|^2 \, \mathrm{e}^{-2\pi t x} \, \mathrm{d}x \cdot \mathrm{e}^{-2\pi\xi\omega} \, \mathrm{d}\omega. \tag{3}$$

For the inner integrand, we note that

$$\mathcal{G}f(x,\omega)\mathrm{e}^{-\pi t x} = \int_{\mathbb{R}} f(s)\mathrm{e}^{-\pi t s} \varphi(s-x) \mathrm{e}^{\pi t(s-x)} \mathrm{e}^{-2\pi \mathrm{i} s\omega} \, \mathrm{d}s = \mathcal{V}_{\psi_t} f_t(x,\omega),$$

where $\varphi := 2^{1/4}\mathrm{e}^{-\pi(\cdot)^2}$ denotes the normalised Gaussian, and $\psi_t := \varphi \cdot \mathrm{e}^{\pi t(\cdot)}$. Moreover,

$$\mathcal{V}_\psi f(x,\omega) := \int_{\mathbb{R}} f(t)\overline{\psi(t-x)}\mathrm{e}^{-2\pi \mathrm{i} t\omega} \, \mathrm{d}t$$

is the short-time Fourier transform with window ψ. According to [1, Equation (3.5) on p. 39], we have

$$\begin{aligned}\int_{\mathbb{R}} |\mathcal{G}f(x,\omega)|^2 \, \mathrm{e}^{-2\pi t x} \, \mathrm{d}x = \int_{\mathbb{R}} |\mathcal{V}_{\psi_t} f_t(x,\omega)|^2 \, \mathrm{d}x &= \int_{\mathbb{R}} \left| \left(\widehat{f_t} \cdot \mathrm{T}_\omega \, \widehat{\psi_t} \right)\widehat{\ }(-x) \right|^2 \, \mathrm{d}x \\ &= \int_{\mathbb{R}} \left| \widehat{f_t}(x) \right|^2 \cdot \left| \mathrm{T}_\omega \, \widehat{\psi_t}(x) \right|^2 \, \mathrm{d}x,\end{aligned}$$

and the splitting is becoming apparent. We can compute the Fourier transform of ψ_t explicitly and obtain

$$\widehat{\psi_t}(\xi) = \mathrm{e}^{\frac{\pi}{4}t^2} \mathrm{e}^{-\pi \mathrm{i} \xi t} \cdot \varphi(\xi), \qquad \xi \in \mathbb{R}.$$

Plugging all of this back into equation (3), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2} |\mathcal{G}f(x,\omega)|^2 \, \mathrm{e}^{-2\pi(tx+\xi\omega)} \, \mathrm{d}(x,\omega) \\ &\quad = \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \widehat{f_t}(x) \right|^2 \cdot \mathrm{e}^{\frac{\pi}{2}t^2} \, |\varphi(x-\omega)|^2 \, \mathrm{d}x \cdot \mathrm{e}^{-2\pi\xi\omega} \, \mathrm{d}\omega \\ &\quad = \mathrm{e}^{\frac{\pi}{2}t^2} \cdot \int_{\mathbb{R}^2} \left| \widehat{f_t}(x) \right|^2 \mathrm{e}^{-2\pi\xi x} \cdot |\varphi(x-\omega)|^2 \, \mathrm{e}^{2\pi\xi(x-\omega)} \, \mathrm{d}(x,\omega) \\ &\quad = \mathrm{e}^{\frac{\pi}{2}t^2} \cdot \int_{\mathbb{R}} \left| \widehat{f_t}(x) \right|^2 \mathrm{e}^{-2\pi\xi x} \cdot \int_{\mathbb{R}} |\varphi(\nu)|^2 \, \mathrm{e}^{2\pi\xi\nu} \, \mathrm{d}\nu \, \mathrm{d}x \\ &\quad = \mathrm{e}^{\frac{\pi}{2}\left(t^2+\xi^2\right)} \cdot \int_{\mathbb{R}} \left| \widehat{f_t}(x)\mathrm{e}^{-\pi\xi x} \right|^2 \, \mathrm{d}x.\end{aligned}$$

The lemma follows from the Paley–Wiener theorem (cf. Appendix A). $\square$

Remark 15. *We suspect that introducing the Gabor transform is not actually necessary and that all of this could also have been explained in terms of the Bargmann transform.*

A The Paley–Wiener theorems

We prove the following theorem (to make sure that all the constants are in the right places):

Theorem 16. *Let $a > 0$ and $f \in L^2(\mathbb{R})$. Then, the following conditions are equivalent:*

1. *There exists a holomorphic function $F : \{z \in \mathbb{C} \,|\, 0 < \operatorname{Im} z < a\} \to \mathbb{C}$ satisfying*
$$\sup_{0<y<a} \int_{\mathbb{R}} |F(x+\mathrm{i}y)|^2 \,\mathrm{d}x < \infty$$
and
$$\lim_{y\downarrow 0} \int_{\mathbb{R}} |F(x+\mathrm{i}y) - f(x)|^2 \,\mathrm{d}x = 0.$$

2. $\widehat{f} \cdot \mathrm{e}^{-2\pi a(\cdot)} \in L^2(\mathbb{R})$.

Proof. We will first prove that item 2 implies item 1: let

$$F(x+\mathrm{i}y) := \int_{\mathbb{R}} \widehat{f}(\xi)\mathrm{e}^{-2\pi y\xi} \cdot \mathrm{e}^{2\pi \mathrm{i}x\xi} \,\mathrm{d}\xi = \int_{\mathbb{R}} \widehat{f}(\xi)\mathrm{e}^{2\pi \mathrm{i}(x+\mathrm{i}y)\xi} \,\mathrm{d}\xi \qquad x+\mathrm{i}y \in \mathbb{C},$$

be the inverse Fourier transform of $\widehat{f} \cdot \mathrm{e}^{-2\pi y(\cdot)}$. Then, F is a well-defined holomorphic function on the strip $\{z \in \mathbb{C} \,|\, 0 < \operatorname{Im} z < a\}$ and Plancherel's theorem implies that

$$\sup_{|y|<a} \int_{\mathbb{R}} |F(x+\mathrm{i}y)|^2 \,\mathrm{d}x = \sup_{|y|<a} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \mathrm{e}^{-4\pi y\xi} \,\mathrm{d}\xi \le \|f\|_2^2 + \|\widehat{f} \cdot \mathrm{e}^{-2\pi a(\cdot)}\|_2^2$$

as well as

$$\lim_{y\downarrow 0} \int_{\mathbb{R}} |F(x+\mathrm{i}y) - f(x)|^2 \,\mathrm{d}x = \lim_{y\downarrow 0} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \cdot |\mathrm{e}^{-2\pi y\xi} - 1|^2 \,\mathrm{d}x = 0.$$

Next, we show that item 1 implies item 2. Consider $\psi \in C_\mathrm{c}^\infty(\mathbb{R})$ such that $0 \le \psi \le 1$, $\psi|_{B_1} = 1$ as well as $\psi|_{B_2^\mathrm{c}} = 0$, and define

$$\widehat{\phi}_n(\xi) := \psi\left(\frac{\xi}{n}\right), \qquad \xi \in \mathbb{R},$$

as well as the holomorphic functions

$$G_n(z) := \int_{\mathbb{R}} \phi_n(t) F(z-t) \,\mathrm{d}t, \qquad \operatorname{Im} z \in (0,a),$$

for $n \in \mathbb{N}$. Denote furthermore $g_{n,y}(x) := G_n(x+\mathrm{i}y)$ and $f_y(x) := F(x+\mathrm{i}y)$ for $x \in \mathbb{R}$ and $y \in (0,a)$. Then, $\widehat{g}_{n,y} = \widehat{\phi}_n \cdot \widehat{f}_y$ shows that $\widehat{g}_{n,y}$ is compactly supported. Therefore,

$$\int_{\mathbb{R}} \widehat{g}_{n,y}(\xi)\mathrm{e}^{2\pi y\xi}\mathrm{e}^{2\pi \mathrm{i}z\xi} \,\mathrm{d}\xi$$

is a holomorphic function and

$$\int_{\mathbb{R}} \widehat{g}_{n,y}(\xi) \mathrm{e}^{2\pi y\xi} \mathrm{e}^{2\pi \mathrm{i}(x+\mathrm{i}y)\xi} \,\mathrm{d}\xi = \int_{\mathbb{R}} \widehat{g}_{n,y}(\xi) \mathrm{e}^{2\pi \mathrm{i}x\xi} \,\mathrm{d}\xi = g_{n,y}(x) = G_n(x+\mathrm{i}y)$$

implies that

$$G_n(z) = \int_{\mathbb{R}} \widehat{g}_{n,y}(\xi) \mathrm{e}^{2\pi y\xi} \mathrm{e}^{2\pi \mathrm{i}z\xi} \,\mathrm{d}\xi, \qquad \operatorname{Im} z \in (0,a),$$

by the identity theorem of complex analysis. Since the above is true for all $y \in (0,a)$, we conclude that

$$\int_{\mathbb{R}} \widehat{g}_{n,y}(\xi) \mathrm{e}^{2\pi y\xi} \mathrm{e}^{2\pi \mathrm{i}z\xi} \,\mathrm{d}\xi = \int_{\mathbb{R}} \widehat{g}_{n,\epsilon}(\xi) \mathrm{e}^{2\pi \epsilon\xi} \mathrm{e}^{2\pi \mathrm{i}z\xi} \,\mathrm{d}\xi$$

for $\epsilon \in (0,a)$ which implies that

$$\widehat{g}_{n,y}(\xi) \mathrm{e}^{2\pi y\xi} = \widehat{g}_{n,\epsilon}(\xi) \mathrm{e}^{2\pi \epsilon\xi}, \qquad \xi \in \mathbb{R}, \tag{4}$$

by the Fourier inversion theorem. Now, let us introduce $g_n := \phi_n * f$ and note that

$$\begin{aligned} &\left\| \widehat{g}_n - \widehat{g}_{n,y} \mathrm{e}^{2\pi y(\cdot)} \right\|_2 \\ &\quad \leq \|\widehat{g}_n - \widehat{g}_{n,\epsilon}\|_2 + \left\| \widehat{g}_{n,\epsilon} \left(1 - \mathrm{e}^{2\pi\epsilon(\cdot)}\right) \right\|_2 + \left\| \widehat{g}_{n,\epsilon} \mathrm{e}^{2\pi\epsilon(\cdot)} - \widehat{g}_{n,y} \mathrm{e}^{2\pi y(\cdot)} \right\|_2 . \end{aligned}$$

For the first term, we have that

$$\|\widehat{g}_n - \widehat{g}_{n,\epsilon}\|_2 = \|g_n - g_{n,\epsilon}\|_2 = \|\phi_n * f - \phi_n * f_\epsilon\|_2 \leq \|\phi_n\|_1 \cdot \|f - f_\epsilon\|_2$$

which goes to zero as $\epsilon \downarrow 0$ once we show that $\phi_n \in L^1(\mathbb{R})$. For the latter, just note that the Fourier transform of ϕ_n is $\psi(\cdot/n)$ which is a smooth compactly supported function. For the second term, we can bound $1 - \mathrm{e}^{2\pi\epsilon(\cdot)}$ uniformly and independently on $\epsilon \in (0,a)$ on the support of of $\widehat{g}_{n,\epsilon}$ and note that the latter is pointwise bounded by $\widehat{f_\epsilon}$ whose L^2-norm can be bounded independently on ϵ as well. Therefore, Lebesgue's dominated convergence implies that

$$\lim_{\epsilon \downarrow 0} \left\| \widehat{g}_{n,\epsilon} \left(1 - \mathrm{e}^{2\pi\epsilon(\cdot)}\right) \right\|_2 = 0.$$

Finally, the third term is zero by equation (4) which implies that

$$\widehat{g}_n(\xi) = \widehat{g}_{n,y}(\xi) \mathrm{e}^{2\pi y\xi}, \qquad \xi \in \mathbb{R}.$$

By construction, $\widehat{f}$ agrees with $\widehat{g}_n$ on B_n and the same is true for f_y and $g_{n,y}$. Since $n \in \mathbb{N}$ was arbitrary, it follows that

$$\widehat{f}(\xi) = \widehat{f_y}(\xi) \mathrm{e}^{2\pi y\xi}, \qquad \xi \in \mathbb{R}.$$

Therefore, we conclude that $\widehat{f} \cdot \mathrm{e}^{-2\pi y(\cdot)} \in L^2(\mathbb{R})$ for all $y \in (0,a)$ with uniform bound on the L^2-norm, which implies item 2. □

Remark 17. *The proof is inspired by [3, Section 7.1 in Chapter VI on pp. 188-189].*

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