The Global Plan

- Integrable and nearly integrable systems, FPU paradox, KAM theorem, Nekhoroshev stability
- Warm up: KAM theorem in low dimension, Newton’s method, Moser-Levi’s proof of KAM
- KAM theorem: Herman-Fejoz’s proof via an implicit function theorem
- Application: stability of planetary motions, the semi-classical asymptotics for Schrödinger operators
Integrable integrable systems and action-angle coordinates

Key examples of nearly integrable systems: the standard map, the planetary \((N + 1)\) body problem, the Fermi-Pasta-Ulam model of the nonlinear spring.

The Fermi-Pasta-Ulam paradox.

KAM theorem: infinite time stability of most solutions.

Nekhoroshev stability: exponentially long stability of all solutions.
Let \((N^{2n}, \omega)\) be a symplectic manifold & \(H : N \rightarrow \mathbb{R}\) be a Hamiltonian.

Call \(F : N \rightarrow \mathbb{R}\) a first integral iff the Poisson bracket \(\{F, H\} \equiv 0\).

Functions \(F_1\) and \(F_2\) are in involution if their Poisson bracket is zero.

A Hamiltonian system is Arnold-Liouville integrable if there are \(n\) independent first integrals in involution.

Arnold-Liouville integrability implies that on open sets \(U \subset T^*T^n \cong T^n \times \mathbb{R}^n \ni (\theta, l)\) there exists a symplectic map \(\Phi : U \subset N^{2n}\) s. t. \(H \circ \Phi(l)\) depends only on \(l\). In particular, \(\Phi(U)\) is foliated by invariant \(n\)-dim’l tori & on each torus \(T^n\) the flow is

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\begin{align*}
\dot{\phi} &= \partial_l(H \circ \Phi)(l) = \omega(l), \\
\dot{l} &= 0.
\end{align*}
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\((\phi, l)\)–action-angle coordinates
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$(\phi, I)$–action-angle coordinates
Integrable systems

- Newtonian two body problem.
- Pendulum \( H = \frac{I^2}{2} - \cos 2\pi \theta, \ (\theta, I) \in T^*T \cong T \times \mathbb{R}. \)
- Harmonic oscillator \( \ddot{x} = -kx, \ x \in \mathbb{R} \) or in action-angle \( H = \frac{I^2}{2}, \ I^2 = (\dot{x})^2 + kx^2, \ \theta = \arg(\sqrt{k} x, \dot{x}/\sqrt{k}) \) (double check!).
- Motion in a central force field \( \ddot{x} = F(\|x\|)x, \ x \in \mathbb{R}^d. \)
- Newtonian two center problem.
- Lagrange’s top.
- Toda lattice: chain of particles \( \cdots < x_0 < x_1 < \cdots \) with exponential potential \( \sum_i \exp(x_i - x_{i+1}) \)
- A geodesic flow on an \( n \)-dimensional ellipsoid with different principal axes.
- A geodesic flow on a surface of revolution.
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Nearly integrable systems: the kicked rotor

The (Chirikov) standard map \((\theta, I) \rightarrow (\theta', I') = (\theta + I', I + \varepsilon \sin 2\pi \theta)\). It is often called the kicked rotator.

The kicked rotator approximates systems studied in mechanics of particles, accelerator physics, plasma physics, and solid state physics.

This is a perturbation of time-one map of the Hamiltonian \(H_0(I) = \frac{I^2}{2}\) of a Harmonic oscillator.
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Nearly integrable systems: a planetary \((N + 1)\) body problem

A planetary \((N + 1)\) body problem is a small perturbation of the sum of \(N\) two body problems.

Let \(q_0, q_1, \ldots, q_N \in \mathbb{R}^3\) — points, \(m_0 \gg m_1, \ldots, m_N > 0\) — masses.

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m_i \ddot{q}_i = \frac{m_0 m_i (q_0 - q_i)}{|q_0 - q_i|^3} + \sum_{j \neq i, 0} \frac{m_j m_i (q_j - q_i)}{|q_0 - q_i|^3}, \ i > 0.
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The harmonic oscillator $\ddot{u} = -ku$ or $H = \frac{p^2}{2} + \frac{ku^2}{2}$, where $u$ is displacement and $V(u) = \frac{ku^2}{2}$ is the potential of interaction.
A nonlinear spring: $\ddot{u} = -u - \alpha u^2$ or $H = \frac{p^2}{2} + \frac{u^2}{2} + \frac{\alpha u^3}{3}$, where $u$ is displacement and $V(u) = \frac{u^2}{2} + \frac{\alpha u^3}{3}$ is the potential of interaction.

A nonlinear spring: chain $0 = u_0 < \cdots < u_i < u_{i+1} < \cdots < u_N = 1$

the potential of interaction is $V(u) = \sum_i \exp(u_i - u_{i+1}) = \sum_i \left(1 + (u_i - u_{i+1}) + \frac{(u_i - u_{i+1})^2}{2!} + \frac{(u_i - u_{i+1})^3}{3!} + \cdots \right)$
Nearly integrable systems: Fermi-Pasta-Ulam model

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For small \( u \) we have a small perturbation of the (linear) harmonic oscillator.

Chain of particles: \( 0 = u_0 < \cdots < u_i < u_{i+1} < \cdots < u_N = 1 \)

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Suppose $u_0 = 0$, $u_N = 1$ (fixed ends). For $i = k, \ldots, N - 1$ we have

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\ddot{u}_k = (u_{k+1} - 2u_k + u_{k-1}) + \alpha((u_{k+1} - u_k)^2 + (u_k - u_{k-1})^2)
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The action-angle coordinates (or normal modes)

$$
\{(u_k, \dot{u}_k)\}_{k=1}^{N-1} \rightarrow \{ (\theta_k, I_k) \}_{k=1}^{N-1}, \quad \text{where } I_k = (\dot{A}_k^2 + \omega_k^2 A_k^2)/2,
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A_k = \sqrt{\frac{2}{N+1}} \sum_{n=1}^{N} u_k \sin \frac{nk\pi}{N+1}, \quad \omega_k = 4 \sin^2(k\pi/(2N+2)).
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Fermi-Pasta-Ulam experiment: $N = 32$ or $64$ and $\alpha = $small. Most solutions turn out to be almost periodic.
Suppose $u_0 = 0$, $u_N = 1$ (fixed ends). For $i = k, \ldots, N - 1$ we have

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Fermi-Pasta-Ulam paradox

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Initially, only mode $A_1$ (blue) is excited. After flowing to other modes, $A_2$ (green), $A_3$ (red), etc., the energy almost fully returns to mode $A_1$: this was a surprise!
Let $H_0(l) -$ integrable and $H_\epsilon(\theta, l) = H_0(l) + \epsilon H_1(\theta, l) -$ a perturbation.

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Kolmogorov - Arnol’d - Moser Theorem (warm up)

Let $H_0(I)$ – integrable and $H_\varepsilon(\theta, I) = H_0(I) + \varepsilon H_1(\theta, I)$ – a perturbation.


**KAM Theorem**

Let $H_0(I)$ be real analytic and nondegenerate, i.e. Hessian $\det \partial^2 H_0 \neq 0$. Let $H_\varepsilon(\theta, I)$ be a real analytic perturbation. Then most initial conditions in an open bounded set $\mathbb{T}^n \times B \subset \mathbb{T}^n \times \mathbb{R}^n$ have quasiperiodic orbits filling an analytic $n$-dimensional torus. In particular, most solutions are bounded.
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Kolmogorov - Arnol’d - Moser (KAM) Theorem

Let $\tau, \gamma > 0$. $\omega \in \mathbb{R}^n$ is called \textbf{(\(\tau, \gamma\))-diophantine} if for all $k \in \mathbb{Z}^n \setminus 0$ we have $|\omega \cdot k| \geq \gamma |k|^{-(n-1)-\tau}$. Denote this set $D_{\tau,\gamma}$.

**Lemma**

There is $c = c(n, \tau) > 0$ s. t. $\text{Leb} (D_{\tau,\gamma} \cap B) \geq (1 - c\gamma) \text{Leb}(B)$.

**KAM Theorem**

Let $H_0(I)$ be nondegenerate, i.e. Hessian $\det \partial^2 H_0 \neq 0$. Let $H_\varepsilon(\theta, I)$ be an analytic perturbation and $\varepsilon$ small.

- for each diophantine $\omega \in D_{\tau,\gamma}$ the system $H_\varepsilon$ has an analytic invar. $n$-dim’l torus $T^n_\omega$ with the flow on $T^n_\omega$ conjugate to $\dot{\theta} = \omega$.
- The union of tori has measure $1 - O(\sqrt{\varepsilon})$ of the total measure.
- Global normal form: There is a symplectic map $\Phi_\varepsilon$ such that $K_\varepsilon(\theta, I) = H_\varepsilon \circ \Phi_\varepsilon(\theta, I) = K_0(I) + \exp(-C\varepsilon^{-1/\rho})R(\theta, I)$, where $\rho > 1$ and $R$ vanishes on $\Phi_\varepsilon^{-1}(\bigcup_{\omega \in D_{\tau,\gamma}} T^n_\omega)$.

V. Kaloshin (University of Maryland)  
Introduction to KAM theory  
July 3, 2013  17 / 20
Let $\tau, \gamma > 0$. $\omega \in \mathbb{R}^n$ is called $(\tau, \gamma)$-diophantine if for all $k \in \mathbb{Z}^n \setminus 0$ we have $|\omega \cdot k| \geq \gamma |k|^{-(n-1)-\tau}$. Denote this set $D_{\tau,\gamma}$.

**Lemma**

There is $c = c(n, \tau) > 0$ s. t. $\text{Leb}(D_{\tau,\gamma} \cap B) \geq (1 - c\gamma) \text{Leb}(B)$.

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Kolmogorov - Arnol’d - Moser (KAM) Theorem

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V. Kaloshin (University of Maryland)
Nekhoroshev stability

Nekhoroshev Stability

Let $H_0(I)$ be nondegenerate, i.e. Hessian $\det \partial^2 H_0 \neq 0$. Let $H_\varepsilon(\theta, I)$ be an analytic perturbation. Then there is $C = C(H_0) > 0$ such that for each initial condition $(\theta_0, I_0)$ in a bounded set $\mathbb{T}^n \times B \subset \mathbb{T}^n \times \mathbb{R}^n$ for a solution $(\theta, I)(t)$ of $H_\varepsilon$ we have

$$\|I(t) - I_0\| \leq C \varepsilon^{1/2n}, \quad \forall |t| \leq C \exp(C \varepsilon^{-1/2n}).$$

Warning: For large dimensional systems exponential stability is not so practical. For example, for the Solar system (the spacial 9 body problem) $2n = 48$.

Remark

KAM Theorem as well as Nekhoroshev stability hold for sufficiently smooth systems with exponential stability replaced by polynomial stability.
Nekhoroshev stability

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Integrable integrable systems and action-angle coordinates.

Key examples of nearly integrable systems: the standard map, the planetary \((N + 1)\) body problem, the Fermi-Pasta-Ulam model of the nonlinear spring.

The Fermi-Pasta-Ulam paradox: most solutions near equilibrium are almost periodic.

KAM theorem: infinite time stability of most solutions.

Nekhoroshev stability: exponentially long stability of all solutions.
The Global Plan

- Integrable and nearly integrable systems, FPU paradox, KAM theorem, Nekhoroshev stability
- Warm up: KAM theorem in low dimension, Newton’s method, Moser-Levi’s proof of KAM
- KAM theorem: Herman-Fejoz’s proof via an implicit function theorem
- Application: stability of planetary motions, the semi-classical asymptotics for Schrödinger operators