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Topological components of spaces of representations

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Let S be a closed oriented surface of genus $g > 1$ and let π denote its fundamental group. Let G be a semisimple Lie group. The purpose of this paper is to investigate the global properties of the space $\mathrm{Hom}(\pi, G)$ of all representations $\pi \to G$, when G is locally isomorphic to either $PSL(2, \mathbf{C})$ or $PSL(2, \mathbf{R})$. The main results of this paper may be summarized as follows:

Theorem A. (i) *Let G be the n-fold covering group of $PSL(2, \mathbf{R})$. Then the number of connected components of* $\mathrm{Hom}(\pi, G)$ *is given by the following formula:*

$$2n^{2g} + (4g-4)/n - 1 \qquad \text{if } n \mid 2g-2$$

$$2\left[\frac{(2g-2)}{n}\right] + 1 \qquad \text{if } n \nmid 2g-2.$$

For example $\mathrm{Hom}(\pi, SL(2, \mathbf{R}))$ *has* $2^{2g+1} + 2g - 3$ *components.*

(ii) *Let $G = SO(3)$ or $PSL(2, \mathbf{C})$. Then* $\mathrm{Hom}(\pi, G)$ *has exactly two connected components. If $G = SU(2)$ or $SL(2, \mathbf{C})$ then* $\mathrm{Hom}(\pi, G)$ *is connected.*

Since π is a finitely generated group, the space $\mathrm{Hom}(\pi, G)$ is a real analytic variety whenever G is a connected Lie group, and is a real affine algebraic variety whenever G is a linear algebraic group over $\mathbf{R}$ [3, 18, 27, 32]. The group G acts on $\mathrm{Hom}(\pi, G)$ by conjugation and the orbit space will be denoted by $\mathrm{Hom}(\pi, G)/G$. Geometrically, the G-orbits on $\mathrm{Hom}(\pi, G)$ parametrize equivalence classes of flat principal G-bundles over S and the space $\mathrm{Hom}(\pi, G)/G$ is the *deformation space* of flat G-bundles over S. The characteristic classes of G-bundles determine invariants of representations $\pi \to G$. When π is the fundamental group of a closed surface and G is a connected Lie group, the only invariants lie in the cohomology group $H^2(S; \pi_1(G)) \cong \pi_1(G)$ (using the orientation on S). There is an *obstruction map*

$$o_2 \colon \mathrm{Hom}(\pi, G) \to H^2(S; \pi_1(G)) \cong \pi_1(G).$$

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When $G=PSL(2,\mathbf{R})$, this invariant equals the Euler number of the associated $\mathbf{RP}^1$-bundle over S, an integer since $\pi_1(G)\cong\mathbf{Z}$. The resulting map

$$e\colon \operatorname{Hom}(\pi, G)\to H^2(S;\mathbf{Z})\cong\mathbf{Z}$$

disregards the flat structure on a flat principal G-bundle yet expresses the isomorphism class of the principal G-bundle. Similarly, when $G=PSL(2,\mathbf{C})$, the resulting invariant

$$w_2\colon \operatorname{Hom}(\pi, G)\to H^2(S;\mathbf{Z}/2)\cong\mathbf{Z}/2$$

is the second Stiefel-Whitney class of the associated $\mathbf{H}^3$-bundle over S.

Theorem B. (i) *Let* $G=PSL(2,\mathbf{R})$. *The connected components of* $\operatorname{Hom}(\pi, G)$ *are the preimages* $e^{-1}(n)$, *where n is an integer satisfying* $|n|\leqq 2g-2$.

(ii) *Let* $G=PSL(2,\mathbf{C})$ *or* $SO(3)$. *The connected components of* $\operatorname{Hom}(\pi, G)$ *are the two preimages of* w_2: $\operatorname{Hom}(\pi, G)\to\mathbf{Z}/2$. *If* $G=SL(2,\mathbf{C})$ *or* $SU(2)$, *then* $\operatorname{Hom}(\pi, G)$ *is connected.*

The inequality $|e(h)|\leqq|\chi(S)|$ is due to Wood [41], based on earlier work by Milnor [31], who treated the case $G=SL(2,\mathbf{R})$. In particular Milnor was the first to notice the existence of flat oriented 2-plane bundles over surfaces which are nontrivial. Theorem B can be used to deduce the following, which was originally proved in the author's doctoral dissertation [8]:

Corollary C. *Let* $\phi\in\operatorname{Hom}(\pi, PSL(2,\mathbf{R}))$. *Equality holds in the Milnor-Wood inequality,* i.e. $|e(\phi)|=|\chi(S)|$, *if and only if* ϕ *is an isomorphism of* π *onto a discrete subgroup of* $PSL(2,\mathbf{R})$.

There are several directions in which Theorem B may be generalized. In his Maryland thesis [24], Jankins determined the components of $\operatorname{Hom}(\pi, PSL(2,\mathbf{R}))$, where π is an arbitrary cocompact Fuchsian group whose quotient is not a sphere; that exceptional case is handled in [25]. A generalization of Corollary C to the case when G is the group of orientation-preserving homeomorphisms of the circle has been given by S. Matsumoto [30] which builds upon work by E. Ghys [7]. For a conjecture generalizing Corollary C to cocompact lattices in semisimple Lie groups, see Goldman [10]. For an extension of Corollary C to representations into $PU(n,1)$, see Toledo [38].

The proof of Theorem B(i) involves certain decompositions of the surface S, and a corresponding relative version of Theorem B for compact oriented surfaces with boundary. Our philosophy (compare [8, 15, 16, 20, 37]) is that compact surfaces of Euler characteristic -1 and -2 are the "building blocks" of compact surfaces of negative Euler characteristic – every surface may be decomposed into subsurfaces of Euler characteristic -1 and two adjacent subsurfaces of Euler characteristic -1 form a subsurface of Euler characteristic -2. If S is a compact surface with nonempty boundary, then $\pi_1(S)$ is a free group of rank $1-\chi(S)$ and $\operatorname{Hom}(\pi, G)\approx G^{1-\chi(S)}$ is connected. Thus to nontrivially generalize Theorem B we must impose boundary conditions on the representations (or the bundles) involved. The most convenient method for imposing boundary conditions for representations $\pi\to G=PSL(2,\mathbf{R})$ is to assume that a representation $\phi \in \mathrm{Hom}(\pi, G)$ maps, for each boundary component C, the corresponding element of $\pi_1(S)$ to a non-elliptic element of G. Since a non-elliptic element of G lies on a unique one-parameter subgroup of G, there is a canonical trivialization of the associated flat bundle over ∂S. Given such a representation ϕ, there is a *relative Euler class* $e(\phi)$ which lies in $H^2(S, \partial S; \mathbf{Z}) \cong \mathbf{Z}$.

Theorem D. *Let S be a compact oriented surface with boundary and let $W(S)$ denote the set of all representations $\phi \in \mathrm{Hom}(\pi, G)$ such that for each boundary component C of S, the corresponding element of π is mapped under ϕ to a hyperbolic element. Then the relative Euler class $e(\phi)$ satisfies an inequality*

$$e(\phi) \leqq |\chi(S)|,$$

with equality holding if and only if ϕ is an isomorphism of π onto a Fuchsian group Γ such that the quotient $\mathbf{H}^2/\Gamma$ is homeomorphic to the interior of S. Furthermore the connected components of $W(S)$ are the preimages of the relative Euler class map

$$e\colon\ W(S) \to H^2(S, \partial S; \mathbf{Z}) \cong \mathbf{Z}.$$

The proof of (ii) in Theorem B is considerably simpler than the proof of (i), due to the better behavior of polynomial maps over the complex numbers. It seems that the following generalization of Theorem B(ii) is plausible:

Conjecture. *Suppose π is the fundamental group of a closed oriented surface S of genus $g > 1$. Let G be a connected complex semisimple Lie group. Then the connected components of* $\mathrm{Hom}(\pi, G)$ *are the preimages of the obstruction map* $o_2\colon \mathrm{Hom}(\pi, G) \to \mathrm{H}^2(S; \pi_1(G)) \cong \pi_1(G)$. *In particular the number of components of* $\mathrm{Hom}(\pi, G)$ *equals the order of* $\pi_1(G)$.

(When G is not necessarily an algebraic group (e.g. when G is the reduced Heisenberg group), then $\mathrm{Hom}(\pi, G)$ may have infinitely many components and therefore cannot be a real algebraic set (Goldman [9]).)

Recently N. Hitchin [22], using quite different analytic techniques, has determined the topological types of the components of $\mathrm{Hom}(\pi, PSL(2, \mathbf{R}))/PSL(2, \mathbf{R})$ and the Betti numbers of $\mathrm{Hom}(\pi, PSL(2, \mathbf{C}))/PSL(2, \mathbf{C})$ where $\pi = \pi_1(S)$ is the fundamental group of a closed orientable surface S of genus $g > 1$. In particular his results imply Theorem B above. Specifically he has shown that for $0 < k \leqq 2g-2$ the component $e^{-1}(2-2g+k)$ is a complex vector bundle over the k-th symmetric power of the surface S. It seems quite plausible that there is a similar description of the components of $W(S)$ where S is a compact surface with boundary.

§1. The Lie groups

In this preliminary section we establish notation and collect basic facts concerning the geometry of the relevant Lie groups. We begin by discussing the complex groups $SL(2, \mathbf{C})$ and $PSL(2, \mathbf{C})$.

1.1. The group $SL(2, \mathbf{C})$ consists of all complex 2×2-matrices having determinant one. Its center is the group of scalar matrices $\pm I$. There are three conjugacy classes of complex algebraic subgroups of $SL(2, \mathbf{C})$ other than $\{I\}$, $\{\pm I\}$ and $SL(2, \mathbf{C})$ itself: the complex Cartan subgroup consisting of all diagonal matrices; the Borel subgroup consisting of upper-triangular matrices; the parabolic one-parameter group consisting of unipotent upper-triangular matrices. An element of $SL(2, \mathbf{C})$ is semisimple if it is diagonalizable over $\mathbf{C}$; equivalently $X\in SL(2, \mathbf{C})$ is semisimple if and only if $X=\pm I$ or $\operatorname{tr} X \neq 2$. Two semisimple elements are conjugate if and only if they have the same trace. A parabolic element of $SL(2, \mathbf{C})$ is one which is not semisimple; one can see that there are two conjugacy classes of parabolic elements $\operatorname{tr}^{-1}(2)-\{I\}$ and $\operatorname{tr}^{-1}(-2)-\{-I\}$.

The symmetric space of $SL(2, \mathbf{C})$ is the real hyperbolic 3-space $\mathbf{H}^3$. A Cartan subgroup can be geometrically understood as the stabilizer of a geodesic in $\mathbf{H}^3$ and the Borel subgroup is the stabilizer of an ideal point in $\mathbf{H}^3$. It is easy to see that if $X, Y\in SL(2, \mathbf{C})$, then the following conditions are equivalent:

(i) X, Y fix a common ideal point on the boundary $\partial\mathbf{H}^3$;

(ii) X, Y generate a reducible representation on $\mathbf{C}^2$;

(iii) X, Y lie in a common Borel subgroup.

The quotient group $PSL(2, \mathbf{C})=SL(2, \mathbf{C})/\{\pm I\}$ acts effectively on $\mathbf{H}^3$ and in fact equals the group of all orientation-preserving isometries of $\mathbf{H}^3$.

1.2. Next we discuss groups locally isomorphic to $PSL(2, \mathbf{R})$. Recall that $PSL(2, \mathbf{R})=SL(2, \mathbf{R})/\{\pm I\}$. For another discussion of this material, see Kulkarni-Raymond [26].

Let G denote the group $PSL(2, \mathbf{R})$. Let $\mathfrak{g}$ denote the Lie algebra of G. We observe that the Killing form on $\mathfrak{g}$ defines an indefinite quadratic function Q: $\mathfrak{g}\to\mathbf{R}$ which is invariant under the adjoint representation. Thus the adjoint representation embeds G in the group $SO(2,1)$, which for the rest of this section we denote by G^*. Note that under this embedding G is the connected component of the identity in G^*.

Geometrically G^* is the full group of isometries of its symmetric space, the hyperbolic plane $\mathbf{H}^2$, which may further be identified (using the upper-half-space model for $\mathbf{H}^2$) with the projective linear group $PGL(2, \mathbf{R})$. Its identity component $PSL(2, \mathbf{R})$ consists of the orientation-preserving isometries of X and the isotropy group of G acting on $\mathbf{H}^2$ is the maximal compact subgroup $PSO(2)\subset PSL(2, \mathbf{R})$ is the image of $SO(2)\subset SL(2, \mathbf{R})$ under the homomorphism $SL(2, \mathbf{R})\to PSL(2, \mathbf{R})$. It follows that G acts simply transitively on the unit tangent bundle T_1X of the hyperbolic plane; hence topologically G is an open solid torus.

Conjugacy classes in $SL(2, \mathbf{R})$ are essentially determined by the trace function $\operatorname{tr}\colon SL(2, \mathbf{R})\to\mathbf{R}$. If the absolute value of the trace of $A\in SL(2, \mathbf{R})$ is greater than 2, then the corresponding element of G acts as a translation along a unique geodesic γ. If the element translates along γ by a distance d, then the corresponding matrix has trace $\pm 2\cosh d/2$. Such a transformation is said to be *hyperbolic*, and the subset of G consisting of hyperbolic elements will be denoted by Hyp. We shall call the distance d the *displacement* of A. Two hyperbolic elements are conjugate in G (resp. G^*) if and only if their displacements are equal. If the absolute value of the trace of A equals 2, then A fixes a unique ideal point of X and translates points along the horocycles centered there. Such a transformation is said to be *parabolic*; the set of all parabolic elements of G will be denoted Par. All parabolic transformations are conjugate in G^*, although they fall into two G-conjugacy classes, denoted Par^+, Par^-, depending on whether elements move ideal points in the positive or negative direction on the circle at infinity. The two G-conjugacy classes of parabolic elements are inverses of each other. If the absolute value of the trace of A is less than 2, then A fixes a point in $\mathbf{H}^2$ about which it acts as a rotation through an angle θ, where the trace of A equals $2\cos\theta/2$; such an element is called *elliptic*. The subset of G consisting of elliptic elements is denoted by Ell.

The adjoint representation leads to the Klein-Beltrami model of hyperbolic geometry as follows. Form the projective space $\mathbf{RP}^2$ of lines in $\mathfrak{g}$. (Since lines in $\mathfrak{g}$ correspond to one-parameter subgroups in G, we may just as well form a "projective space" of one-parameter subgroups in G.) The zero-locus of the homogeneous quadratic function Q is a conic $C\subset\mathbf{RP}^2$, whose complement consists of two components. One complementary component is convex and is a model for the hyperbolic plane; hence we call it $X\subset\mathbf{RP}^2$; the other component is homeomorphic to a Möbius band and will be called X^*. (It points correspond to the geodesics in X, i.e. the lines in $\mathbf{RP}^2$ which intersect X. Dually, the points of X correspond to the lines in X^*.) The conic C then becomes the sphere-at-infinity of the hyperbolic plane X, the points of which are the ideal points of X.

We shall need some notation concerning subsets of the universal covering group $\tilde{G}=\widetilde{SL(2,\mathbf{R})}$ of G. Since $\pi_1(G)=\mathbf{Z}$, the group $\tilde{G}$ is homeomorphic to $\mathbf{R}^3$ and its center is infinite cyclic. Let Z denote the center of G, and let $\mathbf{z}\in Z$ be a generator. Since the action of G on the circle C lifts to an action of $\tilde{G}$ on the universal cover we may specify $\mathbf{z}$ in terms of the action of $\tilde{G}$ on the universal cover $\tilde{C}$ of the circle C: we choose the generator $\mathbf{z}$ so that it translates points on $\tilde{C}$ in the positive direction. Alternatively, let

$$\widetilde{\exp}\colon \mathfrak{g}\to\tilde{G}$$

denote the exponential map for $\tilde{G}$. Then for any integer n, $\mathbf{z}^n$ can be defined in terms of the exponential map by:

$$\mathbf{z}^n=\widetilde{\exp}\begin{bmatrix}0 & -n\pi\\ n\pi & 0\end{bmatrix}.$$

We shall derive a description of the conjugacy classes in $\tilde{G}$ as follows. Say that an element of $\tilde{G}$ is hyperbolic (resp. parabolic, elliptic), if it maps to an element of G which is hyperbolic (resp. parabolic, elliptic). The subsets of $\tilde{G}$ consisting of hyperbolic, elliptic, and parabolic elements fall into infinitely many components, indexed by $\mathbf{Z}$. Let Hyp_0 denote the set of exponential hyperbolic elements, i.e. those hyperbolic elements of $\tilde{G}$ which lie on one-parameter subgroups. For any $n\in\mathbf{Z}$, we define $\mathrm{Hyp}_n=\mathbf{z}^n\,\mathrm{Hyp}_0$. Furthermore associating to a hyperbolic element its invariant axis and its displacement defines a map $\mathrm{Hyp}_n\to X^*\times\mathbf{R}_+$, which one sees easily is a double covering.

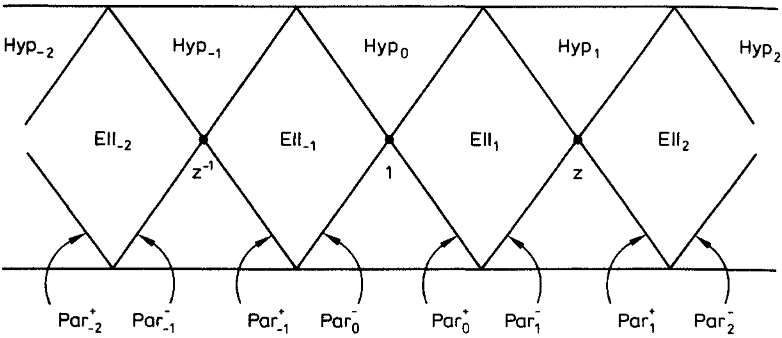

Fig. 1

Similarly we define sets Par_0, Par_0^+, Par_0^- to be the components of the inverse images of the sets Par, Par^+, Par^- which meet one-parameter subgroups. Moreover subsets Par_n, Par_n^+, Par_n^- are defined by $\mathrm{Par}_n^\pm = \mathbf{z}^n \mathrm{Par}_0^\pm$, etc.

Elliptic elements in $\tilde{G}$ all lie on one-parameter subgroups, and every elliptic one-parameter subgroup contains the center Z of $\tilde{G}$. If K is an elliptic one-parameter subgroup, then $K-Z$ has infinitely many components. For example, consider the elliptic one-parameter subgroup consisting of all

$$A(\theta)=\widetilde{\exp}\begin{bmatrix}0 & -\theta\\ \theta & 0\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

where $\theta\in\mathbf{R}$. If $\theta=n\pi$, then $A(\theta)=\mathbf{z}^n$ lies in the center of $\tilde{G}$. If $n>0$ we denote by Ell_n (resp. Ell_{-n}) the subset of all elements of $\tilde{G}$ conjugate to $A(\theta)$ (resp. $A(-\theta)$, where $(n-1)\pi<\theta n\pi$). Note that Ell_0 is undefined. It is easy to see that associating to an elliptic element its fixed point and its rotation angle defines a homeomorphism $\mathrm{Ell}_n \approx X\times(0,2\pi)$.

Thus the group $\tilde{G}$ is decomposed into the following subsets: its center $Z=\{\mathbf{z}^n: n\in\mathbf{Z}\}$, classes of hyperbolic elements Hyp_n, and two classes of parabolic elements Par_n^+ and Par_n^- where $n\in\mathbf{Z}$ and classes of elliptic elements Ell_n, where $n\in\mathbf{Z}$, $n\neq 0$. If $u\in\tilde{G}$, then its trace is defined by $\operatorname{tr} u=\operatorname{tr}\Pi(u)$, where Π: $\tilde{G}\to SL(2,\mathbf{R})$ is the universal covering. (Compare Fig. 1.)

1.3. The following technical lemma will be useful in the sequel.

Lemma. *Let $G=PSL(2,\mathbf{R})$ and suppose that $\{\xi_t\}_{0\leq t\leq 1}$ and $\{\eta_t\}_{0\leq t\leq 1}$ are two piecewise smooth paths in $G-\{I\}$ such that for each $0\leq t\leq 1$, the elements ξ_t and η_t are conjugate. Suppose that each path is transverse to the subset* Par *(e.g. if the maps $t\mapsto|\operatorname{tr}\tilde{\xi}_t|$ and $t\mapsto|\operatorname{tr}\tilde{\eta}|$ have 2 as a regular value). Then there exists a path $\{y_t\}_{0\leq t\leq 1}$ such that $\xi_t=y_t\eta_t y_t^{-1}$.*

Proof. The interval $[0,1]$ can be decomposed into subintervals $[a,b]$ such that either ξ_t (and thus also η_t) is hyperbolic for all $a<t<b$, or ξ_t (and η_t) is elliptic for all $a<t<b$, or ξ_t (resp. η_t) is parabolic for exactly one $t\in[a,b]$. Let $x_0\in X$ and $x_0^*\in X^*$ be arbitrary points. For each of these types of intervals, we find a particular type of path in G: for example, if ξ_t is elliptic for all $a<t<b$,

then choose a path $u_t \in G$ such that u_t maps the unique fixed point $\mathrm{Fix}(\xi_t)$ of ξ_t to $x_0 \in X$. Similarly choose a path $v_t \in G$ such that v_t maps $\mathrm{Fix}(\eta_t)$ to x_0. Then, since an elliptic element is determined by its conjugacy class in G and its fixed point, it follows that $y_t = u_t^{-1} v_t$ conjugates η_t to ξ_t as claimed. In an interval (a, b) where ξ_t and η_t are hyperbolic, let $\mathrm{Fix}(\xi_t)$ and $\mathrm{Fix}(\eta_t)$ denote the unique fixed points of ξ_t and η_t in $X^* \subset \mathbf{RP}^2$. Two conjugate hyperbolic elements which have the same fixed point in X^* are either identical or inverse. Thus $u_t^{-1} v_t$ conjugates η_t to either ξ_t for all t satisfying $a<t<b$ or to ξ_t^{-1} for all t satisfying $a<t<b$. In the former case take $y_t = u_t^{-1} v_t$; in the latter case let $y_t = u_t^{-1} v_t \sigma_t$ where σ_t is a continuous path in G consisting of elliptic elements of order two fixing points on the geodesic invariant under y_t. Then for all $t \in (a, b)$, we have $\xi_t = y_t \eta_t y_t^{-1}$.

Consider next the case that (a, b) is an interval such that ξ_t (and hence η_t) is parabolic for exactly one value of t. We claim that there exists a path $u_t \in G$ such that $u_t \xi_t u_t^{-1}$ is represented by the matrix

$$\begin{bmatrix} 0 & \mp 1 \\ \pm 1 & \operatorname{tr} \tilde{\xi}_t \end{bmatrix}$$

for each $a<t<b$ where $\tilde{\xi}_t$ is a lift of ξ_t to $SL(2, \mathbf{R})$. To this end consider the action of $\tilde{\xi}_t$ on $\mathbf{RP}^1$: there exist paths x_t, y_t in $\mathbf{RP}^1$ such that $x_t \neq y_t$ and such that $\xi_t(x_t) = y_t$. Choose a path w_t such that $w_t^{-1}([1,0]) = x_t$ and $w_t^{-1}([0,1]) = y_t$. Then $w_t \xi_t w_t^{-1}$ takes $[1,0]$ to $[0,1]$ and thus $w_t \tilde{\xi}_t w_t^{-1}$ is represented by a matrix

$$\begin{bmatrix} 0 & -b \\ b^{-1} & \operatorname{tr} \tilde{\xi}_t \end{bmatrix}.$$

Taking

$$u_t = \begin{bmatrix} \sqrt{|b|}^{-1} & 0 \\ 0 & \sqrt{|b|} \end{bmatrix} w_t$$

the claim is proved. Choose a lift $\tilde{\eta}_t$ of η_t such that $\operatorname{tr} \tilde{\eta}_t = \operatorname{tr} \tilde{\xi}_t$. As above, we find a path v_t such that $v_t \xi_t v_t^{-1}$ is represented by the matrix

$$\begin{bmatrix} 0 & \mp 1 \\ \pm 1 & \operatorname{tr} \tilde{\xi}_t \end{bmatrix}$$

for each $a<t<b$. Since η_t and ξ_t are conjugate, it follows that the signs on $u_t \eta_t u_t^{-1}$ and $v_t \xi_t v_t^{-1}$ must be equal, and indeed $u_t \eta_t u_t^{-1} = v_t \xi_t v_t^{-1}$. Letting $y_t = v_t^{-1} u_t$, we have $\xi_t = y_t \eta_t y_t^{-1}$.

Thus we have found an open covering of the interval $[0, 1]$ by intervals (a, b) for which there exist paths y_t such that $\xi_t = y_t \eta_t y_t^{-1}$ for each $t \in (a, b)$. It remains to find a path y_t defined for all $t \in [0, 1]$ such that $\xi_t = y_t \eta_t y_t^{-1}$ over all of $[0, 1]$. We may assume that each point $t \in [0, 1]$ lies in at most two intervals (a, b) and (a', b'), where $a < a' < b < b'$. Furthermore we may assume that for $t \in (a', b)$, the elements ξ_t, η_t are either elliptic or hyperbolic. Let y_t and y'_t be the corresponding paths. For each $t \in (a', b)$ let C_t denote the set of all y such that $y \eta_t y^{-1} = \xi_t$; then the union

$$\bigcup_{a' < t < b} C_t$$

is a principal fibration over (a', b) for which the paths y_t and y'_t are sections. The structure group of this fibration is the centralizer of ξ_t, which is connected; thus there exists a section $\bar{y}_t$ which agrees with y_t for $a' < t < a' + \varepsilon$ and agrees with y'_t for $b < t < b + \varepsilon$ for some $\varepsilon > 0$. This gives a well-defined path $\hat{y}_t$ for $0 \leqq t \leqq 1$ conjugating η_t to ξ_t. □

Path lifting

1.4. All of the spaces we shall be considering in this paper will be semi-analytic sets and therefore locally path-connected (indeed, such sets are triangulable, see Hironaka [21]). Thus for us the notions of connected components and path-connected components agree. It will sometimes be technically easier for us to work with path-connectedness rather than connectedness. The following general lemma will be useful in this regard. We shall say that a map $f: X \to Y$ satisfies the path-lifting property if for every $x \in X$ and path $\{y_t\}_{0 \leqq t \leqq 1}$, with $f(x) = y_0$, there exists a nondecreasing surjective map (a reparametrization of the path) τ: $[0, 1] \to [0, 1]$ and a path $\{x_s\}_{0 \leqq s \leqq 1}$ such that $f(x_s) = y_{\tau(s)}$ for $0 \leqq s \leqq 1$ and $x_0 = x$. In other words, up to possibly reparametrizing the path, every path starting from $f(x)$ and lying in the image of f can be lifted to a path starting from x.

Lemma. *Let X, Y be smooth manifolds and let $f: X \to Y$ be a smooth map. Suppose that Γ is a group acting on X such that $f \circ \gamma = f$ for $\gamma \in \Gamma$. Suppose that for each $y \in Y$, the following two conditions are satisfied:*

(a) *There exists $x \in f^{-1}(y)$ such that the differential of f at x is surjective;*

(b) *Γ acts transitively on the path-components of $f^{-1}(y)$.*

Then $f: X \to Y$ satisfies the path-lifting property. Furthermore, if Y is path-connected and Γ is trivial, then X is path-connected.

Proof. It follows by the implicit function theorem and hypothesis (a) that for each $t \in [0, 1]$, there exists $\tilde{x}_t \in f^{-1}(y_t)$ and $\varepsilon(t) > 0$, and local lifts $\{\tilde{x}^t_s\}_{t - \varepsilon(t) < s < t + \varepsilon(t)}$ such that $\tilde{x}^t_t = \tilde{x}_t$ and $f(\tilde{x}^t_s) = y_s$ whenever $t - \varepsilon(t) < s < t + \varepsilon(t)$. (We extend $\{y_t\}$ to a path on a neighborhood of $[0, 1]$ by making it locally constant outside $[0, 1]$.)

The collection of intervals $(t - \varepsilon(t), t + \varepsilon(t))$ covers $[0, 1]$, so there exists

$$t_0 < 0 < t_1 < \ldots < t_{m-1} < 1 < t_m$$

and a collection of local lifts $\{x^{(i)}_s\}_{t_i < s < t_{i-1}}$. Choose real numbers $u_1, \ldots, u_m$ with

$$u_1 = 0, \quad t_i < u_{i+1} < t_{i+1}, \quad u_m = 1.$$

Then we may define a path $\{x_s\}$ as follows. Since Γ acts transitively on the path components of $f^{-1}(y_0)$, there exists $\gamma_0 \in \Gamma$ and a path from x_0 to $\gamma_0 x_0^{(0)}$. Composing this path with $\{\gamma_0 x_s^{(0)}\}$, we obtain a lifted path x_0 to $x_{u_1}^1$. Since $f^{-1}(y_{u_1})$ is path-connected, there exists $\gamma_1 \in \Gamma$ and a path from $x_{u_1}^1$ to $\gamma_1 x_{u_1}^2$. Compose this path with $\{\gamma_1 x_s^1\}_{u_1 \leq s \leq u_2}$, etc. Continuing in this way we obtain the desired path.

Suppose that Y is path-connected. We claim that Γ acts transitively on the path-components of X. Given $x, x' \in X$, we shall find $\gamma \in \Gamma$ and a path $\{x_s\}_{0 \leq s \leq 1}$ in X such that $x_0 = x$ and $x_1 = \gamma x'$. To this end join $f(x)$ and $f(x')$ by a path $\{y_t\}_{0 \leq t \leq 1}$; by the preceding argument there exists a nondecreasing surjective map $\tau\colon [0,1] \to [0,1]$ and a path $\{x_s\}_{0 \leq s \leq 1}$ such that $f(x_s) = y_{\tau(s)}$ for $0 \leq s \leq 1$ and $x_0 = x$. Since Γ acts transitively on the path components of $f^{-1}(y_1)$, there exists $\gamma \in \Gamma$ such that x_1 can be joined by a path inside $f^{-1}(y_1)$ to $\gamma x'$. Composing with the preceding path, we obtain a path from x to x', proving the claim that Γ acts transitively on the path components of X. If Γ is trivial, then X is path-connected. □

§ 2. Spaces of representations

The purpose of this section is to discuss the general structure of the spaces $\mathrm{Hom}(\pi, G)$, when π is a finitely generated group and G is a Lie group. We then specialize to the case when π is the fundamental group of a surface and G is locally isomorphic to $PSL(2, \mathbf{R})$ or $PSL(2, \mathbf{C})$. Invariants are given which distinguish connected components of the spaces $\mathrm{Hom}(\pi, G)$ in certain cases. The term "connected component" will always refer to connected components in the classical (Hausdorff) topology rather than the Zariski topology, unless otherwise noted.

2.1. Let G denote a Lie group and π a finitely generated group. Suppose that π has a presentation

$$\langle A_1, \dots, A_m \mid R_1(A_1, \dots, A_m) = \dots = I \rangle.$$

Let $\mathrm{Hom}(\pi, G)$ denote the set of all homomorphisms $\pi \to G$. Then the evaluation map on the generators

$$\phi \mapsto (\phi(A_1), \dots, \phi(A_m))$$

defines a map $\mathrm{Hom}(\pi, G) \to G^m$, which is injective since $\{A_1, \dots, A_m\}$ generates π. Furthermore its image is the analytic subvariety of G^m consisting of all m-tuples $(x_1, \dots, x_m) \in G^m$ such that $R_i(x_1, \dots, x_m) = 1$ for each relation R_i. We shall henceforth identify $\mathrm{Hom}(\pi, G)$ with this analytic variety. If G is a linear algebraic group (with entries in a field K, and defined over a field k), then $\mathrm{Hom}(\pi, G)$ is the group of K-points of an algebraic variety defined over k. In particular, if G is a real linear algebraic group, then $\mathrm{Hom}(\pi, G)$ is a (not necessarily irreducible) real algebraic variety. Suppose that f: $G' \to G$ is a local isomorphism of Lie groups. Composition with f defines a map

$$f_*: \operatorname{Hom}(\pi, G) \to \operatorname{Hom}(\pi, G').$$

2.2. **Lemma** (Compare Culler [2]). *The image of* f_*: $\operatorname{Hom}(\pi, G) \to \operatorname{Hom}(\pi, G')$ *is a union of connected components of* $\operatorname{Hom}(\pi, G')$. *Let* C *be a component of the image of* f_*: $\operatorname{Hom}(\pi, G) \to \operatorname{Hom}(\pi, G')$. *Then* $f_*: f_*^{-1}(C) \to C$ *is a covering map with covering group the abelian group* $\operatorname{Hom}(\pi, \operatorname{Ker}(f))$.

Proof. Clearly we may replace G by the image $f(G')$ and hence assume that f is surjective. Then f induces a map f^m: $G'^m \to G^m$ which is a regular covering with covering group $(\operatorname{Ker} f)^m$. Now $\operatorname{Hom}(\pi, G)$ (resp. $\operatorname{Hom}(\pi, G')$) is identified with the subset of G^m (resp. G'^m) consisting of all m-tuples $(x_1, \ldots, x_m)$ satisfying the relations $R_i(x_1, \ldots, x_m) = I$ for each i. We claim that given any $x' = (x'_1, \ldots, x'_m) \in G'^m$ satisfying the relations $R_i(x'_1, \ldots, x'_m) = 1$, and any continuous path

$$x(t) = (x_1(t), \ldots, x_m(t)) \in G^m$$

satisfying $R_i(x_1(t), \ldots, x_m(t)) = 1$, $0 \leqq t \leqq 1$ and $f_*(x_i(0)) = x'_i$, there exists a unique continuous path

$$x'(t) = (x'_1(t), \ldots, x'_m(t)) \in G'^m$$

satisfying $R_i(x'_1(t), \ldots, x'_m(t)) = 1$ with $f_*(x'_i(t)) = x_i(t)$ and $x'_i(0) = x'_i$. Since f^m: $G'^m \to G^m$ is a covering map, there exists a unique lift $x'(t) \in G'^m$ of $x(t) \in G^m$. We must prove that this lift lies in $\operatorname{Hom}(\pi, G')$. Since

$$1 = R_i(x_1(t), \ldots, x_m(t)) = f(R_i(x'_1(t), \ldots, x'_m(t))),$$

it follows that $R_i(x'_1(t), \ldots, x'_m(t))$ describes a continuous path in $(\operatorname{Ker} f)^m$, which must be constant since $\operatorname{Ker} f$ is discrete. Thus for each t,

$$R_i(x'_1(t), \ldots, x'_m(t)) = R_i(x'_1(0), \ldots, x'_m(0)) = 1$$

whence the lift of a $(x_1(t), \ldots, x_m(t))$ is a path of homomorphisms. Suppose that ϕ': $\pi \to G'$ is a homomorphism and that η: $\pi \to \operatorname{Ker} f$ is a homomorphism. Since $\operatorname{Ker} f$ is central in G', there is a homomorphism $\eta\phi'$: $\pi \to G'$ defined by

$$\eta\phi': \ x \mapsto \eta(x)\,\phi'(x).$$

Clearly $f_*(\eta\phi') = f_*(\phi')$. In this way $\operatorname{Hom}(\pi, \operatorname{Ker} f)$ acts on $\operatorname{Hom}(\pi, G')$ leaving invariant the mapping f_*: $\operatorname{Hom}(\pi, G') \to \operatorname{Hom}(\pi, G)$. Conversely, given two homomorphisms $\phi'_1, \phi'_2 \in \operatorname{Hom}(\pi, G')$ such that $f \circ \phi'_1 = f \circ \phi'_2$, there is a homomorphism η: $\pi \to \operatorname{Ker} f$ such that $\phi'_2 = \eta\phi'_1$. For each $x \in \pi$,

$$\eta(x) = \phi'_2(x)\,\phi'_1(x)^{-1} \in \operatorname{Ker} f$$

and since $\operatorname{Ker} f$ is central, η defines a homomorphism. □

2.3. **Corollary.** *Let G be a semisimple Lie group with finite center and π be a finitely generated group. Then* $\mathrm{Hom}(\pi, G)$ *has finitely many connected components.*

Remark. This is not true for groups which are not semisimple or algebraic. In Goldman [9] an example is given of a non-simply connected nilpotent Lie group G for which $\mathrm{Hom}(\pi, G)$ has infinitely many connected components.

Proof. If G is a real linear algebraic group, then $\mathrm{Hom}(\pi, G)$ is a real algebraic variety. Any real algebraic variety has finitely many components by Whitney [40]. Otherwise note that the automorphism group $\mathrm{Aut}(\mathscr{G})$ of the Lie algebra $\mathscr{G}$ of G is a linear algebraic group for which the adjoint representation Ad: $G \to \mathrm{Aut}(\mathscr{G})$ is a local isomorphism with finite kernel. Now apply Lemma 2.2. □

Obstruction classes

2.4. Now we shall suppose that M is a closed oriented surface of genus $g > 1$ with fundamental group π. Then π has a presentation

$$\langle A_1, B_1, \dots, A_g, B_g \mid R_g(A_1, B_1, \dots, A_g, B_g) = I \rangle,$$

where the relation $R_g(A_1, B_1, \dots, A_g, B_g) = [A_1, B_1] \dots [A_g, B_g]$ is a product of commutators $[A, B] = ABA^{-1}B^{-1}$. Hence $\mathrm{Hom}(\pi, G)$ may be identified with the analytic variety

$$\{(a_1, b_1, \dots, a_g, b_g) \in G^{2g} \colon R_g(a_1, b_1, \dots, a_g, b_g) = I\}.$$

When G is semisimple, it is well known (see e.g. Goldman [11]) that the rank of the map $R_g\colon G^{2g} \to G$ at a point $(a_1, b_1, \dots, a_g, b_g) \in G^{2g}$ equals the codimension in G of the subgroup centralizing the set $\{a_1, b_1, \dots, a_g, b_g\}$. Thus $\mathrm{Hom}(\pi, G)$ is smooth at all $\phi \in \mathrm{Hom}(\pi, G)$ such that the centralizer $Z(\phi)$ of the image $\phi(\pi)$ in G is discrete. It is an interesting question for which Lie groups is this subset dense: for Lie groups locally isomorphic to $PSL(2, \mathbf{R})$ or $PSL(2, \mathbf{C})$, it is dense; however for $SU(2, 1)$, it is not dense ([12], see also [17]).

For G connected and locally isomorphic to $PSL(2, \mathbf{R})$ or $PSL(2, \mathbf{C})$, the condition that a homomorphism $\phi \in \mathrm{Hom}(\pi, G)$ satisfy $\dim Z(\phi) > 0$ is equivalent to the condition: $\phi(\pi)$ is abelian and if $G = PSL(2, \mathbf{C})$ then $\phi(\pi)$ is not conjugate to $\mathbf{Z}/2 \oplus \mathbf{Z}/2 \subset SO(3) \subset PSL(2, \mathbf{C})$.

2.5. There is a family of invariants of representations $\phi \in \mathrm{Hom}(\pi, G)$ which provides the key for understanding the topology of $\mathrm{Hom}(\pi, G)$. These obstruction classes arise as follows. Let $\phi \in \mathrm{Hom}(\pi, G)$, M a space with fundamental group π and let X be a space upon which G acts. Then the *flat* (G, X)*-bundle over* M *with holonomy* ϕ is the bundle over M having total space the quotient of $\tilde{M} \times X$ by the action of π given by $\gamma\colon (\tilde{s}, x) \mapsto (\gamma \tilde{s}, \phi(\gamma) x)$. We shall denote this total space by X_ϕ. Then topological invariants of (G, X)-bundles over M are invariants of representations $\phi \in \mathrm{Hom}(\pi, G)$. In particular, the obstructions to the triviality of the bundle (i.e. to finding a section of the principal bundle) are topological invariants of the bundle. (See Steenrod [35] for an account of obstruction theory.) For example, if G is not connected, then the *first obstruction* to the existence of a section of the principal bundle (i.e. a trivialization) is a class $o_1(\phi)$ in $H^1(M; \pi_0(G)) \cong \mathrm{Hom}(\pi, \pi_0(G))$ obtained by composing $\phi: \pi \to G$ with the epimorphism $G \to \pi_0(G)$. For example, if G is $GL(n, \mathbf{R})$ or $O(n)$, then the two components of G are distinguished by whether they preserve or reverse orientation on $\mathbf{R}^n$. The first obstruction $o_1(\phi)$ in this case is just the first Stiefel-Whitney class of the associated flat vector bundle $\mathbf{R}^n_\phi$.

We shall usually assume that G is connected, in which case the first obstruction is identically zero. One may say that the first obstruction is the obstruction to reducing the structure group of G_ϕ from G to its identity component G^0, i.e. reducing the structure group by the inclusion $G^0 \to G$. The second obstruction, then, may be defined as the obstruction to lifting the structure group from G to its universal covering $\tilde{G}$, i.e. reducing the structure group by the homomorphism $\tilde{G} \to G$. The second obstruction is a class $o_2(\phi) \in H^2(S; \pi_1(G))$.

Suppose that M is a closed oriented surface of genus $g > 1$. In that case the fundamental cycle on M determines an isomorphism $H^2(M, \pi_1(G)) \cong \pi_1(G)$. Then the image of the second obstruction in $\pi_1(G)$ may be described group theoretically as follows: Let $A_1, B_1, \ldots, A_g, B_g$ be the generators of π in its standard presentation. Choose lifts $\tilde{\phi}(A_i)$, $\tilde{\phi}(B_i)$ of the images $\phi(A_i)$, $\phi(B_i)$ to $\tilde{G}$. Evaluating the relation

$$R_g(\tilde{\phi}(A_1), \tilde{\phi}(B_1), \ldots, \tilde{\phi}(A_g), \tilde{\phi}(B_g))$$

gives an element of

$$\mathrm{Ker}(\tilde{G} \to G) \cong \pi_1(G),$$

which is independent of the chosen lifts (any two lifts differ by an element of $\pi_1(G) \subset \mathrm{center}(G)$). Thus the second obstruction defines a map

$$o_2: \mathrm{Hom}(\pi, G) \to \pi_1(G)$$

which expresses a topological property of a flat G-bundle. By the covering homotopy property, a family of G-bundles over a contractible space must be trivial and hence since $\mathrm{Hom}(\pi, G)$ is locally contractible, the map o_2 is locally constant (i.e. continuous) (compare [18], 4.5.). Thus o_2 defines an invariant of the connected components of $\mathrm{Hom}(\pi, G)$.

When $G = SL(n, \mathbf{R})$, then $o_2(\phi)$ is the second Stiefel-Whitney class (resp. Euler class) of the associated flat vector bundle $\mathbf{R}^n_\phi$ over M when $n > 2$ (resp. $n = 2$). For other groups, there are similar interpretations as Euler classes and second Stiefel-Whitney classes. If $G = PSL(2, \mathbf{R})$, then $o_2(\phi)$ is the Euler class of the associated oriented circle bundle (where the fiber $\partial X = \mathbf{RP}^1$ has the natural action of G). Alternately, one may form the associated oriented 2-disc bundle X_ϕ; since the fibers are contractible, there exists a smooth section $\sigma: M \to X_\phi$. The exponential map on each fiber defines a fiber diffeomorphism between the normal bundle to $\sigma \in X_\phi$ and X_ϕ. Thus the bundle X_ϕ is a (nonlinear) 2-plane bundle and its Euler class is the second obstruction. If $G = PSL(2, \mathbf{C})$, then the associated $\mathbf{H}^3$-bundle is an oriented 3-disk bundle and similarly its second Stiefel-Whitney class may be defined, which also is a second obstruction class. In all of these cases, we shall write the obstruction classes as $e(\phi)$ or $w_2(\phi)$ rather than $o_2(\phi)$. The proofs that this construction yield the claimed characteristic classes are standard and can be found, e.g. in Milnor [31] or Hirzebruch [23].

§ 3. Surfaces with boundary

In this section we state a generalization of theorem B to surfaces with boundary. We shall need to impose boundary conditions in order to define a relative Euler class of suitable flat bundles over surfaces with boundary.

3.1. Let M be a compact oriented surface with boundary, and let $\pi = \pi_1(M)$ be its fundamental group. If G is a connected Lie group and $\phi \in \mathrm{Hom}(\pi, G)$ is a homomorphism which determines a flat principal G-bundle $G_\phi \to M$, then G_ϕ is trivial as a principal G-bundle (the obstructions take values in the groups $H^i(M; \pi_{i-1}(G))$ which are all zero). Thus there are no characteristic invariants of flat bundles over surfaces with boundary. One may also see this by noting that since π is a free group,

$$\mathrm{Hom}(\pi, G) \approx G^k$$

is already connected. Thus boundary conditions are necessary in order to obtain nontrivial invariants of flat bundles over surfaces with boundary.

The most natural approach is to trivialize the bundle over the boundary and try to extend this trivialization over M. In other words, we fix a section σ of $G_{\phi|\partial M}$ and consider the obstruction to extending σ to a section of G_ϕ over M. This obstruction, of course, will depend on the choice of trivialization over the boundary. Thus it will be useful to find natural conditions under which the flat structure determines a trivialization over the boundary.

Suppose that $C \in \partial M$ is a boundary component. Choose a holonomy homomorphism $\pi_1(C) \to G$ for the flat bundle over C; let $h \in G$ be a generator of the image of this homomorphism. Then using a homeomorphism $C \approx \mathbf{R}/\mathbf{Z}$, the corresponding flat principal bundle P over C may be identified with the quotient of $\mathbf{R} \times G$ by the equivalence relation

$$(t, g) \sim (t+n, h^n g)$$

where $n \in \mathbf{Z}$. A trivialization of P then corresponds to a map $f\colon \mathbf{R} \to G$ which is $\mathbf{Z}$-equivariant in the sense that $f(t+n) = h^{-n} f(t)$ for each $n \in \mathbf{Z}$. We may normalize the trivialization by requiring that $f(0) = I$. Clearly if $h = I$, then the flat bundle is trivial as a flat bundle, and enjoys a natural trivialization. Suppose that $h \neq I$ and that $\psi\colon \mathbf{R} \to G$ is a one-parameter subgroup containing h; by a linear change of parametrization we may assume that $h = \psi(1)$. Clearly taking $f = \psi$ we obtain a trivialization of P. We call such trivializations *special*.

Suppose that $G = PSL(2, \mathbf{R})$. If $h \in G$, then there is a unique homomorphism $\psi\colon \mathbf{R} \to G$ with $h = \psi(1)$ if and only if $h \in \mathrm{Par} \cup \mathrm{Hyp}$; thus there are preferred special trivializations as long as $h \notin \mathrm{Ell}$.

3.2. Suppose that $\phi \in \mathrm{Hom}(\pi, G)$ is a homomorphism such that for each boundary component $C \subset \partial M$, the image of $\pi_1(C) \subset \pi_1(M)$ under $\phi: \pi_1(M) \to G$ contains no elliptic elements. Then there exists a preferred special trivialization σ of the corresponding flat principal G-bundle over C; the obstruction to extending this trivialization to a trivialization of G_ϕ lies in the relative cohomology group $H^2(M, \partial M; \mathbf{Z})$ and is by definition the *relative Euler class* $e(\phi; \sigma)$.

One can describe this relative Euler class more directly in terms of the homomorphism ϕ as follows. Since trivializing a $PSL(2, \mathbf{R})$-bundle is essentially equivalent to lifting the structure group to the universal covering $\widetilde{PSL(2, \mathbf{R})} \to PSL(2, \mathbf{R})$, the relative Euler class can be interpreted in terms of lifting homomorphisms. There is a presentation of π as

$$\langle A_1, B_1, \ldots, A_p, B_p, C_1, \ldots, C_k \mid [A_1, B_1] \ldots [A_p, B_p]\, C_1 \ldots C_k = I \rangle,$$

where $C_1, \ldots, C_k$ correspond to the components of ∂M. By assumption, each $\phi(C_i) \in \{I\} \cup \mathrm{Hyp} \cup \mathrm{Par}$ and thus there is a unique lift $\tilde{\phi}(C_i)$ of $\phi(C_i)$ to the closure

$$\overline{\mathrm{Hyp}_0} = \{I\} \cup \mathrm{Hyp}_0 \cup \mathrm{Par}_0\} \subset \tilde{G}.$$

Choose lifts $\widetilde{\phi(A_i)}, \widetilde{\phi(B_i)} \in \tilde{G}$ of $\phi(A_i), \phi(B_i)$. Since ϕ is a homomorphism the element

$$[\widetilde{\phi(A_1)}, \widetilde{\phi(B_1)}] \ldots [\widetilde{\phi(A_g)}, \widetilde{\phi(B_g)}]\, \tilde{\phi}(C_1) \ldots \tilde{\phi}(C_k)$$

lies in the kernel of the covering homomorphism $\tilde{G} \to G$ and hence equals z^n for some $n \in \mathbf{Z}$. It is a routine exercise (analogous to the closed case) to show that the relative Euler class $e(\phi, \sigma) = n[M]$, where $[M] \in H^2(M, \partial M)$ is the fundamental cohomology class of M.

The main result relating the relative Euler class to components of flat bundles over surfaces which have been trivialized over their boundary is the following:

3.3. **Theorem.** *Let M be a compact oriented surface and let W denote the set of all homomorphisms $\phi: \pi_1(M) \to SL(2, \mathbf{R})$ such that for each closed curve $C \subset \partial M$, the corresponding $\phi(C)$ is hyperbolic. Let e: $W(M) \to \mathbf{Z}$ denote the relative Euler class map as above. Then the connected components of $W(M)$ are precisely the inverse images $e^{-1}(n)$, where $|n| \leqq \chi(M)$.*

The proof of 3.3 will be given in § 10. There is a generalization of Corollary C to surfaces-with-boundary involving the relative Euler class. However, the statement is slightly more complicated, since the boundary must be taken into account. Suppose that M is a compact surface with nonempty boundary and let ϕ: $\pi_1(M) \to G$ be a Fuchsian representation (an isomorphism onto a discrete subgroup $\Gamma \subset G$). Then $\mathbf{H}^2/\Gamma$ is a complete hyperbolic surface, which is homotopy equivalent to M. In general, however, $\mathbf{H}^2/\Gamma$ and M are not homeomorphic. Let $\mathrm{core}(\mathbf{H}^2/\Gamma)$ denote the convex core of $\mathbf{H}^2/\Gamma$. If there exists a diffeomorphism $M \to \mathrm{core}(\mathbf{H}^2/\Gamma)$, we shall say that $\phi \in \mathrm{Hom}(\pi, G)$ is a *holonomy representation for M*.

3.4. **Theorem.** *Let $\phi \in W(M)$. Then $|e(\phi)| \leqq \chi(M)$ with equality* (i.e. $|e(\phi)| = \chi(M)$) *if and only if ϕ is a holonomy representation for M.*

The proof of 3.4 will also be given in § 10. Since we shall need a few calculations of the relative Euler class sooner, we prove the following two results presently.

3.5. **Proposition.** *Let $\phi \in W(M)$ be a holonomy representation for M. Then the relative Euler class $e(\phi)$ equals $\chi(M)$.*

3.6. **Proposition.** *Let $\phi \in W(M)$ be a representation whose image lies inside a Borel subgroup. Then $e(\phi)=0$.*

Proof of 3.5. (Compare [14], Proposition 2.8.) Consider the flat $\mathbf{H}^2$-bundle $\mathbf{H}^2_\phi \to M$ with holonomy ϕ. By hypothesis there exists a diffeomorphism f: $M \to \operatorname{core}(\mathbf{H}^2/\phi(\pi))$ which maps each boundary component of M to a closed geodesic boundary component of $\operatorname{core}(\mathbf{H}^2/\phi(\pi))$. The induced diffeomorphism $\tilde{f}: \tilde{M} \to \mathbf{H}^2$ is equivariant with respect to the action of π on $\tilde{M}$ by deck transformations and the action of π on $\mathbf{H}^2$ via ϕ. The equivariant diffeomorphism $\tilde{f}$ defines a section of the flat bundle $\mathbf{H}^2_\phi \to M$ which over ∂M restricts to the (fiberwise) projection of the special trivialization $\partial M \to (PSL(2, \mathbf{R}))_\phi \to \mathbf{H}^2_\phi$. The relative Euler class of ϕ with respect to the special trivialization may be computed as a relative self-intersection number of a section s over M: the algebraic intersection number of s and a nearby section s' which coincides with s on ∂M. Since $\tilde{f}$ is a local diffeomorphism, the relative self-intersection number of the section corresponding to $\tilde{f}$ is seen to equal in absolute value the Euler class of the tangent bundle of M, i.e. $\pm\chi(M)$. □

Proof of 3.6. If $\{\phi_t\}_{0 \leq t \leq 1}$ is a family of representations in $\operatorname{Hom}(\pi, G)$ such that for no component $C \subset \partial M$ is $\phi(C)$ hyperbolic, then by the covering homotopy property the Euler class of ϕ_t with respect to the special trivialization is constant in t. Suppose that $\phi(\pi)$ lies in a Borel subgroup $B \subset G$. Let $A \subset B$ be a Cartan subgroup. By conjugation by the one-parameter group A one can find a path of representations $\pi \to B$ joining ϕ to a representation $\pi \to A$. Since the representations $\pi \to A$ form a vector space, such a representation can be joined by a path of representations to the trivial representation in such a way that no boundary component is mapped to an elliptic element. Since the relative Euler class of the trivial representation equals zero, the result follows. □

The relative Euler class also enjoys a simple additivity formula, which is a standard fact from obstruction theory (see e.g. Steenrod [35]):

3.7. **Proposition** (Additivity). *Suppose that $M=M_1 \cup M_2$ and $\phi \in \operatorname{Hom}(\pi, G)$ has the property that for each boundary component C of M_1 or M_2 the restriction to $\pi_1(C) \subset \pi$ is not elliptic. Then*

$$e(\phi)=e(\phi|_{\pi_1(M_1)})+e(\phi|_{\pi_1(M_2)})$$

where the relative Euler classes are computed with respect to the special trivializations.

3.8. Clearly 3.3 reduces to Theorem B when M is a closed surface. We first prove 3.3 for surfaces of Euler characteristic -1 and -2; the general case will then follow fairly easily from these special cases. The reduction to these special cases is based on a simple combinatorial argument which we present now. Although the detailed proof will be given in § 10, after the necessary prerequisites are developed, for the purpose of readability we give a broad outline of the basic ideas here. We shall consider decompositions $M=\bigcup_{i=1}^{m} M_i$ satisfying the following properties:

(i) The M_i are subsurfaces any two of which are disjoint or meet in boundary components;

(ii) $\chi(M_i)=-1$;

(iii) The graph dual to $M=\bigcup_{i=1}^{m} M_i$ is a tree.

If M has genus g and b boundary components, then (i) implies that there are $m=-\chi(M)=2-2g+b$ components M_i in the decomposition. Recall that the dual graph has one vertex for each M_i and two vertices are joined by an edge if the corresponding M_i share a common boundary component. We refer to the simple closed curves corresponding to the edges (i.e. the boundary components of the M_i) as *decomposition curves.* Each M_i is either a surface of genus one with one boundary component (a torus minus a disc) or a surface of genus zero with three boundary components (a pair-of-pants). For each genus one M_i there is a unique other M_j such that $\partial M_i \subset \partial M_j$ (unless $M=M_i$). Unless M is a closed surface of genus two, M_j will be a pair-of-pants. It follows that the union of all the genus zero M_i is a connected surface, which has genus zero because of (ii). (Otherwise a nonseparating essential curve would determine a loop in the dual graph.) Thus exactly g of the M_i have genus one and $g-2+b$ have genus zero. We shall call such a decomposition $M=\bigcup_{i=1}^{m} M_i$ a *maximal dual-tree decomposition* of M.

It will turn out to be easier to work with the space $W'(M)\subset W(M)$ consisting of representations ϕ such that each $\phi(\pi_1(M_i))$ is nonabelian. In 8.1 it will be shown that $W'(M)$ is open and dense in $W(M)$. Thus $e^{-1}(n)\cap W(M)$ is connected if and only if $e^{-1}(n)\cap W'(M)$ is connected. Suppose that $\phi,\psi\in W'(M)$ satisfy $e(\phi)=e(\psi)$; we describe how to join them by a path in $W'(M)$. In 10.1 we show how to deform ϕ (resp. ψ) to a representation ϕ' (resp. ψ') in $W'(M)$ which maps each decomposition curve to a hyperbolic element. Thus $\phi',\psi'\in W'(M)\cap\prod_i W'(M_i)$ and we associate to ϕ',ψ' the m-tuples

$$\{e(\phi'|_{\pi_1(M_i)})\}_{i=1,\dots,m},\ \{e(\psi'|_{\pi_1(M_i)})\}_{i=1,\dots,m}\in\{-1,0,1\}^m$$

where each relative Euler class is computed with respect to the special trivializations. Now it follows from the analysis in § 9 of Euler characteristic -2 surfaces that if $M_1\cup M_2\subset M$ is a connected subsurface of M and

$$e(\phi'|_{\pi_1(M_i)})=e(\psi'|_{\pi_1(M_i)}) \quad \text{for } j\neq 1,2$$

and

$$e(\phi'|_{\pi_1(M_1\cup M_2)})=e(\psi'|_{\pi_1(M_1\cup M_2)}),$$

then ϕ' can be deformed to ψ' in $W'(M)$. Thus we reduce the construction of a path from ϕ' to ψ' to the following elementary combinatorial lemma:

3.9. **Lemma.** *Let T be a tree and let $\mathscr{U}$ denote the set of all maps f: $\operatorname{vert}(T)\to\{-1,0,+1\}$, where $\operatorname{vert}(T)$ denotes the set of vertices of T. Let $\Sigma\colon\mathscr{U}\to\mathbf{Z}$ be the map given by*

$$f\mapsto \sum_{v\in\operatorname{vert}(T)} f(v).$$

Then the equivalence relation

$$f\sim f'\Leftrightarrow\Sigma(f)=\Sigma(f')$$

is generated by the relation $\mathscr{R}$ consisting of all $(f,f')\in\mathscr{U}\times\mathscr{U}$ such that there exists an edge $e\subset T$ such that

$$f|_{T-e}=f'|_{T-e}$$

and

$$\sum_{v\in\operatorname{vert}(e)} f(v)=\sum_{v\in\operatorname{vert}(e)} f'(v).$$

Proof. Induction on the number n of vertices of T. Clearly the statement is vacuous for $n=1$ and is trivial for $n=2$. Suppose the statement is known for trees with less than n vertices and let T have n vertices. Suppose that $f,f'\in\mathscr{U}$ satisfy $\Sigma(f)=\Sigma(f')$. Choose a vertex $v\in\operatorname{vert}(T)$.

We claim that there exist sequences $f=f_1,\ldots,f_p\in\mathscr{U}$ and $f'=f'_1,\ldots,f'_q\in\mathscr{U}$ with $f_i\mathscr{R}f_{i+1}$ and $f'_i\mathscr{R}f'_{i+1}$ such that $f_p(v)=f'_q(v)$. For if $\Sigma(f)>0$ (resp. $\Sigma(f)<0$) there exists a vertex v such that $f(v)=1$ (resp. $f(v)=-1$). Let $v=v_1,v_2,\ldots,v=v_p$ be the unique segment joining v and v. By interchanging $f(v_i)$ and $f(v_{i+1})$ one obtains a sequence $f_1,\ldots,f_p$ such that $f_i\mathscr{R}f_{i+1}$ and $f_p(v)=1$ (resp. $f_p(v)=-1$). Applying the same procedure to the sequence $f'_1,\ldots,f'_q$ we may assume that $f'_q(v)=1$ (resp. $f'_q(v)=-1$). If $\Sigma(f)=0$, but there exists $v,v'\in\operatorname{vert}(T)$ such that $f(v),f'(v')\neq 0$, then there exist vertices w,w' such that $f(w)=f'(w')=1$ and the preceding argument can be applied. Thus it remains to consider the case that $f\equiv 0$. In that case, change f on an arbitrary edge to be 1 on one vertex and -1 on the other vertex. Then apply the preceding construction. This proves the claim.

Now each component of $T-\operatorname{star}(v)$ is a tree and $f|_{T-\operatorname{star}(v)}$ and $f'|_{T-\operatorname{star}(v)}$ are equivalent functions $\operatorname{vert}(T)\to\{-1,0,1\}$. Thus by the inductive hypothesis they can be joined by a sequence of elements whose adjacent elements are $\mathscr{R}$-related. Such a sequence extends (by remaining constant on v) to a sequence in $\mathscr{U}$ whose adjacent elements are $\mathscr{R}$-equivalent. The proof of 3.9 is now complete. □

§4. Some invariant theory

In this section we develop the necessary invariant theory which will be used to analyze representations of a free group of rank two into $SL(2,\mathbf{C})$. For more information the reader is referred to the survey article of Magnus [28].

Let Π denote the free group with free generators X, Y. Let G be a Lie group. We may identify $\operatorname{Hom}(\Pi,G)$ with G^2 by the evaluation map $\operatorname{Hom}(\Pi,G)\to G^2$

sending $h \mapsto (h(X), h(Y))$. Consider the action of $GL(2, \mathbf{C})$ on $\mathrm{Hom}(\Pi, SL(2, \mathbf{C})) \approx SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$ by conjugation. The following well known facts are essentially due to Fricke-Klein ([5], pp. 289–291):

4.1. **Proposition.** *Let χ: $SL(2, \mathbf{C}) \times SL(2, \mathbf{C}) \to \mathbf{C}^3$ be the map*

$$\chi(X, Y) = (\mathrm{tr}(X), \mathrm{tr}(Y), \mathrm{tr}(XY)).$$

Let κ: $\mathbf{C}^3 \to \mathbf{C}$ be the polynomial

$$\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2.$$

Then:

(i) *χ is $GL(2, \mathbf{C})$-invariant and surjective;*

(ii) *$\kappa(\chi(X, Y)) = \mathrm{tr}[X, Y]$;*

(iii) *$\kappa(\chi(X, Y)) = 2$ if and only if there exists a line in $\mathbf{C}^2$ invariant under X, Y (i.e. the linear representation generated by X and Y is reducible).*

(iv) *If $u \in \mathbf{C}^3$, $\varkappa(u) \neq 2$, then $\kappa^{-1}(u)$ consists of a single $GL(2, \mathbf{C})$-orbit. If $\kappa(u) = 2$, then there exists a unique G-orbit $o_u \subset \chi^{-1}(u)$ consisting of completely reducible representations such that $\chi^{-1}(u)$ consist of $GL(2, \mathbf{C})$-orbits whose closures contain o_u.*

(v) *Two representations $\phi, \phi' \in \mathrm{Hom}(\pi, SL(2, \mathbf{C}))$ are $GL(2, \mathbf{C})$-conjugate if and only if they are $SL(2, \mathbf{C})$-conjugate.*

(Since the trace of any word $w(X, Y)$ in matrices $X, Y \in SL(2, \mathbf{C})$ may be deduced from the traces of X, Y, and XY (see Magnus [28], or Culler-Shalen [3] 1.4.1), we shall refer to $\chi(X, Y)$ as the *character* of the representation determined by (X, Y).)

Proof. (i) Let $P \in GL(2, \mathbf{C})$. Since

$$\mathrm{tr}(PXP^{-1}) = \mathrm{tr}(X), \quad \mathrm{tr}(PYP^{-1}) = \mathrm{tr}(Y), \quad \mathrm{tr}(PXYP^{-1}) = \mathrm{tr}(XY),$$

χ is $GL(2, \mathbf{C})$-invariant.

Next we prove that χ is surjective. If $(x, y, z) \in \mathbf{C}^3$, choose $\xi, \delta \in \mathbf{C}$ such that $\xi^2 = x^2 - 4$ and $\delta^2 = \kappa(x, y, z) - 2$. If $x \neq \pm 2$, then

$$X = \begin{bmatrix} \frac{x-\xi}{2} & 0 \\ 0 & \frac{x+\xi}{2} \end{bmatrix}, \quad Y = \begin{bmatrix} \frac{1}{2}y + \frac{xy+2z}{\xi} & \frac{\delta}{\xi} \\ -\frac{\delta}{\xi} & \frac{1}{2}y - \frac{xy+2z}{\xi} \end{bmatrix}$$

satisfies $\chi(X, Y) = (x, y, z)$. Similar formulas suffice when $y \neq \pm 2$ and $z \neq \pm 2$. Suppose that $x, y, z = \pm 2$. If an even number of x, y, z equal $+2$, then

$$(X, Y) = \left(\frac{x}{2} I, \frac{y}{2} I\right)$$

satisfies $\chi(X, Y)=(x, y, z)$. Otherwise

$$(X, Y)=\left(\frac{x}{2}\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}, \frac{y}{2}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\right)$$

satisfies $\chi(X, Y)=(x, y, z)$.

(ii)–(iii) See, for example, Magnus [28], 2.1, Culler-Shalen [3], 1.5.2, 1.5.5.

(iv) If $\kappa(u)\neq 2$, then the result follows from Magnus [28], 2.1 or Culler-Shalen [3], 1.5.2. Otherwise there exists $(X, Y)\in\chi^{-1}(u)$ which acts reducibly on $\mathbf{C}^2$, so by conjugation we may assume that X and Y are upper-triangular. For any upper-triangular matrix U, let $U^{(s)}$ denote its semisimple part, i.e. if

$$U=\begin{bmatrix}a & b\\ 0 & c\end{bmatrix}$$

then its semisimple part equals

$$U^{(s)}=\begin{bmatrix}a & 0\\ 0 & c\end{bmatrix}.$$

Then the $GL(2, \mathbf{C})$-orbit o_u of $(X^{(s)}, Y^{(s)})$ is an orbit with the desired properties.

(v) Clearly if ϕ, ϕ' are $SL(2, \mathbf{C})$-conjugate, then they are $GL(2, \mathbf{C})$-conjugate. Conversely, suppose that $\gamma\in GL(2, \mathbf{C})$ conjugates ϕ to ϕ'. Choose a complex number δ satisfying $\delta^{-2}=\det g$; then $(\delta I)\gamma\in SL(2, \mathbf{C})$ conjugates ϕ to ϕ'. □

4.2. The real analogue of Proposition 4.1 is considerably more subtle. One difficulty is that not every real character of a representation into $SL(2, \mathbf{C})$ corresponds to a representation into $SL(2, \mathbf{R})$. Rather, a triple $(x, y, z)\in\mathbf{R}^3$ corresponds to a representation into $SL(2, \mathbf{C})$ which is conjugate to a representation taking values in one of the two real forms of $SL(2, \mathbf{C})$, namely $SL(2, \mathbf{R})$ or $SU(2)$. To determine whether a real character corresponds to a representation in $SL(2, \mathbf{R})$ or $SU(2)$ we shall use the identification of the adjoint group of $SL(2, \mathbf{C})$ with the orthogonal group $SO(3, \mathbf{C})$.

Recall that the adjoint representation of a semisimple Lie group G on its Lie algebra $\mathfrak{g}$ always preserves the nondegenerate symmetric bilinear form (the Killing form) defined by $\mathbf{B}(\alpha, \beta)=\mathrm{tr}(\mathrm{ad}(\alpha)\,\mathrm{ad}(\beta))$. Let $G=SL(2, \mathbf{C})$ and choose an orthonormal basis of the Lie algebra $\mathfrak{sl}(2, \mathbf{C})$, so that we identify $\mathfrak{sl}(2, \mathbf{C})$ with complex Euclidean 3-space, as a three-dimensional complex inner product space. The adjoint representation defines a local isomorphism $SL(2, \mathbf{C})\to SO(3, \mathbf{C})$ whose kernel is the center $\{\pm I\}$ of $SL(2, \mathbf{C})$. We give an alternate construction of a representation in $SL(2, \mathbf{C})$ with a given character, with the eventual goal of constructing representations into $SL(2, \mathbf{R})$ with given real character.

Let $(x, y, z)\in\mathbf{C}^3$ satisfy $\kappa(x, y, z)\neq 2$. Consider the symmetric matrix

$$\mathbf{B}=\begin{bmatrix}2 & z & y\\ z & 2 & x\\ y & x & 2\end{bmatrix}$$

which has determinant $\det \mathbf{B}=2(2-\kappa(x,y,z))\neq 0$. (Compare the discussion of triangle groups in [16].) Thus $\mathbf{B}$ determines a nondegenerate symmetric bilinear form on $\mathbf{C}^3$. Let ρ_1, ρ_2, ρ_3 be the $\mathbf{B}$-orthogonal reflections fixing the coordinate vectors, i.e.

$$\rho_1=\begin{bmatrix} -1 & -z & -y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_2=\begin{bmatrix} 1 & 0 & 0 \\ -z & -1 & -x \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -y & -x & -1 \end{bmatrix}.$$

Then ρ_1, ρ_2, ρ_3 are involutions on $\mathbf{C}^3$ which preserve the symmetric bilinear form $\mathbf{B}$. Let $\hat{X}=\rho_1\rho_2$, $\hat{Y}=\rho_2\rho_3$, whence $\hat{X}\hat{Y}=\rho_1\rho_3$. Since $\mathbf{B}$ is nondegenerate, the group of unimodular isometries of $\mathbf{C}^3$ with respect to $\mathbf{B}$ is conjugate inside $SL(3,\mathbf{C})$ to $SO(3,\mathbf{C})$. We henceforth implicitly identify these two conjugate subgroups with each other as well as with the adjoint group of $SL(2,\mathbf{C})$. Choose lifts $X, Y\in SL(2,\mathbf{C})$ of $\hat{X}$, $\hat{Y}$ under the map $\mathrm{Ad}: SL(2,\mathbf{C})\to SO(3,\mathbf{C})$; a simple calculation shows that

$$\operatorname{tr}(X)=x^2-1, \quad \operatorname{tr}(Y)=y^2-1, \quad \operatorname{tr}(XY)=z^2-1.$$

Furthermore for any matrix $U\in SL(2,\mathbf{C})$, we have $\operatorname{tr}(\mathrm{Ad}(U))=(\operatorname{tr}(U))^2-1$. Thus

$$\operatorname{tr}(X)=\pm x, \quad \operatorname{tr}(Y)=\pm y, \quad \operatorname{tr}(XY)=\pm z.$$

By multiplying X and Y by $\pm I$, we can change our choice of lifts X, Y to assume that $\operatorname{tr} X=x$ and that $\operatorname{tr} Y=y$. If any of x, y, z are zero, then there are at least two choices, and at least one choice will guarantee that $\operatorname{tr} XY=z$. If none of x, y, z are zero, then the lifts X, Y satisfying $\operatorname{tr} X=x$, $\operatorname{tr} Y=y$ are uniquely determined. Furthermore there is a unique lift $Z\in SL(2,\mathbf{C})$ of $(XY)^{-1}$ such that $\operatorname{tr}(Z)=z$. In particular XYZ is a lift of the identity in $SO(3,\mathbf{C})$ under Ad: $SL(2,\mathbf{C})\to SO(3,\mathbf{C})$ and equals $\pm I$. We claim it equals I. For we have constructed a continuous map from

$$\Omega=\{(x,y,z)\in\mathbf{C}^3 \mid xyz\neq 0,\ \kappa(x,y,z)\neq 2\}$$

into $\{\pm I\}$. But Ω is connected, so this locally constant map is constant. Thus it suffices to check that XYZ equals I for a single $(x,y,z)\in\Omega$; the explicit example given for $(-2,-2,-2)$ discussed earlier in the proof of 4.1 (i) will suffice. Thus we have found elements $X, Y\in SL(2,\mathbf{C})$ such that $\operatorname{tr}(X)=x$, $\operatorname{tr}(Y)=y$ and $\operatorname{tr}(XY)=z$. (This gives an alternate proof of the surjectivity in 4.1 (i).)

We shall use the preceding construction to determine the set of characters of representations in $SL(2,\mathbf{R})$.

4.3. **Theorem.** *Let $(x,y,z)\in\mathbf{R}^3$. Then there exists $(X,Y)\in SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ with $\chi(X,Y)=(x,y,z)$ if and only if either $\kappa(x,y,z)\geqq 2$ or one of $|x|, |y|, |z|$ is $\geqq 2$. In that case, unless $\kappa(x,y,z)=2$, the inverse image $\chi^{-1}(x,y,z)$ consists of one $GL(2,\mathbf{R})$-orbit, which is the union of two $SL(2,\mathbf{R})$-orbits. If all of $\kappa(x,y,z)$, $|x|, |y|, |z|$ are <2, there exists $(X,Y)\in SU(2)\times SU(2)$ with $\chi(X,Y)=(x,y,z)$. All such pairs (X,Y) are $SU(2)$-conjugate.*

Remark. It is a general fact that a character of a representation of a finitely generated group in $SL(2, \mathbf{C})$ is real if and only if it is the character of a representation in one of the real forms $SL(2, \mathbf{R})$ or $SU(2)$. For an elegant proof of this fact, the reader is referred to Morgan-Shalen [32], Proposition III.1.1.

Proof. We first recall basic facts about the adjoint representations of $SL(2, \mathbf{R})$ and $SU(2)$. Suppose that $\Gamma \in SL(2, \mathbf{R})$. Then clearly the adjoint representation restricted to Γ stabilizes the Lie algebra $\mathfrak{sl}(2, \mathbf{R})$ of $SL(2, \mathbf{R})$. The restriction of the Killing form on $\mathfrak{sl}(2, \mathbf{C})$ to $\mathfrak{sl}(2, \mathbf{R})$ is nondegenerate and indefinite. With respect to a suitable basis of $\mathfrak{sl}(2, \mathbf{R})$, the adjoint representation defines a local isomorphism $SL(2, \mathbf{R}) \to SO(1,2)^0$. Indeed, the stabilizer of any indefinite totally real subspace is conjugate in $SL(2, \mathbf{C})$ to $SL(2, \mathbf{R})$. On the other hand, any subgroup of $SU(2)$ stabilizes the Lie algebra $\mathfrak{su}(2)$, which is a totally real subspace to which the restriction of the bilinear form is definite. Any subgroup of $SL(2, \mathbf{C})$ which stabilizes a totally real definite subspace of $\mathbf{C}^3$ is conjugate to a subgroup of $SU(2)$.

Suppose $(x, y, z) \in \mathbf{R}^3$. By 4.1 (i), there exists $(X, Y) \in SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$ such that $\chi(X, Y) = (x, y, z)$. Using the above criteria for subgroups to be conjugate to $SL(2, \mathbf{R})$ or $SU(2)$, we shall determine when (x, y, z) is the character of $SL(2, \mathbf{R})$-representation.

Consider the symmetric bilinear form $\mathbf{B}$ as above. Since $\mathbf{B}$ is real, the real subspace $\mathbf{R}^3 \subset \mathbf{C}^3$ is invariant under X, Y. Assume that $\kappa(x, y, z) \neq 2$. Then

$$\det \mathbf{B} = 2(2 - \kappa(x, y, z)) \neq 0$$

so the bilinear form is nondegenerate. Since the restriction of $\mathbf{B}$ to each coordinate line is positive, $\mathbf{B}$ cannot be negative definite. The restriction of $\mathbf{B}$ to the first coordinate plane is given by the matrix $\begin{bmatrix} 2 & z \\ z & 2 \end{bmatrix}$, which is positive definite if and only if $|z| < 2$. Thus necessary conditions that $\mathbf{B}$ be positive definite are that $|x|, |y|, |z|$ are all < 2. A further necessary condition is that $\det \mathbf{B} > 0$, i.e. that $\kappa(x, y, z) < 2$. We claim these necessary conditions are sufficient. For if $\mathbf{B}$ is indefinite, either there exists a two-dimensional subspace of $\mathbf{R}^3$ which is definite or there does not. If there does, then $\det \mathbf{B} < 0$, whence $\kappa(x, y, z) > 2$. If there does not exist a two-dimensional definite subspace, then no coordinate plane is definite, whence none of $|x|, |y|, |z|$ are < 2. Thus $\mathbf{B}$ is positive definite if and only if $\kappa(x, y, z) < 2$, $|x| < 2$, $|y| < 2$ and $|z| < 2$.

It follows that $\operatorname{Ad} X$ and $\operatorname{Ad} Y$ are conjugate to $SO(1,2)$ (resp. $SO(3)$) if and only if X and Y are conjugate to $SL(2, \mathbf{R})$ (resp. $SU(2)$) if and only if one of $\kappa(x, y, z)$, $|x|, |y|, |z|$ is $\geqq 2$ (resp. $\kappa(x, y, z) < 2$, $|x| < 2$, $|y| < 2$, and $|z| < 2$).

If $\kappa(x, y, z) = 2$, then (x, y, z) is the character of a reducible representation, whose semisimple part is a representation lying on a one-parameter subgroup. Any such character is the character of a representation in $SL(2, \mathbf{R})$. Furthermore if all of $|x|, |y|, |z|$ are $\leqq 2$ (this will be the case if just one of $|x|, |y|, |z|$ is < 2) then there will be a representation in $SU(2)$ as well having this character.

Suppose that $(X, Y), (X', Y') \in SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ satisfy $\chi(X, Y) = \chi(X', Y')$. Suppose furthermore that $\kappa(\chi(X, Y)) \neq 2$. Then X and Y generate an irreducible representation on $\mathbf{R}^2$, as do X' and Y'. By 4.1 (iv) there exists $\gamma \in GL(2, \mathbf{C})$ such that $\gamma X \gamma^{-1} = X'$ and $\gamma Y \gamma^{-1} = Y'$. Since X and Y act irreducibly, the Burnside lemma implies that the $\mathbf{R}$-algebra they generate is the full algebra $M_2(\mathbf{R})$ of real matrices, and similarly for X' and Y'. Thus γ normalizes the algebra of real matrices and thus $\mathrm{Ad}\,\gamma \in PGL(2, \mathbf{R})$. We may replace γ by a real matrix of determinant ± 1 determining the same inner automorphism $\mathrm{Ad}\,\gamma$. It follows that (X, Y) and (X', Y') are $PGL(2, \mathbf{R})$-conjugate. If $\det \gamma = 1$, then (X, Y) and (X', Y') are conjugate in $SL(2, \mathbf{R})$. Suppose then that $\det \gamma = -1$; we claim that (X, Y) and (X', Y') cannot be conjugate in $SL(2, \mathbf{R})$. Otherwise there would exist $\gamma, \eta \in GL(2, \mathbf{R})$ such that $\det \eta = 1$, $\det \gamma = -1$ but the composition $\mathrm{Ad}\,\eta \circ \mathrm{Ad}\,\gamma$ conjugates (X, Y) to itself, i.e., $\eta \circ \gamma$ commutes with X, Y. Since $\det \eta \circ \gamma = -1$, it follows that there exists a decomposition of $\mathbf{R}^2$ as the direct sum of two lines, one of which is fixed pointwise by $\eta \circ \gamma$, and the other upon which $\eta \gamma$ acts by -1. Since X and Y commute with $\eta \circ \gamma$, it follows that X and Y must leave invariant each of these lines, contradicting $\kappa(\chi(X, Y)) \neq 2$.

Finally suppose that $(X, Y), (X', Y') \in SU(2) \times SU(2)$ satisfies $\chi(X, Y) = \chi(X', Y')$. Then by 4.1 (iv), (X, Y) and (X', Y') are $GL(2, \mathbf{C})$-conjugate. If (X, Y) determines an irreducible representation, then the conjugating element must preserve (up to scaling) the Hermitian structure invariant under X and Y. It follows as above that (X, Y) and (X', Y') are $U(2)$-conjugate, and by the same argument as in 4.1 (v), are $SU(2)$-conjugate. If (X, Y) and (X', Y') satisfy $\kappa = 2$, then a separate argument shows that they are $SU(2)$-conjugate as well. The proof of Theorem 4.3 is now complete. □

4.4. **Proposition.** *Let $(X, Y) \in SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$. Then $d_{\chi(X,Y)}$ is surjective if and only if X and Y do not commute.*

Proof. Let $\xi, \eta \in \mathfrak{sl}(2, \mathbf{C})$. *Then an elementary calculation gives*

$$d_{\chi(X,Y)}(\xi, \eta) = (\mathrm{tr}(X\chi), \mathrm{tr}(Y\eta), \mathrm{tr}(X\eta) + \mathrm{tr}(Y\xi)).$$

We show that either X, Y commute or $\dim \mathrm{Ker}\, d_{\chi(X,Y)} = 3$. If $U \in SL(2, \mathbf{C})$, then the set of all $v \in \mathfrak{sl}(2, \mathbf{C})$ such that $\mathrm{tr}(Uv) = 0$ has dimension two, unless $U = \pm I$. Suppose that $(\xi, \eta) \in \mathrm{Ker}\, d_{\chi(X,Y)}$. Then $\mathrm{tr}(X\xi) = \mathrm{tr}(Y\eta) = 0$ and $\mathrm{tr}(X\eta) + \mathrm{tr}(Y\xi) = 0$. If either X or Y equals $\pm I$, then $\mathrm{Ker}\, d_{\chi(X,Y)}$ has dimension at least four. Furthermore X and Y commute. If $X, Y \neq \pm I$, then the set $X^\perp$ of all ξ such that $\mathrm{tr}(X\xi) = 0$ has dimension two and the map $X^\perp \to \mathbf{C}$ given by $\xi \mapsto \mathrm{tr}(Y\xi)$ is nonzero unless X, Y commute. Similarly $Y^\perp$ has dimension two and $Y^\perp \to \mathbf{C}$, $\eta \mapsto \mathrm{tr}(X\eta)$ is nonzero unless X, Y commute. Thus the set consisting of $(\xi, \eta) \in X^\perp \times Y^\perp$ which satisfy $\mathrm{tr}(X\eta) = -\mathrm{tr}(Y\xi)$ has dimension three unless X, Y commute. □

4.5. **Corollary.** (a) *Let*

$$\Omega_{\mathbf{C}} = \{(X, Y) \in SL(2, \mathbf{C}) \times SL(2, \mathbf{C}) \mid [X, Y] \neq I\}.$$

Then $\chi\colon \Omega_{\mathbf{C}} \to \mathbf{C}^3$ satisfies the path-lifting property.

(b) *Let*

$$\Omega_{\mathbf{R}}=\{(X, Y)\in SL(2,\mathbf{R})\times SL(2,\mathbf{R}) \mid [X, Y]\neq I\}.$$

Then

$$\chi\colon \Omega_{\mathbf{R}}\to\mathbf{R}^3-([-2,2]^3\cap\kappa^{-1}([-2,2]))$$

satisfies the path-lifting property.

Proof of (*a*). Let $\Gamma=\mathbf{Z}/2$ act on $\Omega_{\mathbf{C}}$ by $(X, Y)\mapsto(X^{-1}, Y^{-1})$; clearly $\chi\colon\Omega_{\mathbf{C}}\to\mathbf{C}^3$ is Γ-invariant. By Lemma 3.9, it suffices to show that Γ acts transitively on the path-components of each preimage of $\chi\colon\Omega_{\mathbf{C}}\to\mathbf{C}^3$. If $\kappa(x, y, z)\neq 2$, then the preimage $\chi^{-1}(x, y, z)$ is a $SL(2,\mathbf{C})$-orbit and is hence path-connected. If $\kappa(x, y, z)=2$, then $\chi^{-1}(x, y, z)$ contains a nonabelian representation if and only if not all of x, y, z equal ± 2; thus the image of $\chi\colon\Omega_{\mathbf{C}}\to\mathbf{C}^3$ equals the connected set $\mathbf{C}^3-\mathscr{I}$ where

$$\mathscr{I}=\{(2,2,2),(2,-2,-2),(-2,2,-2),(-2,-2,2)\}.$$

It suffices to show that if $(x, y, z)\notin\mathscr{I}$ and $\kappa(x, y, z)=2$, then Γ permutes the path-components of $\Omega_{\mathbf{C}}\cap\chi^{-1}(x, y, z)$. Consider the commutator map

$$R_1\colon \Omega_{\mathbf{C}}\cap\chi^{-1}(x, y, z)\to SL(2,\mathbf{C})$$

defined by $(X, Y)\mapsto[X, Y]=XYX^{-1}Y^{-1}$; since $\operatorname{tr}[X, Y]=2$ but $[X, Y]\neq I$ it follows that X, Y can be conjugated to upper-triangular matrices. The image of $R_1\colon\Omega_{\mathbf{C}}\cap\chi^{-1}(x, y, z)\to SL(2,\mathbf{C})$ consists of all elements of $SL(2,\mathbf{C})-\{I\}$ which have trace 2; all such elements of $SL(2,\mathbf{C})$ are conjugate. This map is $SL(2,\mathbf{C})$-equivariant and hence is a fibration; we shall show that Γ permutes the two components of each fiber. To this end, represent X and Y respectively by matrices $\begin{bmatrix} a & \xi \\ 0 & a^{-1}\end{bmatrix}$ and $\begin{bmatrix} b & \eta \\ 0 & b^{-1}\end{bmatrix}$ where $(a, b)\in\mathbf{C}^*\times\mathbf{C}^*$ is one of the two solutions of

$$a+a^{-1}=x, \qquad b+b^{-1}=y, \qquad ab+(ab)^{-1}=z.$$

The fiber $R_1^{-1}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix}\right)$ is then easily seen to be the set of solutions of

$$a(1-b^2)\,\xi-b(1-a^2)\,\eta=1$$

which is clearly a connected set, for fixed a, b. Since there are two choices of (a, b) which are permuted under Γ, the result follows. □

Proof of (*b*). The proof of (b) follows similar lines, although it is slightly more complicated. Let $\Gamma=Z/2\times PGL(2,\mathbf{R})$, where $\mathbf{Z}/2$ acts as above and $PGL(2,\mathbf{R})$ acts by conjugation. The image $\chi(\Omega_{\mathbf{R}})$ equals the complement in $\mathbf{R}^3$ of $[-2,2]^3\cap\kappa^{-1}[-2,2]$. Given these modifications, the proof follows that of part (a). □

The previous discussion of representations of a free group of rank two into $SL(2,\mathbf{R})$ easily gives a proof of the simplest nontrivial case of Theorem 3.3. Let M be a pair-of-pants with boundary components A, B, C and let the corresponding elements of $\pi = \pi_1(M)$ also be denoted A, B, C so that π has a presentation of the form

$$\langle A, B, C \mid ABC = I \rangle.$$

Let $G = PSL(2, \mathbf{R})$. Let $W(M)$ be the subset of $\mathrm{Hom}(\pi, G)$ consisting of representations ϕ such that $\phi(A)$, $\phi(B)$, $\phi(C)$ are hyperbolic. Let $e\colon W(M) \to \mathbf{Z}$ be the relative Euler class map.

4.6. **Proposition.** *The components of $W(M)$ are the inverse images $e^{-1}(n)$ where $n = -1, 0, 1$. Furthermore the evaluation map $\mathrm{ev}_C\colon W(M) \to \mathrm{Hyp}$ defined by $\phi \mapsto \phi(C)$ satisfies the path-lifting property.*

Proof. We may find unique lifts $\tilde{\phi}(A)$, $\tilde{\phi}(B)$ of $\phi(A)$, $\phi(B)$ to $\mathrm{Hyp}_0 \subset \tilde{G}$. Then $\tilde{\phi}(C) = \tilde{\phi}(B)^{-1}\tilde{\phi}(A)^{-1} \in \mathrm{Hyp}_n$ where $n = e(\phi)$. We shall assume that every representation in $W(M)$ has been so lifted; in particular note that the character of such a representation lies in $(2, \infty) \times (2, \infty) \times ((-\infty, -2) \cup (2, \infty))$.

Suppose that $\phi \in W(M)$ has character $\chi(\phi) \in (2, \infty)^3$, i.e. $\tilde{\phi}(C) \in \mathrm{Hyp}_{2m}$. We shall show that $e(\phi) = 0$ and that the set of all such ϕ is connected. Suppose first that the image of ϕ does not lie in a Borel subgroup, so that $\kappa(\chi(\phi)) \neq 2$. Choose $(x_1, y_1, z_1) \in (2, \infty)^3$ such that $\kappa(x_1, y_1, z_1) = 2$. Then by 4.5 there exists a path $\{\phi_t\}_{0 \leq t \leq 1}$ from $\phi = \phi_0$ to a representation ϕ_1 such that $\chi(\phi_1) = (x_1, y_1, z_1)$ and such that $\phi_t \in W(M)$ has nonabelian image for each t. Since $\kappa(x_1, y_1, z_1) = 2$ the image of ϕ_1 lies in a Borel subgroup and by 3.6 $e(\phi) = 0$. We next show that $e^{-1}(0)$ is connected: by the preceding argument it suffices to show that $W(M) \cap (\kappa \circ \chi)^{-1}(2)$ is connected. To this end, note that every $\phi_1 \in W(M)$ whose image is nonabelian yet lies in a Borel subgroup has the property that there exists an abelian representation ϕ_1' and a one-parameter subgroup $\{h_t\}_{t \in \mathbf{R}}$ such that $h_t \phi_1 \to \phi_1'$ as $t \to \infty$. In particular one can join ϕ_1 to ϕ_1' by a path lying in $\chi^{-1}(\chi(\phi_1))$. Thus it suffices to show that the subspace of abelian representations in $W(M)$ is connected. Since every abelian representation is $PSL(2, \mathbf{R})$-conjugate to a representation whose image lies in a fixed hyperbolic one-parameter group H and since $PSL(2, \mathbf{R})$ and $\mathrm{Hom}(\pi, H) \approx \mathbf{R}^2$ are both connected, the result follows.

Next suppose that $\phi \in W(M)$ satisfies $\mathrm{tr}\,\tilde{\phi}(C) < -2$, i.e. that $\tilde{\phi}(C) \in \mathrm{Hyp}_{2m+1}$. We shall show that $e(\phi) = \pm 1$ and for $n = \pm 1$, the inverse image $e^{-1}(n)$ is connected. To this end, choose $\varepsilon > 0$ and consider the path $\{(a_t, b_t, c_t)\}_{0 \leq t \leq 1}$ where

$$a_t = \mathrm{tr}\,\tilde{\phi}(A), \qquad b_t = \mathrm{tr}\,\tilde{\phi}(B), \qquad c_t = (1-t)\,\mathrm{tr}\,\tilde{\phi}(C) + t(2+\varepsilon).$$

Then (after a possible reparametrization) there exists a path $\{\phi_t\}_{0 \leq t \leq 1}$ such that

$$\chi(\phi_t) = (a_t, b_t, c_t), \qquad \phi_0 = \phi.$$

By the preceding paragraph, $\phi_1(C) \in \mathrm{Hyp}_0$. Since the set of t such that $\phi_t(C) \in \mathrm{Ell}$ is connected, it follows that $\phi_1(C) \in \mathrm{Hyp}_{\pm 1}$. Conjugating by an element of $PGL(2, \mathbf{R})$ not in $PSL(2, \mathbf{R})$ (if necessary) we may assume that $\phi_1(C) \in \mathrm{Hyp}_1$. Suppose that $\psi \in W(M)$ satisfies $e(\psi) = 1$. Then there exists a path in $(2, \infty) \times (2, \infty) \times (-\infty, -2)$ from $\chi(\phi)$ to $\chi(\psi)$, which can be lifted to a path from ϕ to $\psi' \in \chi^{-1}(\chi(\psi))$. We claim that ψ and ψ' are $PSL(2, \mathbf{R})$-conjugate, and (since $PSL(2, \mathbf{R})$ is connected) can be joined by a path inside a $PSL(2, \mathbf{R})$-orbit, and hence inside $e^{-1}(1)$. For by 4.3 they are conjugate by an element of $PGL(2, \mathbf{R})$. Since the component of $PGL(2, \mathbf{R})$ not equal to $PSL(2, \mathbf{R})$ conjugates $e^{-1}(n)$ to $e^{-1}(-n)$, it follows that ψ and ψ' are $PSL(2, \mathbf{R})$-conjugate.

Finally we show that ev_C satisfies the path-lifting property. Since ev_C is a submersion (it is the restriction to the open set $W(M) \approx \mathrm{Hyp} \times \mathrm{Hyp} \subset G \times G$ of the multiplication map $(A, B) \mapsto AB$, which is evidently a submersion), it suffices to show that each inverse image of

$$\mathrm{ev}_C \colon W(M) \to \bigcup_{i=-1}^{1} \mathrm{Hyp}_i$$

is path-connected. Given $\phi, \psi \in W(M)$ with $\phi(C) = \psi(C)$, we must construct a path from ϕ to ψ inside the fiber of ev_C. If $e(\phi) = e(\psi) = 0$, then one can join ϕ and ψ to abelian representations as above (simply lift paths from $\chi(\phi)$ and $\chi(\psi)$ to $(c^2 - 2, c, c)$ where $c = \operatorname{tr} \tilde{\phi}(C)$) and use the fact that the abelian representations in $W(M)$ are path-connected. If $e(\phi) = e(\psi) = \pm 1$, then lift a path from $\chi(\tilde{\phi})$ to $\chi(\tilde{\psi})$ as above to join the representations. □

4.7. *Remark*. These results on the relative Euler class were proved in [8] using another interpretation of representations of a free group. We say that a representation $\phi \in \mathrm{Hom}(\pi, G)$ is "stable" if its image is neither abelian nor lies in a Borel subgroup and that ϕ is "semistable" if either it is stable or its image lies in a hyperbolic or an elliptic one-parameter subgroup. It is shown in [8], Theorem 5.2, by a geometric argument that a semistable representation $\phi \colon \pi \to G$ extends to a representation $\phi^* \colon \pi^* \to PGL(2, \mathbf{R})$ where $\pi^* \cong \mathbf{Z}/2 * \mathbf{Z}/2 * \mathbf{Z}/2$ is the two-fold extension of $\pi \cong \mathbf{Z} * \mathbf{Z}$. The embedding of π in

$$\pi^* = \langle \rho_1, \rho_2, \rho_3 \mid \rho_1^2 = \rho_2^2 = \rho_3^2 = I \rangle$$

is given by

$$A = \rho_2 \rho_3, \qquad B = \rho_3 \rho_1, \qquad C = \rho_1 \rho_2.$$

The fixed points of $\tilde{\phi}(\rho_1)$, $\tilde{\phi}(\rho_2)$, $\tilde{\phi}(\rho_3)$ in $\mathbf{RP}^2$ (using the Klein projective model for $\mathbf{H}^2$) define a triangle, which completely determines the representation ϕ. (If ϕ is stable, then this triangle is unique). The triangles satisfy certain properties (such as no side being tangent to the conic $\partial \mathbf{H}^2$) and by deforming the allowable triangles, one can produce deformations just as by using trace coordinates. See [8] for details. (See Magnus [29], Lemma 2.1 for an algebraic proof that any semistable representation $\pi \to PSL(2, \mathbf{C})$ extends to a representation $\pi^* \to SL(2, \mathbf{C})$.) The present approach using trace coordinates is also taken in Jankins-Neumann [25].

§5. The complex case

In this section we compute the components of the space of representations of a surface group π into groups locally isomorphic to $SL(2, \mathbf{C})$. Throughout this section G will denote $PSL(2, \mathbf{C})$ and $\tilde{G}$ will denote the double cover $SL(2, \mathbf{C})$. Our goal is to prove the following result, stated in the introduction as part (ii) of Theorem B:

5.1. **Theorem.** *Let π be the fundamental group of a closed oriented surface of genus $g>1$. Then* $\mathrm{Hom}(\pi, \tilde{G})$ *is connected. The connected components of* $\mathrm{Hom}(\pi, G)$ *are the two preimages of the Stiefel-Whitney map* $w_2\colon \mathrm{Hom}(\pi, G) \to \mathbf{Z}/2$.

The standard presentation of π is

$$\pi = \langle A_1, B_1, \ldots, A_g, B_g \mid R_g(A_1, B_1, \ldots, A_g, B_g) = I \rangle$$

where

$$R_g(A_1, B_1, \ldots, A_g, B_g) = [A_1, B_1] \ldots [A_g, B_g].$$

For $C \in \tilde{G}$, let $X_g(C) \subset \tilde{G}^{2g}$ denote the inverse image $R_g^{-1}(C)$ so that $\mathrm{Hom}(\pi, \tilde{G}) = X_g(I)$. Let $\tilde{R}_1\colon G \times G \to \tilde{G}$ denote the canonical lift of the commutator map $R_1\colon G \times G \to G$, and let $\Pi\colon \tilde{G} \to G$ denote the covering projection. Define

$$\tilde{R}_g\colon G^{2g} \to \tilde{G}$$
$$(A_1, B_1, \ldots, A_g, B_g) \mapsto \tilde{R}_1(A_1, B_1) \ldots \tilde{R}_1(A_g, B_g).$$

For $u \in \mathbf{Z}/2$ the fiber $w_2^{-1}(u) \subset \mathrm{Hom}(\pi, G)$ is identified with

$$(\tilde{R}_g)^{-1}((-I)^u) = \Pi(X_g((-I)^u)).$$

If $X_g((-I)^u)$ is known to be connected, then so is its continuous image $\Pi(X_g((-I)^u)) = (\tilde{R}_g)^{-1}((-I)^u)$. Thus Theorem 5.1 is a special case of the following result:

5.2. **Proposition.** *Let $C \in \tilde{G}$. Then $X_g(C)$ is connected.*

The proof will proceed most smoothly if we restrict attention to a dense open subset of $X_g(C)$. For each $1 \leqq i < j \leqq g$, $(i, j) \neq (1, g)$, let

$$f_{(i,j)}(A_1, B_1, \ldots, A_g, B_g) = [A_i, B_i] \ldots [A_j, B_j].$$

Let $X'_g(C)$ denote the subset of $X_g(C)$ consisting of all $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ such that if $1 \leqq i < j \leqq g$, $(i, j) \neq (1, g)$, then

$$f_{(i,j)}(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) \neq I.$$

Proposition 5.2 follows from the next two lemmas.

5.3. **Lemma.** *$X'_g(C)$ is a dense open subset of $X_g(C)$ for every $g \geqq 1$ and $C \in \tilde{G}$.*

5.4. **Lemma.** *$X'_g(C)$ is connected for every $g \geqq 1$ and $C \in \tilde{G}$ and for $g > 1$ the map $f_{(1,g-1)}\colon X'_g(C) \to \tilde{G} - \{I\}$ satisfies the path-lifting property.*

Proof of 5.3. The map $f_{(i,j)}\colon X_g(C) \to \tilde{G}$ is clearly a polynomial map from the algebraic set $X_g(C) \to \tilde{G}$; thus $f_{(i,j)}^{-1}(I)$ is closed. Now

$$X_g(C)' = \bigcup_{(i,j) \neq (1,g)} f_{(i,j)}^{-1}(\tilde{G} - \{I\})$$

is evidently open and it suffices to show that for each (i, j), $f_{(i,j)}^{-1}(\tilde{G}-\{I\})$ is dense. To this end, suppose that $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) \in X_g(C)$ satisfies

$$f_{(i,j)}(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g) = I.$$

Since the i-fold commutator map $R_i\colon \tilde{G}^{2i} \to \tilde{G}$ is a polynomial map on the irreducible variety $\tilde{G} \times \tilde{G}$ whose differential is surjective on the Zariski dense subset consisting of all $(\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i)$ such that $\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i$ do not commute, it follows that there is an open set $\Omega_{(1,i-1)} \subset \tilde{G}^{2g}$ whose closure contains $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ which is mapped submersively under $f_{(1,i-1)}$ to an open subset in $\tilde{G}-\{I\}$ whose closure contains

$$f_{(1,i-1)}(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g).$$

Similarly there exist open sets $\Omega_{(i,j)}$ and $\Omega_{(j+1,g)}$ whose closure contains $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ which are mapped submersively under $f_{(i,j)}$ and $f_{(j+1,g)}$. Thus there are points

$$(\alpha'_1, \beta'_1, \ldots, \alpha'_g, \beta'_g) \in \Omega_{(1,i-1)} \cap \Omega_{(i,j)} \cap \Omega_{(j+1,g)}$$

arbitrarily close to $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ for which

$$f_{(i,j)}(\alpha'_1, \beta'_1, \ldots, \alpha'_g, \beta'_g) \neq I$$

and

$$R_g(\alpha'_1, \beta'_1, \ldots, \alpha'_g, \beta'_g) = C.$$

It follows that

$$f_{(i,j)}^{-1}(\tilde{G}-\{I\}) \cap X_g(C)$$

is open and dense and thus the finite intersection of open dense sets $X'_g(C)$ is also open and dense. □

Next we prove 5.4. The initial case $g=1$ will be proved separately:

5.5. **Proposition.** *Let* $C \in \tilde{G}$. *Then* $X_1(C) = \{(\alpha, \beta) \in \tilde{G} \times \tilde{G} \mid [\alpha, \beta] = C\}$ *is nonempty and connected.*

Proof of 5.4 assuming 5.5. Induction on g. The initial case $g=1$ of 5.4 is precisely 5.5. Suppose inductively that $n>1$ and that 5.4 has been proved for all $C \in \tilde{G}$ and for $g<n$. We show that for $C \in \tilde{G}$, $X'_n(C)$ is nonempty and connected. The fiber of the map

$$f_{(1,n-1)}\colon X'_n(C) \to \tilde{G}-\{I\}$$

is

$$f_{(1,n-1)}^{-1}(v) = X'_{n-1}(v) \times X_1(v^{-1}C),$$

which by the induction hypothesis is connected. Furthermore since $f_{(1,n-1)}$ is a submersion, it follows from 3.9 that $f_{(1,n-1)}$ satisfies the path-lifting property and $X'_n(C)$ is connected. □

Proof of 5.5. For $t \in \mathbf{C}$, let

$$R(t) = \bigcup_{\mathrm{tr}(C)=t} R_1^{-1}(C)$$

and

$$C(t) = \{C \in \tilde{G} \mid \mathrm{tr}(C) = t\}.$$

The map $R_1 \colon R(t) \to C(t)$ is clearly equivariant with respect to the natural actions of $\tilde{G}$ by conjugation.

Suppose $t \neq 2$. Then the representation determined by any $(\alpha, \beta) \in R(t)$ is irreducible, and by 5.1 G acts properly and freely on the set $R(t)$. By 4.1 the quotient space $R(t)/G$ may be identified with the cubic surface

$$S_t = \{(x, y, z) \in \mathbf{C}^3 \mid x^2 + y^2 + z^2 - x\,y\,z = t + 2\}$$

under the map $R(t) \to S_t$ given by

$$(\alpha, \beta) \mapsto (\mathrm{tr}(\alpha), \mathrm{tr}(\beta), \mathrm{tr}(\alpha\beta)).$$

Clearly the polynomial $x^2 + y^2 + z^2 - x\,y\,z - t - 2$ is irreducible, whence S_t is an irreducible complex affine algebraic surface and is thus connected.

Suppose further that $t \neq -2$. Then G acts transitively on $C(t)$ with isotropy group a maximal torus G_1 of G. Since G is connected and $R(t)/G \approx S_t$ is connected, it follows from the exact homotopy sequence of the fibration $G \to R(t) \to R(t)/G$ that $R(t)$ is connected. Since G acts transitively on $C(t)$, the G-equivariant map $R_1 \colon R(t) \to C(t)$ is a fibration. Since $C(t) \approx G/G_1$ is simply connected, the exact homotopy sequence of the fibration $R_1^{-1}(C) \to R(t) \to C(t)$ implies that $R_1^{-1}(C)$ is connected whenever $\mathrm{tr}(C) = t \neq 2$.

Suppose next that $t = -2$. In that case there are two G-orbits on $C(t)$, one consisting of the single point $-I$ and the other consisting of parabolic elements. We show that $R_1^{-1}(C)$ is connected in each case. First suppose that $C = -I$. If $R_1(\alpha, \beta) = -I$, then $\alpha\beta\alpha^{-1}\beta^{-1} = -I$, whence

$$\alpha\beta\alpha^{-1} = -\beta, \qquad \alpha\beta = -\beta\alpha, \qquad \beta\alpha^{-1}\beta^{-1} = -\alpha^{-1}.$$

By taking traces we obtain that $\mathrm{tr}(\alpha) = \mathrm{tr}(\beta) = \mathrm{tr}(\alpha\beta) = 0$, and hence by 4.1 the fiber $R_1^{-1}(-I)$ consists of a single G-orbit. Thus $R_1^{-1}(C)$ is connected.

Now G acts transitively on $R(-2) - \{-I\}$ and the quotient $(R(-2) - \{-I\})/G$ is identified with the set

$$\{(x, y, z) \in \mathbf{C}^3 \mid x^2 + y^2 + z^2 - x\,y\,z = 0,\ (x, y, z) \neq (0, 0, 0)\}.$$

The origin is the only singularity in the surface S_{-2}, and the defining equation for S_{-2} has nondegenerate quadratic leading term. Thus the link of the singular point is connected (it is homeomorphic to a 2-torus) and it follows (by a Mayer-Vietoris argument) that the complement of the origin in the connected cubic surface S_{-2} is also connected. Applying the homotopy exact sequence as in the previous case, we conclude that $R_1^{-1}(C)$ is connected whenever $\mathrm{tr}(C) = -2$.

It remains to consider the case $\mathrm{tr}(C) = 2$. As before, $C(t)$ splits into two orbits, namely $\{I\}$ and an orbit of parabolic elements. Clearly $R_1^{-1}(I)$ consists of all pairs $(\alpha, \beta) \in \tilde{G} \times \tilde{G}$ such that α and β commute, and it is easy to see that two elements of $\tilde{G}$ which commute must belong to a common abelian subgroup of the form $\{\pm I\} \cdot U$, where U is either a parabolic one-parameter subgroup or to a common Cartan subgroup. In the former case we can easily find a path starting at (α, β) lying within $(\{\pm I\} \cdot U) \times (\{\pm I\} \cdot U) \subset R_1^{-1}(I)$ ending at a point in $\{\pm I\} \times \{\pm I\}$. Such a point lies in some (in fact every) Cartan subgroup. If α, β lie in a common Cartan subgroup G_1, then (since G_1 is connected) there exists a path from (α, β) to (I, I) which lies in $R_1^{-1}(I)$. Hence every point in $R_1^{-1}(I)$ can be connected by a path in $R_1^{-1}(I)$ to (I, I) and it follows that $R_1^{-1}(I)$ is path-connected.

Finally we show that $R_1^{-1}(C)$ is connected when $\mathrm{tr}(C) = 2$, $C \neq I$. Applying an inner automorphism of $\tilde{G}$ we may assume that C is represented by the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

from which it follows that any $\alpha, \beta \in SL(2, \mathbf{C})$ such that $[\alpha, \beta] = C$ will be represented by upper-triangular matrices. Writing

$$\alpha = \begin{bmatrix} a & \xi \\ 0 & a^{-1} \end{bmatrix}, \quad \beta = \begin{bmatrix} b & \eta \\ 0 & b^{-1} \end{bmatrix}$$

with $a, b \in \mathbf{C}^*$, $\xi, \eta \in \mathbf{C}$, we see that $[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$ is represented by the matrix

$$\begin{bmatrix} 1 & a(1-b^2)\,\xi - b(1-a^2)\,\eta \\ 0 & 1 \end{bmatrix}.$$

Hence $R_1^{-1}(C)$ may be identified with

$$X = \{(a, b, \xi, \eta) \in \mathbf{C}^4 \mid a, b \neq 0,\ a(1-b^2)\,\xi - b(1-a^2)\,\eta = 1\},$$

which is easily seen to be connected: the map (a, b): $X \to \mathbf{C}^2$ is a submersion onto $\mathbf{C}^* \times \mathbf{C}^* - (\{\pm 1\} \times \{\pm 1\})$, which is connected, and its fibers are complex lines. Thus by Lemma 1.4 X is connected. □

§6. The topology of κ

6.1. In this section we discuss the topology of the polynomial function $\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$ which arises in our setting as the trace of a commutator of a pair of elements of $SL(2, \mathbf{R})$ or $SL(2, \mathbf{C})$. We are particularly interested in the real algebraic surfaces S_t in $\mathbf{R}^3$ defined by the equations $\kappa(x, y, z) = t$, for $t \in \mathbf{R}$. Observe that the group of orthogonal transformations leaving κ invariant is generated by the symmetric group on $\{x, y, z\}$ and the subgroup Λ consisting of diagonal matrices

$$\begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix},$$

$\varepsilon_i = \pm 1$ having determinant $\varepsilon_1 \varepsilon_2 \varepsilon_3 = 1$. We shall see that:

(a) If $t < -2$, then S_t is homeomorphic to the disjoint union of four discs, which are permuted by Λ;

(b) S_{-2} is homeomorphic to the disjoint union of four discs permuted by Λ and the origin;

(c) If $-2 < t < 2$, S_t is homeomorphic to the disjoint union of four discs permuted by Λ and a 2-sphere contained in $[-2, 2]^3$ (this compact component consists of characters of representations in $SU(2)$);

(d) S_2 is homeomorphic to the union of four disjoint discs (freely permuted by Λ) and a 2-sphere which meets each of the discs in one of the four points

$$(2, 2, 2), (2, -2, -2), (-2, 2, -2), (-2, -2, 2);$$

(e) For $t > 2$, S_t is homeomorphic to a 2-sphere minus four discs.

The case (d) is special. The four singular points are the characters of representations of Π into the center $\{\pm I\}$ of $SL(2, \mathbf{R})$. The complex points have a rational parametrization:

$$\mathbf{C}^* \times \mathbf{C}^* \to \mathbf{C}^3$$
$$(\lambda, \mu) \mapsto (\lambda + \lambda^{-1}, \mu + \mu^{-1}, \lambda\mu + \lambda^{-1}\mu^{-1}).$$

Taking $\lambda, \mu \in \mathbf{R}$, one obtains rational parametrizations of four pieces of S_2 corresponding to characters of abelian representations consisting of hyperbolic elements; taking $\lambda, \mu \in \mathbf{C}$, $|\lambda| = |\mu| = 1$, one obtains a rational parametrization of the bounded piece of S_2 consisting of characters of abelian representations consisting of elliptic elements. (Compare Fig. 2.)

To understand the topology of S_t, we shall decompose S_t into the level sets $S_t(z)$ of the coordinate function $z: S_t \to \mathbf{R}$. Thus we shall rewrite the defining equation

$$x^2 + y^2 + z^2 - xyz = t + 2$$

as follows:

$$\frac{2-z}{4}(x-y)^2 + \frac{2+z}{4}(x+y)^2 = 2 + t - z^2.$$

One can therefore see that each level set is a (possibly degenerate) conic in a plane $z =$ constant. There are the following possibilities:

(i) $S_t(z)$ is a nondegenerate hyperbola. This occurs whenever the coefficients of the quadratic terms $\frac{2-z}{4}, \frac{2+z}{4}$ are nonzero and have opposite signs (i.e. when $|z| > 2$) and the constant term $2 + t - z^2 \neq 0$;

(ii) $S_t(z)$ is a nondegenerate ellipse. This occurs whenever the coefficients of the quadratic terms $\frac{2-z}{4}, \frac{2+z}{4}$ are both positive (i.e. when $|z| < 2$) and the constant term $2 + t - z^2 > 0$.

(iii) $S_t(z)$ is empty. This occurs when $|z| \leqq 2$ and $2 + t - z^2 < 0$.

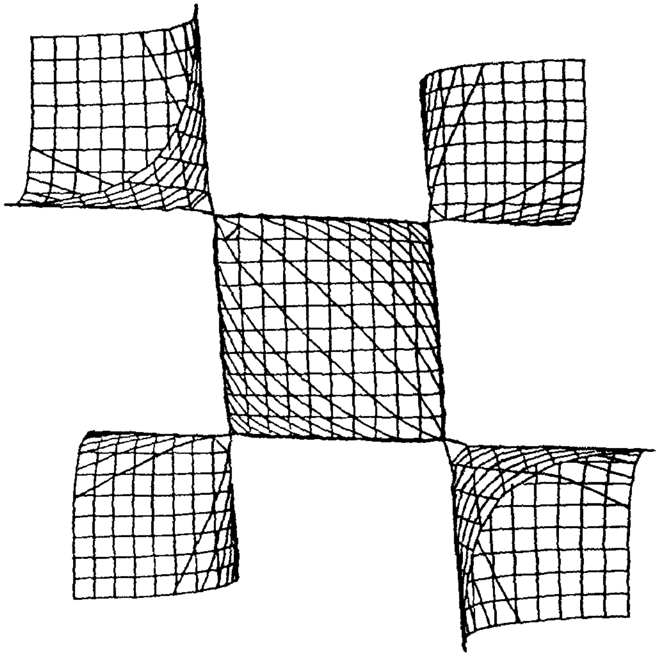

Fig. 2

(iv) $S_t(z)$ is a union of two parallel lines. This occurs when one of the coefficients of the quadratic terms vanishes (i.e. when $|z|=2$) but the constant term $2+t-z^2>0$.

(v) $S_t(z)$ is a union of intersecting lines. This occurs when $|z|>2$ but $2+t-z^2=0$.

(vi) $S_t(z)$ is a single line. This occurs when $|z|=2$ and $2+t-z^2=0$.

(vii) $S_t(z)$ is a single point. This occurs when $|z|<2$ and $2+t-z^2=0$.

The degenerate cases therefore occur only when

$$z=\pm 2,\ \pm\sqrt{2+t}.$$

The cases (i), (ii), (iii) are nondegenerate (although (iii) includes a degenerate case as well). By analyzing these various cases and how the level curves bifurcate, we determine the topological type of S_t.

6.2. Suppose that $t>2$. For $z>2$, $S_t(z)$ is a hyperbola, except at the degenerate case $z=\sqrt{2+t}$, where $z\colon S_t\to\mathbf{R}$ has a saddle point. For $z=2$, the level set is two parallel lines, which is a limiting case of hyperbolas with coalescing asymptotes. For $-2<z<2$, the level sets are ellipses, whose eccentricities become unbounded as $t\nearrow 2$. From the preceding discussion, we see that the part of S_t with $t\geqq 0$ is homeomorphic to a pair of pants, with one boundary component

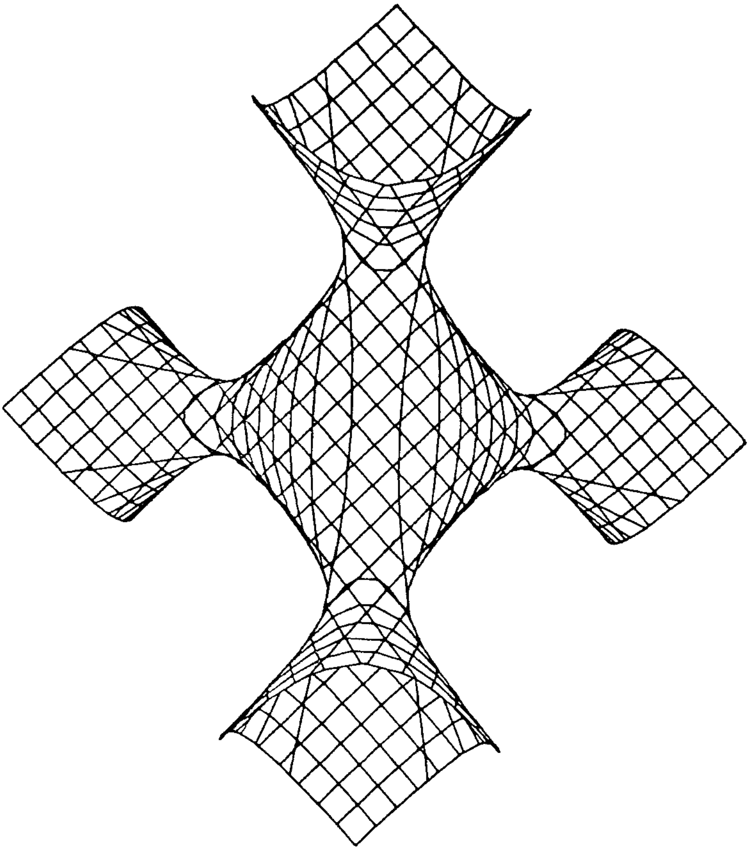

Fig. 3

the circle $z=0$. The level sets for $z<0$ are arranged similarly, and it is seen that S_t is homeomorphic to a sphere minus four discs. (Compare Fig. 3.)

Now suppose $-2<t<2$. Then $S_t(z)$ is a hyperbola for $|z|>2$, is empty for $2\geqq|z|>\sqrt{2+t}$, and is an ellipse for $|z|<\sqrt{2+t}$ (and a point for $|z|=\sqrt{2+t}$). Thus the portion of S_t with $z>2$ is a union of two discs, and the portion with $-2<z<2$ is a 2-sphere. (Compare Fig. 4.)

The case when $t\leqq -2$ is similar, except that no point of S_t satisfies $|z|<2$. One can prove that if $t<-2$, each point in S_t is the character of a holonomy representation for a torus minus a disc. If $t=-2$, then each point of S_t is the character of a faithful representation of Π onto a discrete subgroup of $SL(2,\mathbf{R})$ such that the quotient surface is complete, has finite area, and is homeomorphic to a torus-minus-a-disc.

§ 7. Commutators in $SL(2, \mathbf{R})$

The purpose of this section is to analyze the lifted commutator map $\tilde{R}_1$: $G \times G \to \tilde{G}$, where $G=PSL(2,\mathbf{R})$. We shall compute its image and show that its preimages are connected.

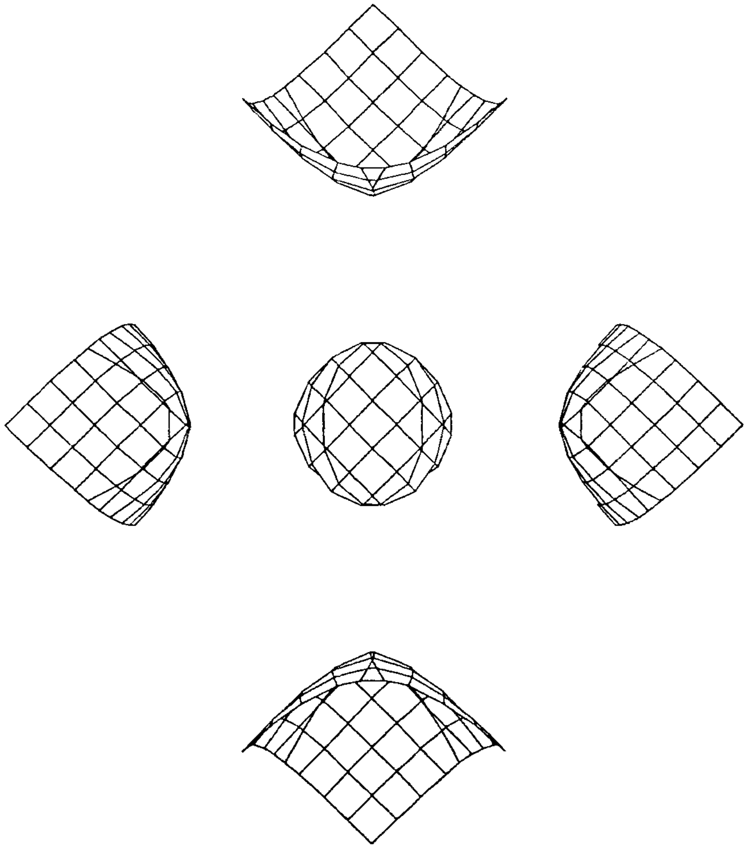

Fig. 4

7.1. **Theorem.** (i) *The image of the lifted commutator map* $\tilde{R}_1\colon G\times G\to\tilde{G}$ *equals the set*

$$\mathfrak{I}=\{I\}\cup \mathrm{Ell}_{\pm 1}\cup \mathrm{Par}_0^{\pm}\cup \mathrm{Hyp}_0\cup \mathrm{Par}_{\mp 1}^{\pm}\cup \mathrm{Hyp}_{\pm 1}.$$

(ii) *For each* $C\in\tilde{G}$, *the preimage* $(\tilde{R}_1)^{-1}(C)$ *is connected.*

Remark. Theorem 7.1 (i) is not new; see Milnor [31], Wood [41], and Eisenbud-Hirsch-Neumann [4] for proofs using different ideas than those presented here.

Proof. The image of $\tilde{R}_1$ is clearly a connected subset of $\tilde{G}$, which is invariant both under conjugation and inversion (since $\tilde{R}_1(B,A)=\tilde{R}_1(A,B)^{-1}$). Furthermore, $\tilde{R}_1$ is submersive at any (A,B) such that $[A,B]\neq I$ and is open at (I,I), so that the image of $\tilde{R}_1$ is an open subset of $\tilde{G}$.

Let κ and χ be as in § 4. Then the following diagram

$$\begin{array}{ccc} SL(2,\mathbf{R})\times SL(2,\mathbf{R}) & \xrightarrow{R_1} & SL(2,\mathbf{R}) \\ \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle\mathrm{tr}} \\ \mathbf{R}^3 & \xrightarrow{\kappa} & \mathbf{R} \end{array}$$

commutes. Furthermore the image of χ consists of the complement in $\mathbf{R}^3$ of the set of all $(x, y, z) \in [-2, 2]^3$ such that $\kappa(x, y, z) < 2$, and if $k(x, y, z) \neq 2$, then $\chi^{-1}(x, y, z)$ consists of a single $GL(2, \mathbf{R})$-orbit.

7.2. **Lemma.** *The image $\mathfrak{I}$ of $\tilde{R}_1: G \times G \to \tilde{G}$ intersects*

$$\bigcup_{m \in \mathbf{Z}} (\mathrm{Hyp}_{2m} \cup \mathrm{Par}_{2m})$$

in $\mathrm{Hyp}_0 \cup \mathrm{Par}_0$.

Proof. Let $C \in \mathrm{Hyp}_{2m} \cup \mathrm{Par}_{2m}$ and $X, Y \in G$ satisfy $\tilde{R}_1(X, Y) = C$. Then $\operatorname{tr} C \geqq 2$. Suppose first that $\operatorname{tr} C = 2$. Then by 4.1(iii), either $C = I$ or X, Y lie in a solvable subgroup, and there exists a one-parameter group $\{\theta_u \mid u \in \mathbf{R}\}$ such that

$$\theta_u \tilde{R}_1(X, Y)\, \theta_u^{-1} \to I$$

as $u \to \infty$. It follows that $\tilde{R}_1(X, Y) \in \mathrm{Par}_0$. Now suppose that $\operatorname{tr} C > 2$; by the path-lifting property for $\kappa \circ \chi$, there exists a path $\{(X_s, Y_s)\}_{0 \leqq s \leqq 1}$ such that $X_1 = X$, $Y_1 = Y$, with

$$\operatorname{tr} \tilde{R}_1(X_s, Y_s) = 2 + s(\operatorname{tr} C - 2).$$

In particular $\{\tilde{R}_1(X_s, Y_s)\}_{1 \geqq s \geqq 0}$ determines a path from $\tilde{R}_1(X, Y)$ to Par_0 which never meets Ell. It follows that $\tilde{R}_1(X, Y) \in \mathrm{Hyp}_0$.

It remains to show that $\mathrm{Hyp}_0 \cup \mathrm{Par}_0 \subset \mathrm{Image}\, \tilde{R}_1$. To this end we note that if X, Y are noncommuting elements of a Borel subgroup, then $\tilde{R}_1(X, Y) \in \mathrm{Par}_0$. Since the image of $\tilde{R}_1$ is invariant under $PGL(2, \mathbf{R})$, which acts transitively on Par_0, we have $\mathrm{Par}_0 \subset \mathrm{Image}\, \tilde{R}_1$. Since the image of $\kappa \circ \chi$: $SL(2, \mathbf{R}) \times SL(2, \mathbf{R}) \to \mathbf{R}$ contains $(2, \infty)$ and $\mathrm{Image}\, \tilde{R}_1$ is invariant under conjugation, it follows that the image $\tilde{R}_1$ contains Hyp_0. □

7.3. We claim that for each $C \in \mathrm{Hyp}_0$ the preimage $\tilde{R}_1^{-1}(C) \in G \times G$ is connected. To prove this, we shall use the description of the level sets of κ obtained in §6. First note that the natural homomorphism $SL(2, \mathbf{R}) \times SL(2, \mathbf{R}) \to G \times G$ is a covering map with covering group $\mathbf{Z}/2 \times \mathbf{Z}/2$. We shall first show that the set of all pairs $(X, Y) \in SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ such that $[X, Y] = C$ is connected. This in turn will be deduced from the fact that the set of $PGL(2, \mathbf{R})$-conjugacy classes of such pairs is connected.

Let $t = \operatorname{tr} C$ and let $\mathrm{Hyp}_0(t) = \mathrm{Hyp}_0 \cap \operatorname{tr}^{-1}(t)$; recall that the set of $PGL(2, \mathbf{R})$-conjugacy classes of pairs $(X, Y) \in SL(2, \mathbf{R}) \times SL(2, \mathbf{R})$ such that $\operatorname{tr}[X, Y] = t$ may be identified with the cubic surface $\kappa^{-1}(t)$, which is connected (it is homeomorphic to a sphere minus four discs). Furthermore the evaluation map R_1: $(\operatorname{tr} R_1)^{-1}(t) \to \mathrm{Hyp}_0(t)$ given by $(X, Y) \mapsto [X, Y]$ is equivariant with respect to the actions of $SL(2, \mathbf{R})$ by conjugation; the action on $\mathrm{Hyp}_0(t)$ is clearly transitive. Thus there are fibrations

$$\begin{array}{ccccc} & & PGL(2,\mathbf{R}) & & \\ & & \downarrow & & \\ R_1^{-1}(C) & \longrightarrow & (\operatorname{tr} R_1)^{-1}(t) & \xrightarrow{R_1} & \mathrm{Hyp}_0(t) \\ & & \downarrow{\scriptstyle \chi} & & \\ & & \kappa^{-1}(t) & & \end{array}$$

The vertical fibration determines an exact sequence

$$\pi_1(\kappa^{-1}(t)) \to \pi_0(PGL(2,\mathbf{R})) \to \pi_0((\operatorname{tr} R_1)^{-1}(t)) \to \pi_0(\kappa^{-1}(t))$$

so that to prove that $(\operatorname{tr} R_1)^{-1}(t)$ is connected it will suffice to prove

7.4. **Lemma.** *The connecting homomorphism*

$$\pi_1(\kappa^{-1}(t)) \to \pi_0(PGL(2,\mathbf{R}))$$

is surjective.

Proof. Since $PGL(2,\mathbf{R})$ has two components, it suffices to construct a path from a point (X_0, Y_0) to $\rho(X_0, Y_0)$ in $(\operatorname{tr} R_1)^{-1}(t)$, where $\rho \in PGL(2,\mathbf{R})$ does not lie in the identity component $PSL(2,\mathbf{R})$ of $PGL(2,\mathbf{R})$, which covers a closed loop in $\kappa^{-1}(t)$. To this end, we let

$$\rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Y_0 = \begin{bmatrix} 0 & u \\ -u^{-1} & 0 \end{bmatrix}$$

where $u^2 + u^{-2} = t$. Then our choice of ρ satisfies $\rho X_0 \rho^{-1} = -X_0$, $\rho Y_0 \rho^{-1} = -Y_0$. We define a continuous path $\{(X_s, Y_s)\}_{0 \leqq s \leqq 1}$ in $(\operatorname{tr} R_1)^{-1}(t)$ as follows. For $0 \leqq s \leqq \frac{1}{2}$, let

$$X_s = X_0$$

$$Y_0 = Y_0 \begin{bmatrix} \cos 2s\pi & -\sin 2s\pi \\ \sin 2s\pi & \cos 2s\pi \end{bmatrix}.$$

Then $X_{\frac{1}{2}} = X_0$ and $Y_{\frac{1}{2}} = -Y_0$. For $\frac{1}{2} \leqq s \leqq 1$, let

$$X_s = X_0 \begin{bmatrix} \cos(2s-1)\pi & -u\sin(2s-1)\pi \\ u^{-1}\sin(2s-1)\pi & \cos(2s-1)\pi \end{bmatrix}$$

$$Y_s = Y_{\frac{1}{2}}.$$

Note that the commutator $[X_s, Y_s]$ remains constant throughout this deformation. Moreover $X_1 = -X_0 = \rho X_0 \rho^{-1}$ and $Y_1 = -Y_0 = \rho Y_0 \rho^{-1}$ as desired. Finally the path $\{(X_s, Y_s)\}_{0 \leqq s \leqq 1}$ covers a closed loop in $\kappa^{-1}(t)$ (a union of two semicircles). Thus the proof of Lemma 7.4 is complete. □

Remark. Each of the two segments of the path of representations constructed above is a trajectory of a natural kind of flow on the spaces $\mathrm{Hom}(\pi, G)$, extensively discussed in [13]. The particular representations which occur in this example are solvable representations generated by two elliptic involutions. The particular path discussed above may also be found in [24], p. 64.

We have so far proved that $(\operatorname{tr} R_1)^{-1}(t) \subset SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ is connected. To show that $R_1^{-1}(C)$ is connected, consider the homotopy exact sequence of the horizontal fibration:

$$\pi_1((\operatorname{tr} R_1)^{-1}(t)) \to \pi_1(\mathrm{Hyp}_0(t)) \to \pi_0(R_1^{-1}(C)) \to \pi_0((\operatorname{tr} R_1)^{-1}(t)).$$

The fundamental group of $\mathrm{Hyp}_0(t)$ is generated by the homotopy class of a loop of the form $\{\theta_s C_0 \theta_s^{-1}\}_{0 \le s \le 1}$, where $\{\theta_s\}_{0 \le s \le 1}$ is an elliptic one-parameter subgroup and $C_0 \in \mathrm{Hyp}_0(t)$. Choose $(X_0, Y_0) \in R_1^{-1}(C_0)$. Then $\{(\theta_s X_0 \theta_s^{-1}, \theta_s Y_0 \theta_s^{-1})\}_{0 \le s \le 1}$ defines a closed loop in $(\operatorname{tr} R_1)^{-1}(t)$ which maps to $\{\theta_s C_0 \theta_s^{-1}\}_{0 \le s \le 1}$. It follows that $\pi_1((\operatorname{tr} R_1)^{-1}(t)) \to \pi_1(\mathrm{Hyp}_0(t))$ is surjective, whence $\pi_1(\mathrm{Hyp}_0(t)) \to \pi_0(R_1^{-1}(C))$ is zero. Since $(\operatorname{tr} R_1)^{-1}(t)$ is connected, it follows that $R_1^{-1}(C) \subset SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ is connected.

Now $(\tilde{R}_1)^{-1}(C) \subset G \times G$ is the continuous image of $R_1^{-1}(C)$ under the covering map $SL(2,\mathbf{R}) \times SL(2,\mathbf{R}) \to G \times G$, and is therefore itself connected.

7.5. We next discuss pairs $(X, Y) \in SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ with $\operatorname{tr}[X, Y] = 2$. By 7.2, this implies that either X and Y commute or $[X, Y] \in \mathrm{Par}_0$.

Two elements of $G = PSL(2,\mathbf{R})$ commute if and only if they lie on a common one-parameter subgroup S. Thus given any $(X, Y) \in G \times G$ with $[X, Y] = I$, there exists a path in $S \times S$ from (X, Y) to (I, I).

Next suppose that $C \in \mathrm{Par}_0$. If $(X, Y) \in G \times G$ satisfy $[X, Y] = C$, then any two lifts $\tilde{X}, \tilde{Y}$ of X, Y to $SL(2,\mathbf{R})$ satisfy $\operatorname{tr}[\tilde{X}, \tilde{Y}] = 2$. It follows that $\tilde{X}, \tilde{Y}$ normalize the one-parameter subgroup of $\tilde{G}$ containing C. Conjugating by an element of $GL(2,\mathbf{R})$ we may represent $\tilde{C}$ by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It follows from 4.1(iii) that X and Y must also be represented by upper-triangular matrices. As in the proof of 5.5, we may identify the set of upper-triangular matrices $(X, Y) \in SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ with $[X, Y] = C$ with

$$\{(a, b, \xi, \eta) \in \mathbf{R}^4 \mid a, b \neq 0,\ a(1-b^2)\,\xi - b(1-a^2)\,\eta = 1\}.$$

By multiplying X and Y by $\pm I$, we obtain the different lifts of (X, Y) to $SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$. It follows that $\tilde{R}_1^{-1}(C) \subset G \times G$ may be identified with the set

$$\{(a, b, \xi, \eta) \in \mathbf{R}^4 \mid a, b > 0,\ a(1-b^2)\,\xi - b(1-a^2)\,\eta = 1\},$$

which by an argument similar to 5.5 is connected.

7.6. Now suppose that $C \in \mathrm{Ell}$. We show that if $X, Y \in G$, $\tilde{R}_1(X, Y) = C$, then $C \in \mathrm{Ell}_1 \cup \mathrm{Ell}_{-1}$ and that the set of all such pairs (X, Y) forms a connected set.

For the first assertion, we use the path-lifting property as above to find a path $\{(X_s, Y_s)\}_{0 \leqq s \leqq 1}$ such that

$$X_1 = X, \qquad Y_1 = Y, \qquad \operatorname{tr} \tilde{R}_1(X_s, Y_s) = 2 + s(\operatorname{tr} C - 2).$$

Since $\tilde{R}_1(X_s, Y_s)$ is elliptic for every $s > 0$, and for $s = 0$ lies in $\{I\} \cup \mathrm{Par}_0$, we see that $\tilde{R}_1(X, Y) \in \mathrm{Ell}_{\pm 1}$.

Now we show that $\tilde{R}_1^{-1}(C)$ is connected. By conjugation by an element of $PGL(2, \mathbf{R})$, we may suppose that $C \in \mathrm{Ell}_1$. Let $t = \operatorname{tr} C$. The set $\mathrm{Ell}_1(t) = \{u \in \mathrm{Ell}_1 \mid \operatorname{tr} u = t\}$ equals the set of G-conjugates of C, and is homeomorphic to a disc; the set $\mathrm{Ell}(t) = \{u \in \mathrm{Ell} \mid \operatorname{tr} u = t\}$ is then the disjoint union of two discs. Thus the fibration

$$\tilde{R}_1^{-1}(C) \to (\operatorname{tr} \tilde{R}_1)^{-1}(t) \to \mathrm{Ell}(t)$$

is trivial, whence $\tilde{R}_1^{-1}(C)$ is connected if and only if $(\operatorname{tr} \tilde{R}_1)^{-1}(t)$ has two components. In particular it suffices to show that the set of $PGL(2, \mathbf{R})$-conjugacy classes of pairs $(X, Y) \in G \times G$ such that $\operatorname{tr} \tilde{R}_1(X, Y) = t$ is connected. By 4.1 the set

$$\{(\tilde{X}, \tilde{Y}) \in SL(2, \mathbf{R}) \times SL(2, \mathbf{R}) \mid \operatorname{tr} R_1(\tilde{X}, \tilde{Y}) = t\} / PGL(2, \mathbf{R})$$

may be identified with the union of the four noncompact components of the cubic surface $\kappa^{-1}(t)$. Each noncompact component is a 2-disc, and they are freely permuted by the covering group Λ of the covering space $SL(2, \mathbf{R}) \times SL(2, \mathbf{R}) \to G \times G$; thus the quotient

$$\{(X, Y) \in G \times G \mid \operatorname{tr} R_1(\tilde{X}, \tilde{Y}) = t\} / PGL(2, \mathbf{R})$$

is homeomorphic to a disc and thus connected as claimed. □

7.7. The case that $C \in \mathrm{Hyp}_{2m+1}$ is analogous to the preceding case. By a path-lifting argument, one can see that if $C = \tilde{R}_1(X, Y)$, then $C \in \mathrm{Hyp}_{\pm 1}$. By considering fibrations one can see that $\tilde{R}_1^{-1}(C) \in G \times G$ is connected if and only if

$$\{(X, Y) \in G \times G \mid \operatorname{tr} \tilde{R}_1(X, Y) = \operatorname{tr} C\} / PGL(2, \mathbf{R})$$

is connected if and only if the quotient $\kappa^{-1}(\operatorname{tr} C) / \Lambda$ is connected. Since $\kappa^{-1}(\operatorname{tr} C)$ consists of four discs freely permuted by Λ, the result follows as above.

The last remaining case is that of $C \in \mathrm{Par}_{2m+1}$, when $\operatorname{tr} C = -2$; path-lifting gives that if $C = \tilde{R}_1(X, Y)$, then $C \in \mathrm{Par}_1^{\pm}$. The fact that $\mathrm{Image}\, \tilde{R}_1$ is open and meets Hyp in the set $\mathrm{Hyp}_0 \cup \mathrm{Hyp}_{\pm 1}$ implies that in fact $C \in \mathrm{Par}_1^{\pm}$. The rest of the argument is as above.

This concludes the proof of Theorem 7.1. □

7.8. **Corollary.** *Let M be a surface homeomorphic to a torus-minus-a-disc. Let C denote both the boundary of M and the corresponding element of $\pi_1(M)$. Then the components of $W(M)$ are the inverse images $e^{-1}(n)$ where $n = -1, 0, 1$. Furthermore the evaluation map ev_C: $W(M) \to \mathrm{Hyp}$ defined by $\phi \mapsto \phi(C)$ satisfies the path-lifting property.*

Proof. As each $\phi \in W(M)$ has nonabelian image ($\phi(C)$ is hyperbolic), the differential of ev_C is surjective. Let $\widetilde{\mathrm{ev}}_C\colon W(M) \to \tilde{G}$ denote the canonical lift of ev_C, i.e. $\widetilde{\mathrm{ev}}_C(\phi) = \tilde{\phi}(C)$ for any lift $\tilde{\phi}\colon \pi \to \tilde{G}$ of ϕ. Then for each $n = \pm 1, 0$, the preimage $e^{-1}(n) = (\widetilde{\mathrm{ev}}_C)^{-1}(\mathrm{Hyp}_n)$. By 7.1, the inverse images of ev_C are path-connected; now apply 1.4 and the result follows. □

7.9. An easier application of the analysis in § 6 deals with representations of surface groups in $SU(2)$ and $SO(3)$. One needs only apply the techniques of the proof of Theorem 5.1 to representations in $SO(2)$ and $SO(3)$. All that is needed is that the set of pairs $(A, B) \in SU(2) \times SU(2)$ such that $[A, B] = C$ is connected for each $C \in SU(2)$; this follows from the identification of the set of conjugacy classes of pairs (A, B) with $\mathrm{tr}[A, B] = t$ with $\kappa^{-1}(t) \cap [-2, 2]^3$. If $t > -2$, then this set is homeomorphic to a 2-sphere; for $t = -2$, it is a single point. In either case it is connected, and by similar arguments to § 5, it can be shown that for a closed surface group π, the space $\mathrm{Hom}(\pi, SU(2))$ is connected and the space $\mathrm{Hom}(\pi, SO(3))$ has two components, distinguished by the Stiefel-Whitney map $w_2\colon \mathrm{Hom}(\pi, SO(3)) \to \mathbf{Z}/2$. For more information on these spaces of representations, the reader is referred to Newstead [33]. For an interesting application of the map κ and the relation of $SU(2)$-representations to dynamical systems, see Fried [6].

§ 8. Generic properties

In this section we discuss several properties which describe open and dense subsets of spaces of representations.

The proofs will be simplified by working in these open and dense subsets. The result we shall use is the following:

8.1. **Lemma.** *Let $G = PSL(2, \mathbf{R})$ or $PSL(2, \mathbf{C})$ and M be a compact surface of negative Euler characteristic. Suppose $W(M)$ is as in Theorem 3.3. Suppose that $M = \bigcup_{i=1}^{2g-2+b} M_i$ is a decomposition of M into Euler characteristic -1 subsurfaces. Let $W'(M)$ denote the subset of $W(M)$ consisting of representations $\phi\colon \pi_1(M) \to G$ such that for each M_i, the image of the representation*

$$\pi_1(M_i) \to \pi_1(M) \xrightarrow{\phi} G$$

is nonabelian. Then $W'(M)$ is an open dense subset of $W(M)$.

Proof. We shall show that for each M_i, the set of representations $\phi \in W(M)$ such that the image of the representation

$$\pi_1(M_i) \to \pi_1(M) \xrightarrow{\phi} G$$

is abelian is a proper closed subvariety of $\mathrm{Hom}(\pi_1(M), G)$. To this end, choose $A_i, B_i \in \pi_1(M_i) \hookrightarrow \pi_1(M)$ which generate $\pi_1(M_i)$ and consider the map C_i: $\mathrm{Hom}(\pi_1(M), G) \to G$ defined by

$$C_i(\phi) \mapsto [A_i, B_i].$$

Clearly C_i is a polynomial map and either $C_i^{-1}(I)$ is a proper closed subvariety or contains an irreducible component of $\mathrm{Hom}(\pi_1(M), G)$.

Consider first the case that M has nonempty boundary. Then $\pi_1(M)$ is a free group on $1-\chi(M)$ generators and $\mathrm{Hom}(\pi_1(M), G) \approx G^{1-\chi(M)}$. In particular $\mathrm{Hom}(\pi_1(M), G)$ is a connected smooth manifold, and therefore an irreducible variety. For each integer $n>0$, there exists a faithful representation of a free group of rank n into G; since $[A_i, B_i] \neq I$ in $\pi_1(M)$, it follows that C_i is not identically I. Thus $C_i^{-1}(I)$ is a proper closed subvariety and hence nowhere dense. The union of all $C_i^{-1}(I)$ over M_i is also a proper closed subvariety and is nowhere dense; thus the set U of all $\phi \in \mathrm{Hom}(\pi_1(M), G)$ such that $\phi(\pi_1(M_i))$ is nonabelian is open and dense. Since $W(M)$ is an open subset of $\mathrm{Hom}(\pi_1(M), G)$, it follows that $W'(M) = W(M) \cap U$ is open and dense in $W(M)$.

Next consider the case that M is a closed surface. By 4.1 the irreducible components of $\mathrm{Hom}(\pi_1(M), G)$ are the two preimages of the Stiefel-Whitney map w_2: $\mathrm{Hom}(\pi_1(M), G) \to \mathbf{Z}/2$. Thus to show that $\bigcup_{i=1}^{2g-2} C_i^{-1}(I)$ is a proper closed subvariety, it suffices to exhibit one representation in each of the two irreducible components for which $C_i \neq I$. This is accomplished as follows. For each $g>1$ there exists a Fuchsian representation ϕ which necessarily satisfies $w_2(\phi)=0$ (since $e(\phi) = \pm(2-2g)$); thus $w_2^{-1}(0)$ contains an injective representation. In particular $C_i(\phi) \neq I$. In the other component we shall take a representation which has Euler class $3-2g$; that this suffices is an immediate consequence of the following:

8.2. **Lemma.** *For every integer $g>1$ let M be a closed surface of genus g and let $M_0 \subset M$ be a subsurface with $\chi(M_0) = -1$. Then there exists a representation ϕ: $\pi_1(M) \to PSL(2, \mathbf{R})$ with $e(\phi) = 3-2g$ such that the composition*

$$\pi_1(M_0) \hookrightarrow \pi_1(M) \xrightarrow{\phi} PSL(2, \mathbf{R})$$

is injective.

Proof. Write M as the union $M_1 \cup T$ where $M_1 \supset M_0$ has Euler characteristic $3-2g$ and T has Euler characteristic -1. Choose a Fuchsian representation ϕ_1: $\pi_1(M_1) \to PSL(2, \mathbf{R})$ which corresponds to a finite-volume complete hyperbolic structure on M_1; necessarily for each boundary component $C \subset M_1$, the image of

$$\pi_1(C) \hookrightarrow \pi_1(M_1) \to PSL(2, \mathbf{R})$$

is parabolic; the Euler class of ϕ_1 with respect to the standard trivializations over ∂M_1 equals $\chi(M_1) = 3-2g$. Choose a solvable representation ϕ_T: $\pi_1(T)$

$\to PSL(2,\mathbf{R})$ such that for each boundary component C of ∂M_1, the corresponding element of $\pi_1(T)$ is mapped to a conjugate of $\phi_1(C)$. Since

$$\pi_1(M)=\pi_1(M_1)\coprod_{\pi_1(C)}\pi_1(T),$$

there exists a representation $\phi\colon \pi_1(M)\to PSL(2,\mathbf{R})$ such that ϕ restricts to ϕ_1 on $\pi_1(M_1)$ and restricts to a conjugate of ϕ_T on $\pi_1(T)$. It follows that

$$e(\phi)=e(\phi_1)+e(\phi_T)=(3-2g)+0=3-2g$$

as claimed. Since ϕ_1 is injective, its restriction to $\pi_1(M_0)$ is injective as claimed. □

The proof of Lemma 8.1 is now complete. □

Remark. The results we have stated are only as strong as they are needed for this paper. Stronger results are available concerning the density of injective representations using the Baire category theorem; such a result is the following, proved in [8]. Since there exists a faithful representation of a closed surface group π in $SL(2,\mathbf{C})$, it follows that faithful representations are dense in $\mathrm{Hom}(\pi, SL(2,\mathbf{C}))$ and $\mathrm{Hom}(\pi, SL(2,\mathbf{R}))$. We do not know, however, if there exists a faithful representation in the $w_2\neq 0$ component. If such a representation exists, then it follows that faithful representations are dense in all of $\mathrm{Hom}(\pi, PSL(2,\mathbf{C}))$ as well as in $\mathrm{Hom}(\pi, PSL(2,\mathbf{R}))$. Riley [34] and Sullivan [36] have shown that for any finitely generated group Γ, either a representation $\phi\in\mathrm{Hom}(\pi, PSL(2,\mathbf{C}))$ is in the closure of the set of non-faithful representations or is an isomorphism onto a geometrically finite discrete group containing no parabolic elements.

§ 9. Surfaces of Euler characteristic -2

In this section we prove Theorem B for surfaces of Euler characteristic -2. There are three topological types of such surfaces: the closed surface of genus two; the surface of genus one with two boundary components; the surface of genus zero with four boundary components. In addition to proving Theorem B, we shall prove Lemma 9.3, which will be an essential step in the proof for the general case. In this section G will denote $PSL(2,\mathbf{R})$ and $\tilde{G}$ will denote its universal covering $\widetilde{SL(2,\mathbf{R})}$.

We begin with the special case when M is a closed genus two surface:

9.1. **Theorem.** *Let π be the fundamental group of a surface of genus two. Then the connected components of* $\mathrm{Hom}(\pi, G)$ *are the inverse images $e^{-1}(n)$, where $n=0, \pm 1, \pm 2$ and e:* $\mathrm{Hom}(\pi, G)\to\mathbf{Z}$ *is the Euler class map.*

Proof. We identify $\mathrm{Hom}(\pi, G)$ with the set of all $(A_1, B_1, A_2, B_2)\in G^4$ satisfying

$$[A_1, B_1]=[B_2, A_2].$$

The inverse image $e^{-1}(n)$ is the set of $(A_1, B_1, A_2, B_2) \in G^4$ satisfying

$$\tilde{R}_1(A_1, B_1) = \mathbf{z}^n \tilde{R}_2(B_2, A_2).$$

Since $\mathfrak{I} \cap \mathbf{z}^n \mathfrak{I}$ is nonempty for only $n=0, \pm 1, \pm 2$, the image of e: $\mathrm{Hom}(\pi, G) \to \mathbf{Z}$ equals $\{0, \pm 1, \pm 2\}$. By 8.1, the set W' of $(A_1, B_1, A_2, B_2) \in \mathrm{Hom}(\pi, G)$ satisfying $[A_1, B_1] \neq I$ is open and dense in $\mathrm{Hom}(\pi, G)$; thus it suffices to show that for each n, $W' \cap e^{-1}(n)$ is connected. Consider the lifted commutator map $\tilde{R}_1$: $G \times G \to \tilde{G}$; since

$$\tilde{R}_1\colon \tilde{R}_1^{-1}(G - \{I\}) \to (G - \{I\})$$

is a submersion, it suffices (by 1.4 and 7.1) to show that for each $n \in \{0, \pm 1, \pm 2\}$, the set $\mathfrak{I} \cap \mathbf{z}^n \mathfrak{I}$ is connected. Since $\mathfrak{I}$ is connected, it readily follows that $e^{-1}(0)$ is connected. Since

$$\mathfrak{I} \cap \mathbf{z}\,\mathfrak{I} = \mathrm{Hyp}_0 \cup \mathrm{Par}_0^+ \cup \mathrm{Ell}_1 \cup \mathrm{Par}_1^- \cup \mathrm{Hyp}_1$$

is a connected set, $e^{-1}(1)$ is connected. Since $\mathfrak{I} \cap \mathbf{z}^2 \mathfrak{I} = \mathrm{Hyp}_1$ is connected, it follows that $e^{-1}(2)$ is connected. Since the component of $PGL(2, \mathbf{R})$ not containing the identity element interchanges $e^{-1}(n)$ and $e^{-1}(-n)$ for $n \neq 0$, it follows that $e^{-1}(-1)$ and $e^{-1}(-2)$ are likewise connected. □

9.2. **Proposition.** *Let M be a surface of genus one with two boundary components A and B. Let A and B also denote the corresponding elements of $\pi = \pi_1(M)$. Let $W(M)$ be the subset of* $\mathrm{Hom}(\pi, G)$ *consisting of representations ϕ such that $\phi(A)$ and $\phi(B)$ are hyperbolic. Let e: $W(M) \to \mathbf{Z}$ be the relative Euler class. Then the components of $W(M)$ are the preimages $e^{-1}(n)$ where $n = \pm 2, \pm 1, 0$. If $\alpha, \beta \subset G$ are two hyperbolic conjugacy classes and $W(M; \alpha, \beta)$ denotes the subset of $W(M)$ for which $\phi(A) \in \alpha$ and $\phi(B) \in \beta$, then the connected components of $W(M; \alpha, \beta)$ are the sets $e^{-1}(n) \cap W(M; \alpha, \beta)$.*

The proof will be based on the following lemma:

9.3. **Lemma.** *Let M be a surface of genus one with two boundary components and let π and A, B be as above. Let $C \subset M$ be a separating simple closed curve which represents the element (also denoted C) $AB \in \pi$. Let $P \subset M$ be the pair-of-pants bounded by A, B, C and let $T \subset M$ be the torus-minus-disc bounded by C. Suppose $\phi \in$* $\mathrm{Hom}(\pi, G)$ *is such that $\phi(\pi_1(P))$ and $\phi(\pi_1(T))$ are nonabelian. Then there exists a path $\{\phi_t\}_{0 \leq t \leq 1}$ in* $\mathrm{Hom}(\pi, G)$ *such that:*

(i) $\phi_0 = \phi$;

(ii) *$\phi_t(A)$ is conjugate to $\phi(A)$ and $\phi_t(B)$ is conjugate to $\phi(B)$ for all $0 \leq t \leq 1$;*

(iii) *$\phi_1(C)$ is hyperbolic.*

Furthermore if $\phi(A)$ and $\phi(B)$ are both hyperbolic, such a path exists such that the relative Euler class of ϕ_1 restricted to P equals ± 1.

Proof of 9.3. Choose a lift of the representation $\tilde{\phi}$: $\pi \to \tilde{G}$. Let $a = \mathrm{tr}\,\tilde{\phi}(A)$, $b = \mathrm{tr}\,\tilde{\phi}(B)$, $c = \mathrm{tr}\,\tilde{\phi}(C)$. If $\phi(C)$ is already hyperbolic, there is nothing to prove. Thus $-2 \leq c \leq 2$. For $\varepsilon > 0$, consider the path $\{(a_t, b_t, c_t)\}_{0 \leq t \leq 1}$, defined by

$$a_t = a, \qquad b_t = b, \qquad c_t = t(2+\varepsilon) + (1-t)\,c.$$

If (a_t, b_t, c_t) never meets the set

$$[-2,2]^3 \cap \kappa^{-1}((-2,2))$$

then by 4.5 (after possibly reparametrizing $\{(a_t, b_t, c_t)\}$) there exists a path $(A_t, B_t) \in \tilde{G} \times \tilde{G}$ such that

$$[A_t, B_t] \neq I, \quad A_0 = \tilde{\phi}(A), \quad B_0 = \tilde{\phi}(B) \quad \text{and} \quad \chi(A_t, B_t) = (a_t, b_t, c_t).$$

If (a_t, b_t, c_t) does meet $[-2,2]^3 \cap \kappa^{-1}((-2,2))$, then $-2 \leqq a, b \leqq 2$ and

$$c \leqq \frac{a\,b}{2} - \frac{\sqrt{(4-a^2)(4-b^2)}}{2}.$$

In that case the alternate path $\{(a_t, b_t, c_t)\}_{0 \leqq t \leqq 1}$ defined by

$$a_t = a, \quad b_t = b, \quad c_t = t(-2-\varepsilon) + (1-t)\,c$$

can be lifted to $\{(A_t, B_t)\}_{0 \leqq t \leqq 1} \subset \tilde{G} \times \tilde{G}$ such that

$$[A_t, B_t] \neq I, \quad A_0 = \tilde{\phi}(A), \quad B_0 = \tilde{\phi}(B), \quad \chi(A_t, B_t) = (a_t, b_t, c_t).$$

Since $\operatorname{tr} A_t$ and $\operatorname{tr} B_t$ are constant and A_t and B_t are never equal to $\pm I$ (otherwise A_t and B_t commute), it follows that A_t is conjugate to A_0 and B_t is conjugate to B_0 for all t. Let $C_t = A_t B_t$. Let $X, Y \in \pi$ represent nonseparating simple loops which satisfy the relation $[X, Y] = C$, and let $X_0 = \phi(X)$, $Y_0 = \phi(Y)$. Any path $\gamma_t \in \tilde{G}$ which begins in $\mathrm{Ell}_{\pm 1}$, such that γ_t is elliptic only when $\gamma_t \in \mathrm{Ell}_{\pm 1}$, lies completely within $\mathfrak{J}$. By 7.7 there exists a path $(X_t, Y_t) \in G \times G$ such that $\tilde{R}_1(X_t, Y_t) = C_t$. Since

$$\pi = \langle A, B, X, Y, C \mid AB = C = [X, Y] \rangle$$

it follows that $\{(A_t, B_t, X_t, Y_t)\}_{0 \leqq t \leqq 1}$ defines a path of representations with the desired properties.

Suppose now that $\phi(A)$ and $\phi(B)$ are both hyperbolic. If $\phi(C)$ is hyperbolic and the relative Euler class of ϕ over P is nonzero, there is nothing to prove. If $\phi(C)$ is not hyperbolic then since the line $\{a\} \times \{b\} \times \mathbf{R}$ misses $[-2,2]^3 \cap \kappa^{-1}((-2,2))$ both of the above paths $\{(a_t, b_t, c_t)\}$ can be lifted to paths in $\tilde{G} \times \tilde{G}$. In particular, c_1 may be chosen to either be positive or negative. If an even number of a, b, c_1 are negative, the corresponding representation is Fuchsian and the relative Euler class equals ± 1 as claimed. Consider finally the case that $\phi(C)$ is hyperbolic and the relative Euler class of ϕ over P is zero. Then, choosing lifts $\tilde{A}, \tilde{B}$ of $\phi(A)$, $\phi(B)$ to Hyp_0, the product $\tilde{C} = \tilde{A}\tilde{B} \in \mathrm{Hyp}_0$. In that case there is a path $\{a_t, b_t, c_t\}_{0 \leqq t \leqq 1}$ in $\mathbf{R}^3 - ([-2,2]^3 \cap \kappa^{-1}((-2,2)))$ with

$$a_t = \operatorname{tr} \tilde{\phi}(A), \quad b_t = \operatorname{tr} \tilde{\phi}(B), \quad c_0 = \operatorname{tr} \tilde{\phi}(C), \quad c_1 = -c_0.$$

By 4.5 such a path can be lifted to a path in $\tilde{G} \times \tilde{G}$. As above, a path $\{(X_t, Y_t)\}$ exists determining a path $\{\phi_t\}_{0 \leqq t \leqq 1}$ in $W(M)$. Since the relative Euler class of ϕ over P is zero, an odd number of a, b, c are positive. Thus an even number of $\operatorname{tr}\phi_1(A)$, $\operatorname{tr}\phi_1(B)$, $\operatorname{tr}\phi_1(C)$ are positive, whence the relative Euler class of ϕ_1 over P is nonzero. □

Proof of 9.2. Let $\phi, \psi \in W(M)$ satisfy $e(\phi)=e(\psi)$. We shall find a path in $W(M)$ joining ϕ and ψ. By 9.3 ϕ can be deformed to ϕ_1 such that $\phi_1(C)$ is hyperbolic and the relative Euler class of ϕ_1 over P is nonzero. Similarly deform ψ and ψ_1 so that $\psi_1(C)$ is hyperbolic and the relative Euler class of ψ_1 over P is nonzero. Let $T=M-P$ be the torus-minus-disc bounded by C. Suppose that $e(\phi|_{\pi_1(P)})=e(\psi|_{\pi_1(P)})$. Then since

$$e(\phi_1)=e(\phi)=e(\psi)=e(\psi_1)$$

and

$$e(\phi_1)=e(\phi_1|_{\pi_1(P)})+e(\phi_1|_{\pi_1(T)})\, e(\psi_1)=e(\psi_1|_{\pi_1(P)})+e(\psi_1|_{\pi_1(T)}),$$

it follows that $e(\phi_1|_{\pi_1(T)})=e(\psi_1|_{\pi_1(T)})$. That ϕ and ψ can be joined by a path in $W'(M)$ is a consequence of the following general fact:

9.4. **Proposition.** *Let M be a compact surface and let $M=\bigcup_{i=1}^{-\chi(M)} M_i$ be a maximal dual-tree decomposition. Let $W'(M)$ denote the subset of $W(M)$ consisting of representations ϕ such that for each M_i, the image of the representation*

$$\pi_1(M_i)\hookrightarrow \pi_1(M)\xrightarrow{\phi} G$$

is nonabelian. Suppose that $\phi, \psi \in W'(M)$ satisfy the condition that for each M_i the relative Euler class of ϕ over M_i equals the relative Euler class of ψ over M_i. Then there exists a path in $W'(M)$ joining ϕ to ψ.

Proof. Let M_1 be a subsurface in the decomposition such that $M'=M-M_1$ is connected. By the induction hypothesis the restrictions of ϕ and ψ to $\pi_1(M')$ lie in the same path-component of $W'(M')$. Let $C=\partial M_1 \cap \partial M'$ be the common boundary component of these two subsurfaces, and let C also denote the corresponding elements of $\pi_1(M_1)$, $\pi_1(M')$, $\pi_1(M)$. Let $\{\phi_t'\}_{0\leq t\leq 1}$ denote a path in $W'(M')$ joining $\phi|_{\pi_1(M')}$ and $\psi|_{\pi_1(M')}$. By the path-lifting properties 4.6 and 7.8, there exists a path $\{\eta_t\}_{0\leq t\leq 1}$ of representations in $W'(M_1)$ such that $\eta_t(C)=\phi_t(C)$ and $\eta_0=\phi|_{\pi_1(M_1)}$. Thus the pair of paths of representations defines a path of representations in $W'(M)$ joining ϕ and ψ. □

It remains to consider the case that the relative Euler class of ϕ on P equals the negative of the relative Euler class of ψ on P. In that case necessarily $e(\phi)=e(\psi)=0$ so that there exist lifts $\tilde{\phi}, \tilde{\psi}\colon \pi\to\tilde{G}$ with $\tilde{\phi}(A), \tilde{\phi}(B), \tilde{\psi}(A), \tilde{\psi}(B)\in \mathrm{Hyp}_0$. We shall deform ϕ_1 to a representation $\phi_2\in W(M)$ such that the relative Euler classes of ϕ_2 and ψ_1 agree on each of T, P. Suppose that the relative Euler class of ϕ_1 on P equals 1; then it follows that $\tilde{\phi}_1(C)\in \mathrm{Hyp}_1$. There exists a path $\{C_t\}_{1\leq t\leq 2}$ lying completely in $\mathfrak{Z}$ such that $C_0=\tilde{\phi}_1(C)$ and $C_1\in \mathrm{Hyp}_{-1}$. By the path-lifting properties 4.6 and 7.8 there exists a path $\{\phi_1\}_{1\leq t\leq 2}$ with $\phi_t(A), \phi_t(B)\in \mathrm{Hyp}_0$ such that $\tilde{\phi}_1(C)=C_t$. Thus we may assume that $e(\phi|_{\pi_1(P)})=e(\psi|_{\pi_1(P)})$ and $e(\phi|_{\pi_1(T)})=e(\psi|_{\pi_1(T)})$, the case which has already been treated.

For the last assertion, simply observe that in all of the above deformations, the traces of $\phi_t(A)$ and $\phi_t(B)$ can be arranged to be constant. Thus we may fix the conjugacy class of the image of each boundary component throughout the deformation.

This concludes the proof of 9.2. □

9.5. **Proposition.** *Let M be a surface of genus zero with four boundary components A, B, C, D and denote the corresponding elements of $\pi = \pi_1(M)$ also by A, B, C, D. Let $W(M)$ be the subset of $\mathrm{Hom}(\pi, G)$ consisting of representations ϕ such that*

$$\phi(A),\ \phi(B),\ \phi(C),\ \phi(D)$$

are each hyperbolic. Let $e: W(M) \to \mathbf{Z}$ be the relative Euler class. Then the components of $W(M)$ are the preimages $e^{-1}(n)$ where $n = \pm 2, \pm 1, 0$. Furthermore if $\alpha, \beta, \gamma, \delta$ are hyperbolic conjugacy classes in G, and

$$W(M; \alpha, \beta, \gamma, \delta) = \{\phi \in W(M) \mid \phi(A) \in \alpha,\ \phi(B) \in \beta,\ \phi(C) \in \gamma,\ \phi(D) \in \delta\},$$

then the components of $W(M; \alpha, \beta, \gamma, \delta)$ are the sets $e^{-1}(n) \cap W(M; \alpha, \beta, \gamma, \delta)$.

A crucial step in the proof is the following:

9.6. **Lemma.** *Let M, $\pi = \pi_1(M)$, A, B, C, D be as above and let $X \subset M$ be a simple closed curve which separates M into two pair-of-pants P_1, P_2 such that $\partial P_1 = A \cup B \cup X$ and $\partial P_2 = X \cup C \cup D$. Choose elements (also denoted A, B, C, D, X) of π corresponding to the curves A, B, C, D, X satisfying the relations $AB = X = (CD)^{-1}$. Suppose $\phi \in \mathrm{Hom}(\pi, G)$ is such that $\phi(\pi_1(P_1))$ and $\phi(\pi_1(P_2))$ are nonabelian. Suppose furthermore that $\phi(A)$ is hyperbolic. Then there exists a path $\{\phi_t\}_{0 \leqq t \leqq 1}$ in $\mathrm{Hom}(\pi, G)$ such that:*

(i) *$\phi_0 = \phi$;*

(ii) *$\phi_t(A)$ is conjugate to $\phi(A)$, $\phi_t(B)$ is conjugate to $\phi(B)$, $\phi_t(C)$ is conjugate to $\phi(C)$ and $\phi_t(D)$ is conjugate to $\phi(D)$ for all $0 \leqq t \leqq 1$;*

(iii) *$\phi_1(X)$ is hyperbolic.*

Furthermore if in addition $\phi(B)$, $\phi(C)$, $\phi(D)$ are all hyperbolic, such a path exists such that the relative Euler class of ϕ_1 restricted to P_1 equals ± 1.

Proof of 9.6. The proof is similar to that of 9.3. If $\phi(X)$ is already hyperbolic, there is nothing to prove. Lift the representation ϕ to $\tilde{\phi}: \pi \to \tilde{G}$. Let

$$a = \mathrm{tr}\,\tilde{\phi}(A), \quad b = \mathrm{tr}\,\tilde{\phi}(B), \quad c = \mathrm{tr}\,\tilde{\phi}(C), \quad d = \mathrm{tr}\,\tilde{\phi}(D), \quad x = \mathrm{tr}\,\phi(X).$$

As in the proof of 9.3, there exists a linear path $\{(c_t, d_t, x_t)\}_{0 \leqq t \leqq 1}$ such that $c_t = c$ and $d_t = d$ remain constant, $|x_1| > 2$ and

$$(c_t, d_t, x_t) \notin ([-2, 2]^3 \cap \kappa^{-1}([-2, 2])).$$

There exists a path $\{(C_t, D_t)\}_{0 \leqq t \leqq 1}$ in $\tilde{G} \times \tilde{G}$ such that $C_0 = \tilde{\phi}(C)$, $D_0 = \tilde{\phi}(D)$, $[C_t, D_t] \neq I$ and $\chi(C_t, D_t) = (c_t, d_t, x_t)$. Next consider the path $\{(a_t, b_t, x_t)\}_{0 \leqq t \leqq 1}$, where $a_t = a$ and $b_t = b$ remain constant. Since $|a| > 2$, the path completely misses

$$[-2, 2]^3 \cap \kappa^{-1}([-2, 2])$$

and therefore there exists a path $\{(A_t, B_t)\}_{0 \leq t \leq 1}$ in $\tilde{G} \times \tilde{G}$ such that

$$A_0 = \tilde{\phi}(A), \quad B_0 = \tilde{\phi}(B), \quad [A_t, B_t] \neq I, \quad \chi(A_t, B_t) = (a_t, b_t, x_t).$$

By 1.3 there exists a path $\{U_t\}_{0 \leq t \leq 1}$ in $\tilde{G}$ such that $A_t B_t = U_t X_t U_t^{-1}$. The desired path in $\mathrm{Hom}(\pi, \tilde{G})$ is given by:

$$\phi_t(A) = A_t, \quad \phi_t(B) = B_t, \quad \phi_t(C) = U_t C_t U_t^{-1}, \quad \phi_t(D) = U_t D_t U_t^{-1}.$$

Now suppose that $\phi(B)$, $\phi(C)$, $\phi(D)$ are all hyperbolic. By lifting a path (as above) in $\{a\} \times \{b\} \times \mathbf{R}$ and a corresponding path in $\{c\} \times \{d\} \times \mathbf{R}$ we see that $\operatorname{tr} \tilde{\phi}_1(X)$ can be made to be greater than 2 as well as less than -2. Thus we may arrange that an odd number of $\operatorname{tr} \tilde{\phi}_1(A)$, $\operatorname{tr} \tilde{\phi}_1(B)$, $\operatorname{tr} \tilde{\phi}_1(C)$ are negative, whence the relative Euler class of ϕ_1 over P_1 equals ± 1. □

Proof of 9.5. By 8.1, $W'(M)$ is open and dense in $W(M)$. It suffices to show that if $\phi, \psi \in W'(M)$ satisfy $e(\phi) = e(\psi)$, then there exists a path in $W'(M)$ joining them. The proof follows the exact same lines as that of 9.2 and will be abbreviated: First deform ϕ and ψ so that X becomes hyperbolic; if the relative Euler classes of ϕ and ψ on P_1 and P_2 agree, then by 9.4 the representations lie in the same path-component of $W'(M)$. Otherwise a similar deformation argument to that of 9.2 constructs a path ending at a representation where the relative Euler classes all agree. All of the deformations involved can be arranged so that the trace of each boundary component remains constant, whence the last claim follows. □

9.7. *Remark.* Lemma 9.6 may fail if none of $\phi(A)$, $\phi(B)$, $\phi(C)$, $\phi(D)$ are hyperbolic. Indeed, for some choices of elliptic conjugacy classes $\alpha, \beta, \gamma, \delta$ in G, the space of conjugacy classes of quadruples $(A, B, C, D) \in G^4$ such that $ABCD = I$ and $A \in \alpha$, $B \in \beta$, $C \in \gamma$, $D \in \delta$ has a compact component in which AB will never be hyperbolic. Such a case arises, for example, when the traces $a = \operatorname{tr}(A)$, $b = \operatorname{tr}(B)$, $c = \operatorname{tr}(C)$, $d = \operatorname{tr}(D)$ satisfy the inequality

$$ab + \sqrt{(4-a^2)(4-b^2)} < cd - \sqrt{(4-c^2)(4-d^2)}.$$

§ 10. The general case

Using the results of the previous sections, we conclude the proofs of Theorems 3.3, 3.4, and Theorem A. In this section G will denote $PSL(2, \mathbf{R})$, $\tilde{G}$ will denote its universal cover, M will denote a compact surface of genus g with b boundary components such that $\chi(M) = 2 - 2g + b < -2$ and $\pi = \pi_1(M)$. As usual $W(M)$ denotes the space of homomorphisms $\phi \in \mathrm{Hom}(\pi, G)$ such that for each boundary component $C \subset \partial M$ the image $\phi(C)$ is hyperbolic where C also denotes (as usual) a corresponding generator of $\pi_1(C)\hookrightarrow\pi$. Let e: $W(M)\to\mathbf{Z}$ denote the relative Euler class map. Choose a maximal dual-tree decomposition (as in 3.8)

$$M=M_1\cup\ldots\cup M_{-\chi(M)}.$$

Let $W'(M)$ denote the subset of $W(M)$ consisting of representations which, for each M_i restrict to a nonabelian representation on $\pi_1(M_i)\hookrightarrow\pi_1(M)$. By 8.1 $W'(M)$ is open and dense in $W(M)$; hence to prove that $e^{-1}(n)$ is connected it suffices to show that $e^{-1}(n)\cap W'(M)$ is connected. Our first goal is to prove the generalization of 9.3 and 9.5 to all surfaces:

10.1. **Lemma.** *Let $\phi\in W'(M)$. Then there exists a path in $W'(M)$ from ϕ to $\phi'\in W'(M)$ such that for each M_i, the restriction of ϕ' to each component $C\in\partial M_i$ is hyperbolic.*

Proof. We prove this result by induction on $\chi(M)$; the case $\chi(M)=-1,-2$ has already been treated in § 9. After possibly reindexing the M_i, assume that M_1, M_2 are subsurfaces in the given decomposition satisfying the following properties:

(1) If $\partial M\neq\emptyset$, then ∂M_1 contains at least one component of ∂M;

(2) $M_1\cup M_2$ is a connected subsurface M' of M.

The subsurface $\bigcup_{i>2} M_i$ has either one or two connected components N_j. Let C_j denote the curve $\partial M_2\cap\partial N_j$ and let $C'=\partial M_1\cap\partial M_2$. Then ϕ restricted to $\pi_1(M')$ satisfies the hypotheses of 9.3 or 9.6, so there exists a path $\{\phi_t\}_{0\leq t\leq 1}$ in $W'(M')$ from $\phi_0=\phi|_{\pi_1(M')}$ to a representation ϕ_1 in $W'(M')$ such that each $\phi_1(C')$ is hyperbolic and for given C_j, $\phi_t(C_j)$ remains conjugate to $\phi(C_j)$ for all t. Let $\{U_t^{(j)}\}_{0\leq t\leq 1}$ be a path such that

$$\phi_t(C)=U_t^{(j)}\,\phi_t(C)\,(U_t^{(j)})^{-1}.$$

Extend ϕ_t from $\pi_1(M')$ to $\pi_1(M)$ by requiring that

$$\phi_t(\gamma)=U_t^{(j)}\,\phi(\gamma)\,(U_t^{(j)})^{-1}$$

for $\gamma\in\pi_1(N_j)$. This defines a path $\{\phi_t\}_{0\leq t\leq 1}$ of representations in $W'(M)$ such that $\phi_1(C')$ is hyperbolic. Thus the restriction of ϕ to $\pi_1(\bigcup_{i>1} M_i)$ lies in $W'(\bigcup_{i>1} M_i)$. By the induction hypothesis, there exists a path in $W'(\bigcup_{i>1} M_i)$ from this representation to one which maps each component of ∂M_i (where $i>1$) to a hyperbolic element. Furthermore by the path-lifting property 4.6 for representations of fundamental groups of Euler characteristic -1 surfaces, this path may be extended to a path in

$$(W'(M_1)\times W'(\bigcup_{i>1} M_i))\cap W'(M).$$

Such a path now has the desired properties. □

10.2. Suppose that $\phi, \psi \in W'(M)$. By 10.1 both ϕ, ψ may be deformed to representations which lie in $\prod_{i=1}^{-\chi(M)} W'(M_i)$. Let

$$\tilde{e}: \prod_{i=1}^{-\chi(M)} W'(M_i) \to \mathbf{Z}^{-\chi(M)}$$

be given by

$$\tilde{e}(\sigma_1, \ldots, \sigma_{-\chi(M)}) = (e(\sigma_1), \ldots, e(\sigma_{-\chi(M)})).$$

According to 9.4, if $\tilde{e}(\phi) = \tilde{e}(\psi)$ the representations ϕ and ψ lie in the same path-component of $W'(M)$. Next suppose that M_1 and M_2 are two adjacent subsurfaces and that the relative Euler classes of ϕ and ψ restricted to $M_1 \cup M_2$ are equal. By 9.2 and 9.5, there exists a path joining $\phi|_{\pi_1(M_1 \cup M_2)}$ and $\psi|_{\pi_1(M_1 \cup M_2)}$ such that the restriction to the fundamental group of the boundary components of $M_1 \cup M_2$ stay in a fixed conjugacy class. It follows that this path extends to a path of representations in $W'(M)$ joining ϕ and ψ. Thus two such representations lie in the same component of $W'(M)$.

Theorem 3.3 now follows immediately from the preceding remarks and Lemma 3.9 by taking T to be the tree dual to the decomposition of M, $f = \tilde{e}(\phi)$ and $f' = \tilde{e}(\psi)$.

10.3. *Proof of 3.4*. The fact that if $\phi: \pi \to G$ is a holonomy representation for M then $e(\phi) = \pm\chi(M)$ has been proved in 3.5. Thus it remains to show that if $e(\phi) = \pm\chi(M)$ then ϕ is a holonomy representation for M.

We begin by assuming that M is a closed surface. In that case ϕ is a holonomy representation for M if and only if ϕ is Fuchsian, i.e., if it is discrete and faithful; necessarily the image of such a ϕ is a cocompact discrete subgroup of G. It follows from Weil [39] that the set of such representations is open inside $\mathrm{Hom}(\pi, G)$. On the other hand, it follows from Chuckrow [1] (see also [32, 18]) that this set is also closed inside $\mathrm{Hom}(\pi, G)$. Thus the set of holonomy representations is a union of connected components of $\mathrm{Hom}(\pi, G)$. By 3.5 this set is contained in the union $e^{-1}(\chi(M)) \cup e^{-1}(-\chi(M))$. Since there exist Fuchsian representations in both $e^{-1}(\chi(M))$ and $e^{-1}(-\chi(M))$, it follows that the set of Fuchsian representations equals $e^{-1}(\chi(M)) \cup e^{-1}(-\chi(M))$.

We can reduce the case of a surface with boundary to the closed case by a doubling construction. This is most succinctly stated in the language of orbifolds (see e.g. [19] or [36]). Suppose that M is a surface with boundary and that $\phi \in W(M)$. Let $\bar{M}$ denote the orbifold whose underlying topological space is M with singular set ∂M and whose fundamental group is generated by $\pi_1(M)$ together with one reflection ρ_C for each boundary component $C \subset \partial M$ and satisfying the relations that $\rho_C^2 = I$ and $\rho_C C \rho_C = C$ (where, as usual, we abuse notation by letting C also denote a generator of the fundamental group of the boundary component C). This orbifold $\bar{M}$ has a double covering which is the double $2M$ of M; thus $\bar{M}$ is the quotient orbifold of $2M$ by the involution $\imath$ which fixes ∂M with fundamental domain M.

Now suppose that $\phi \in W(M)$ satisfies $e(\phi) = \pm\chi(M)$. Let $\mathbf{H}_\phi^2$ denote the flat $\mathbf{H}^2$-bundle over M with holonomy ϕ. By the above construction, there exists a flat $\mathbf{H}^2$-bundle $\bar{\mathbf{H}}^2$ over the orbifold $\bar{M}$. By passing to a double covering one obtains a flat $\mathbf{H}^2$-bundle $2(\mathbf{H}_M^2)$ over $2M$ which restricts to $\mathbf{H}_M^2$ over M and to $\iota^*(\mathbf{H}_M^2)$ over $\iota(M)$. Let $2\phi\colon \pi_1(2M)\to G$ denote the holonomy representation of $2(\mathbf{H}_M^2)$. By the additivity property 3.7 we have

$$e(2\phi)=e(\phi)+e(\iota_*\circ\phi\circ\rho_{\phi(C)})=2e(\phi)=\pm 2\chi(M)=\pm\chi(2M)$$

(where C is any boundary component of M). By the closed case of 3.4, 2ϕ is a holonomy representation for $2M$, whose restriction ϕ to $\pi_1(M)$ is necessarily a holonomy representation for M. The proof of 3.4 is now complete. □

10.4. *Proof of Theorem A(i).* We must show that if $G^{(n)}$ denotes the n-fold covering group of G, then the number of connected components of $\mathrm{Hom}(\pi, G^{(n)})$ is given by the formula

$$\begin{array}{ll} 2n^{2g}+(4g-4)/n-1 & \text{if } n \mid 2g-2 \\ 2\left[\dfrac{(2g-2)}{n}\right]+1 & \text{if } n \nmid 2g-2. \end{array}$$

The proof will be based on the following lemma. Let $f\colon G^{(n)}\to G$ denote the covering projection and let $f_*\colon \mathrm{Hom}(\pi, G^{(n)})\to\mathrm{Hom}(\pi, G)$ denote the induced map (as in 2.2). Recall that a Fuchsian representation is a homomorphism ϕ: $\pi\to G$ which maps π isomorphically onto a discrete (necessarily cocompact) subgroup of G. Theorem 3.4 asserts, when π is the fundamental group of a closed surface that the Fuchsian representations comprise the two connected components $e^{-1}(\pm(2g-2))$ of $\mathrm{Hom}(\pi, G)$.

10.5. **Lemma.** *Suppose that* $\phi, \phi'\in\mathrm{Hom}(\pi, G^{(n)})$ *are such that* $f_*(\phi)=f_*(\phi')$ *is not a Fuchsian representation* $\pi\to G$. *Then* ϕ *and* ϕ' *lie in the same connected component of* $\mathrm{Hom}(\pi, G^{(n)})$.

Assuming 10.5, the proof of Theorem A(i) proceeds as follows. As the obstruction to lifting a representation $\phi\colon\pi\to G$ to $\pi\to G^{(n)}$ is the reduction modulo n of the Euler class, the image of f_*: $\mathrm{Hom}(\pi, G^{(n)})\to\mathrm{Hom}(\pi, G)$ consists of all $\phi\in\mathrm{Hom}(\pi, G^{(n)})$ such that $n\mid e(\phi)$. Let $k\in\mathbf{Z}$, $|nk|<2g-2$; then $e^{-1}(nk)$ is a connected component of $\mathrm{Hom}(\pi, G)$. By 2.2, f_*: $f_*^{-1}(e^{-1}(nk))\to e^{-1}(nk)$ is a covering map and it follows by 10.5 that $f_*^{-1}(e^{-1}(nk))$ is a connected component of $\mathrm{Hom}(\pi, G^{(n)})$.

If $n\nmid 2g-2$ then every component of $\mathrm{Hom}(\pi, G^{(n)})$ is of this form; thus corresponding to the $2\left[\dfrac{2g-2}{n}\right]+1$ integers k such that $|nk|\leqq 2g-2$ are the $2\left[\dfrac{2g-2}{n}\right]+1$ components of $\mathrm{Hom}(\pi, G^{(n)})$. Suppose next that $n\mid 2g-2$; then there are $(4g-4)/n-1$ connected components of representations which map under f_* to non-Fuchsian representations. By 3.4, there are two components of Fuchsian representations in $\mathrm{Hom}(\pi, G)$, namely $e^{-1}(2-2g)$ and $e^{-1}(2g-2)$. Let $\mathscr{C}$ denote one of these components. By 2.2, the map $f_*\colon f_*^{-1}(\mathscr{C})\to\mathscr{C}$ is a covering space with covering group $\mathrm{Hom}(\pi, \mathrm{Ker}\, f)\cong H^1(S; \mathbf{Z}/n)$. Furthermore the action of $\mathrm{Hom}(\pi, \mathrm{Ker}\, f)$ on $\mathrm{Hom}(\pi, G^{(n)})$ commutes with the action of $G^{(n)}$ by conjugation and hence f_* induces a covering space $f_*^{-1}(\mathscr{C})/G^{(n)}\to\mathscr{C}/G$ with covering group $\mathrm{Hom}(\pi, \mathrm{Ker}\, f)$. Since $\mathscr{C}/G$ is the Teichmüller space of S which is simply connected, $\mathrm{Hom}(\pi, \mathrm{Ker}\, f)$ acts simply transitively on the components of $f_*^{-1}(\mathscr{C})/G$ and hence also on the components of $f_*^{-1}(\mathscr{C})$. Since $H^1(S; \mathbf{Z}/n)$ has n^{2g} elements, it follows that $f_*^{-1}(\mathscr{C})$ has n^{2g} components. Since there are two such components $\mathscr{C}$ consisting of Fuchsian representations, there are $2n^{2g}$ components of $\mathrm{Hom}(\pi, G^{(n)})$ consisting of Fuchsian representations. As there are $(4g-4)/n-1$ components of non-Fuchsian representations, there is a total of $2n^{2g}+(4g-4)/n-1$ components of $\mathrm{Hom}(\pi, G^{(n)})$. □

Proof of 10.5. Since $f_*\colon \mathrm{Hom}(\pi, G^{(n)})\to\mathrm{Hom}(\pi, G)$ is a covering space onto its image whose covering group is $\mathrm{Hom}(\pi, \mathrm{Ker}\, f)$, it suffices to show that for a fixed generating set Λ for $\mathrm{Hom}(\pi, \mathrm{Ker}\, f)$, there exists a path in $\mathrm{Hom}(\pi, G^{(n)})$ joining ϕ and $\gamma\phi$ for each $\gamma\in\Lambda$. We denote by $\mathbf{z}$ a generator of the center of $G^{(n)}$. Let $A_1, B_1, \ldots, A_g, B_g$ denote a standard set of generators for π; then there is a generating set Λ consisting of $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ such that

$$\alpha_i(A_j)=\mathbf{z}^{\delta_{ij}}, \qquad \alpha_i(B_j)=I$$

$$\beta_i(B_j)=I, \qquad \beta_i(B_j)=\mathbf{z}^{\delta_{ij}}.$$

To join ϕ and $\gamma\phi$ where $\gamma=\alpha_1$, for example, we proceed as follows. By [12], p. 104, there exists a path σ in $\mathrm{Hom}(\pi, G)$ joining $f_*\phi$ to ψ, where $\psi(\beta_1)$ is elliptic. As on [12], p. 104, there is a path $\{\psi_t\}_{0\leqq t\leqq 1}$ of representations in $\mathrm{Hom}(\pi, G^{(n)})$ joining $\psi=\psi_0$ to $\gamma\psi=\psi_1$ defined by

$$\psi_t(\alpha_1)=\psi(\alpha_1)\,\zeta_t$$

$$\psi_t(\alpha_i)=\psi(\alpha_i) \qquad \text{if } i>1$$

$$\psi_t(\beta_j)=\psi(\beta_j) \qquad \text{if } j\geqq 1$$

where $\{\zeta_t\}_{0\leqq t\leqq 1}$ is the elliptic one-parameter subgroup in $G^{(n)}$ centralizing $\psi(\beta_1)$, normalized so that $\zeta_0=I$ and $\zeta_1=\mathbf{z}$. Then $\psi_1=\gamma\psi$ as claimed and the composition of paths $\sigma*\{\psi_t\}*\sigma^{-1}$ is a path joining ϕ to $\gamma\phi$ as desired. Similar paths can be constructed joining ϕ to $\gamma\phi$ for the other $\gamma\in\Lambda$. □

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