# ISOSPECTRALITY OF FLAT LORENTZ 3-MANIFOLDS

TODD A. DRUMM AND WILLIAM M. GOLDMAN

ABSTRACT. Complete flat Lorentz 3-manifolds with nonamenable fundamental group bear a striking resemblance to hyperbolic Riemann surfaces. For example, every nonparabolic closed curve is freely homotopic to a unique closed geodesic, which is necessarily spacelike. In his seminal papers on the subject, Margulis introduced a function $\alpha : \pi_1(M) \longrightarrow \mathbb{R}$ which associates the signed Lorentzian length of this geodesic to a conjugacy class in $\pi_1(M)$. In this paper we show that the conjugacy class of the linear holonomy representation $\pi_1(M) \longrightarrow \mathrm{SO}(2,1)$ and Margulis's invariant completely determine $M$ up to isometry.

## 1. INTRODUCTION

In this paper we consider actions of groups of isometries of Minkowski $2+1$-space $\mathbb{E}$. Minkowski space is a complete simply-connected flat Lorentzian manifold, which identifies with an affine space whose underlying vector space is a 3-dimensional real vector space $\mathbb{R}^{2,1}$ with a nondegenerate symmetric bilinear form of index 1. Explicitly we take $\mathbb{R}^{2,1}$ to be $\mathbb{R}^3$ with inner product:

$$\mathbb{B}(\mathsf{x},\mathsf{y}) := x_1y_1 + x_2y_2 - x_3y_3$$

so that $\mathbb{E}$ identifies with $\mathbb{R}^3$ with Lorentzian metric tensor

$$(dx_1)^2 + (dx_2)^2 - (dx_3)^2.$$

The automorphism group of $\mathbb{R}^{2,1}$ is the orthogonal group $\mathrm{O}(2,1)$ consisting of linear isometries of $\mathbb{E}$. In general, an isometry of $\mathbb{E}$ is an affine transformation

$$\begin{aligned} h : \mathbb{E} &\longrightarrow \mathbb{E} \\ x &\longmapsto g(x) + u \end{aligned}$$

where the linear part $g = \mathbb{L}(h) \in \mathrm{O}(2,1)$ is a linear isometry. The intersection $\mathrm{SO}(2,1) = \mathrm{O}(2,1) \cap \mathrm{SL}(3,\mathbb{R})$ consists of orientation-preserving linear isometries. The *nullcone*

$$\mathfrak{N} := \{\mathsf{x} \in \mathbb{R}^{2,1} \mid \mathbb{B}(\mathsf{x},\mathsf{x}) = 0\}$$

is invariant under $\mathrm{O}(2,1)$. The complement $\mathfrak{N} - \{0\}$ consists of two components (the *future* and the *past*)

$$\mathfrak{N}_+ := \{\mathsf{x} \in \mathfrak{N} \mid x_3 > 0\}, \mathfrak{N}_- := \{\mathsf{x} \in \mathfrak{N} \mid x_3 < 0\}.$$

The subgroup $\mathrm{SO}(2,1)^0$ of $\mathrm{SO}(2,1)$ stabilizing either $\mathfrak{N}_+$ or $\mathfrak{N}_-$ is the identity component of the Lie group $\mathrm{O}(2,1)$. The group $\mathrm{Isom}(\mathbb{E})$ of affine isometries of $\mathbb{E}$ equals the semidirect product $\mathrm{O}(2,1) \ltimes \mathbb{R}^{2,1}$ and the quotient projection

$$\mathbb{L} : \mathrm{Isom}(\mathbb{E}) \longrightarrow \mathrm{O}(2,1))$$

assigns to an affine isometry $h \in \mathbb{L} : \mathrm{Isom}(\mathbb{E})$ its linear part $g = \mathbb{L}(h) \in \mathrm{O}(2,1)$:

$$h(x) = g(x) + u$$

where $u \in \mathbb{R}^{2,1}$ is the translational part of $h$.

An element of $\mathrm{O}(2,1)$ is *hyperbolic* if and only if it has three distinct real eigenvalues. Since an isometry's eigenvalues occur in reciprocal pairs, a hyperbolic element of $\mathrm{SO}(2,1)$ must have 1 as an eigenvalue. If $g \in \mathrm{SO}(2,1)^0$ is hyperbolic, then the other two eigenvalues are necessarily positive. Margulis associated to a hyperbolic element $g \in \mathrm{SO}(2,1)^0$ a canonical basis as follows. Let the eigenvalues of $g$ be $\lambda^{-1} < 1 < \lambda$. Then there exist unique eigenvectors $\mathsf{x}^-(g), \mathsf{x}^0(g), \mathsf{x}^+(g)$ such that

- $g\mathsf{x}^\pm(g) = \lambda^{\pm 1}\mathsf{x}^\pm(g)$ and $g\mathsf{x}^0(g) = \mathsf{x}^0(g)$;
- $\mathsf{x}^\pm(g) \in \mathfrak{N}_+$ and $\|\mathsf{x}^\pm(g)\| = 1$;
- $(\mathsf{x}^-(g), \mathsf{x}^0(g), \mathsf{x}^+(g))$ is a right handed basis for $\mathbb{R}^{2,1}$.

Since $\mathsf{x}^0(g)$ is fixed under the orthogonal linear transformation $g$,

$$\mathbb{B}(gu - u, \mathsf{x}^0(g)) = 0 \tag{1}$$

for all $u \in \mathbb{R}^{2,1}$.

An affine isometry $h$ of $E$ is called *hyperbolic* if its linear part $g = \mathbb{L}(h)$ is hyperbolic.

## 2. The Margulis invariant of hyperbolic affine isometries

Suppose that $h \in \mathrm{Isom}^0(\mathbb{E})$ is a hyperbolic affine isometry. Following Margulis, define

$$\alpha(h; x) = \mathbb{B}(hx - x, \mathsf{x}^0(g)) \tag{2}$$

for any $x \in \mathbb{E}$. For any $y \in \mathbb{E}$, let $u = y - x$. Then (1) implies

$$\alpha(h; x) - \alpha(h; y) = \mathbb{B}\big((g - \mathbb{I})u, \mathsf{x}^0(g)\big) = 0$$

so that $\alpha(h; x) = \alpha(h)$ is independent of $x$. The foliation of $\mathbb{E}$ by lines parallel to $\mathsf{x}^0(g)$ is invariant under $h$ and therefore there is an induced affine transformation $h'$ on the leaf space $\mathbb{E}' = \mathbb{E}/\mathsf{x}^0(g)$. Since the linear part $g'$ has no fixed vectors, $h'$ has a unique fixed point in $\mathbb{E}'$. Therefore $h$ leaves invariant a unique line $C_h$ parallel to $\mathsf{x}^0(g)$.

The restriction of $h$ to $C_h$ is translation $\tau$ by $\alpha(h)\mathsf{x}^0(g)$. In particular $\alpha(h) = 0$ if and only if $h$ fixes a point $x \in \mathbb{E}$. In this case the set of fixed points is exactly the line $C_h$. In general the planes parallel to the orthogonal complement $\mathsf{x}^0(g)^\perp$ (which is spanned by $\mathsf{x}^\pm(g)$) define a foliation whose leaf space identifies to $C_h$ under the quotient map $\Pi : \mathbb{E} \longrightarrow C_h$. The diagram

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\;h\;} & \mathbb{E} \\ {\scriptstyle\Pi}\big\downarrow & & \big\downarrow{\scriptstyle\Pi} \\ C_h & \xrightarrow[\tau]{} & C_h \end{array}$$

commutes. Suppose that $\langle h \rangle$ acts freely on $\mathbb{E}$. In this case, the projection $\mathbb{E} \longrightarrow C_h$ is equivariant and $C_h$ projects to the unique closed geodesic in $\mathbb{E}/\langle h \rangle$. Because $\mathsf{x}^0(g)$ has unit (Lorentzian) length, $|\alpha(h)|$ equals as the *Lorentzian length of the unique closed geodesic in* $\mathbb{E}/\langle h \rangle$. Let $\Gamma_0$ be a subgroup of $\mathrm{SO}(2,1)^0$. An affine deformation of $\Gamma_0$ is a representation

$$\phi : \Gamma_0 \longrightarrow \mathrm{Isom}(\mathbb{E}) \cong \mathrm{SO}(2,1)^0 \ltimes \mathbb{R}^{2,1}$$

such that $\mathbb{L} \circ \phi$ is the identity map of $\Gamma_0$. For $\gamma \in \Gamma_0$, write

$$\phi(\gamma)(x) = \mathbb{L}(\gamma)x + u(\gamma)$$

where $\mathbb{L}(\gamma) \in \Gamma_0$ and $u(\gamma) \in \mathbb{R}^{2,1}$. (When there is no danger of confusion, the symbol $\phi$ will be omitted.) Then $u$ is a cocycle of $\Gamma_0$ with coefficients in the $\Gamma_0$-module $\mathbb{R}^{2,1}$ corresponding to the linear action of $\mathbb{L} : \Gamma_0 \longrightarrow \mathrm{SO}(2,1)^0$. In this way affine deformations of $\Gamma_0$ correspond to cocycles in $Z^1(\Gamma_0, \mathbb{R}^{2,1})$ and translational conjugacy classes of affine deformations correspond to cohomology classes in $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

**Lemma 1.** *$\alpha$ is a class function on $\pi$.*

*Proof.* Let $\gamma, \eta \in \pi$. Then $\mathsf{x}^0(\eta\gamma\eta^{-1}) = \mathbb{L}(\eta)\mathsf{x}^0(\gamma)$ and

$$u(\eta\gamma\eta^{-1}) = \mathbb{L}(\eta)u(\gamma) + \big(I - \mathbb{L}(\eta\gamma\eta^{-1})\big)u(\eta).$$

Therefore

$$\begin{aligned} \alpha(\eta\gamma\eta^{-1}) &= \mathbb{B}\big(u(\eta\gamma\eta^{-1}), \mathsf{x}^0(\eta\gamma\eta^{-1})\big) \\ &= \mathbb{B}(\mathbb{L}(\eta)u(\gamma), \mathbb{L}(\eta)\mathsf{x}^0(\gamma)) + \mathbb{B}\Big(\big(I - \mathbb{L}(\eta\gamma\eta^{-1})\big)u(\eta), \mathbb{L}(\eta)\mathsf{x}^0(\gamma)\Big) \\ &= \mathbb{B}(u(\gamma), \mathsf{x}^0(\gamma)) = \alpha(\gamma) \end{aligned}$$

by (1). $\square$

## 3. Radiance

Margulis's invariant can be interpreted homologically. Each element $\gamma \in \Gamma$ defines a homomorphism

$$\begin{aligned} i_\gamma : \mathbb{Z} &\longrightarrow \Gamma \\ n &\longmapsto \gamma^n \end{aligned}$$

which induces

$$i_\gamma^* : H^1(\Gamma_0, \mathbb{R}^{2,1}) \longrightarrow H^1(\mathbb{Z}, \mathbb{R}^{2,1}).$$

Inner product with $\mathsf{x}^0(\gamma)$

$$\begin{aligned} \mathbb{B}(., \mathsf{x}^0(\gamma)) : \mathbb{R}^{2,1} &\longrightarrow \mathbb{R} \\ v &\longmapsto \mathbb{B}(v, \mathsf{x}^0(\gamma)) \end{aligned}$$

is a homomorphism of $\mathbb{Z}$-modules inducing an isomorphism

$$\mathbb{B}(., \mathsf{x}^0(\gamma))_* : H^1(\mathbb{Z}, \mathbb{R}^{2,1}) \longrightarrow H^1(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}.$$

The composition

$$H^1(\Gamma, \mathbb{R}^{2,1}) \longrightarrow H^1(\mathbb{Z}, \mathbb{R}^{2,1}) \longrightarrow H^1(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}.$$

maps the cohomology class $[u] \in H^1(\Gamma, \mathbb{R}^{2,1})$ to $\alpha(\gamma)$.

## 4. Main theorem

The purpose of this note is to prove:

**Theorem 1.** *Suppose that $\Gamma_0$ is a discrete subgroup of $\mathrm{SO}(2,1)^0$ freely generated by $g_1, g_2$. Suppose that $u, v \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$ define affine deformations with $\alpha(u) = \alpha(v)$. Then $[u] = [v]$.*

Thus the classification of affine deformations reduces from $\mathbb{R}^{2,1}$-valued cohomology classes $[u]$ of $\Gamma$ to ordinary $\mathbb{R}$-valued class functions $\alpha(u)$ on $\Gamma$. The invariant $\alpha(u)$ depends linearly on $u$. Therefore it suffices to show that the cohomology class $[u] \in H^1(\Gamma, \mathbb{R}^{2,1})$ correponding to an affine deformation $\Gamma_u$ with $\alpha_u = 0$ must vanish. In this case we say that $\Gamma_u$ is *radiant,* that is, there exists a point $x \in \mathbb{E}$ fixed by $\Gamma$. (The terminology arises since an affine transformation is radiant if and only if it preserves a radiant vector field

$$\sum_{i=1}^{n} (x_i - p_i) \frac{\partial}{\partial x_i}$$

"radiating" from $p \in \mathbb{E}$.) We shall in fact show a much stronger statement:

**Lemma 2.** *Let* $h_1, h_2 \in \mathrm{Isom}^0(\mathbb{E})$ *be hyperbolic whose linear parts* $g_1, g_2$ *generate a nonsolvable subgroup* $\Gamma_0$ *of* $\mathrm{SO}(2,1)^0$*. Suppose that* $h_1, h_2$ *and their product* $h_2h_1$ *are radiant. Then* $\Gamma = \langle h_1, h_2\rangle$ *is radiant.*

An alternative statement is that if $\alpha(h_1) = \alpha(h_2) = \alpha(h_2h_1) = 0$, then $\alpha(w(h_1, h_2)) = 0$ for any word $w \in \mathbb{F}_2$.

*Proof.* Since $h_1, h_2$ are radiant, their invariant lines consist of their respective fixed points. For hyperbolic $h \in \mathrm{Isom}^0(\mathbb{E})$, let $E^{\pm}(h)$ denote the affine subspace containing $C_h$ and parallel to the linear subspace spanned by $\mathsf{x}^{\pm}(h)$ and $\mathsf{x}^{o}(h)$. Since $h_1$ and $h_2$ are assumed to be transversal and hyperbolic, the four vectors $\{\mathsf{x}^{\pm}(h_1), \mathsf{x}^{\pm}(h_1)\}$ are all distinct. Since the line $C_{h_1}$ is transverse to the plane $E^+(h_2)$, they intersect at a point $q$. Furthermore since $h_1$ and $h_2$ share no fixed points, $q \notin C_{h_2}$. Since $q \in E^+(h_2) - C_{h_2}$, there exists $c \neq 0$ such that

$$h_2(q) - q = c\mathsf{x}^+(g_2).$$

Since $g_2g_1$ and $g_2$ share no eigenspaces, $\mathbb{B}(\mathsf{x}^+(g_2), \mathsf{x}^0(g_2g_1)) \neq 0$. Therefore:

$$\begin{aligned}
\alpha(h_2h_1) &= \mathbb{B}(h_2h_1(q) - q, \mathsf{x}^0(g_2g_1)) \\
&= \mathbb{B}(h_2(q) - q, \mathsf{x}^0(g_2g_1)) \\
&= c\mathbb{B}(\mathsf{x}^+(g_2), \mathsf{x}^0(g_2g_1)) \neq 0
\end{aligned}$$

as desired. ☐

The converse is not true: If $g_1, g_2$ are hyperbolic linear isometries which share a null eigenvector, then it is easy to construct a nonradiant affine deformation such that $\alpha(h_1) = \alpha(h_2) = \alpha(h_1h_2) = 0$. For example, choose $p_1, p_2 \neq 0$ and

$$g_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh(p_i) & \sinh(p_i) \\ 0 & \sinh(p_i) & \cosh(p_i) \end{bmatrix}$$

for $i = 1, 2$, and translational parts

$$u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

It can be shown that $\alpha(\gamma) = 0$ for any $\gamma \in \langle h_1, h_2\rangle$. However, the line $l = \{(t, 0, 0) | t \in \mathbb{R}\}$ is the fixed point set for $g_1$ and $g_2$, but $C_{h_1} = l$ and $C_{h_2} = (e^{p_2} - 1)^{-1}(u_2) + l$. Since $C_{h_1} \cap C_{h_2} = \emptyset$, the group $\langle h_1, h_2\rangle$ is nonradiant.

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Swarthmore College, Swarthmore, PA 19081
*E-mail address*: tad@swarthmore.edu

University of Maryland, College Park, MD 20742
*E-mail address*: wmg@math.umd.edu