Write legibly and show all work. No partial credit can be given for an unjustified, incorrect answer. Put your name in the top right corner and sign the honor pledge at the end of the exam. If you need more room than what’s given, please continue onto the back.

1. (a) (Five points) Let \( A \) be an \( m \times n \) matrix. Define the null space of \( A \). What vector space is the null space naturally a subspace of?

   **Solution.** The null space is the set of vectors \( \vec{x} \in \mathbb{R}^n \) (the domain of the transformation \( \vec{x} \mapsto A\vec{x} \)) satisfying \( A\vec{x} = \vec{0} \). It is a subspace of \( \mathbb{R}^n \).

(b) (Ten points) Let

\[
A = \begin{bmatrix}
1 & 0 & 0 & -1 \\
2 & -1 & -2 & -7 \\
3 & -1 & -2 & 4 \\
\end{bmatrix}
\]

Find bases for the null space of \( A \) and the column space of \( A \).

   **Solution.** The row-reduced echelon form row-equivalent to \( A \) is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

From this we can see that the first, second, and fourth columns of \( A \) are the pivot columns. Therefore a basis of the column space is

\[
\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -7 \\ 4 \end{bmatrix} \right\}
\]

From the rref we see that \( A\vec{x} = \vec{0} \) has the same solutions as the system of equations \( x_1 = 0, x_2 + 2x_3 = 0, \) and \( x_4 = 0, \) with \( x_3 \) free. A basis for the null space is therefore

\[
\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}
\]

(c) (Five points) Do the columns of \( A \) form a basis of \( \mathbb{R}^3 \)? Explain why or why not.

   **Solution.** They do not. \( \mathbb{R}^3 \) is three-dimensional, but \( A \) has four columns. If you have more vectors than the dimension, the set is automatically going to be linearly dependent. As a basis is by definition linearly independent, the columns can’t comprise a basis. (A subset of the columns do form a basis, of course, since any spanning set can be pruned down to a basis.)
2. Let $A$ be the $3 \times 3$ matrix

$$A = \begin{bmatrix} 4 & -2 & -1 \\ 3 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$ 

I’ll tell you that the characteristic polynomial for $A$ is $(1 - \lambda)(2 - \lambda)^2$.

(a) (Ten points) Find a basis for the 2-eigenspace of $A$.

Solution. $A - 2I$ can be row-reduced to the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Therefore a basis for the 2-eigenspace is $\{[1,1,0]\}$.

(b) (Five points) Is $A$ diagonalizable? Briefly explain why or why not. (I’m not asking you to diagonalize it.)

Solution. $A$ is not diagonalizable. To be diagonalizable, we’d need to be able to find three linearly independent eigenvectors for it. $\lambda = 1$ can only contribute one, and in the previous part we saw that $\lambda = 2$ also only contributes one. So we fall one dimension of eigenspace short.

(c) (Five points) Is $A$ similar to the matrix

$$B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Briefly explain why or why not.

Solution. If two matrices are similar, they’d have the same eigenvalues. Since $B$ is triangular, its eigenvalues are its diagonal entries: 3, 1, and 1. The eigenvalues of $A$ are 2, 2, and 1. That’s not a match.
3. Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the linear transformation whose standard matrix is
\[
A = \begin{bmatrix} 2 & -1 & 0 \\ 5 & -2 & 0 \\ -3 & 1 & 1 \end{bmatrix}.
\]

Let \( W \) be the plane in \( \mathbb{R}^3 \) spanned by \( \vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \) and \( \vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \).

(a) (Eight points) Come up with a basis for \( T(W) \), the image of \( W \). [Hint. You already have a basis for \( W \), use that somehow.]

\textbf{Solution.} A basis for \( T(W) \) will be \( \{ T(\vec{w}_1), T(\vec{w}_2) \} \). This works out to
\[
\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}
\]
after some matrix multiplication.

(b) (Twelve points) \( T \) is also a function between \( W \) and \( T(W) \). We can identify \( W \) with \( \mathbb{R}^2 \) using the basis \( \{ \vec{w}_1, \vec{w}_2 \} \) and we can identify \( T(W) \) with \( \mathbb{R}^2 \) using the basis from the previous part. Convince me that, in these bases, \( T \) is acting as a 90 degree rotation. [Hints. What \( 2 \times 2 \) matrix represents \( T \)? What’s the matrix of a 90 degree rotation?]

\textbf{Intended solution.} As a transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), the \( 2 \times 2 \) matrix that describes \( T \) should have as its columns the images of the vectors \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). In this case the domain \( \mathbb{R}^2 \) is the set of coordinate vectors for the basis \( \mathcal{B} = \{ \vec{w}_1, \vec{w}_2 \} \), so \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) corresponds to \( \vec{w}_1 \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) corresponds to \( \vec{w}_2 \). In the previous part we saw that \( T(\vec{w}_1) = \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( T(\vec{w}_2) = -\vec{w}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \). Thus the matrix is
\[
[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]
This is the matrix of a 90-degree counter-clockwise rotation,\(^1\) so that’s what \( T \) is doing to \( W \) in these bases.

\textbf{Mea culpa.} The problem statement should not say to use the basis for \( T(W) \) that came from the previous part, because if you do that then \( \vec{w}_1 \) goes to the first basis vector and \( \vec{w}_2 \) goes to the second basis vector, in which case the matrix for \( T \) should be the identity. Zero people out of 24 told me this, so I assume you all tried to answer the question as it was intended. The problem should have said to use \( \{ \vec{w}_1, \vec{w}_2 \} \) as the basis for both \( W \) and \( T(W) \). If you think your grade is adversely affected by this error, let’s talk during office hours.

\(^1\)A rotation of \( \theta \) radians counterclockwise has matrix
\[
\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.
\]
4. (Five points each) Each part of this problem is a question whose answer is “yes, always”, “sometimes yes but also sometimes no”, and “no, never”. Circle the correct choice of the three. No work is required for these problems: I’ll give full credit if you circle the correct choice. If you want to explain yourself with examples and/or counterexamples, they may be worth partial credit if your answer is incorrect.

(a) Suppose $V$ is a finite-dimensional vector space and $W$ is a subspace of $V$. Suppose further $\dim W = \dim V$. Does $V = W$?

Yes, always  Sometimes yes but sometimes no  No, never

Explanation. Suppose the common dimension is $d$. Then $W$ is the span of $d$ vectors. Those vectors are also in $V$ since $W$ is a subset of $V$. Those $d$ vectors are still linearly independent in $V$. But then that’s a linearly independent set in $V$ whose size is the same as the dimension, so that set is also a basis for $V$. Therefore $V$ and $W$ are the span of the same set and so are equal.

(b) Suppose $\vec{v}$ is an eigenvector for $A$ with eigenvalue 3 and $\vec{v}$ is an eigenvector for $B$, also with eigenvalue 3. Is $\vec{v}$ an eigenvector for $AB$ with eigenvalue 3?

Yes, always  Sometimes yes but sometimes no  No, never

Explanation. $AB\vec{v} = A(3\vec{v}) = 3(A\vec{v}) = 9\vec{v}$. The eigenvalue is necessarily 9, not 3.

(c) Suppose $\dim V = 4$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set inside $V$. Does there exist an $\vec{x} \in V$ such that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{x}\}$ is a basis for $V$?

Yes, always  Sometimes yes but sometimes no  No, never

Explanation. Just like how every spanning set can be pruned down to a basis, every linearly independent set can be extended to a basis.

(d) Suppose $A$ is an $n \times n$ matrix with $n$ distinct, real eigenvalues. Is $A$ invertible?

Yes, always  Sometimes yes but sometimes no  No, never

Explanation. If $\lambda = 0$ is one of those eigenvalues for $A$, then it is not, and otherwise it is. I never said whether or not zero was an eigenvalue.
5. (Ten points) If $A$ is a $3 \times 3$ matrix, then the kernel of the transformation $\vec{x} \mapsto A\vec{x}$ is a subspace of $\mathbb{R}^3$ and the image of the transformation $\vec{x} \mapsto A\vec{x}$ is a subspace of $\mathbb{R}^3$. For the moment, set aside the issue of whether these two copies of $\mathbb{R}^3$ are different or not—let’s treat them as “the same”, so that we can view the image and kernel as subspaces of the same space. Is it possible to choose $A$ so that the image and kernel wind up being the same subspace of $\mathbb{R}^3$? Why or why not?

**Solution.** The rank-nullity theorem says that the dimension of the kernel plus the dimension of the image has to be the dimension of the domain, 3 in this case. Our options for the dimensions of the kernel and the image are $(3,0)$, $(2,1)$, $(1,2)$, and $(0,3)$. In no case can we have the dimensions match, so there’s no hope for them ever to be the exact same space.

6. (Ten points) Let $V$ be the collection of all $2 \times 2$ matrices. With the usual operation of matrix addition and scalar multiplication, $V$ is a 4-dimensional vector space (you don’t need to check this). Let $W$ be the subset of upper-triangular matrices. Prove that $W$ is a subspace of $V$.

**Solution.** There are three things we need to check: is $W$ closed under addition, is it closed under scalar multiplication, and does it contain zero. For the first question, the zero matrix is upper-triangular, so it is in $W$. (Remember a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is upper triangular when $c = 0$.) If $A$ and $B$ are two matrices in $W$, then they’re both upper triangular, and their sum

$$A + B = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 & a_{22} + b_{22} \end{bmatrix}$$

is as well, so $A + B \in W$. Lastly, if $A \in W$ and $c$ is any scalar, then

$$cA = c \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ 0 & ca_{22} \end{bmatrix}$$

is again upper triangular, so $W$ is closed under scaling.