Write legibly and show all work. No partial credit can be given for an unjustified, incorrect answer. Put your name in the top right corner and sign the honor pledge at the end of the exam. If you need more room than what’s given, please continue onto the back.

**Scoring note.** Based on what is written in the syllabus, this exam is meant to account for 25% of the course. In order for one point here to be the same value as one point in other places this summer, this exam is out of 167 total.

1. Let $Q$ be the quadratic form $x_1^2 - 6x_1x_2 + x_2^2$.
   
   (a) (Six points) What symmetric matrix is associated to $Q$?
   
   (b) (Eleven points) Find a diagonal quadratic form that could arise from $Q$ by changing variables.

   (c) (Eleven points) Determine a change of variables $(x_1 = ay_1 + by_2, x_2 = cy_1 + dy_2)$ that will take $Q$ to the diagonal form you found in the previous part.

   (d) (Six points) Is $Q$ positive-definite? Negative-definite? Indefinite? What information determines this?
2. Let $\ell$ be the line spanned by $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the function which takes $\vec{x}$ to $\text{proj}_\ell(\vec{x})$.

(a) (Five points) Give a formula for $T$. [Your formula can have dot products in it.]

(b) (Ten points) Prove that $T$ is a linear transformation. [Hint. I’m hoping that you’ll use your formula from the last part in some way in your explanation.]

(c) (Eight points) Find the standard matrix for $T$. 

This problem continues on the next page.
(d) (Ten points) Let \( \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \). Find a nonzero vector orthogonal to \( \vec{v} \).

(e) (Seven points) Call the vector you found in the previous part \( \vec{w} \), and let \( \mathcal{B} = \{ \vec{v}, \vec{w} \} \) be a basis for \( \mathbb{R}^2 \). Find the matrix \([T]_\mathcal{B}\) of \( T \) in the basis \( \mathcal{B} \) from the previous part.
3. Below I’ve given you a $3 \times 5$ matrix $A$ as well as its reduced row-echelon form.

\[
A = \begin{bmatrix}
0 & 0 & 2 & 4 & 0 \\
2 & 2 & 0 & 6 & 4 \\
1 & 1 & 2 & 7 & 2
\end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix}
1 & 1 & 0 & 3 & 2 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(a) (Six points) What is the rank of $A$?

(b) (Six points) Let $T$ be the linear transformation $\mathbb{R}^3 \to \mathbb{R}^5$ given by $\vec{x} \mapsto A^T \vec{x}$. What is the dimension of the kernel of $T$?

(c) (Six points) Find a basis for the column space of $A$.

(d) (Fourteen points) Find an orthonormal basis for the column space of $A$. 

4. (Five points each) Each part of this problem is a question whose answer is “yes, always”, “sometimes yes but also sometimes no”, and “no, never”. Circle the correct choice of the three. No work is required for these problems: I’ll give full credit if you circle the correct choice. If you want to explain yourself with examples and/or counterexamples, they may be worth partial credit if your answer is incorrect.

There are six of these, unlike previous exams that only had four. Three are on the back.

(a) Suppose that $A$ and $B$ are both invertible $n \times n$ matrices. Is $AB$ invertible?

- Yes, always
- Sometimes yes but sometimes no
- No, never

(b) Let $A = \begin{bmatrix} 2 & a \\ 0 & 3 \end{bmatrix}$. I forgot what $a$ was but it’s a nonzero number. Despite my forgetfulness, the eigenvalues of $A$ are still 2 and 3. Are the 2-eigenvector for $A$ and the 3-eigenvector for $A$ are orthogonal to each other?

- Yes, always
- Sometimes yes but sometimes no
- No, never

(c) Suppose $A$ is a $4 \times 4$ matrix whose characteristic polynomial is $\lambda^3(\lambda - 1)$. Is the transformation $\vec{x} \mapsto A\vec{x}$ from $\mathbb{R}^4$ to $\mathbb{R}^4$ surjective?

- Yes, always
- Sometimes yes but sometimes no
- No, never
(d) Let $A$ be an $n \times n$ matrix and let $\vec{b} \in \mathbb{R}^n$ be any vector. Let $\vec{y}$ be a least-squares solution to the equation $A\vec{x} = \vec{b}$. Does $A\vec{y} = \vec{b}$?

Yes, always
Sometimes yes
but sometimes no
No, never

(e) Suppose $A$ is an orthogonal matrix. Is $A^2$ orthogonal?

Yes, always
Sometimes yes
but sometimes no
No, never

(f) Suppose that $S$ is a set consisting of $m$ vectors in $\mathbb{R}^n$, and that $m > n$. (There are more vectors than there are entries in any one of them.) Does $S$ span $\mathbb{R}^n$?

Yes, always
Sometimes yes
but sometimes no
No, never
5. (a) (Eight points) Let $S$ be a set of vectors in $\mathbb{R}^n$. Define what it means for $S$ to be an orthogonal set.

(b) (Eight points) Let $A$ be an $m \times n$ matrix. Define the null space of $A$. What vector space is the null space naturally a subspace of?

6. (Fifteen points) Suppose that $A$ is an $n \times n$ symmetric matrix. Show that $A$ “floats through dot products”, in that if $\vec{x}$ and $\vec{y}$ are vectors in $\mathbb{R}^n$, then $(A\vec{x}) \cdot \vec{y} = \vec{x} \cdot (A\vec{y})$. [The technical term for this is that $A$ is self-adjoint.]

Please copy the honor pledge below and sign your name next to it.
“I pledge on my honor that I have not given or received any unauthorized assistance on this examination.”