Write legibly and show all work. No partial credit can be given for an unjustified, incorrect answer. Put your name in the top right corner.

1. Let \( A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \), \( B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & -3 \end{bmatrix} \), \( \vec{v} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \), and \( \vec{w} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \).

   (One point each) Either calculate the following expressions, or say that they don’t make sense.

   (a) \( A\vec{v} \)
   (b) \( \vec{w}A \)
   (c) \( B\vec{v} \)
   (d) \( B\vec{w} \)

   **Solution.** The only one here that makes sense is (c). You can only multiply a \( m \times n \) matrix by an \( n \times 1 \) vector (in that order), so you can multiply \( B \) (which is \( 2 \times 3 \)) by \( \vec{v} \) (which is \( 3 \times 1 \)). The calculation works out to

   \[
   B\vec{v} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 + 9 + 20 \\ -2 - 3 - 15 \end{bmatrix} = \begin{bmatrix} 28 \\ -20 \end{bmatrix}.
   \]

2. (One point) Suppose that \( A \) is a \( 3 \times 4 \) matrix and its columns span \( \mathbb{R}^3 \). Write down a matrix that could be \( \text{rref}(A) \). [**Hint.** Not every entry of \( A \) is determined, so you’ll have to make some of them up.]

   **Solution.** As suggested, answers may vary, but the one thing that is crucial is that there must be a pivot in every row. So a couple of options for how to do it might look like

   \[
   \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}
   \]

   or

   \[
   \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
   \]

   or something along these lines.
3. (Five points) Let $M = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & -2 \end{bmatrix}$ and consider the homogeneous linear system represented by the matrix equation $M\vec{x} = \vec{0}$. Determine a set of vectors which spans the solution set of this equation.

**Solution.** The first two columns are pivot columns here, so $x_3$ and $x_4$ are both going to be free variables. Therefore we expect that we’re going to need two different vectors in order to span the solution space. Written out as equations, the system looks like

$$
\begin{align*}
  x_1 + 3x_4 &= 0 \\
  x_2 + 2x_3 - 2x_4 &= 0,
\end{align*}
$$

so that $x_1 = -3x_4$ and $x_2 = -2x_3 + 2x_4$. Therefore the generic solution vector is

$$
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_4 \\ -2x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix},
$$

or in other words the solution set is spanned by the two vectors $\begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$. 
