1. Let \( T : P_3 \to \mathbb{R}^2 \) be the function that takes a polynomial \( p(x) \) to the vector \( \begin{bmatrix} p(1) \\ p(-1) \end{bmatrix} \).

(a) (Four points) Show that \( T \) is a linear transformation.

**Solution.** There are two things we have to check. First, does \( T \) break up over addition? It does, because if \( p \) and \( q \) are two polynomials, then
\[
T(p+q) = \begin{bmatrix} (p+q)(1) \\ (p+q)(-1) \end{bmatrix} = \begin{bmatrix} p(1) + q(1) \\ p(-1) + q(-1) \end{bmatrix} = \begin{bmatrix} p(1) \\ p(-1) \end{bmatrix} + \begin{bmatrix} q(1) \\ q(-1) \end{bmatrix} = T(p) + T(q).
\]

Second, does \( T \) respect scaling? Again, yes, because if \( p \) is a polynomial and \( c \) is a scalar, then
\[
T(cp) = \begin{bmatrix} (cp)(1) \\ (cp)(-1) \end{bmatrix} = \begin{bmatrix} cp(1) \\ cp(-1) \end{bmatrix} = c \begin{bmatrix} p(1) \\ p(-1) \end{bmatrix} = cT(p).
\]

(b) (One point) Find a nonzero vector in the kernel of \( T \). (I don’t want a spanning set for the kernel. Just a single vector is fine.)

**Solution.** We’re looking for an element in the domain, \( P_3 \), which is taken to the zero vector in \( \mathbb{R}^2 \). Elements of \( P_3 \) are polynomials, and we’ll get \( \vec{0} \) exactly when \( p(1) = 0 \) and \( p(-1) = 0 \). Thus we want a polynomial which has roots at 1 and \(-1\). One possibility, then, is \( p(x) = (x-1)(x+1) = x^2 - 1 \). There are many other answers.

2. (Five points) Suppose \( A \) is an \( n \times n \) matrix. Consider the set
\[
W = \{ \vec{v} \in \mathbb{R}^n : A\vec{v} = 5\vec{v} \}.
\]
Show that \( W \) is a subspace of \( \mathbb{R}^n \).

**Solution.** The condition we have to look for is “does multiplication by \( A \) count as multiplication by 5?” With that in mind, there are three things to check. First, is the zero vector in \( W \)? Yes—it satisfies the condition, since
\[
A\vec{0} = \vec{0} = 5 \cdot \vec{0},
\]
so multiplication by \( A \) counts as multiplication by 5. Next, closure under addition. Suppose that \( \vec{v} \) and \( \vec{w} \) are in \( W \). Then is \( \vec{v} + \vec{w} \) in \( W \)? Well, we see what happens if we multiply by \( A \):
\[
A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = 5\vec{v} + 5\vec{w} = 5(\vec{v} + \vec{w})
\]
(the middle = is because \( \vec{v} \in W \) and \( \vec{w} \in W \)). So multiplication by \( A \) counts as multiplication by 5, and \( W \) is closed under addition. Finally, is it closed under scaling? Suppose \( \vec{v} \in W \) and \( c \) is any scalar. We get
\[
A(c\vec{v}) = c(A\vec{v}) = c(5\vec{v}) = 5(c\vec{v}),
\]
so again we could replace the \( A \) with a 5, completing the checks.