Write legibly and show all work. No partial credit can be given for an unjustified, incorrect answer. Put your name in the top right corner.

**Special instruction.** I decided that if I made you do both of these questions this quiz would take forever. So I ask that you only do one of the two of them. I’ll write up solutions to both.

1. (Zero or ten points) Diagonalize the matrix

\[ A = \begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}, \]

that is, find a diagonal matrix \( D \) and an invertible matrix \( P \) so that \( A = PDP^{-1} \).

**Solution.** The characteristic polynomial of this matrix is

\[ (-\lambda)(5 - \lambda) + 6 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3), \]

so the eigenvalues are 2 and 3. We now need eigenvectors for each of the eigenvalues. Those come from the null space of \( A - \lambda I \). For \( \lambda = 2 \), we get

\[ A - 2I = \begin{bmatrix} -2 & 3 \\ -2 & 3 \end{bmatrix} \]

which corresponds to the equations \(-2x_1 + 3x_2 = 0, x_2 \) free. We could use \( \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \) or \( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \) as our vector in the null space here. Next

\[ A - 3I = \begin{bmatrix} -3 & 3 \\ -2 & 2 \end{bmatrix} \]

which corresponds to \( x_1 = x_2 \) after dividing by three, so \( \begin{bmatrix} 1 \end{bmatrix} \) is the natural choice. The columns of \( P \) are the eigenvectors and \( D \) is the diagonal matrix of eigenvalues, so we get

\[ P = \begin{bmatrix} 3/2 & 1 \\ 1 & 1 \end{bmatrix} \]

and

\[ D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}. \]
2. (Zero or ten points) Let $T$ be the transformation $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose standard matrix is
\[
\begin{bmatrix}
-3 & -2 \\
8 & 5
\end{bmatrix},
\]
and let $B$ be the basis $\{ \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \}$ for $\mathbb{R}^2$. Determine the matrix $[T]_B$, the matrix for $T$ in the basis $B$. Put another way, fill in the question mark in the following diagram, where the top row is in the standard basis and the bottom row is in the basis $B$:

\[
\begin{array}{ccc}
\mathbb{R}^2 & \xrightarrow{\begin{bmatrix} -3 & -2 \\ 8 & 5 \end{bmatrix}} & \mathbb{R}^2 \\
\begin{bmatrix} \frac{1}{2} & 1 \end{bmatrix} & P^{-1} & P \begin{bmatrix} -1 & 1 \\ -1 \end{bmatrix} \\
\mathbb{R}^2 & \xrightarrow{?} & \mathbb{R}^2
\end{array}
\]

**Solution.** There are a couple of ways to do this. The most literal way is to say that the columns of $[T]_B$ are $T\vec{b}_1$ and $T\vec{b}_2$, each written in basis $B$. So since
\[
T\vec{b}_1 = \begin{bmatrix} -3 & -2 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix},
\]
and the right side of this equation in basis $B$ is $\begin{bmatrix} \frac{1}{2} \end{bmatrix}_B$, the first column of the matrix $[?]$ is going to be $\begin{bmatrix} \frac{1}{2} \end{bmatrix}$. Similar logic says that because
\[
T\vec{b}_1 = \begin{bmatrix} -3 & -2 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix},
\]
and $\begin{bmatrix} -1 \end{bmatrix} = 2\vec{b}_1 + \vec{b}_2$, the second column of the matrix is going to be $\begin{bmatrix} ? \end{bmatrix}$. Therefore
\[
[T]_B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.
\]

There is another way. The matrix $[?]$ is supposed to take us from the bottom left $\mathbb{R}^2$ in the diagram to the bottom right $\mathbb{R}^2$. The diagram offers us another way to get from the same place to the same place: we can go up (against the flow of the $P^{-1}$ arrow), then across (using the standard matrix), then down (against the flow of the $P$ arrow). Going against the flow is doing the inverse. So we’re saying first do $P$, then do $\begin{bmatrix} -3 & -2 \\ 8 & 5 \end{bmatrix}$, then do $P^{-1}$. Matrix multiplication is from right to left, so we get
\[
[?] = P^{-1}TP = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.
\]
If you do this matrix multiplication you again come to $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. 