No review sheet can cover everything that is potentially fair game for an exam, but I tried to hit on all of the topics with these questions, as well as show you some of the different kinds of things that I could ask.

1. Explain what it means for a matrix to be invertible. (I think this is basically the only new definition we have for this exam.)

   **Solution.** A square matrix (say $n \times n$) $A$ is invertible when there is another $n \times n$ matrix $C$ which makes the equations $AC = I_n$ and $CA = I_n$ simultaneously hold.$^1$

2. Find the inverse of the matrix

   \[
   \begin{pmatrix}
   2 & 1 & 0 \\
   9 & 1 & 1 \\
   0 & -3 & 1
   \end{pmatrix}
   \]

   **Solution.** I’ll add in some more detail later, but the answer is

   \[
   \begin{pmatrix}
   -4 & 1 & -1 \\
   9 & -2 & 2 \\
   27 & -6 & 7
   \end{pmatrix}
   \]

3. Find a nonzero $2 \times 2$ matrix $B$ so that

   \[
   \begin{pmatrix}
   1 & 2 \\
   2 & 4
   \end{pmatrix}
   B = 0.
   \]

   **Solution.** There’s two ways I can think of to do this one. The first is the plug-and-chug method: Let’s just write $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, and then do the matrix multiplication:

   \[
   \begin{pmatrix}
   1 & 2 \\
   2 & 4
   \end{pmatrix}
   \begin{pmatrix}
   b_{11} & b_{12} \\
   b_{21} & b_{22}
   \end{pmatrix}
   = \begin{pmatrix}
   b_{11} + 2b_{21} & b_{12} + 2b_{22} \\
   2b_{11} + 4b_{21} & 2b_{12} + 4b_{22}
   \end{pmatrix}.
   \]

   Now set those entries equal to zero and see what equations we get. For instance, in the top corner we have $b_{11} + 2b_{21} = 0$, so $b_{11} = -2b_{21}$. It turns out that the bottom left entry gives the same condition. Similarly, both of the entries in the right column tell us that $b_{12} = -2b_{22}$. So okay, with nothing better to do let’s just throw some values out there, like $b_{21} = b_{22} = 1$. Then $b_{11} = b_{12} = -1$, and so our matrix $B$ is $\begin{pmatrix} -2 & -2 \\ -1 & 1 \end{pmatrix}$.

   Sure enough,

   \[
   \begin{pmatrix}
   1 & 2 \\
   2 & 4
   \end{pmatrix}
   \begin{pmatrix}
   -2 & -2 \\
   1 & 1
   \end{pmatrix}
   = \begin{pmatrix}
   0 & 0 \\
   0 & 0
   \end{pmatrix}.
   \]

   The second way produces the same matrix but involves a bit more forethought. Recall that the columns of a product $AB$ look like $A\vec{v}_1$, $A\vec{v}_2$, and so on. So if we can find a vector for which $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \vec{v} = \vec{0}$, then if we use that vector as every column of $B$, then we’ll get the zero matrix out of the product $AB$. If we try to solve that matrix equation we’re lead to the augmented matrix

   \[
   \begin{pmatrix}
   1 & 2 & 0 \\
   2 & 4 & 0
   \end{pmatrix}
   \]

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$^1$There are other things equivalent to being invertible—for instance, the map $\vec{x} \mapsto A\vec{x}$ must be injective and surjective, or $A$ must have a pivot in each row and each column, etc.. But the meaning of invertibility is originally that the matrix has an inverse.
and consequently the condition \( x_1 = -2x_2 \) that we had before. \( x_2 \) is a free variable, so setting it equal to 1 recovers the vector \( \vec{v} = [-2, 1] \) and thus the answer from the previous paragraph.

4. (a) Find the \( LU \)-factorization of the matrix

\[
A = \begin{bmatrix}
-1 & 1 & 0 \\
-1 & 0 & 7 \\
2 & 7 & -1 \\
\end{bmatrix}.
\]

(b) Use your answer from part (a) to solve the system \( A\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} \).

**Solution.** I owe you more detail here, but so that the answer is available it’s

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
-2 & -9 & 1 \\
\end{bmatrix} \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 7 \\
0 & 0 & 62 \\
\end{bmatrix}.
\]

The vector equation has the solution \( \vec{x} = [-3, 1, 0] \).

5. Find the cofactor \( C_{43} \) for the matrix

\[
\begin{bmatrix}
2 & 1 & 3 & 0 \\
0 & 3 & 4 & 0 \\
1 & 0 & 1 & 2 \\
1 & 3 & 0 & -2 \\
\end{bmatrix}.
\]

**Solution.** You strike out the fourth row and third column, then multiply by \((-1)^{4+3}\) and take the determinant. So you get

\[
-\det \begin{bmatrix}
2 & 1 & 0 \\
0 & 3 & 0 \\
1 & 0 & 2 \\
\end{bmatrix} = -2 \det \begin{bmatrix}
2 & 1 \\
0 & 3 \\
\end{bmatrix} = -12,
\]

where the first step is what becomes of doing the determinant by performing a cofactor expansion down the third column.

6. Write down a \( 4 \times 4 \) elementary matrix that represents each of the following row operations:

(a) Replace \( R_3 \) with \( R_3 - 3R_1 \).
(b) Scale \( R_2 \) by a factor of 3.
(c) Swap rows 1 and 4.
(d) Replace \( R_1 \) with \( 4R_4 + R_1 \).

**Solution.** Man, I hope I get these right...
(a) \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-3 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]. We replaced R3, so the action is in the third row.

(b) \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\].

(c) \[
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\].

(d) \[
\begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\].

7. Find the volume of the parallelepiped in $\mathbb{R}^3$ whose sides are determined by the vectors $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

**Solution.** Since we’re given the sides of the parallelepiped as vectors, the answer will be the absolute value of the determinant of the matrix with those vectors as columns. I’ll do a cofactor expansion across the first row:

$$
\begin{vmatrix}
2 & 2 & 0 \\
0 & 3 & 2 \\
1 & 0 & 2
\end{vmatrix} = 2 \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} - 2 \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = |12 + 2| = |14|.
$$

8. Find the area of the parallelogram in $\mathbb{R}^2$ whose vertices are (3, 1), (7, 4), (4, 10), and (8, 13). *(Hint. We don’t have any vectors in this problem yet. You have to express the sides of the parallelogram as vectors before you can use any formula.)*

**Solution.** As the hint says, we don’t have vectors yet, but we can get them. The bottom of the parallelogram is the side that goes from (3, 1) to (7, 4). That’s a displacement of 4 in the $x$ direction and 3 in the $y$ direction, hence is represented with the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Similarly the left edge of the parallelogram is the side that goes from (3, 1) to (4, 10). Here we change $x$ up by one and $y$ up by nine, so the vector expressing that side is $\begin{bmatrix} 1 \\ 9 \end{bmatrix}$. Now we can answer this question just like the previous one:

$$
\text{area} = \left| \begin{vmatrix} 1 & 4 \\ 9 & 3 \end{vmatrix} \right| = |3 - 36| = 33.
$$
9. Each part of this problem is a question whose answer is “yes, always”, “sometimes yes but also sometimes no”, and “no, never”. Circle the correct choice of the three.

(a) Let \( A \) and \( B \) both be \( n \times n \) invertible matrices. Does \( (AB)^T = (A^T)(B^T) \)?

Yes, always

Sometimes yes

but sometimes no

No, never

Explanation. Both operations that we’re doing here—transpose and inverse—will reverse the order of multiplication, so we switch them twice and \( A \) winds up on the left.

(b) Suppose \( A \) is a square matrix, \( \det A = 0 \), and that the equation \( A\vec{x} = \vec{b} \) has a solution. Is that solution unique?

Yes, always

Sometimes yes

but sometimes no

No, never

Explanation. If \( \det A = 0 \), then \( A \) is not injective, so the equation \( A\vec{x} = \vec{0} \) has infinitely many solutions. If \( \vec{u} \) is one solution to \( A\vec{x} = \vec{b} \), then adding to \( \vec{u} \) any of the infinitely many solutions to \( A\vec{x} = \vec{0} \) will produce a new solution.

(c) Suppose \( A \) is an \( m \times n \) matrix with a zero row, and \( B \) is an \( n \times p \) matrix. Does the product \( AB \) have a zero column?

Yes, always

Sometimes yes

but sometimes no

No, never

Explanation. I mean, sure, maybe \( B \) is the zero matrix and then \( AB \) has only zero columns. But in general for this to happen a miracle has to occur. Observe an example where both \( A \) and \( B \) are 2 \( \times \) 2:

\[
\begin{bmatrix}
0 & 0 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
7 & 10
\end{bmatrix}.
\]

What is true is that \( AB \) will have a zero row. (If you think about \( A \) as some sort of projection, then you can make sense of this fact geometrically.)

(d) If \( A \) and \( B \) are row equivalent, does \( \det A = \pm \det B \)?

Yes, always

Sometimes yes

but sometimes no

No, never

Explanation. Again, maybe \( A = B \) and then sure, \( \det A = + \det B \). But this is typically false. All invertible matrices are row-equivalent to each other, so if this were true then the determinant of everything would be the same.

10. Suppose that \( A \) and \( P \) are \( n \times n \) invertible matrices. Explain why \( PAP^{-1} \) is invertible.

Solution. There are a couple of ways to answer this one. One option is to note that we observed in class that if you multiply together invertible matrices, the resulting
product is invertible. Since $P$, $A$, and $P^{-1}$ are all invertible ($P^{-1}$ is invertible because its inverse is $P!$), then the expression $PAP^{-1}$ is invertible.

A second way is to say that since $A$ is invertible, $\det A \neq 0$, and since $P$ is invertible, $\det P \neq 0$. Then

$$\det(PAP^{-1}) = (\det P)(\det A)(\det P^{-1}) = \frac{(\det P)(\det A)}{(\det P)} = \det A \neq 0,$$

and when a matrix’s determinant is nonzero, it is invertible.

11. Suppose that $A$ is an $n \times n$ matrix and that for some $b \in \mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a unique solution. Is $A$ invertible?

**Solution.** Yes it is. Since the equation has a unique solution, there can’t be any free variables in the system. That means that there’s a pivot in every column. Since $A$ is square, there’s also a pivot in every row. But then $\text{rref}(A) = I_n$, i.e., $A$ is invertible.

12. Let

$$J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$  

(So $J$ is $4 \times 4$.) Let $A$, $B$, $C$, and $D$ be $2 \times 2$ matrices. Lastly, let $M$ be the $4 \times 4$ block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$ 

(a) What is $M^T$? Express it as a block matrix where the blocks are in terms of $A$, $B$, $C$, and $D$.

(b) Calculate $M^TJM$.

(c) Write down a set of equations that $A$, $B$, $C$ and $D$ have to satisfy in order for it to satisfy $M^TJM = J$. (Such a matrix that satisfies this identity is called **symplectic**. They show up in later courses in linear algebra.)

**Solutions.** (a) To transpose the matrix, make the rows the columns and vice versa. Doing that switches $B$ and $C$ and also transposes all four of the blocks. (This might take writing down a $4 \times 4$ matrix to see.) You get

$$M^T = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix}.$$ 

(b) Multiplying three things together involves multiplying together the first two, then
multiplying that product times the third one. You get

$$M^T J M = \begin{bmatrix}
A^T & C^T \\
B^T & D^T
\end{bmatrix}
\begin{bmatrix}
0 & I_2 \\
-I_2 & 0
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}$$

$$= \begin{bmatrix}
-C^T & A^T \\
-D^T & B^T
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}$$

$$= \begin{bmatrix}
-C^T A + A^T C & -C^T B + A^T D \\
-D^T A + B^T C & -D^T B + B^T D
\end{bmatrix}.$$ 

(c) The equations we get are: $A^T C = C^T A$, $A^T D = I_2 + C^T B$, $B^T C + I_2 = D^T A$, and $B^T D = D^T B$. (Often we assume that $A$, $B$, $C$, and $D$ have some relationships with their transposes so that we can make further simplifications, but this is far enough for now.)