No review sheet can cover everything that is potentially fair game for an exam, but I tried to hit on all of the topics with these questions, as well as show you some of the different kinds of things that I could ask.

0. I’ll use this first item to list some of the definitions you should be familiar with.

- You *don’t* have to know the axioms that make up a vector space.
- But you should know how to tell if a subset of a vector space is a subspace.
- You should know how to check if a function is a linear transformation.
- You should be able to define what is meant by the span of a set of vectors and what is meant by a set of vectors being linearly independent.
- You should know what the rank, row space, column space, and null space of a matrix are,
- and you should know what the image and kernel of a linear transformation are.
- You should know what a basis of a vector space is
- and what is meant by dimension.
- You should know what $B$-coordinate vectors are.
- You should know what eigenvalues, eigenvectors, and eigenspaces are.
- You should be able to explain what it means for two matrices to be similar,
- or for a matrix to be diagonalizable.
- You should know what the characteristic polynomial of a matrix is.

1. Let $A = \begin{bmatrix} -2 & 0 \\ 7 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ -3 & -5 \end{bmatrix}$.

   (a) Calculate the characteristic polynomials of $A$ and $B$.

   (b) Explain why the previous part is enough to tell you that $A$ and $B$ are similar.

**Solution.** (a) Each of the two has as its characteristic polynomial $(-2 - \lambda)(1 - \lambda)$.

(b) Since the characteristic polynomial has distinct roots, $A$ and $B$ are both diagonalizable. They are therefore each similar to the matrix $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$, and thus are similar to each other.
2. For this problem I’ve supplied you with $A$ and $\text{rref}(A)$.

$$A = \begin{bmatrix} 4 & 1 & -5 & 2 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & 5 & 3 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ 

(a) How big is the largest set of linearly independent vectors you can find within the row space of $A$?

(b) Write down a basis $\mathcal{B}$ for the row space of $A$. If $\mathcal{B}$ has fewer than four vectors in it, determine vectors that you can append to the end of $\mathcal{B}$ in order to form a basis for $\mathbb{R}^4$. [So if $\mathcal{B}$ has one vector, you need to add in three more. If it has two vectors, you need to add in two more. If it has three vectors, you have to add in one more.]

(c) The transpose of $A$ determines a transformation $\vec{x} \mapsto A^T \vec{x}$ which goes from $\mathbb{R}^3$ to $\mathbb{R}^4$. Is that function injective? Is it surjective?

**Solutions.**

(a) This is asking for the dimension of the row space. There are three pivots, so that dimension is 3.

(b) One basis for the row space consists of the rows of $\text{rref}(A)$, i.e.,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ 

This is one vector short of being a basis for $\mathbb{R}^4$. One possible way to extend this set into a basis for $\mathbb{R}^4$ is to add in a vector that has a pivot in the missing horizontal position, i.e., the third spot—thus, you could choose $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Note that many many vectors work here; ultimately the thing that we need to have happen is that the matrix

$$\begin{bmatrix} 1 & 0 & 0 & | & \vec{v} \\ 0 & 1 & 0 & | & \vec{v} \\ -2 & 3 & 0 & | & \vec{v} \\ 0 & 0 & 1 & | & \vec{v} \end{bmatrix}$$

needs to be invertible.

(c) The transpose of $A$ has the same rank as $A$ does, since the rank is the number of pivots. (Transposing the matrix doesn’t change how many pivots there are.) So the rank of $A^T$ is three, meaning the column space is 3-dimensional. Since the dimension of the null space plus the dimension of the column space is the dimension of the domain (3 in this case), we see that the null space for $A^T$ has dimension zero. Therefore $\vec{x} \mapsto A^T \vec{x}$ is injective. The column space has dimension 3 but the target has dimension 4, so it cannot be surjective.
3. Each part of this problem is a question whose answer is “yes, always”, “sometimes yes but also sometimes no”, and “no, never”. Circle the correct choice of the three.

(a) Suppose the characteristic polynomial of $A$ is $(5 - \lambda)^3(1 - \lambda)(3 - \lambda)$. Is $A$ diagonalizable?

| Yes, always | Sometimes yes but sometimes no | No, never |

Explanation. It depends on the dimension of the 5-eigenspace. That dimension could be 1, 2, or 3. If it’s 3, then it will be diagonalizable.

(b) Suppose the characteristic polynomial of $A$ is $(5 - \lambda)^3(1 - \lambda)(3 - \lambda)$. Is $A$ invertible?

| Yes, always | Sometimes yes but sometimes no | No, never |

Explanation. In fact I can tell you $\det A = 5^3 \cdot 1 \cdot 3 = 375$, and that’s not zero. Alternatively, the null space is the 0-eigenspace, and since 0 isn’t an eigenvalue that space must have dimension zero. Hence the null space is $\{\vec{0}\}$ which means that $\vec{x} \mapsto A\vec{x}$ is injective, but injective plus square is invertible.

(c) Suppose $A$ is a $5 \times 8$ matrix and that the transformation $\vec{x} \mapsto A\vec{x}$ is surjective. Can the null space of $A$ be four-dimensional?

| Yes, always | Sometimes yes but sometimes no | No, never |

Explanation. Since the transformation is surjective, the column space of $A$ (the image of the transformation) has dimension 5. But then the null space has dimension 3 (they have to add up to eight, the dimension of the domain).

(d) Suppose $\{f_1, f_2, f_3\}$ is a basis for a vector space $W$. Is $\{f_1, f_1 + f_2, f_1 + f_2 + f_3\}$ a basis?

| Yes, always | Sometimes yes but sometimes no | No, never |

Explanation. Since $\{f_1, f_2, f_3\}$ is a basis, we know that $W$ is 3-dimensional. So any set of size 3 is a basis as long as it is linearly independent. We can check if $\{f_1, f_1 + f_2, f_1 + f_2 + f_3\}$ is linearly independent by trying to solve the equation

$$c_1(f_1) + c_2(f_1 + f_2) + c_3(f_1 + f_2 + f_3) = 0.$$ 

If they are linearly independent then the only solution will be $c_1 = c_2 = c_3 = 0$. Rearranging,

$$(c_1 + c_2 + c_3)f_1 + (c_1 + c_2)f_2 + c_3f_3 = 0,$$

and since the $f$-s are linearly independent, the only possible way you can take a linear combination of them and get zero is if all the weights are zero. Therefore all the coefficients in this second centered equation are zero, i.e., $c_1 + c_2 + c_3 = 0$, $c_1 + c_2 = 0$, and $c_3 = 0$. These equations “cascade” and we see that all the $c$-s are zero, meaning that our set is linearly independent and therefore is a basis.
(e) Suppose $A$ and $B$ are two $2 \times 2$ matrices and that there is a basis $B = \{ \vec{b}_1, \vec{b}_2 \}$ with the property that $[A]_B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $[B]_B = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$. Is $2$ an eigenvalue of $A + B$?

Yes, always

Sometimes yes but sometimes no

No, never

Explanation. If $P$ is the matrix whose columns are $\vec{b}_1$ and $\vec{b}_2$, then $A = P [2 1 \choose 0 3] P^{-1}$ and $B = P [2 -1 \choose 0 4] P^{-1}$. But then

$$A + B = P \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} P^{-1} + P \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} P^{-1}$$

$$= P \left( \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} \right) P^{-1}$$

$$= P \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix} P^{-1}.$$}

This shows that $A + B$ is similar to $[4 0 \choose 0 7]$, and so has eigenvalues $4$ and $7$.

4. Find an invertible matrix $P$ and a diagonal matrix $D$ so that

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} = PDP^{-1}.$$}

Solution. The eigenvalues of $A$ are $1$, $1$, and $2$. The $1$-eigenspace is the null space of

$$A - 1I = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

which has $x_1$ and $x_3$ free, and $x_2 = x_3$. So a basis for the $1$-eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. For $\lambda = 2$, there is one dimension of eigenspace, and one possible choice of basis for the $2$-eigenspace winds up being $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Therefore we get

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$}

5. Let $P_3$ be the space of polynomials of degree at most $3$ and let $T : P_3 \to \mathbb{R}^1$ be the function that takes $f(x) \in P_3$ to the $1 \times 1$ vector $[f(4)]$. What is the dimension of the kernel of $T$?

Solution. This map is surjective, since $\mathbb{R}^1$ is one-dimensional, spanned by $[1]$ (among other things), and $T(x-3) = [1]$. The dimension of the kernel plus the dimension of the image is the dimension of the domain, and $\dim P_3 = 4$. Since the map is surjective the image is the target which has dimension $1$. So the kernel is apparently $3$-dimensional.
6. Let \( B = \{ \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \} \) and \( C = \{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \} \) be two different bases for \( \mathbb{R}^2 \) besides the standard basis. I’ll write \( \mathbb{R}^2 \) to mean \( \mathbb{R}^2 \) with the standard basis, \( V \) to mean \( \mathbb{R}^2 \) with basis \( B \), and \( W \) to mean \( \mathbb{R}^2 \) with basis \( C \). What change of coordinates matrices describe the transformations \( a, b, c, d, e, \) and \( f \)?

**Solution.** The easiest two are \( f \) and \( b \), since those are the “decoding” matrices passing from \( B \) and \( C \) respectively back to the standard basis. Those matrices have as columns the constituents of the basis. Better yet, \( e \) and \( a \) are the inverses of \( f \) and \( b \) respectively. So we now know four of the six: the pair

\[
f = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix} \quad e = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}
\]

changes back and forth from \( B \), while the pair

\[
b = \begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix} \quad a = \begin{bmatrix} 3 & -\frac{1}{2} \\ -2 & \frac{1}{2} \end{bmatrix}
\]

does the same for \( C \).

There are two ways to get \( c \), which we also have called in the past \( P_{B \to C} \). One way is to row-reduce the matrix \( \begin{bmatrix} B \\ C \end{bmatrix} \) down to \( \begin{bmatrix} I_2 \\ c \end{bmatrix} \). The other way uses the fact that we have this diagram: \( c \) represents a way to get from \( W \) to \( V \), but so does first doing \( b \), then doing \( e \). So those two things have to be equal. Remember that as a product we have to write this in the other order, so we get

\[
c = eb = \begin{bmatrix} -2 & -4 \\ -7 & -13 \end{bmatrix} \quad d = af = c^{-1} = \begin{bmatrix} \frac{13}{7} & -2 \\ -\frac{7}{2} & 1 \end{bmatrix}.
\]